Derivation and solutions of some fractional Black–Scholes equations in coarse-grained space and time. Application to Merton’s optimal portfolio

Guy Jumarie *

Department of Mathematics, University of Québec at Montréal, P.O. Box 8888, Downtown Station, Montréal, Qc, H3C 3P8, Canada

ARTICLE INFO

Keywords:
Fractional stochastic differential equation
Fractional Brownian motion
Fractional Black–Scholes equation
Fractional Taylor’s series
Fractional Merton’s portfolio

ABSTRACT

By using the new fractional Taylor’s series of fractional order \( f(x+h) = E_{\alpha}(h^{\alpha}D_{x}^{\alpha})f(x) \) where \( E_{\alpha}(.) \) denotes the Mittag–Leffler function, and \( D_{x}^{\alpha} \) is the so-called modified Riemann–Liouville fractional derivative which we introduced recently to remove the effects of the non-zero initial value of the function under consideration, one can meaningfully consider a modeling of fractional stochastic differential equations as a fractional dynamics driven by a (usual) Gaussian white noise. One can then derive two new families of fractional Black–Scholes equations, and one shows how one can obtain their solutions. Merton’s optimal portfolio is once more considered and some new results are contributed, with respect to the modeling on one hand, and to the solution on the other hand. Finally, one makes some proposals to introduce real data and virtual data in the basic equation of stock exchange dynamics.

1. Introduction

It is by now taken for granted that the volatility of stock exchange variations can be suitably represented by a time variation of order \((dt)^{H}\) where \(H\), referred to as the Hurst exponent, is a real-valued parameter which fulfils the condition \(0 < H < 1\). The kinds of function so obtained which are defined everywhere but are nowhere differentiable necessarily exhibits random-like features in the sense that they are not reproducible, in such a manner that, as a result, in quite a natural way, one comes across fractional Brownian motion. In other words, one way to take account of large volatility in stock exchange market is to use a modeling via stochastic processes of fractional order.

Stochastic differential equation of fractional order, thought of as a generalization of Itô stochastic differential equation, involves a considerable amount of theoretical difficulties to define its solution, and recently, in an applied mathematical approach, recently we have suggested to rather consider a modeling in the form of non-random fractional dynamics subject to a standard (usual) Brownian motion. This has been made possible because we now have at hand a fractional calculus parallel with the standard one, and above all, we have a fractional Taylor’s series which appears to be a powerful tool to obtain new results. In the preceding paper, we outlined how one can derive new families of Black–Scholes equations, and our purpose in the following, is to expand this question, and to get new results on this equation.

The article is organized as follows. Since the Taylor’s series of fractional order is a recent result not yet largely diffused, we begin with a brief background, for the convenience of the reader (Section 2). Again for the sake of self-containing, we give a brief background on the Blach–Scholes equation and we shall take this opportunity to bring new points of view (Section 3). On duplicating this calculus, but in the fractional framework, we shall derive two families of fractional

* Tel.: +1 514 987 3000; fax: +1 514 987 8935.
E-mail address: jumarie.guy@uqam.ca.

© 2009 Elsevier Ltd. All rights reserved.

doi:10.1016/j.camwa.2009.05.015
Black–Scholes equations (Sections 4 and 5), and then we shall look for their solutions (Sections 6 and 7). Then we shall once more consider Merton’s model of optimal portfolio in the presence of fractal noises, we shall carefully make the difference between long-range dependence and short range noise, and we shall compare with previous results which we obtained on the same question (Sections 8 and 9). The last contributing Section 10, which deals with modeling in mathematical finance, displays some possible approaches to introduce real data and virtual data in these kinds of problem.

The present paper is the continuation of the Ref [1]. In this previous study, mainly we considered a new general modeling by using fractal dynamics subjected to (standard) white noise, and we obtained a fractional Black–Scholes equation ultimately as a consequence of this model. Here, we shall once more consider this problem and we shall obtain two different classes of Black–Scholes equation.

2. Background on fractional Taylor’s series

2.1. Fractional derivative revisited

In order to circumvent some drawbacks involved in the classical Riemann–Liouville definition of fractional derivative [2–5], we propose the following approach.

Fractional derivative via fractional difference

Definition 2.1. Let \( f: \mathbb{R} \to \mathbb{R}; x \to f(x) \), denote a continuous (but not necessarily differentiable!) function and let \( h > 0 \) denote a constant discretization span. Define the forward operator \( FW(h) \), i.e.

\[
FW(h)f(x) := f(x+h);
\]

then the fractional difference of order \( \alpha, 0 < \alpha \leq 1 \), of \( f(x) \) is defined by the expression

\[
\Delta^\alpha f(x) := (FW - 1)^\alpha f(x)
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f [x + (\alpha - k)h]. \tag{2.1}
\]

Lemma 2.1. The following equality holds,

\[
f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \tag{2.2}
\]

The proof can be obtained by using Laplace transform and Z-transform and then making \( h \) tend to zero. See for instance [6].

The direct consequence of this definition is that the fractional derivative of a constant is zero for any \( \alpha \) such that \( 0 < \alpha \leq 1 \). In addition, there is continuity of the definition around the value \( \alpha = 1 \), in other words making \( \alpha = 1 \) into (3.3) yields the classical derivative definition.

Definition 2.2. (Riemann–Liouville definition revisited). Refer to the function of Proposition 2.1.

(i) Assume that \( f(x) \) is a constant \( K \). Then its fractional derivative of order \( \alpha \) is

\[
D_x^\alpha K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}, \quad \alpha \leq 0,
\]

\[
= 0, \quad \alpha > 0. \tag{2.3}
\]

(ii) When \( f(x) \) is not a constant, then one will set

\[
f(x) = f(0) + (f(x) - f(0)),
\]

and its fractional derivative will be defined by the expression

\[
f^{(\alpha)}(x) = D_x^\alpha f(0) + D_x^\alpha (f(x) - f(0))
\]

in which, for negative \( \alpha \), one has

\[
D_x^\alpha (f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi) d\xi, \quad \alpha < 0. \tag{2.5}
\]
Assume that the continuous function $f$ satisfies the following conditions:

$$D_x^\alpha (f(x) - f(0)) = D_x^\alpha f(x)$$

$$= (f'^{(\alpha - 1)}(x))^\prime, \quad 0 < \alpha < 1,$$

$$= \frac{1}{\Gamma (1 - \alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} (f(\xi) - f(0)) \, d\xi.$$

When $n < \alpha \leq n + 1$, one will set

$$f^{(n)}(x) := \left(f^{(\alpha - n)}(x)\right)^{(n)}, \quad n < \alpha \leq n + 1, \quad n \geq 1. \quad \blacksquare$$

For further results on the development and application of this derivative, see [7–9]. For further complements on the history of fractional derivative, see for instance [3–5, 10–12].

We shall refer to this fractional derivative as to the modified Riemann Liouville derivative.

First of all, it is in order to point out that the Definitions 2.1 and 2.2 are quite equivalent. Moreover our definition and the so-called Caputo–Djrbashian’s definition [12, 10] yield the same result when the function is differentiable, but here we deal with the $\alpha$-th derivative only, $0 < \alpha \leq 1$, without referring to the derivative itself!

With this definition, the Laplace transform $L\{\cdot\}$ of the fractional derivative is

$$L \{f^{(\alpha)}(x)\} = s^\alpha L \{f(x)\} - s^{\alpha - 1} f(0), \quad 0 < \alpha \leq 1.$$  \hspace{1cm} (2.8)

### 2.2. Background on Taylor’s series of fractional order

**Main definition**

A generalized Taylor expansion of fractional order (F-Taylor series in the following) reads as follows

**Proposition 2.1.** Assume that the continuous function $f : \mathbb{R} \to \mathbb{R}, x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer $k$ and any $\alpha$, $0 < \alpha \leq 1$, then the following equality holds, which

$$f(x + h) = \sum_{k=0}^\infty \frac{h^{\alpha k}}{\Gamma (1 + \alpha k)} f^{(\alpha k)}(x), \quad 0 < \alpha \leq 1.$$  \hspace{1cm} (2.9)

where $f^{(\alpha k)}$ is the derivative of order $\alpha k$ of $f(x)$. \hspace{1cm} \blacksquare

With the notation

$$\Gamma (1 + \alpha k) =: (\alpha k)!,$

one has the formula

$$f(x + h) = \sum_{k=0}^\infty \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha \leq 1$$  \hspace{1cm} (2.10)

which looks like the classical one.

**Proof.** For the proof, see the reference [7,9]. \hspace{1cm} \blacksquare

**Corollary 2.1.** Assume that $m < \alpha < m + 1$, $m \in \mathbb{N} - \{0\}$ and that $f(x)$ has derivatives of order $k$ (integer), $1 \leq k \leq m$, then, integrating $m$ times the serial expansion

$$f^{(m)}(x + h) = \sum_{k=0}^\infty \frac{h^{k(\alpha - m)}}{\Gamma [1 + k(\alpha - m)]} D^{k(\alpha - m)} f^{(m)}(x), \quad m < \alpha \leq m + 1,$$

with respect to $h$ yields

$$f(x + h) = \sum_{k=0}^m \frac{h^k}{k!} f^{(k)}(x) + \sum_{k=1}^\infty \frac{h^{k(\beta + m)}}{\Gamma (k\beta + m + 1)} f^{(k\beta + m)}(x), \quad \beta := \alpha - m.$$  \hspace{1cm} (2.11)

In the special case when $m = 1$, one has the series

$$f(x + h) = f(x) + hf'(x) + \sum_{k=1}^\infty \frac{h^{k+1}}{\Gamma (k\beta + 2)} f^{(k\beta + 1)}(x), \quad \beta := \alpha - 1.$$  \hspace{1cm} (2.12)
Mc-Laurin series of fractional order
Let us make the substitution \( h \leftarrow x \) and \( x \leftarrow 0 \) into (2.11), we so obtain the fractional Mc-Laurin series
\[
\begin{align*}
\alpha 
\sum_{k=0} f^{(\alpha k)}(0), 
\end{align*}
\]
As a direct application of the fractional Taylor’s series, one has the following

**Useful relations**
The Eq. (2.11) provides the useful relation
\[
\Delta_0^\alpha f \equiv \Gamma(1 + \alpha) \Delta f, \quad 0 < \alpha < 1,
\]
or in a differential form \( d^\alpha f \equiv \Gamma(1 + \alpha) df \), between fractional difference and finite difference.

**Corollary 2.2.** The following equalities hold, which are
\[
\begin{align*}
D^\alpha x^\gamma &= \Gamma(\gamma + 1)\Gamma^{-1}(\gamma + 1 - \alpha)x^{\gamma - \alpha}, \quad \gamma > 0; \\
(u(x)v(x))^{(\alpha)} &= u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \\
(f[u(x)])^{(\alpha)} &= f_u^{(\alpha)}(u)u^{(\alpha)}(x), \\
&= f_u^{(\alpha)}(u)(u')^{\alpha}. 
\end{align*}
\]

**Proof.** For the proof, see the reference [13,14].

**Corollary 2.3.** Assume that \( f(x) \) and \( x(t) \) are two \( \mathbb{R} \rightarrow \mathbb{R} \) functions which both have derivatives of order \( \alpha, 0 < \alpha \leq 1 \), then one has the chain rule
\[
\begin{align*}
\alpha 
\left( 1 + \frac{1}{\alpha!} \right) \xi^{2\alpha} D_x^{2\alpha} + \frac{1}{(2\alpha)!} \xi^{2\alpha} D_x^{2\alpha} 
\end{align*}
\]
\[
\begin{align*}
\left( 1 + \frac{1}{\alpha!} \right) \eta^{2\alpha} D_y^{2\alpha} + \frac{1}{(2\alpha)!} \eta^{2\alpha} D_y^{2\alpha} 
\end{align*}
\]
\[
\begin{align*}
f(x, y) \quad \text{(2.19)}
\end{align*}
\]
\[
\begin{align*}
\text{(2.20)}
\end{align*}
\]
\[
\begin{align*}
\text{(2.21)}
\end{align*}
\]
so we obtain the fractional Mc-Laurin series
\[
\begin{align*}
f(x) = \sum_{k=0} \frac{x^{\alpha k}}{\Gamma(1 + \alpha k)} f^{(\alpha k)}(0), \quad 0 < \alpha \leq 1. 
\end{align*}
\]

**Corollary 2.2.** The following equalities hold, which are
\[
\begin{align*}
D^\alpha x^\gamma &= \Gamma(\gamma + 1)\Gamma^{-1}(\gamma + 1 - \alpha)x^{\gamma - \alpha}, \quad \gamma > 0; \\
(u(x)v(x))^{(\alpha)} &= u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \\
(f[u(x)])^{(\alpha)} &= f_u^{(\alpha)}(u)u^{(\alpha)}(x), \\
&= f_u^{(\alpha)}(u)(u')^{\alpha}. 
\end{align*}
\]

**Proof.** For the proof, see the reference [13,14].

This result which generalizes the Kolwankar fractional Rolles formula [15,16], is quite different from the Osler’s fractional Taylor’s series [5], because the latter involves the classical Riemann–Liouville definition. For further details on the differences between these two formulae, see [17].

**2.3. Multivariable fractional Taylor’s series**

This fractional Taylor’s series can be generalized in a straightforward way to multivariable functions, and for instance, for a two variables, one has the series
\[
\begin{align*}
f(x, y) &= E_\alpha(\xi^{2\alpha} D_x^{2\alpha})E(\eta^{2\alpha} D_y^{2\alpha})f(x, y). 
\end{align*}
\]
In the approximation of order \( 2\alpha, \) this series provides the equality
\[
\begin{align*}
f(x + \xi, y + \eta) \cong & \left( 1 + \frac{1}{\alpha!} \xi^{2\alpha} D_x^{2\alpha} + \frac{1}{(2\alpha)!} \xi^{2\alpha} D_x^{2\alpha} \right) \\
&\left( 1 + \frac{1}{\alpha!} \eta^{2\alpha} D_y^{2\alpha} + \frac{1}{(2\alpha)!} \eta^{2\alpha} D_y^{2\alpha} \right) f(x, y), 
\end{align*}
\]
or more explicitly,
\[
\begin{align*}
f(x + \xi, y + \eta) \cong f(x, y) + \frac{1}{\alpha!} \left( f_x^{(\alpha)}(x, y)\xi^{\alpha} + f_y^{(\alpha)}(x, y)\eta^{\alpha} \right) \\
&\quad + \frac{1}{(2\alpha)!} \left( f_x^{(2\alpha)}(x, y)\xi^{2\alpha} + f_y^{(2\alpha)}(x, y)\eta^{2\alpha} \right) \\
&\quad + \frac{1}{(\alpha!)^2}f_{xy}^{(2\alpha)}(x, y)\xi^{\alpha}\eta^{\alpha}. 
\end{align*}
\]

**2.4. Integration with respect to \( (dt)^\alpha \)**

The solution of the equation
\[
\begin{align*}
dx = f(t)(dt)^\alpha, \quad t \geq 0, \quad x(0) = x_0, \quad 0 < \alpha \leq 1
\end{align*}
\]
is defined by the following result:

**Lemma 2.2.** Let \( f(t) \) denote a continuous function, then the solution of the Eq. (2.21) is defined by the equality

\[
\int_0^t f(\tau) (d\tau)^\alpha = \alpha \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha \leq 1. \tag{2.22}
\]

Some examples

(i) On making \( f(\tau) = \tau^\gamma \) in (2.21) one obtains

\[
\int_0^t \tau^\gamma (d\tau)^\alpha = \frac{\Gamma(\alpha + 1) \Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha+\gamma}, \quad 0 < \alpha \leq 1. \tag{2.23}
\]

(ii) We shall need the following result: with the definition of modified Riemann–Liouville’s sderivative (and this point is of importance), the solution of the fractional differential equation

\[
x^{(\alpha)}(t) = \lambda x(t), \quad t \geq 0, \quad x(0) = x_0, \quad 0 < \alpha \leq 1,
\]

\( \lambda \) constant, is

\[
x(t) = x_0 E_\alpha(\lambda t^\alpha),
\]

where \( E_\alpha(\cdot) \) is the Mittag–Leffler function.

(ii) The solution of the fractional differential equation

\[
x^{(2\alpha)}(t) = \lambda x(t), \quad t \geq 0, \quad x(0) = x_0, \quad x^{(\alpha)}(0) = x^{(\alpha)}_0, \quad 1 < 2\alpha \leq 2,
\]

\( \lambda \) constant, is

\[
x(t) = AE_\alpha\left(\sqrt{\lambda} t^\alpha\right) + BE_\alpha\left(-\sqrt{\lambda} t^\alpha\right),
\]

where \( A \) and \( B \) denote two constants defined by the initial conditions on \( x(0) \) and \( x^{(\alpha)}(0) \).

2.5. **Cascaded combination of fractional derivatives**

On the order of the derivations in cascaded derivatives

Assume that we want to calculate \( D^{\alpha+\beta} f(x) \), \( 0 < \alpha, \beta < 1 \), by applying \( D^\alpha \) and \( D^\beta \) in any order. At first glance, one could use either \( D^\alpha D^\beta f(x) \) or \( D^\beta D^\alpha f(x) \), but the results so obtained are sensibly different, since then in terms of Laplace’s transform (see Eq. (2.7)) one has

\[
L\{D^\beta D^\alpha f(x)\} = s^{\alpha+\beta} F(s) - s^{\alpha+\beta-1} f(0) - s^{\beta-1} f^{(\alpha)}(0),
\]

\[
\text{and}
\]

\[
L\{D^\alpha D^\beta f(x)\} = s^{\alpha+\beta} F(s) - s^{\alpha+\beta-1} f(0) - s^{\alpha-1} f^{(\beta)}(0).
\]

The same problem occurs when \( \theta \), for instance, is a positive integer \( n \), and here again one has \( D^n D^\alpha f(x) \neq D^\alpha D^n f(x) \). For instance, when \( f(x) = x^2 \), \( n = 3 \) and \( \alpha = 0.5 \); one obtains

\[
D^{0.5} D^3(x^2) = 0
\]

and

\[
D^3 D^{0.5}(x^2) = KD^3(x^{1.5}) = -1.5(0.5)^2 K x^{-1.5},
\]

with \( K \) denoting a constant.

Once more, we are facing the same problem when we try to define \( D^\alpha f(x) \) with \( n < \alpha < n + 1 \), in which case we have to set either \( D^\alpha := D^n D^{\alpha-n} \) or \( D^\alpha := D^{\alpha-n} D^n \).

As a result, we have to choose a model, and we select the following one:

**Definition 2.3.** Principle of derivative increasing orders. The fractional derivative of fractional order \( D^{\alpha+\beta} \) expressed in terms of \( D^\alpha \) and \( D^\beta \) is defined by the equality

\[
D^{\alpha+\beta} f(x) := D^{\max(\alpha, \beta)} \left(D^{\min(\alpha, \beta)} f(x)\right). \tag{2.30}
\]
On doing so, we merely follows the practical rule in accordance of which we increase the derivation order rather than the opposite. Or again, we start from low order derivative to define large order derivative.

On the decomposition of fractional derivatives
Let be \( \alpha \) positive, and assume that \( 0 < 3\alpha < 1 \). There are two different manners to calculate \( D^{3\alpha} f(x) \). One can calculate \( D^{\alpha} D^{2\alpha} D^{\alpha} f(x) \) to obtain the Laplace’s transform

\[
L\{D^{\alpha} D^{2\alpha} D^{\alpha} f(x)\} = s^{3\alpha} F(s) - s^{3\alpha-1} f(0) - s^{2\alpha-1} f'(0) - s^{\alpha-1} f''(0).
\]

or else calculate \( D^{3\alpha} f(x) \) to obtain

\[
L\{D^{3\alpha} f(x)\} = s^{3\alpha} F(s) - s^{3\alpha-1} f(0),
\]

in such a manner that one will have

\[
D^{\alpha} D^{2\alpha} D^{\alpha} f(x) \neq D^{3\alpha} f(x), \quad 0 < 3\alpha < 1.
\]

For instance \( f(x) = x^{2\alpha} \) yields

\[
D^{\alpha} D^{2\alpha} D^{\alpha} (x^{2\alpha}) = 0
\]

and

\[
D^{3\alpha} (x^{2\alpha}) = \frac{\Gamma(1 + \alpha)}{\Gamma(1 - 2\alpha)} x^{-2\alpha}.
\]

This pitfall can be easily circumvented if we carefully define the framework. When the problem which we are dealing with involves \( D^{\alpha} \) as the basic derivative, then we shall necessarily refer to \( D^{\alpha} D^{2\alpha} D^{\alpha} \). Otherwise, if the smaller derivative so involved in the problem is \( D^{3\alpha} \), then we shall use the modified Riemann–Liouville expression for the later.

3. Stock exchange dynamics of fractional order

3.1. Consistency with fractional Taylor’s series prerequisite

The general model of fractional stochastic systems is usually selected in the form

\[
ds x = f(x, t) dt + g(x, t) db(t, \alpha),
\]

where \( b(t, \alpha) \) is a fractional Gaussian white noise with zero mean and the covariance function \([18–20]\)

\[
cov (b(t, \alpha), b(\tau, \alpha)) = \frac{\sigma^2}{2} (\tau^{2\alpha} + \tau^{2\alpha} - (t - \tau)^{2\alpha})
\]

with \( t \geq \tau > 0, \sigma > 0 \) and \( b(0, \alpha) = 0 \) p.s. An alternative is to represent the fractional Brownian motion by using the Maruyama’s notation

\[
 db(t, \alpha) = \sigma \, w(t)(dt)^\alpha
\]

to have

\[
ds x = f(x, t) + g(x, t) \, w(t)(dt)^\alpha,
\]

but then in such a case, the general form of the dynamical equation of stock exchange should be fully consistent with the fractional Taylor’s series.

Indeed assume that \( 0 < \alpha \leq 1 \). Then the fractional Taylor’s series of a given function \( x(t) \) is necessarily in the form

\[
ds x = a(t)(dt)^\alpha + b(t)(dt)^{2\alpha}, \quad 0 < \alpha \leq 1,
\]

or merely

\[
ds x = a(t)(dt)^\alpha,
\]

if we restrict ourselves to an accuracy of order \( 2\alpha \). This is the only approximation which is fully consistent with the fractional Taylor’s series. If instead we have \( 1 < \alpha \leq 2 \), then the fractional Taylor’s approximation reads

\[
ds x = a(t)dt + b(t)(dt)^\alpha.
\]

Our claim is that there should be consistency between the Eq. (3.2) on one hand, and the Eq. (3.3) together with (3.5) on the other hand. So, if we try to put in evidence this equivalence, then we come across the following models.
3.2. Wiener–Langevin model of fractional stock exchange

We come back to the basic standard dynamics (3.2) to state the following result.

**Lemma 3.1.** Given $\alpha$ such that $0 < \alpha \leq 1$, the equation

$$dx = \mu x dt + \sigma x w(t)(dt)^{\alpha},$$

is equivalent to the equation

$$x^{(\alpha)}(t) = \Gamma^{-1}(2 - \alpha) \mu x t^{1 - \alpha} + \Gamma(1 + \alpha) \sigma x w(t),$$

or in the differential form

$$d^\alpha x = \Gamma^{-1}(2 - \alpha) \mu x t^{1 - \alpha} (dt)^\alpha + \Gamma(1 + \alpha) \sigma x w(t)(dt)^\alpha, \quad 0 < \alpha \leq 1. \quad \Box$$

**Proof.** For the proof, see the Ref [1]. \Box

To summarize, we shall say that the standard model (3.6) is consistent with this new model defined by the Eq. (3.8), for any $\alpha$ such that $0 < \alpha \leq 1$.

Remark that if we set $\sigma = 0$ in (3.7), we come across the equation $x^{(\alpha)}(t) = \Gamma^{-1}(2 - 2\alpha) \mu x t^{1 - \alpha}$ which is exactly $\dot{x}(t) = \mu x(t)$!

3.3. Itô–Maruyama model of fractional stock exchange

Once more we come back to the Eq. (3.6) and we state the following result:

**Lemma 3.2.** Given $\alpha$, such that $0 < \alpha \leq 1$, the equation

$$dx = \mu x dt + \sigma x w(t)(dt)^{\alpha}$$

is equivalent to the equation

$$d^\alpha x = \Gamma^{-1}(2 - \alpha) \mu x t^{1 - \alpha} (dt)^\alpha + \alpha! \sigma x w(t)(dt)^{\alpha/2}, \quad 0 < \alpha \leq 1. \quad (3.10)$$

**Proof.** For the proof, see the Ref [1]. \Box

Further remarks and comments

(i) The Eq. (3.8) is the extension of the equation referred to as the Langevin’s equation in physics and which reads

$$\dot{x}(t) = \psi(x, t) dt + \psi(x, t) w(t),$$

whilst (3.10) should be rather considered in the wake of the Itô’s or Stratonovich modeling, clearly

$$dx = \psi(x, t) dt + \psi(x, t)(dt)^{1/2}.$$

(ii) These two models (3.8) and (3.10) yield the same dynamical equation

$$D^\alpha t E \{x(t)\} = \Gamma^{-2}(2 - \alpha) \mu t^{1 - \alpha} E \{x(t)\}$$

for the state average value, and the discrepancy appears only in the order of the disturbing noisy term.

(iii) Another difference between these models lies in their respective practical meaning. In the Eq. (3.8) the modeling of $x^{(\alpha)}(t)$ comprises the noisy term (see (3.7)) whilst in (3.10), according to the fractional Taylor’s expansion one would have simultaneously (formally, of course!)

$$x^{(\alpha/2)}(t) = (\alpha/2)! \sigma x(t) w(t)$$

and

$$x^{(\alpha)}(t) = \Gamma^{-1}(2 - \alpha) \mu x(t) t^{1 - \alpha}.$$
4. Background on Black–Scholes equation

4.1. Derivation of the equation

For the convenience of the reader, let us briefly bear in mind the essential of the famous Black and Scholes article. Loosely speaking the framework is as follows. The actual real value \( x(t) \) of a share is assumed to satisfy the stochastic differential equation

\[
dx = rx dt + \sigma x w(t) \sqrt{dt},
\]

where \( x \) is the price, \( r \) is the mean rate of increase, and \( \sigma \) is positive. This equation can be thought of as picturing the non-random dynamics \( dx = rx dt \) disturbed by the additive random noise \( \sigma x w(t) \sqrt{dt} \). An alternative is to assume that we are dealing with the observed dynamics \( dx = rt \) where \( t \) is the proper random time of the system defined by the equation \( dt = \sqrt{\sigma w(t)} \). On the stock market, for various reasons, for instance speculation, the value of this share is not \( x(t) \), but rather a function \( P(x, t) \) of \( x \). As we know it, the market may be over-valued, clearly \( P(x, t) \geq x \), or on the contrary under valued with \( P(x, t) \leq x \).

An investor who is mainly interested in saving for his own retirement will consider the matter as follows. At the instant zero, he pays \( P(x, 0) := P_0(x) \) to buy a share and he would like to have a criterion to determine at the instant \( t \) whether he gains or lose money. The simplest criterion is obtained by referring to the interest rate dynamics

\[
dP_t = rP(x, t) dt,
\]

where \( r \) here is the interest rate assumed to be constant; in such a manner that the limiting value of \( P(x, t) \) would satisfy the condition

\[
dP(x, t) = rP(x, t) dt.
\]

Under this assumption, deriving the partial differential equation of \( P(x, t) \) is straightforward. Indeed, the Itô’s lemma applied to (9.1) yields (the subscript hold for the derivative)

\[
dP = \left( P_t + rxP_x + \frac{1}{2} \sigma^2 x^2 P_{xx} \right) dt,
\]

and on equating to \( rP dt \) and dividing both sides by \( dt \), we obtain the PDE

\[
P_t = rP - rxP_x - \frac{1}{2} \sigma^2 x^2 P_{xx},
\]

with the initial condition \( P(x, 0) = P_0(x) \).

The Eq. (4.5) has been obtained by Black and Scholes [21] in the framework of call option, in which case \( P(x, t) \) denotes the value of the option as a function of the stock price \( x \) and time \( t \). Instead of initial condition, the PDE (4.5) then comes with a terminal condition on the magnitude of \( x \) compared with the exercise price. Clearly, if \( T \) is the maturity date of the option, the terminal condition will be

\[
P(x, T) = \begin{cases} x - c, & x \geq c \\ 0, & x < c \end{cases}
\]

(4.6)

4.2. Solution of the Black–Scholes equation

Black and Scholes proposed the solution of this Eq. (4.5) with the boundary condition (4.6) in the form

\[
P(x, t) = xN(d_1) - ce^{r(T-t)}N(d_2),
\]

with \( d_1 := \frac{\ln \frac{x}{c} + (r + \frac{1}{2} \sigma^2) (T-t)}{\sigma \sqrt{T-t}} \) \( \sqrt{T-t} \) and

\[
d_2 := \frac{\ln \frac{x}{c} + (r - \frac{1}{2} \sigma^2) (T-t)}{\sigma \sqrt{T-t}}
\]

where \( N(d) \) is the cumulative Laplace–Gaussian normal density function.
5. Derivation of two classes of fractional Black–Scholes equations

5.1. On the stochastic modeling of fractional dynamics

As we mentioned in the Section 4.1, in the presence of fractional white noise, the dynamics of the asset is usually assumed to be defined by the standard modeling parallel of (4.2), which now reads

\[ \frac{dx}{dt} = rx + \sigma w(t)(dt)^{\alpha/2}, \quad 0 < \alpha \leq 1. \] (5.1)

Unfortunately, as we pointed out, the exponents 1 and $\alpha/2$ of $dt$ in this equation are not consistent with fractional Taylor’s series, and to rectify this feature, we shall use the two conversion formulae

\[ \frac{d^\alpha x}{dx} = \frac{1}{(1-\alpha)!} x^{1-\alpha}, \quad 0 < \alpha \leq 1. \] (5.2)

and

\[ \frac{d^\alpha x}{dx} = \frac{x^{1-\alpha}}{\alpha!(1-\alpha)!} \] (5.3)

So, on applying (5.2) to the pair $(d^\alpha t, dt)$, we obtain

\[ \frac{dx}{t} = r(\alpha^{-1} x dt + \sigma w(t)(dt)^{\alpha/2}, \] (5.4)

and on applying (5.3) to the pair $(d^\alpha t, dt^{\alpha})$, we eventually have the modeling

\[ \frac{dx}{t} = \frac{r}{\alpha!(1-\alpha)!} x^{1-\alpha}(dt)^{\alpha} + \sigma w(t)(dt)^{\alpha/2} \] (5.5)

which is now fully consistent with the fractional Taylor’s series of order $\alpha/2$ of $x(t)$.

Notice that combining (5.2) and (5.3) yields the conversion formula

\[ \frac{d^\alpha x}{dx} = \frac{x^{1-\alpha}}{\alpha!(1-\alpha)!} \] (5.6)

which will be useful in the following.

5.2. Derivation of fractional Black–Scholes equations

First family of fractional B–S equations

We now generalize the above derivation to fractal processes, and according to our model (3.10), we select the dynamical equation of the stock value $x(t)$ in the form

\[ d^\alpha x = \hat{r} x t^{1-\alpha}(dt)^{\alpha} + \hat{\sigma} x w(t)(dt)^{\alpha/2}, \] (5.7)

with

\[ \hat{r} := \Gamma^{-1}(1-\alpha)r =: r/(1-\alpha)! \] (5.8)

and

\[ \hat{\sigma} := (\alpha!)\sigma, \] (5.9)

and where $w(t)$ is a normalized Gaussian white noise i.e. with zero mean and the unit variance. In addition, we denote by $r$ the interest rate, whilst $P(x, t)$ is the price of the stock option. Our problem is to determine the fractional partial differential equation which defines $P(x, t)$. To this end, we make the following assumption:

**Assumption H1.** We assume that the function $P(x, t)$ has a fractional derivative of order $\alpha$ with respect to $t$ and is twice differentiable with respect to $x$. ■

**Principle of the method.** By using the Eq. (5.7) combined with the Itô’s lemma, we shall obtain a first expression for the fractional differential $d^\alpha P$, and then later we shall equate it to the variational fractional increment $d^\alpha P$ provided by the equation

\[ dP = rP dt \] (5.10)

to obtain the result.

**Detail of the derivation.** (i) Multiplying both sides of (5.10) by $\alpha!$ yields

\[ d^\alpha P = \alpha! rP dt, \]
and combining with (5.6) which provides \( dt \) in terms of \((dt)\alpha\), we obtain
\[
d^{\alpha}P = \frac{rP}{(1-\alpha)!} t^{1-\alpha} (dt)^\alpha.
\] (5.11)

This being the case, for convenience, we shall refer to the equivalent form of (5.7), that is to say the usual model
\[
dx = rxdt + \sigma xw(t) (dt)^{\alpha/2}.
\] (5.12)

Under the Assumption H1, one has successively (the subscripts denote the derivatives)
\[
dP = (\alpha')^{-1}P_t^{(\alpha)} (dt)^\alpha + P_x dx + (1/2)P_{xx} (dx)^2
\]
\[
= (\alpha')^{-1}P_t^{(\alpha)} (dt)^\alpha + rxP_x dt + (1/2)\sigma^2 x^2 P_{xx} (dt)^\alpha.
\] (5.13)

As usual, we switch from \( dt \) to \((dt)\alpha\) by using (5.6) in such a manner that we can re-write \( dP \) in the form
\[
dP = \left( \frac{1}{\alpha!}P_t^{(\alpha)} + \frac{r}{\alpha!(1-\alpha)!} xt^{1-\alpha} P_x + \frac{\alpha^2}{2} x^2 P_{xx} \right) (dt)^\alpha.
\] (5.14)

We multiply both sides of this equation by \( \alpha' \) to obtain a first expression for \( d^{\alpha}P \), and we equate it to the \( d^{\alpha}P \) provided by (5.11) to have
\[
d^{\alpha}P = \left( P_t^{(\alpha)} + \frac{r}{(1-\alpha)!} xt^{1-\alpha} P_x + \frac{\alpha^2}{2} x^2 P_{xx} \right) (dt)^\alpha
\]
\[
= \frac{r}{(1-\alpha)!} t^{1-\alpha} P (dt)^\alpha,
\]
therefore the fractional B–S equation
\[
\frac{\partial^\alpha P}{\partial t^\alpha} = \left( \frac{rP - \frac{\partial P}{\partial x}}{(1-\alpha)!} \right) t^{1-\alpha} P_x + \frac{\alpha^2}{2} x^2 \frac{\partial^2 P}{\partial x^2}.
\] (5.15)

In the Black–Scholes problem of the pricing of options, we shall still have the boundary condition (4.6).

**Second family of fractional B–S equations**

Here, we make the following

**Assumption H2.** We assume that the function \( P(x, t) \) has a partial derivative of order \( \alpha \) with respect to \( t \) and partial derivative of orders \( \alpha \) and \( 2\alpha \) with respect to \( x \).

**Detail of the derivation.** Again, we refer to the Eqs. (5.10)–(5.12).

(i) This being the case, the fractional Taylor’s series of \( P(x, t) \) yields the equality (up to a remaining error term)
\[
dP(x, t) = (\alpha')^{-1}P_t^{(\alpha)} (dt)^\alpha + (\alpha')^{-1}P_x^{(\alpha)} (dx)^\alpha + ((2\alpha)!)^{-1} P_{xx}^{(\alpha)} (dx)^2,
\]
and once more using (5.3), we can re-write it in the form
\[
dP(x, t) = \frac{1}{\alpha!}P_t^{(\alpha)} (dt)^\alpha + \frac{1}{(1-\alpha)!} x^{\alpha-1} P_x^{(\alpha)} dx + \frac{(1-\alpha)!}{(2\alpha)!} x^{2\alpha-2} P_{xx}^{(\alpha)} (dx)^2.
\] (5.16)

(ii) According to the Itô’s lemma, we shall identify \( w^2(t) \) with its variance which is the unity, to write (drawn from (5.7))
\[
(d^{\alpha}x)^2 = (\alpha!)^2 \sigma^2 x^2 (dt)^\alpha,
\]
and on substituting into (5.16), we obtain the equality
\[
\alpha! dP = \left( P_t^{(\alpha)} + (1-\alpha)! \frac{\alpha!}{(2\alpha)!} x^{\alpha} P_x^{(\alpha)} + x^\alpha \frac{(1-\alpha)!}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \sigma^2 x^2 P_{xx}^{(\alpha)} \right) (dt)^\alpha.
\] (5.17)

(iii) But we have that \( \alpha! dP = d^{\alpha}P \), and on equating (5.17) to (5.11) and then dividing both sides by \((dt)^\alpha\), we eventually obtain the fractional Black–Scholes equation
\[
\frac{\partial^\alpha P}{\partial t^\alpha} = \left( \frac{r}{(1-\alpha)!} t^{\alpha-\alpha} \frac{\alpha!}{(2\alpha)!} P_x^{(\alpha)} + \frac{(\alpha!)^3}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} \right),
\]
or again
\[
\frac{\partial^\alpha P}{\partial t^\alpha} = \left( \frac{r}{(1-\alpha)!} P - \alpha \frac{\alpha!}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} \right) t^{\alpha-\alpha} - \frac{(\alpha!)^3}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \frac{(1-\alpha)!}{(2\alpha)!} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2}.
\] (5.18)

with \( 0 < \alpha \leq 1. \)
5.3. Further remarks and comments

On the meaning of these equations
The milestone of our modeling is the fractalization of the Eq. (5.10), i.e. the equation \( dP = rP \, dt \). Here, we used the Eq. (5.11) which reads

\[
P^{(\alpha)}(t) = r^{(1/2)}(2 - \alpha) r^{(1-\alpha)} P(t), \quad P(0) = P_0.
\]

(5.19)

(i) But at first glance, we could have as well generalized (5.10) in the form

\[
P^{(\alpha)}(t) = rP(t), \quad P(0) = P_0,
\]

(5.20)

and the question of course is to soundly determine which one is the most suitable for the fractal modeling.

Solving (5.19) yields

\[
\int_0^t \frac{d^\alpha P}{P} = \frac{r}{(1-\alpha)!} \int_0^t \tau^{1-\alpha} (d\tau)^\alpha,
\]

(5.21)

and by using the formula (see (2.23))

\[
\int_0^t \tau^\beta (d\tau)^\alpha = \frac{\alpha! \beta!}{(\alpha + \beta)!} t^{\alpha+\beta},
\]

(5.21) yields the solution

\[
\ln_a \left( \frac{P}{P_0} \right) = \alpha! r t,
\]

(5.22)

where \( \ln_a \) \( P \) is the inverse function of the Mittag–Leffler function \( E_\alpha(P) \). We eventually obtain

\[
P(t) = P_0E_\alpha(\alpha! r t).
\]

(5.23)

(ii) This being the case, the solution of (5.20) is

\[
P(t) = P_0E_\alpha(rt^\alpha),
\]

(5.24)

in other words, clearly, we have to select between (5.23) and (5.24). To this end, we shall refer to the continuity of the interest rate process as it is experienced on a practical standpoint. Indeed, if we start from the principle that we should approximate a continuous process by a continuous process, then we shall select the model ((5.19) and (5.23))

Comparison with previous modeling
As we mentioned it, Wyss [22] proposed a B–S equation in the form

\[
P^{(\alpha)}(x, t) = rP - rxP_x - (1/2)\sigma^2x^2 P_{xx},
\]

(5.25)

and we believe that it gives rise to some questions at least with respect to its practical meaning. Indeed, can we ascertain that it is fully meaningful?

On way to obtain this equation by using a calculation parallel with the above one (Section 5.2) is to start from the following equations: the fractional stochastic dynamics (5.12) which we re-write below for convenience,

\[
dx = rx \, dt + \sigma x \omega(t)(dt)^{\alpha/2},
\]

(5.26)

the fractional Taylor’s series

\[
dP = P_t^{(\alpha)}(dt)^{\alpha} + P_x dx + (1/2)P_{xx}(dx)^2,
\]

(5.27)

and the interest rate dynamical equation

\[
dP = rP(dt)^{\alpha}.
\]

(5.28)

Indeed, on substituting (5.26) into (5.27), we obtain the increment

\[
dP = \left( P_t^{(\alpha)} + rxP_x + (1/2)\sigma^2 P_{xx} \right) (dt)^{\alpha},
\]

and equating to \( rP(dt)^{\alpha} \) provides the Eq. (5.25).

There is absolutely no sound reason to reject the local Taylor’s series (5.27). Of course, our Taylor’s series involves the term \( (\alpha!)^{-1} P_t^{(\alpha)}(x, t) \) instead of \( P_t^{(\alpha)}(x, t) \), but on the surface, in the absence of any unified theory, we cannot disqualify (5.27).

The problem appears really with the Eq. (5.28) which is hard to ascribe with a practical meaning. This equation is meaningless to the bank which uses only \( dP = rP(dt) \) on the spot, and all we can do is to convert it into a continuous fractal dynamics, and it is this conversion which involved the term \( t^{1-\alpha} \) (see the above remarks on the fractalization of the interest rate process).
6. Solution of the fractional Black–Scholes equation, model 1

6.1. Derivation of a fractional heat equation

In this section we focus on the solution of the first B–S equation which we re-write below for convenience,

\[ p_t^{(\alpha)}(x, t) = (rP - rxP_x) \frac{t^{1-\alpha}}{(1 - \alpha)!} - \frac{a_1}{2} \sigma^2 x^2 P_{xx} \]  \hspace{1cm} (6.1)

with the terminal condition \( P(x, T) \) defined by (4.6).

Due to the presence of the fractional derivative with respect to time, the method for solution is slightly different from that of the B–S-equation, and we detail it below as we believe it could be of help for further problem in the area. In addition, we shall be in a position to more easily compare with the B–S-solution.

(Step 1) Deleting the \( rP \)-term.

We have to consider two different instances depending upon the differentiability of \( P(x, t) \).

If \( P(x, t) \) is differentiable w.r.t. time (in addition to be \( \alpha \)-th differentiable, of course), then we make the change of variable

\[ P(x, t) = e^{-r(T-t)} \tilde{P}(x, t), \]  \hspace{1cm} (6.2)

but if \( P(x, t) \) is not differentiable w.r.t. time, then we shall set

\[ P(x, t) = E_\alpha (-r(T-t)^\alpha) \tilde{P}(x, t). \]  \hspace{1cm} (6.3)

we notice that one has the equality

\[ p_t^{(\alpha)}(x, t) = (D_t^\alpha e^{-r(T-t)}) \tilde{P}(x, t) + e^{-r(T-t)} \tilde{p}_t^{(\alpha)}(x, t) \]
\[ = re^{-r(T-t)} \frac{t^{1-\alpha}}{(1 - \alpha)!} \tilde{P}(x, t) + e^{-r(T-t)} \tilde{p}_t^{(\alpha)}(x, t), \]

and on substituting into (6.1), we obtain the equation

\[ \tilde{p}_t^{(\alpha)}(x, t) = -rx \frac{t^{1-\alpha}}{(1 - \alpha)!} \tilde{P}_x(x, t) - \frac{a_1}{2} \sigma^2 x^2 \tilde{P}_{xx}(x, t). \]  \hspace{1cm} (6.4)

with the terminal condition

\[ \tilde{P}(x, T) = P(x, T). \]  \hspace{1cm} (6.5)

The same Eq. (6.4) is obtained with the transformation (6.3), by virtue of (2.24) and (2.25).

(Step 2) Derivation of a fractional PDE with constant coefficient. The presence of \( x \tilde{P}_x \) and \( x^2 \tilde{P}_{xx} \) in the Eq. (6.4) suggests to make the change of variable

\[ y := \ln x + a, \]  \hspace{1cm} (6.6)

where \( a \) denotes a constant, and to look for a solution in the form

\[ \tilde{P}(x, t) \equiv Q(y, t). \]  \hspace{1cm} (6.7)

with the terminal condition

\[ Q(y, T) = P(x, T) = P(e^{y-a}, T). \]  \hspace{1cm} (6.8)

And indeed, on substituting (6.6) into (6.4) yields

\[ \tilde{Q}_t^{(\alpha)}(y, t) + \left( \frac{y}{(1 - \alpha)!} \frac{t^{1-\alpha}}{r} - \frac{a_1}{2} \sigma^2 \right) \tilde{Q}_y(y, t) = 0. \]  \hspace{1cm} (6.9)

(Step 3) Solution of the first order fractional PDE. In order to be able to guess the general form of the solution of (6.9), we first consider the special case

\[ \tilde{Q}_t^{(\alpha)}(y, t) + \left( \frac{y}{(1 - \alpha)!} \frac{t^{1-\alpha}}{r} - \frac{a_1}{2} \sigma^2 \right) \tilde{Q}_y(y, t) = 0. \]  \hspace{1cm} (6.10)

and to obtain its solution, we apply the suitable extension of the Lagrange's characteristics method which we proposed recently for this kind of equations [17], and we consider the linear system associated with (6.10) which reads

\[ \frac{(dt)^{\alpha}}{1} = \frac{d^a y}{r \frac{t^{1-\alpha}}{(1 - \alpha)!} - \frac{a_1}{2} \sigma^2} = \frac{d \tilde{Q}}{0}. \]  \hspace{1cm} (6.11)
The equality on the right yields the first integral
\[ \tilde{Q}(y, t) = \text{constant}. \] (6.12)

The equality on the left, re-written in the form
\[ r \frac{t^{1-\alpha}}{(1-\alpha)!} (dt)^\alpha - \frac{\alpha^2}{2} \sigma^2 (dt)^\alpha = dy, \] (6.13)
provides the second integral
\[ y - rt + (1/2) \sigma^2 t^\alpha = \text{constant} \]
therefore the general solution
\[ \tilde{Q}(y, t) = \Phi \left( y - rt + \frac{1}{2} \sigma^2 t^\alpha \right). \] (6.14)

(Step 4) Derivation of a fractional heat equation. This result (6.14) suggests to look for \( Q(y, t) \) in the form
\[ Q(y, t) = R(u, t), \] (6.15)
with
\[ u := y - \ln c + r(T - t) - \frac{1}{2} \sigma^2 (T^\alpha - t^\alpha), \] (6.16)
clearly, the constant \( a \) in (6.6) is selected in the form
\[ a = -\ln c + rT - (1/2) \sigma^2 T^\alpha. \]

On substituting (6.14) into (6.9), we obtain the fractional heat equation
\[ R^{(\alpha)}_t(\xi, t) = -\rho^2 R_{uu}(u, t), \] (6.17)
with
\[ \rho^2 := \alpha! \sigma^2 / 2, \]
and the terminal condition
\[ R(u, T) = Q(y, T) = P(x, T), \] (6.18)
which provides
\[ R(u, T) = c \left( e^u - 1 \right). \] (6.19)

6.2. Solution of the fractional heat equation

Denoting by
\[ \hat{R}(\xi, t) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi u} R(u, t) du \]
the Fourier's transform of \( R(u, t) \), and taking the Fourier's transform of the Eq. (6.17), we come across the fractional differential equation
\[ \hat{R}^{(\alpha)}_t(\xi, t) = \rho^2 \xi^2 \hat{R}(\xi, t). \] (6.20)

The solution of the Eq. (6.20) which accounts for the terminal condition (6.19) is
\[ \hat{R}(\xi, t) = E_\alpha \left( -\rho^2 \xi^2 (T - t)^\alpha \right) \hat{R}(\xi, T), \] (6.21)
therefore the expression
\[ R(u, t) = \int_{-\infty}^{+\infty} \Psi(u - v), (T - t)R(v, T) dv, \] (6.22)
where \( \Psi(u) \) is defined by the expression
\[ \Psi(u, T - t) = \int_{-\infty}^{+\infty} e^{-i\xi u} E_\alpha \left( \xi^2 (T - t)^\alpha \right) d\xi. \] (6.23)
7. Solution of fractional Black–Scholes equation, model 2

We work exactly as with Model 1. 

(i) First, still on making the transformation (6.6), we obtain the equation

\[ \tilde{P}_1^{(\alpha)}(x, t) = -r x^{\alpha} \tilde{P}_x^{(\alpha)}(x, t) - \gamma(\alpha) \sigma^{2} x^{\alpha} \tilde{P}_{xx}(x, t), \tag{7.1} \]

with

\[ \gamma(\alpha) := \frac{(\alpha!)^{3}}{(2\alpha)!} \Gamma(1- \alpha) \Gamma(1-2\alpha). \tag{7.2} \]

(ii) Again, the variable transformation (6.6) yields

\[ Q_1^{(\alpha)}(y, t) = - \left( \gamma(\alpha) \frac{\Gamma(1- \alpha)}{\Gamma(1-2\alpha)} \sigma^{2} + rt^{1-\alpha} \right) Q_y(y, t) - \gamma(\alpha) \sigma^{2} Q_{yy}(y, t). \tag{7.3} \]

(iii) In order to guess the general form of \( Q(y, t) \) we shall refer to the equation

\[ Q_1^{(\alpha)}(y, t) + \left( \gamma(\alpha) \frac{\Gamma(1- \alpha)}{\Gamma(1-2\alpha)} \sigma^{2} + rt^{1-\alpha} \right) Q_y(y, t) = 0, \]

and by using the same technique as for the Eq. (6.10), we find that its solution is

\[ Q(y, t) = \Phi \left( \alpha! y - \alpha!(1- \alpha)! rt - \gamma(\alpha) \frac{\Gamma(1- \alpha)}{\Gamma(1-2\alpha)} \sigma^{2} t^{\alpha} \right). \]

This result suggests making the transformation

\[ u = y - \ln c - \alpha!(1- \alpha)! r(T-t) - \gamma(\alpha) \frac{\Gamma(1- \alpha)}{\Gamma(1-2\alpha)} \sigma^{2}(T^{\alpha} - t^{\alpha}), \tag{7.4} \]

with \( Q(y, t) = R(u, t) \) and the terminal condition (6.18) and (6.19). Here one has the Fourier's transform equation

\[ \hat{R}_1^{(\alpha)}(\xi, t) = \gamma(\alpha) \xi^{2} \sigma^{2} \hat{R}(\xi, t). \tag{7.5} \]

The solution of the Eq. (7.5) which accounts for the terminal condition (6.19) is

\[ \hat{R}(\xi, t) = E_{\alpha} \left( -\gamma(\alpha) \sigma^{2} \xi^{2} (T-t)^{\alpha} \right) \hat{R}(\xi, T), \tag{7.6} \]

therefore the expression

\[ R(u, t) = \int_{-\infty}^{+\infty} \Psi(u - v), (T-t)R(v, T)dv, \tag{7.7} \]

where \( \Psi(u) \) is defined by the expression

\[ \Psi(u, T-t) = \int_{-\infty}^{+\infty} e^{-i\xi u} E_{\alpha} \left( \sigma^{2} \xi^{2} (T-t)^{\alpha} \right) d\xi. \tag{7.8} \]

8. Application to fractional Merton’s portfolio

8.1. Preliminary background

Background on the classical Merton’s portfolio

The basic Merton’s model of optimal portfolio in a Black–Scholes market driven by fractional Brownian motion is defined by the dynamics [23]

\[ dx = [(1-u_1) r + \mu u_1] x dt + \sigma u_1 x w(t) (dt)^{\alpha} - u_2 dt, \quad x(0) = x_0 \quad \text{a.s.} \tag{8.1} \]

with the expected discounted utility, to be maximized,

\[ J = E \left\{ \int_{0}^{T} e^{-\rho t} u_2^p dt \right\}, \quad 0 < \gamma < 1. \tag{8.2} \]
\( x(t) \) is the wealth at time \( t \), \( u_1(t) \) with \( 0 \leq u_1 \leq 1 \), is the fraction of wealth invested in the risky asset at time \( t \), \( 1 - u_1 \) is the fraction of wealth invested in the risk free asset, \( u_2(t) \geq 0 \) is the consumption rate; \( \alpha, r, \mu \) and \( \sigma \) are three constants such that \( 0 < \alpha \leq 1, r < \mu, \sigma > 0; r \) is the interest rate, and \( w(t) \) is the normalized Gaussian white noise.

If we look for a closed loop control in the form
\[
    u_1(t) = u(t); \quad u_2(t) = v(t)x(t),
\]
then we have the new dynamics
\[
    dx = [(1 - u) r + (\mu u - v)] x \, dt + \sigma x w(t) \, (dt)^\alpha
    =: f(u, v) x \, dt + g(u) x w(t) \, (dt)^\alpha,
\]
with
\[
    f(u, v) := (1 - u)r + (\mu u - v); \quad g(u) := \sigma u(t),
\]
and the new utility function
\[
    J = E \left\{ \int_0^T e^{-\rho t} \gamma r \, x' \, d\tau \right\}.
\]

**Lemma 8.1.** Assume that \( 0 < 2\alpha < 1 \), then, with the definition of integral w.r.t \((dt)^\alpha\), which is expressed by the equality
\[
    \int_0^t x(\tau) \,(d\tau)^\alpha = \alpha \int_0^t (t - \tau)^{\alpha - 1} x(\tau) \, d\tau, \quad 0 < \alpha < 1,
\]
the following equality holds
\[
    \int_0^t x(\tau) \,(d\tau)^{2\alpha} = 2 \int_0^t (t - \tau)\alpha x(\tau) \,(d\tau)^{\alpha}. \quad \blacksquare
\]

**Proof.** We have only to calculate both sides of (8.8) by using (8.7) and to verify the results so obtained. \( \blacksquare \)

### 8.2. Fractional Merton’s portfolio. New results

According to the Section 3, in the general case when \( 0 < \alpha \leq 1 \), one can convert the Eq. (8.4) either into the form
\[
    x^{(\alpha)}(t) = \Gamma^{-1}(2 - \alpha) f(u, v) t^{1-\alpha} x + \alpha \Gamma(\alpha) g(u) x w(t), \quad 0 < \alpha \leq 1,
\]
which amounts to
\[
    d^\alpha x = \Gamma^{-1}(2 - \alpha) f(u, v) t^{1-\alpha} x (dt)^\alpha + \alpha \Gamma(\alpha) g(u) x w(t) (dt)^\alpha, \quad 0 < \alpha \leq 1,
\]
or into the form
\[
    d^\alpha x = \Gamma^{-1}(2 - \alpha) f(u, v) t^{1-\alpha} x (dt)^\alpha + \alpha \Gamma(\alpha) g(u) x w(t) (dt)^{\alpha/2}, \quad 0 < \alpha \leq 1.
\]

For solving the problem defined by the Eqs. (8.1) and (8.2), we have introduced \([23]\) the mathematical expectation
\[
    y(t) := E \left\{ x' \, (t) \right\}.
\]

so as to convert the initial stochastic problem into a non-random one with the state variable \( y(t) \). Here, we shall follow the same procedure, and in order to have the fractional differentials \( d^\alpha y(t) \) and \( d^{2\alpha} y(t) \), we shall calculate the mathematical expectation
\[
    d^\alpha E \left\{ x' \right\} = E \left\{ (x + d^\alpha x)^\gamma - x' \right\}
    = E \left\{ \gamma x^{\gamma - 1} d^\alpha x + 2^{-1} \gamma (\gamma - 1) x^{\gamma - 2} (d^\alpha x)^2 \right\},
\]
and on taking account of the expressions of \( d^\alpha x \) as provided by (8.9) and (8.10), we obtain the following equations:

**First model or Wiener–Langevin model**

As a result of the expression (8.9) of \( d^\alpha x \), one has the equations
\[
    d^\alpha y = \gamma \frac{ft^{1-\alpha}}{\Gamma(2 - \alpha)} y (dt)^\alpha + \gamma (\gamma - 1) \left( \frac{\alpha t^2}{2} + \frac{f^2 t^2 (1-\alpha)}{\Gamma^2(2 - \alpha)} \right) y (dt)^{2\alpha}, \quad 0 < \alpha \leq 1/2,
\]
\[
    d^{2\alpha} y = \gamma \frac{ft^{1-\alpha}}{\Gamma(2 - \alpha)} y (dt)^\alpha, \quad 1/2 < \alpha \leq 1,
\]
Let us recall that we switch from (8.13) to (8.14) by noticing that one can drop the \((dt)^{2\alpha}\)-term when \(1/2 < \alpha \leq 1\). The difference between (8.13) and (8.14) is due to the fact that when \(1/2 < \alpha \leq 1\), then one can drop the \((dt)^{2\alpha}\)-term.

**Second model or Itô–Maruyama model**

Here, we use \(d\alpha\) in the Eq. (8.10) to obtain

\[
d^{\alpha}y = \left( \frac{\gamma}{\Gamma(2 - \alpha)} ft^{1-\alpha} + \frac{\gamma'(\gamma - 1)}{2} \Gamma^2(1 + \alpha)g^2 \right) y(dt)^{\alpha} + \frac{\gamma'(\gamma - 1)}{2}\Gamma^2(2 - \alpha) y(2\alpha) y(dt)^{2\alpha}, \quad 0 < \alpha \leq 1/2, \tag{8.15}
\]

\[
d^{\alpha}y = \left( \frac{\gamma}{\Gamma(2 - \alpha)} ft^{1-\alpha} + \frac{\gamma'(\gamma - 1)}{2} \Gamma^2(1 + \alpha)g^2 \right) y(dt)^{\alpha}, \quad 1/2 < \alpha \leq 1. \tag{8.16}
\]

Here again, we switch from Eq. (8.15) to (8.16) by deleting the \((dt)^{2\alpha}\)-term which is not significant in the latter.

### 8.3. Further remarks and comments

(i) On making \(\alpha = 1/2\) in Eq. (8.14), we obtain the equation corresponding to the (usual) Brownian motion.

(ii) To summarize, the various models apply as follows:

- Eqs. (8.13) and (8.15) when \(0 < \alpha \leq 1/4\).
- Eqs. (8.13) and (8.15) when \(1/4 < \alpha \leq 1/2\).
- Eqs. (8.14) and (8.16) when \(1/2 < \alpha \leq 1\).

(iii) All the matter is related to the accuracy involved in the term \((dt)^{\alpha}\). It is clear that the Eq. (7.14) holds also when \(0 < \alpha \leq 1/4\), but then it is less accurate than (7.3).

(iv) It is well known that the fractal Brownian motion exhibits two very different behaviors according to whether \(\alpha \leq 1/2\), short range memory process, or \(1/2 < \alpha \leq 1\), long range memory process. Luckily we come across this value \(\alpha = 1/2\) here.

When \(\alpha > 1/2\), we have to refer to the Eq. (7.12), which is the simpler model, and on which the solution is

\[
y(t) = y(0) E_{\alpha} \left( \Gamma^{-1}(2 - \alpha) \int_0^t f(\tau) e^{1-\alpha}(d\tau)^{\alpha} \right). \tag{8.17}
\]

(v) So to solve our optimal portfolio problem, we shall have to consider each of the dynamics above, one at a time, and at first glance this could be done by directly duplicating our preceding approach [23].

(vi) The reader could be puzzled by the fact that we have several possible solutions to the same equation, but this should not be surprising at all, and on the contrary, is quite rightly so. The fractional Brownian motion is a function which is highly discontinuous, and as a result one is entitled to expect that the definition of the solution of the corresponding fractional differential equation is likely not to be unique. For instance, it is classical to have Itô’s solution and Stratonovich’s solution for the same given equation.

### 9. Optimal fractional Merton’s portfolio

#### 9.1. General equations of optimality

For the sake of simplicity, we shall introduce a new parameter \(\beta\) to re-write the dynamical equation of \(y\) in the form

\[
d^{\alpha}y = \gamma y \varphi(u, v, t)(dt)^{\alpha} + \beta \psi(u, v, t)(dt)^{2\alpha}. \tag{9.1}
\]

with

\[
\varphi(u, v, t) := \frac{\gamma}{(1 - \alpha)!} f(u, v) t^{1-\alpha} + \frac{\gamma'(\gamma - 1)}{2}(\alpha!)^2 g^2(u), \tag{9.2}
\]

and

\[
\psi(u, v, t) := \frac{\gamma'(\gamma - 1)}{2}\Gamma^2(2 - \alpha) f^2(u, v) t^{2(1-\alpha)}. \tag{9.3}
\]

When \(\beta = 0\), we deal with long-range dependence noises, whilst \(\beta = 1\) refers to short range dependence noise.

The utility function (8.6) now turn to be

\[
J = \int_0^T e^{-\rho \tau} v^y y d\tau, \tag{9.4}
\]

which we re-write in the form

\[
J = \int_0^T e^{-\rho \tau} v^y y \frac{t^{1-\alpha}}{\alpha!(1-\alpha)!}(d\tau)^{\alpha}
\]

\[
= \int_0^T v^y y h(\tau)(d\tau)^{\alpha}. \tag{9.5}
\]

with the obvious definition for \(h(\tau)\).
Introducing the Lagrange parameter function \( \lambda(t) \), the augmented Lagrangian \( J_a \) is

\[
J_a = \int_0^T v^\alpha yh(\tau)(d\tau)^\alpha + \lambda(\tau) \left( y\varphi(d\tau)^\alpha + \beta y\psi(d\tau)^{2\alpha} - d^\alpha y \right)
\]

\[
= -\alpha! [\lambda y]_0^T + \int_0^T \left( v^\alpha yh d^\alpha + \lambda y(\varphi d\tau^\alpha + \beta \psi d\tau^{2\alpha}) + yd^\alpha \lambda \right).
\]

Using the equality (8.8), we have as well,

\[
J_a = -\alpha! [\lambda y]_0^T + \int_0^T \left( v^\alpha yh d^\alpha + \lambda y\varphi(d\tau)^\alpha + 2\lambda \beta y(T - t)^\alpha \psi(d\tau)^\alpha + yd^\alpha \lambda \right)
\]

where \( H \) denotes the Hamiltonian

\[
H := v^\alpha y + \lambda y(\varphi + 2\beta(T - t)^\alpha \psi).
\]

The optimality conditions for \( J_a \) are obtained in writing that \( \delta J_a = 0 \), and they read

\[
\partial_u H = 0,
\]

\[
\partial_\lambda H = 0,
\]

\[
\lambda^{(\alpha)}(t) = -\partial_t H, \quad \lambda(T) = 0.
\]

Expliciting these equations yields

\[
\left( \gamma (1 - \alpha)^2 + 2\beta(\mu - r) - \frac{\gamma(\gamma - 1)}{\Gamma^2(2 - \alpha)}(T - t)^2 \right) u - 2\beta(\mu - r) - \frac{\gamma(\gamma - 1)}{\Gamma^2(2 - \alpha)}(T - t)^2 v
\]

\[
= e^{-\rho t}e^{-r(T-t)^\alpha} - 2\lambda \beta(\mu - r) - \frac{\gamma(\gamma - 1)}{\Gamma^2(2 - \alpha)}(T - t)^2 u + 2\lambda \beta - \frac{\gamma(\gamma - 1)}{\Gamma^2(2 - \alpha)}(T - t)^2 v = 0,
\]

(9.6)

\[
\lambda^{(\alpha)}(t) = -\lambda \left( \frac{\gamma}{(1 - \alpha)!} \right) (\mu - r)u + r - v) t^{1-\alpha}
\]

\[
+ \beta \gamma (\gamma - 1)(T - t)^\alpha \left( (\alpha!)^2 \sigma^2 u^2 + \frac{(\mu - r)u + r - v}{\Gamma^2(2 - \alpha)} t^{2(1-\alpha)} \right) \right) - e^{-\rho t} e^{t^{1-\alpha}/\alpha!} - \rho \right).
\]

(9.7)

9.2. Fractional noises with long range dependence: \( 1 < 2\alpha < 2 \)

In the presence of noise with long range dependence, one has \( \beta = 0 \), and the equations above turn to be

\[
u = \left( \lambda(t)e^{\alpha t} \right) \left( (\alpha!)^2 \sigma^2 u^2 + \frac{(\mu - r)u + r - v}{\Gamma^2(2 - \alpha)} t^{2(1-\alpha)} \right)
\]

(9.9)

\[
\lambda^{(\alpha)}(t) = -\lambda \left( \frac{\gamma}{(1 - \alpha)!} t^{1-\alpha} \left( r + \frac{(\mu - r)^2 t^{1-\alpha}}{(1 - \gamma)^2 (1 - \alpha)!^2} \right) \right)
\]

(9.10)

\[
\lambda^{(\alpha)}(t) = -\lambda \left( \frac{\gamma}{(1 - \alpha)!} t^{1-\alpha} \left( r + \frac{(\mu - r)^2 t^{1-\alpha}}{(1 - \gamma)^2 (1 - \alpha)!^2} \right) \right)
\]

(9.11)

where the Eq. (9.11) is obtained by combining (9.9) and (9.8).

In order to obtain the expression of \( \lambda(t) \), we introduce the new function

\[
A(t) := \lambda^{-1/(\gamma - 1)}
\]

which satisfies the differential equation

\[
(1 - \gamma) A^{(\alpha)}(t) = -\frac{\gamma}{(1 - \alpha)!} \left( r + \frac{(\mu - r)^2 t^{1-\alpha}}{(1 - \gamma)^2 (1 - \alpha)!^2} \right) A + \left( \gamma - \frac{1}{\alpha!} \right) t^{1-\alpha} e^{-\rho t}.
\]

(9.12)
The solution of this equation can be obtained as follows. Define the function $\chi(T, t)$ by the expression

$$
\chi_\alpha(T, t) := E_\alpha \left\{ \frac{\gamma}{(1-\gamma)(1-\alpha)!} \int_t^T \left( r + \frac{(\mu - r)^2 t^{1-\alpha}}{(1-\gamma)\sigma^2(1-\alpha)!}\alpha!^2 \right) (dr)^\alpha \right\}.
$$

(9.13)

then one has (see the Appendix)

$$
A(t) = A(T)\chi_\alpha(T, t) - \frac{\gamma - (\alpha!)^{-1}}{(1-\gamma)(1-\alpha)!} \int_t^T e^{-\frac{\rho t}{\tau}} t^{1-\alpha} \chi_\alpha(T, \tau)(dr)^\alpha.
$$

(9.14)

and since $A(T) = 0$, we eventually have

$$
A(t) = -\frac{\gamma - (\alpha!)^{-1}}{(1-\gamma)(1-\alpha)!} \int_t^T e^{-\frac{\rho t}{\tau}} t^{1-\alpha} \chi_\alpha(T, \tau)(dr)^\alpha.
$$

(9.15)

One has that $\alpha! < 1$ when $0 < \alpha < 1$, in such a manner that $\gamma - (\alpha!)^{-1}$ is positive on the same range of variation. But in the integrand of $A(t)$, the exponential is dominant so that this integrand is a decreasing function, and that, as a result, $A(t)$ is positive, and we can so meaningfully write

$$
v(t) = \left( -\frac{\gamma - (\alpha!)^{-1}}{(1-\gamma)(1-\alpha)!} \int_t^T e^{-\frac{\rho t}{\tau}} t^{1-\alpha} \chi_\alpha(T, \tau)(dr)^\alpha \right)^{-1} e^{-\frac{\rho t}{\tau}}.
$$

(9.16)

Further remarks and comments

The consumption vanishes at $t = T$, which is quite meaningful from a practical standpoint. (If $T$ is the time at which I shall die, I need not to consume at that time!). The result which, at first glance, may appear as disturbing is the fact that $u(T) = O(T^{1-\alpha})$ where $O(\cdot)$ denotes Landau’s symbol. Troublesome because, here again, if I have to die at time $T$, then I do not care to invest in a risky asset! Moreover, with such a rationale, we would not need to invest in the risk free asset as well! As a matter of fact, with this point of view, we would be dealing with an optimal control problem involving the terminal condition $x(T) = 0$. Instead, if we address the problem with a free terminal condition on $x(T)$, then there is absolutely no sound argument to necessarily have $u(T) = 0$. One way to ascribe a meaning to the condition $u(T) = 0$ is to say that it is an optimal strategy regarding the inheritance of the portfolio.

9.3. Fractional noise with short range dependence: $0 < 2\alpha < 1$

In this case, the complete optimality conditions are obtained in setting $\beta = 1$ in the Eqs. (9.6)–(9.8), and the result so appears quite untractable from a practical standpoint. At first glance, only a numerical computation could provide useful results. Nevertheless we can make the following comments.

First, when $t$ is close to $T$, the terms in $\beta(T - t)^\alpha$ can be dropped, and we so directly come across the solution corresponding to noises with long-range dependence.

Next, for small $t$ (close to zero), these Eqs. (9.6)–(9.8) reduce to

$$
\gamma(\gamma - 1)(\alpha!)^2\sigma^2 u + \frac{\gamma(\mu - r)}{(1-\alpha)!} t^{1-\alpha} = 0,
$$

$$
\frac{\gamma}{(1-\alpha)!} e^{-\rho t} t^{1-\alpha} \lambda^{-1} - \frac{\gamma}{(1-\alpha)!} t^{1-\alpha} \lambda = 0,
$$

$$
\chi_\alpha(t) = -\frac{\gamma}{(1-\alpha)!} \left( r + \frac{(\mu - r)^2 t^{1-\alpha}}{(1-\gamma)\sigma^2(1-\alpha)!}\alpha!^2 \right) t^{1-\alpha} + \left( \frac{1}{\alpha!} \right) e^{-\frac{\rho t}{\tau}} t^{1-\alpha} \lambda^{-\gamma},
$$

which are exactly the equations corresponding to long-range dependence.

So if we denote by $\Phi_\beta(t) \equiv (\gamma(t), \lambda(t))$ the solution with long-range dependence, then one can try to find an approximation in the form

$$
(\eta(t)(T - t)^\alpha) e^{-\eta} + \beta(t) t^\alpha e^{-\beta(T - t)} \Phi_\beta(t),
$$

where $\eta(t)$ and $\beta(t)$ could be determined, for instance, by minimizing a quadratic error term.

9.4. Comparison with results we previously obtained

In the reference [23] we started with the stock exchange dynamics

$$
dx = xf(u, v)dt + xg(u)w(t)(dt)^\alpha,
$$

which provides
\[ dy = yf(u, v)dt + \frac{\gamma(y - 1)}{2}g^2(u)yd\tau, \]
and we maximized the utility function (9.4), by introducing the Lagrange parameter function \( \lambda(t) \). To this end, we converted the initial (\( dt^\alpha, dt \))-problem into a (\( dt, dt \))-problem, by using the formula
\[ \frac{\gamma(y - 1)}{2} \int_0^T \lambda(t) \sigma^2 u^2 y dt = \frac{\gamma(y - 1)}{2} 2\alpha \int_0^T (T - t)^{2\alpha - 1} \sigma^2 u^2 y dt, \]
which holds for short-range dependence noise with \( 0 < 2\alpha < 1 \), and we so obtained the Hamiltonian
\[ H = e^{-\rho t} v' y + \lambda y [ (r - v) + (\mu - r) u + \alpha \sigma^2 (y - 1)(T - t)^{2\alpha - 1} u^2], \]
therefore we obtained the optimal solution
\[ u^* = \frac{\mu - r}{(1 - y)\sigma^2} \frac{(T - t)^{1 - 2\alpha}}{2\alpha}, \quad 0 < 2\alpha < 1, \quad (9.17) \]
\[ v^* = \left[ e^{\rho t} \lambda^* (t) \right]^{1/(y - 1)}. \quad (9.18) \]
\[ \lambda^* (t) = \left[ (1 - y) e^{-\psi(T - t)} \int_t^T e^{\rho t - \psi(T - t)} d\tau \right]^{-1/y}, \quad (9.19) \]
where \( \psi(T - t) \) and \( c \) are respectively defined by the expressions
\[ \psi(T - t) := \frac{r\gamma}{1 - \gamma} (T - t) + \frac{c \gamma}{1 - \gamma} (T - t)^{2(1 - \alpha)} \]
and
\[ c := \frac{(\mu - r)^2}{4\alpha(1 - y)\sigma^2}. \]

We can complete this result as follows. For long-range dependence, that is to say when \( 1 < 2\alpha < 2 \), the term in (\( dt^{2\alpha} \)) can be dropped, and the Hamiltonian turns to be
\[ H = e^{-\rho t} v' y + \lambda y [ (r - v) + (\mu - r) u]. \]
The formal expression of the optimal \( v^* (t) \), Eq. (9.18) still holds, but now the Hamiltonian is a linear function with respect to the control \( u(t) \), and the problem turns to be a linear programming.

As a matter of fact, we have two alternatives: convert the initial (\( dt^\alpha, dt \))-problem into either a (\( dt, dt \))-problem or into a (\( dt^\alpha, dt^\alpha \))-problem. In the first case, by using the formula (5.6), one would have the equation
\[ dy = yf(ydt + (1/2)\gamma(y - 1)(2\alpha)!/(1 - 2\alpha))g^2(u)y^{2\alpha - 1}dt, \]
in which, clearly, the coefficient of \( dt \) is not differentiable at \( t = 0 \), which gives rises to some problems regarding the definition of its solution. This pitfall is removed with the (\( dt^\alpha, dt^\alpha \))-model.

In other words, the preceding solution which we so proposed, should be thought of as an approximation. It is well known that the dynamics of fractal processes are very different depending upon whether they involve long-range dependence or short-range dependence [11,24], and fortunately we came across this property.

In the next section, we shall comment on the modeling of virtual data and real data in some problems of mathematical finance, and mainly we shall make some suggestions for further research.

10. On real data and virtual data in finance

10.1. Random time and proper time in finance

The dynamical equation of stock prices is usually selected in the form
\[ dx = rx dt + \sigma x db, \quad (10.1) \]
where \( x \) is the price, \( r \) is the mean rate of increase, \( \sigma \in \mathcal{R} \) is positive and \( b \) is a normalized Brownian motion. Some of the practical significances of this model are as follows:

(i) One may assume that the stock market is not driven by the time \( t \), but by its proper time \( \tau \) defined by the expression
\[ d\tau := dt + (\sigma/r)db, \quad (10.2) \]
to yield
\[ dx = r x dt. \] (10.3)

(ii) One can also re-write (10.1) in the form
\[ dx = (r + \sigma w(t)) x dt, \] (10.4)
where \( w(t) \) is a Gaussian white noise and hence assume that it is \( r \) (and not the proper time) which is affected by a random disturbance.

(ii) Another possible meaning is the following one. The actual value \( x_r \) of the equity is driven by the nominal reference equation
\[ dx_r = r x_r dt, \] (10.5)
and it is this \( dx_r \) which should be used by the investor to make any decision. Unfortunately, because of various factors (for instance he does not have at hand sound data about the enterprise) the increment which is observed is not \( dx_r \) but
\[ dx = dx_r + d\varepsilon \] (10.6)
where \( d\varepsilon \) is an error term.

In quite a natural way, one is led to assume that
\[ d\varepsilon = \sigma x db, \] (10.7)
\((x \text{ and not } x_r) \) to yield
\[ dx = rx_r dt + \sigma x db. \] (10.8)

Next, if we approximate \( x_r \) by \( x \), we eventually obtain the Eq. (10.1). In the following we shall try to get more insight in this approach. We shall refer to the Eq. (10.5) as to the reference stock exchange dynamics (RSED) equation, and to the Eq. (10.8) as to the observed stock exchange dynamics (OSED) equation. The Eq. (10.1) could be thought of as a virtual stock exchange dynamics (VSED) equation.

10.2. Virtual data in stock exchange dynamics. Model 1

**Derivation of the basic equation**

This model is defined by the Eqs. (10.5) and (10.8) which we re-write below on doing the substitution \((x_r \leftarrow x, \ x \leftarrow y)\)
\[ dx = rx dt, \] (10.9)
\[ dy = dx + \sigma y db. \] (10.10)

(i) Taking account of (10.9), the Eq. (10.10) can be re-written as
\[ dy = rx_0 \exp(rt) dt + \sigma y db. \] (10.11)

(ii) Itô's lemma yields
\[ dy = (y_t + (1/2)y_{bb}) dt + y_b db, \] (10.12)
and comparing with (10.11) we obtain
\[ y_b = \sigma y, \] (10.13)
\[ y_t + (1/2)y_{bb} = rx_0 \exp(rt). \] (10.14)

(iii) The Eq. (10.13) yields \( y_{bb} = \sigma y_b \) and on substituting into (10.14) we obtain
\[ y_t + (1/2)\sigma y_b = rx_0 \exp(rt). \] (10.15)

(iv) Looking for a solution in the form \( y = g(b) \exp(rt) \), we eventually obtain
\[ y = x_0 e^{rt} + (y_0 - x_0) e^{(r - 2\sigma b) t}. \] (10.16)

**Application to the Black–Scholes equation**

Let \( P(y, t) \) denote the price of an option at time \( t \). Classically Itô's lemma yields
\[ dP = (P_t + r x P_y + (1/2)\sigma^2 y^2 P_{yy}) dt + \sigma y P_y db, \] (10.17)
and on equating to
\[ dP = rP dt, \] (10.18)
we have the modified version of the Black–Scholes equation which reads

\[ \frac{\partial P}{\partial t} = rP - rX_0e^{rt} \frac{\partial P}{\partial y} - \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 P}{\partial y^2}. \quad (10.19) \]

Assuming that \( \sigma \) is a “small parameter” we can find a solution to (2.11) in the form

\[ P = P_0 + \sigma P_1 + \sigma^2 P_2 + \sigma^3 P_3 + \cdots. \]

On substituting into (10.19) and equating the exponent-like terms we obtain the set of first order PDE’s

\[ \partial_t P_k = rP_k + x \partial_y P_k, \quad k = 0, 1, \quad (10.20) \]

\[ \partial_t P_n = rP_n - x \partial_y P_n - (1/2)y^2 \partial^2_y P_{n-2}, \quad n \geq 2. \quad (10.21) \]

10.3. Virtual data in stock exchange dynamics. Model II

Derivation of the basic equation

The model is now defined by the equations

\[ dx = rx dt + \sigma x dB, \]
\[ dy = dx + \eta y d\beta. \quad (10.23) \]

\( x \) is the reference state of the system, \( y \) is the observed state of the system, \( \eta \) and \( \sigma \) are positive real-valued parameters, \( b \) and \( \beta \) are two normalized Brownian motions which are mutually independent.

(i) The well known solution to (10.22) is

\[ x = x_0 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma b \right). \quad (10.24) \]

(ii) Itô’s lemma yields

\[ dy = \left[ y_t + \frac{1}{2}(y_{bb} + y_{\beta\beta}) \right] dt + y_b dB + y_\beta d\beta. \quad (10.25) \]

and on equating to (10.23) we have the system

\[ y_t + \frac{1}{2}(y_{bb} + y_{\beta\beta}) = rx, \quad (10.26) \]
\[ y_b = \sigma x, \quad (10.27) \]
\[ y_\beta = \eta y. \quad (10.28) \]

(iii) (10.27) and (10.28) yield respectively (remark that according (10.22) one has \( x_b = \sigma x \))

\[ y_{bb} = \sigma^2 x, \quad (10.29) \]
\[ y_{\beta\beta} = \eta y, \quad (10.30) \]

and on substituting into (10.26), we get

\[ y_t + \frac{1}{2} \eta y_\beta = \left( r - \frac{\sigma^2}{2} \right) x. \quad (10.31) \]

(iv) On looking for a solution in the form

\[ y(t, b, \beta) = \tilde{y}(t, \beta) \exp \{ \sigma b \}, \]

we find that \( \tilde{y}(t, \beta) \) is defined by the equation

\[ \tilde{y}_t + \frac{1}{2} \eta \tilde{y}_\beta = \left( r - \frac{\sigma^2}{2} \right) x. \quad (10.32) \]

therefore

\[ y(t, b, \beta) = x(t, b) \left[ \left( \frac{y_0}{x_0} - 1 \right) e^{-\frac{1}{2} \left( r - \frac{\sigma^2}{2} \right) \beta} + 1 \right], \quad (10.33) \]

where \( x \) is defined by the Eq. (10.24).

Application to the Black–Scholes equation

Here, Itô’s lemma yields the variation

\[ dP(y, t) = P_t dt + r x P_y dt + (1/2) (\sigma^2 x^2 + \eta^2 y^2) P_{yy} dt, \quad (10.34) \]

and on equating to

\[ dP(y, t) = rP(y, t) dt, \]
we obtain
\[
\frac{\partial P}{\partial t} = rP - r\alpha P - \frac{1}{2}(\sigma^2 x^2 + \eta^2 y^2) \frac{\partial^2 P}{\partial y^2}.
\] (10.35)

Assume that \( \eta \) is a "small parameter" and let us look \( P \) in the form
\[
P = P_0 + \eta P_1 + \eta^2 P_2 + \eta^3 P_3 + \cdots.
\]

On substituting into (10.35) and identifying the exponent-like terms gets
\[
\frac{\partial t}{\partial P_k} = rP_k - r\alpha P - \frac{1}{2}(\sigma^2 x^2) \frac{\partial^2 P_k}{\partial y^2}, \quad k = 0, 1
\]
(10.36)
\[
\frac{\partial t}{\partial P_n} = rP_n - r\alpha P_n - \frac{1}{2}(\sigma^2 x^2) \frac{\partial^2 P_n}{\partial y^2} + \eta y^2 \frac{\partial^2 P_n}{\partial y^2}, \quad k \geq 2.
\]
(10.37)

In the discussion above, we restricted ourselves to (standard) fractional Brownian motion, but it is clear that they should be improved by introducing fractal noises.

11. Concluding remarks

We have suggested recently defining fractional stochastic systems [27] as non-random fractional dynamics subject to (standard) Brownian motion input, and here, in this framework, we have carried on our investigation in this topic. Fractional stochastic differential equations involve discontinuities of utmost grades, in such a manner that defining their solutions soundly is rather a controversial task. Even for the standard stochastic differential equation with standard Brownian motion \( w(t) \sqrt{dt} \) we already have two meaningful solutions, which are the Ito's definition and the Stratonovich's [25,26]. On the strict theoretical standpoint, there is no sound reason to prefer one model instead of the other one, and it is only later, that it became clear that Ito's definition is the more useful. Our contribution herein is two-fold.

First, we have carefully derived the Black–Scholes equation in the presence of fractal noises, putting in evidence two new families of such equations; and mainly, we have shown that it is not sufficient to replace the time-derivative by a fractal time-derivative in the classical equation to get the suitable result. And by using the Lagrange technique for solving linear partial differential equations as we extended it recently to fractal linear partial differential equations, we have been able to obtain the explicit expressions of the solutions of these Black–Scholes equations.

Our second contribution is related to the Merton's optimal portfolio, which we once more considered here, in this new framework defined by non-random fractional dynamics subject to Brownian motion. The solution can be easily obtained for long range dependence systems, whilst, at first glance, we need computerized techniques to grasp the solution in the case of short-range systems.

And finally, we made some suggestions about how we could deal with the concept of virtual or suggestive data in finance problems. But we stayed only at the level of suggestions, and there remains to investigate this topic for the future.

Appendix

A.1. Solution of fractional linear differential equation

The solution of the fractional differential equation
\[
y^{(\alpha)}(t) = a(t)y(t) + b(t), \quad y(T) = Y_T,
\]
(A.1)
can be obtained as follows.

(i) First of all, one seeks the solution of the homogeneous equation
\[
y^{(\alpha)}(t) = a(t)y(t), \quad y(T) = y_T,
\]
and to this end, we rewrite it in the form
\[
\frac{d\alpha}{y} = a(t)(dt)^\alpha,
\]
whereby one obtains (\( \ln_{\alpha} y \) denotes the inverse function of the Mittag–Leffler function of order \( \alpha \))
\[
\ln_{\alpha} y(t) = \ln_{\alpha} y(T) - \int_t^T a(\tau)(d\tau)^\alpha.
\]

Therefore
\[
y(t) = y(T)E_{\alpha} \left\{ -\int_t^T a(\tau)(d\tau)^\alpha \right\}.
\] (A.2)
(ii) Using the Lagrange technique of constant variation, we look for a special solution of the complete nonhomogeneous equation in the form

\[ y(t) = C(t)E_\alpha \left\{ - \int_{t}^{T} a(\tau)(d\tau)^\alpha \right\}. \]

On substituting into (A.1) we find that \( C(t) \) is provided by the equation

\[ C^{(\alpha)}(t) = b(t)E_\alpha^{-1} \left\{ - \int_{t}^{T} a(\tau)(d\tau)^\alpha \right\}, \]

therefore

\[ C(t) = C(T) - \frac{1}{\alpha!} \int_{t}^{T} b(\tau)E_\alpha^{-1} \left\{ - \int_{\tau}^{T} a(u)(du)^\alpha \right\} (d\tau)^\alpha. \]

(iii) On putting these results together, we obtain the general solution

\[ y(t) = y(T)E_\alpha \left\{ - \int_{t}^{T} a(\tau)(d\tau)^\alpha \right\} - \frac{1}{\alpha!} E_\alpha \left\{ - \int_{t}^{T} a(\tau)(d\tau)^\alpha \right\} \int_{t}^{T} b(\tau)E_\alpha^{-1} \left\{ - \int_{\tau}^{T} a(u)(du)^\alpha \right\} (d\tau)^\alpha. \]  \hspace{1cm} (A.3)

References