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The minimum upper bound on the first ambiguous power of an irreducible, nonpowerful ray or sign pattern

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ABSTRACT

Let A be an $n \times n$ irreducible ray or sign pattern matrix. If A is a sign pattern, it is shown that either A is powerful or else A^k has an ambiguous entry for some $k \leqslant n^2 - 2n + 2$, and further, sign patterns based on the Wielandt graph show that this bound is the best possible. If A is a ray pattern, partial results for the same bound are given.

1. Introduction

In [4], Li et al. conjectured that for an irreducible sign pattern A, if $A^{\ell(|A|)+2h(A)}$ contains no ambiguous entries, where $\ell(|A|)$ denotes the base of $|A|$ and where $h(A)$ denotes the index of imprimitivity of $|A|$, then A is powerful. In [6], You et al. extended the concepts of base and period to nonpowerful sign patterns. In particular, they proved that an $n \times n$ primitive, nonpowerful sign pattern A has base $\ell(A) = \min\{k : A^k = \#J\}$ where J is the matrix all of whose entries are 1, and they determined bounds on $\ell(A)$ in terms of n and the structure of the digraph. For imprimitive, nonpowerful sign patterns, they proved analogous results for the base, and proved that the period was the index of imprimitivity. In this paper, we investigate related questions for both sign patterns and ray patterns. We will show that if A is an $n \times n$ irreducible sign pattern that is not powerful, then A^k contains an ambiguous entry for some positive integer k with $k \leqslant n^2 - 2n + 2$. We also show that there is an $n \times n$ sign pattern associated with the Wielandt graph, for which the first ambiguous power is indeed $n^2 - 2n + 2$, and

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hence that the upper bound we give is, in fact, the minimum upper bound. In the case that A is a ray pattern, we determine certain cases in which the analogous results hold. The ray pattern associated with the Wielandt graph shows that any lower bound on k must be at least $n^2 - 2n + 2$.

Let G be a directed graph on n vertices. When A is a square matrix, $G = G(A)$ will denote the directed graph of A. A *walk* of *length* k in G is a sequence of directed edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1})$ from G. A *path* of *length* k is a walk of length k such that all of the v_j are distinct. A *circuit* of *length* k is a sequence of directed edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1})$ from G such that $v_{k+1} = v_1$. A *cycle* of length k is a circuit of length k such that $v_1, v_2, \ldots, v_k$ are all distinct. (Note that some authors use the terms cycle and simple cycle for what we call circuits and cycles, respectively.) For any walk W in G, the walk product, denoted $\wp(W)$, is the product of the entries from A whose index pairs are the coordinates of the edges in W. That is, if W consists of the edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1})$, then

$$\wp(W) = \prod_{j=1}^{k} a_{v_j, v_{j+1}}.$$

Given an $n \times n$ matrix A with entries in the set $\{-1, 0, 1\}$, the *sign pattern* corresponding to A is the class of all $n \times n$ real matrices of the form $A \circ X$ where $\circ$ denotes the hadamard product and where X ranges over all $n \times n$ entrywise positive, real matrices. Following standard practice, we will regard A as the canonical representative of the class, and call A the sign pattern. Similarly, given an $n \times n$ matrix A with entries in the set $\{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$, the *ray pattern* corresponding to A is the class of all $n \times n$ complex matrices of the form $A \circ X$ where X ranges over all $n \times n$ entrywise positive, real matrices. Again following standard practice, we will regard A as the canonical representative of the class, and call A the ray pattern. We adopt all of the standard conventions for sign patterns and ray patterns; see [1,2,4] or [5] for details. When an ambiguous entry occurs in a product of sign or ray patterns, we will denote such an entry by #. When working with ray patterns, if $a \in \mathbb{C}$ and k is a positive integer such that a^k is a positive real number, we will replace a^k with 1.

2. A useful lemma on powers of cycle products

In this section, we show that if A is an irreducible ray pattern with two cycles whose product weights raised to certain powers differ, then A^k has an ambiguous entry for some $k \leqslant n^2 - 2n + 2$. We begin with a short lemma that will be used repeatedly in the proof of the useful lemma that follows it.

Lemma 1. *Let A be an $n \times n$ irreducible ray pattern. If there exist circuits γ_1 and γ_2, with lengths l_1 and l_2, respectively, such that γ_1 and γ_2 share a common vertex, such that $l_1 + l_2 \leqslant 2n - 2$, and such that $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, where $m = \operatorname{lcm}(l_1, l_2)$, then A^m has an ambiguous entry and $m < n^2 - 2n + 2$.*

Proof. Since $l_1 + l_2 \leqslant 2n - 2$, we see that $m = \operatorname{lcm}(l_1, l_2) \leqslant l_1 l_2 \leqslant (n-1)^2 < n^2 - 2n + 2$. Let v_p be a common vertex between γ_1 and γ_2. For $j = 1, 2$, let β_j be the circuit through v_p obtained by following γ_j exactly $\frac{m}{l_j}$ times. Then β_j has length m and weight $\wp(\gamma_j)^{\frac{m}{l_j}}$. Since $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, it follows that $(A^m)_{pp} = \#$. $\square$

Lemma 2. *Let A be an $n \times n$ irreducible ray pattern. If there exist cycles γ_1 and γ_2 with lengths l_1 and l_2, respectively, such that $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, where $m = \operatorname{lcm}(l_1, l_2)$, then A^k has an ambiguous entry for some $k \leqslant n^2 - 2n + 2$.*

Proof. Case I: Suppose that γ_1 and γ_2 contain at least one common vertex; call it v_p.

By Lemma 1, we need only consider the case where $l_1 + l_2 > 2n - 2$. Since γ_1 and γ_2 are cycles on at most n vertices we see that $l_1 + l_2 \leqslant 2n$. Thus as a multiset, $\{l_1, l_2\}$ is either the multiset $\{n, n\}$ or the set $\{n, n-1\}$. We thus assume without loss of generality that $l_1 = n$, and that l_2 is either n or $n - 1$. If $l_2 = n$, then there are two cycles of length n through v_p with different product weights, and hence,

$(A^n)_{pp} = \#$. Thus we assume for the remainder of Case I that $l_2 = n-1$, and hence, $m = n(n-1)$. Let H be the subgraph of $G(A)$ whose edges are precisely the edges common to γ_1 and γ_2.

Suppose first that H is a path α of length $n-2$. Let v_q be the first vertex in α and let v_r be the last vertex in α. Going around γ_1 exactly $n-1 = \frac{m}{l_1}$ times and around γ_2 exactly $n = \frac{m}{l_2}$ times, we see that $(A^{n(n-1)})_{rr} = \#$. By backtracking through the $n-2$ common vertices along α, we see that $(A^{n(n-1)-(n-2)})_{rq} = \#$. Note that $n(n-1)-(n-2) = n^2-2n+2$.

Next we consider the case where H is not a path with length $n-2$. In this case, there are at least two disjoint edges in γ_1 that are not in γ_2. We can assume without loss of generality that the n-cycle γ_1 has edges labeled (v_j, v_{j+1}) for $j = 1, \ldots, n-1$ and edge (v_n, v_1). We also assume without loss of generality that (v_1, v_2) and (v_h, v_{h+1}) are not edges in γ_2 for some h with $2 < h < n$. Since γ_2 has $n-1$ vertices, at least three of the vertices v_1, v_2, v_h, v_{h+1} are in γ_2; we can assume without loss of generality that v_1 and v_2 are vertices of γ_2. Let (v_1, v_k) be an edge in γ_2. Notice $k \neq 2$. Then γ_1 can be decomposed into three paths: $\alpha_1 = (v_1, v_2)$, α_2 from v_2 to v_k, and α_3 from v_k to v_1. Similarly γ_2 can be decomposed into three paths: $\beta_1 = (v_1, v_k)$, β_2 from v_k to v_2, and β_3 from v_2 to v_1. Then $\gamma_1\gamma_2 = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$. By following the same edges in a different order, we get three circuits, $\gamma_3 = \alpha_1\beta_3$, $\gamma_4 = \alpha_2\beta_2$, and $\gamma_5 = \alpha_3\beta_1$, with lengths l_3, l_4, and l_5, respectively.

Notice that $l_3 \leqslant 1+n-3 = n-2$ and $l_5 \leqslant 1+n-2 = n-1$. Let $m_j = \mathrm{lcm}(l_2, l_j)$ for $j = 3, 4, 5$. Since γ_2 has vertices in common with both of γ_3 and γ_5, by Lemma 1 we need to consider only the case where

$$\wp(\gamma_3)^{\frac{m_3}{l_3}} = \wp(\gamma_2)^{\frac{m_3}{l_2}} \quad \text{and} \quad \wp(\gamma_5)^{\frac{m_5}{l_5}} = \wp(\gamma_2)^{\frac{m_5}{l_2}},$$

and hence,

$$\wp(\gamma_3)^{l_2} = \wp(\gamma_2)^{l_3} \quad \text{and} \quad \wp(\gamma_5)^{l_2} = \wp(\gamma_2)^{l_5}.$$

If in addition,

$$\wp(\gamma_4)^{l_2} = \wp(\gamma_2)^{l_4},$$

then

$$\begin{aligned} \wp(\gamma_1)^{l_2}\wp(\gamma_2)^{l_2} &= \wp(\gamma_1\gamma_2)^{l_2} \\ &= \wp(\gamma_3\gamma_4\gamma_5)^{l_2} \\ &= \wp(\gamma_3)^{l_2}\wp(\gamma_4)^{l_2}\wp(\gamma_5)^{l_2} \\ &= \wp(\gamma_2)^{l_3+l_4+l_5} \\ &= \wp(\gamma_2)^{l_1+l_2}. \end{aligned}$$

Hence, $\wp(\gamma_1)^{l_2} = \wp(\gamma_2)^{l_1}$. Since $\gcd(l_1, l_2) = \gcd(n, n-1) = 1$, it follows that $m = \mathrm{lcm}(l_1, l_2) = l_1 l_2$, and hence that

$$\wp(\gamma_1)^{\frac{m}{l_1}} = \wp(\gamma_2)^{\frac{m}{l_2}},$$

which contradicts one of our main assumptions. Thus for the remainder of Case I, we assume that

$$\wp(\gamma_4)^{l_2} \neq \wp(\gamma_2)^{l_4},$$

which implies

$$\wp(\gamma_4)^{\frac{m_4}{l_4}} \neq \wp(\gamma_2)^{\frac{m_4}{l_2}}.$$

By Lemma 1, we need only consider the case where $l_4 \geqslant n$.

Since γ_4 does not go through v_1, it has at least n edges on at most $n-1$ vertices, and hence is not a cycle. Decompose γ_4 into cycles $\gamma_6 \ldots \gamma_q$. Since γ_4 is made up of two paths α_2 and β_2, each γ_j for $j = 6, \ldots, q$ contains at least one vertex from β_2, and hence, from γ_2. Let $m_j = \mathrm{lcm}(l_2, l_j)$ for $j = 6, \ldots, q$. If

$$\wp(\gamma_j)^{\frac{m_j}{l_j}} = \wp(\gamma_2)^{\frac{m_j}{l_2}},$$

for $j = 6, \ldots, q$, then it is easy to see that

$$\wp(\gamma_j)^{l_2} = \wp(\gamma_2)^{l_j},$$

and hence that

$$\wp(\gamma_4)^{l_2} = \wp(\gamma_2)^{l_4},$$

which is a contradiction. Thus there must exist $j \in \{6, \ldots, q\}$ such that $\wp(\gamma_j)^{\frac{m_j}{l_j}} \neq \wp(\gamma_2)^{\frac{m_j}{l_2}}$. Since γ_j is a cycle on at most $n-1$ vertices, $l_j \leqslant n-1$, and hence, $l_2 + l_j \leqslant 2(n-1)$. By Lemma 1, there exists $k \leqslant n^2 - 2n + 2$ such that A^k contains an ambiguous entry.

Case II: Suppose that γ_1 and γ_2 have no vertices in common. Since A is irreducible, there is a shortest path β from some vertex v_p in γ_1 to some vertex v_q in γ_2 such that v_p is the only common vertex for γ_1 and β and such that v_q is the only common vertex for γ_2 and β. Let l_3 be the length of β. Then $l_3 \leqslant n$. Consider two walks from v_p to v_q, W_1 consisting of m/l_1 laps around γ_1 followed by β, and W_2 consisting of β followed by m/l_2 laps around γ_2. Thus W_1 and W_2 have length $m + l_3$, and

$$\wp(W_1) = \wp(\gamma_1)^{\frac{m}{l_1}} \wp(\beta) \neq \wp(\beta)\wp(\gamma_2)^{\frac{m}{l_2}} = \wp(W_2).$$

Since γ_1 and γ_2 are cycles with no vertex in common, $l_1 + l_2 \leqslant n$. Thus

$$m + l_3 \leqslant l_1 l_2 + l_3 \leqslant l_1(n - l_1) + l_3 \leqslant \left(\frac{n}{2}\right)^2 + n$$

It is easy to check that for $n \geqslant 4$,

$$\left(\frac{n}{2}\right)^2 + n < n^2 - 2n + 2.$$

Thus when $n \geqslant 4$, there is a $k \leqslant n^2 - 2n + 2$ such that A^k contains a #. It remains to examine the $n = 2$ and $n = 3$ cases. Either γ_1 and γ_2 are both disjoint loops, in which case the result is immediate, or one is a loop and the other is a 2-cycle. Since the number of vertices is at most three, it is easy to confirm these cases. □

Lemma 3. *Suppose A is an $n \times n$ irreducible ray pattern. Suppose that γ_1 and γ_2 are cycles in $G(A)$ such that $\wp(\gamma_1)^{\frac{m}{l_1}} = \wp(\gamma_2)^{\frac{m}{l_2}}$ where l_j is the length of γ_j for $j = 1, 2$, and where $m = \operatorname{lcm}(l_1, l_2)$. Let $g = \gcd(l_1, l_2)$. Let $a \in \mathbb{C}$ satisfy $\wp(\gamma_1) = a^{l_1}$. Then $\wp(\gamma_2) = a^{l_2} \exp(2\pi i t g/l_1)$ for some integer t with $0 \leqslant t < l_1$.*

Proof. Let $b \in \mathbb{C}$ satisfy $\wp(\gamma_2) = b^{l_2}$. Then $\wp(\gamma_2)^{\frac{m}{l_2}} = \wp(\gamma_1)^{\frac{m}{l_1}}$ is equivalent to $a^m = b^m$, so $b = a \exp(2\pi i r/m)$ for some integer r. Thus $\wp(\gamma_2) = a^{l_2} \exp(2\pi i l_2 r/m)$. Employ $l_2/m = g/l_1$. Finally, writing $r = kl_1 + t$ where $0 \leqslant t < l_1$, then $rg \equiv tg \pmod{l_1}$. □

The next result follows from the preceding lemma.

Lemma 4. *Suppose A is an $n \times n$ irreducible ray pattern. Suppose that γ is a cycle in $G(A)$ with length l. Let $a \in \mathbb{C}$ satisfy $\wp(\gamma) = a^l$. Suppose that for all cycles γ_1 and γ_2 in $G(A)$ with lengths l_1 and l_2, respectively, $\wp(\gamma_1)^{\frac{m}{l_1}} = \wp(\gamma_2)^{\frac{m}{l_2}}$ holds when $m = \operatorname{lcm}(l_1, l_2)$. Then for every cycle γ' in $G(A)$ with length h, $\wp(\gamma') = a^h \exp(2\pi i t_h g_h/l)$ where $g_h = \gcd(l, h)$ and t_h is some nonnegative integer with $0 \leqslant t_h < l$. Since $G(\bar{a}A) = G(A)$, the weight on γ' with respect to $\bar{a}A$ is $\wp(\gamma') = \exp(2\pi i t g/l)$. Finally, if γ is a loop, then the ray pattern $\bar{a}A$ has all of its cycle products equal to the ray 1.*

Note that if l_1 and l_2 are positive integers, $g = \gcd(l_1, l_2)$ and $m = \operatorname{lcm}(l_1, l_2)$, then $m/l_1 = l_2/g$ and $m/l_2 = l_1/g$. Consequently all of the results in this section that depend on m/l_1 and m/l_2 can be restated in terms of l_1/g and l_2/g.

3. An upper bound on the first ambiguous power

Let p be an integer such that $p > 1$. The integer l is called *p-odd* if $l \not\equiv 0 \bmod p$. If $l \equiv u \bmod p$ for some integer u with $0 < u < p$, then l is called *p-odd of modular class u (with respect to p).* If $l \equiv 0 \bmod p$, then l is called *p-even.* If γ is a cycle in $G(A)$ for some sign or ray pattern A, γ will be called p-odd (p-even) if its length is p-odd (p-even). Note that when $p = 2$, p-odd and p-even mean odd and even in the traditional sense.

Lemma 5. *Let p be a prime number. Let $\eta = \exp(2\pi i/p)$. For $j = 1, 2$, let t_j be an integer satisfying $0 \leqslant t_j < p$. Let l_1 and l_2 be positive integers. Let $m = \operatorname{lcm}(l_1, l_2)$ and let $g = \gcd(l_1, l_2)$. Suppose*

$$\left(\eta^{t_1}\right)^{\frac{l_2}{g}} = \left(\eta^{t_2}\right)^{\frac{l_1}{g}}. \tag{1}$$

(i) If l_1/g and l_2/g are p-odd of the same modular class, then $t_1 = t_2$.
(ii) If l_1/g is p-odd and l_2/g is p-even, then $t_2 = 0$.

Proof. (*All equivalences are modulo p*). Since p is prime, $0 < u < p$, implies u^{-1} exists modulo p.

Suppose that the hypotheses of (i) hold with modular class u. Since η is a pth root of unity, equality (1) is equivalent to $\eta^{t_1 u} = \eta^{t_2 u}$. Thus $ut_1 \equiv ut_2$. Multiply by u^{-1} to obtain $t_1 \equiv t_2$. Since $0 \leqslant t_j < p$ for $j = 1, 2$, $t_1 = t_2$. Thus (i) holds.

Suppose that the hypotheses of (ii) hold. Since η is a pth root of unity, condition 1 is equivalent to $\eta^0 = \eta^{t_2 u}$. Now proceed as the proof of part (i). $\square$

The preceding lemma can be extended to composite p.

Lemma 6. *Let A be an $n \times n$ irreducible ray pattern such that A^{n^2-2n+2} is unambiguous. Suppose that there exists a prime number p such that for every cycle γ in $G(A)$ with length l, $\wp(\gamma) = \exp(2\pi it/p)$ for some t with $0 \leqslant t < p$. Let γ_1 and γ_2 be cycles with lengths l_1 and l_2, respectively. Then the following hold:*

(i) If l_1 and l_2 are p-odd of the same modular class, then $\wp(\gamma_1) = \wp(\gamma_2)$.
(ii) If l_1 is p-odd and l_2 is p-even, then $\wp(\gamma_2) = 1$.
(iii) For a fixed positive integer r and a fixed integer u with $0 < u < p$, all cycles whose lengths are of the form $p^r q$ with $q \equiv u \bmod p$ have the same product ray.
(iv) If h is the gcd *of the lengths of all p-even cycles, and if $h = p^r q$ where q is p-odd, then every cycle whose length is divisible by p^{r+1} must have product* 1.

Proof. Let $g = \gcd(l_1, l_2)$. Apply Lemma 2 using the following cases.

Suppose l_1 is p-odd. Then g is p-odd. Since p is prime, g^{-1} exists modulo p. Let u be an integer with $0 < u < p$ such that $l_1 \equiv u \bmod p$. Then $l_1/g \equiv ug^{-1} \bmod p$, and since p is prime, l_1/g is p-odd. If l_2 is p-odd of modular class u, then the same argument as that for l_1 shows that l_2/g is p-odd of the same modular class as l_1/g. If l_2 is p-even, then $l_2 \equiv 0 \bmod p$, so $l_2/g \equiv 0g^{-1} \equiv 0 \bmod p$, which is to say, l_2/g is p-even. Apply parts (i) and (ii) of the previous lemma to obtain parts (i) and (ii) of this result.

Suppose that there is a positive integer r such that $l_1 = p^r b_1$ and $l_2 = p^r b_2$ where $b_1 \equiv b_2 \equiv u \bmod p$ for some integer u with $0 < u < p$. Then $g = p^r d$ where $d \equiv v \bmod p$ for some v with $0 < v < p$. Since p is prime, $l_1/g \equiv l_2/g \equiv uv^{-1} \not\equiv 0 \bmod p$. Thus, part ($iii$) follows from part ($i$) of the preceding lemma.

If there are p-even cycles in $G(A)$, then there must be a cycle, call it γ_1, such that $l_1 = 2^r a$ where a is p-odd. If γ_2 has length $l_2 = p^{r+1} b$ for some positive integer b, then $g = p^r c$ where c is relatively prime to p, and hence, l_1/g is p-odd and l_2/g is p-even. Hence, part (iv) follows from part (ii) of the preceding lemma. $\square$

The proof of the following well-known result is included for completeness.

Lemma 7. *Let A be an irreducible sign or ray pattern. If all cycles in A have product* 1, *then A must be powerful.*

Proof. An irreducible sign or ray pattern has a power with an ambiguous entry if and only if it has a power with an ambiguous entry on its diagonal. Consequently, an irreducible sign or ray pattern A is not powerful if and only if there are two circuits of the same length through a common vertex with conflicting products. Since circuits are constructed by traversing cycles, if all cycles in $G(A)$ have product 1, then clearly, A must be powerful. $\square$

The next result is a trivial consequence of the definition of a powerful ray (sign) pattern.

Lemma 8. *Let A be an $n \times n$ irreducible sign or ray pattern. If A is powerful then A^{n^2-2n+2} does not contain an ambiguous entry.*

Theorem 9. *Let A be an $n \times n$ irreducible ray pattern. Suppose that there exist $a \in \mathbb{C}$ and a prime number p such that for every cycle γ in $G(A)$, $\wp(\gamma) = a^l \eta^t$ where $\eta = \exp(2\pi i/p)$, where l is the length of γ, and where t is an integer with $0 \leqslant t < p$. Then A is powerful if and only if A^{n^2-2n+2} does not contain an ambiguous entry.*

Proof. By the preceding lemma, one direction is clear. We prove that if A^{n^2-2n+2} does not contain an ambiguous entry, then A is powerful. Since A is powerful if and only if $\overline{a}A$ is powerful, and since A^{n^2-2n+2} does not contain an ambiguous entry if and only if $(\overline{a}A)^{n^2-2n+2}$ does not contain an ambiguous entry, we assume without loss of generality that $a = 1$. By Lemma 7, the result is clear if every cycle in $G(A)$ has product 1, so assume that A contains at least one cycle whose product is not 1.

Since A is irreducible and A^{n^2-2n+2} does not contain an ambiguous entry, it follows that A^k cannot contain an ambiguous entry for any positive integer $k \leqslant n^2 - 2n + 2$. By Lemma 2, whenever γ_1 and γ_2 are cycles in $G(A)$, it follows that $\wp(\gamma_1)^{\frac{m}{l_1}} = \wp(\gamma_2)^{\frac{m}{l_2}}$ where l_j is the length of γ_j for $j = 1, 2$, and where $m = \operatorname{lcm}(l_1, l_2)$.

By part (*ii*) of Lemma 6, if $G(A)$ contains a p-odd cycle, then it must contain a p-odd cycle γ whose product is not 1. Thus the proof consists of two cases: (*I*) $G(A)$ contains a p-odd cycle whose product is not 1; and (*II*) all cycles in $G(A)$ are p-even, and there is a p-even cycle whose product is not 1.

Case I: It follows from Lemma 6 that all p-odd cycles in the same modular class have the same product, and that if there is a p-even cycle, its product must be 1. Consequently, even cycles have no effect on the product for a circuit that contains them; that is, the product for a circuit is determined only by the products for the odd cycles contained in the circuit.

Let C_1 and C_2 be two circuits in $G(A)$ with the same length. For $\sigma = 1, 2$, let $n_{\sigma 0}$ count the number of p-even cycles in C_σ, and for $0 < j < p$, let $n_{\sigma j}$ be the number of p-odd cycles of modular class j in C_σ. Since the two circuits have the same length l,

$$l \equiv \sum_{j=1}^{p-1} j n_{1j} \equiv \sum_{j=1}^{p-1} j n_{2j} \mod p. \tag{2}$$

By part (*i*) of Lemma 6, for each j with $0 < j < p$, there is an integer θ_j with $0 \leqslant \theta_j < p$ such that every p-odd cycle of modular class j in $G(A)$ has product η^{θ_j}. (If there is no p-odd cycle of modular class j in $G(A)$, set $\theta_j = 0$.) Further, since there is a p-odd cycle whose product is not 1, some $\theta_j \neq 0$. Let j_* denote the smallest value of j for which $\theta_j \neq 0$, and let α be a p-odd cycle whose length l_α is in modular class j_*. Then $\wp(\alpha) = \eta^{\theta_{j_*}} \neq 1$. Suppose that there is a p-odd cycle β whose length l_β is of modular class $k \neq j_*$ for some integer k with $0 < k < p$. Then $\wp(\beta) = \eta^{\theta_k}$. Then $g = \gcd(l_\alpha, l_\beta)$ is p-odd. By Lemma 2,

$$\wp(\alpha)^{\frac{l_\beta}{g}} = \wp(\beta)^{\frac{l_\alpha}{g}}.$$

That is, $\theta_{j_*} l_\beta / g \equiv \theta_k l_\alpha / g \bmod p$. Then $\theta_{j_*} l_\beta \equiv \theta_k l_\alpha \bmod p$, and hence, $\theta_{j_*} k \equiv \theta_k j_* \bmod p$. Since p is prime, $\theta_k \equiv \theta_{j_*} j_*^{-1} k$ for each integer k with $0 < k < p$ for which $G(A)$ contains a p-odd cycle whose length is in modular class k.

Observe that for $\sigma = 1, 2$,

$$\begin{aligned}\wp(C_\sigma) &= \left(\eta^{\theta_1}\right)^{n_{\sigma 1}} \left(\eta^{\theta_2}\right)^{n_{\sigma 2}} \cdots \left(\eta^{\theta_{p-1}}\right)^{n_{\sigma p-1}} \\ &= \eta^{\sum_{j=1}^{p-1} \theta_j n_{\sigma j}}.\end{aligned}$$

Thus $\wp(C_\sigma)$ is determined by $\left(\sum_{j=1}^{p-1} \theta_j n_{\sigma j}\right) \bmod p$. Using equivalence (2),

$$\begin{aligned}\sum_{j=1}^{p-1} \theta_j n_{1j} &\equiv \sum_{j=1}^{p-1} \left(\theta_{j_*} j_*^{-1} j\right) n_{1j} \bmod p \\ &\equiv \theta_{j_*} j_*^{-1} \sum_{j=1}^{p-1} j n_{1j} \bmod p \\ &\equiv \theta_{j_*} j_*^{-1} \sum_{j=1}^{p-1} j n_{2j} \bmod p \\ &\equiv \sum_{j=1}^{p-1} \theta_{j_*} j_*^{-1} j n_{2j} \bmod p \\ &\equiv \sum_{j=1}^{p-1} \theta_j n_{2j} \bmod p.\end{aligned}$$

Thus $\wp(C_1) = \wp(C_2)$.

Case II: Let h be the gcd of the cycle lengths of all cycles in $G(A)$. Since every cycle is p-even, $h = p^r a$ where r is a positive integer and where a is p-odd. Then by Lemma 6, every cycle whose length is divisible by p^{r+1} must have product 1. Call each cycle in $G(A)$ for which p^{r+1} does not divide the length of the cycle a *minimally p-even cycle*. By Lemma 6, every minimally p-even cycle of length l must have its product determined solely by the modular class of l/h, which is p-odd For each j with $0 < j < p$, there is an integer θ_j with $0 \leqslant \theta_j < p$ such that every minimally p-even cycle of modular class j in $G(A)$, where $j \equiv l/h \bmod p$, has product η^{θ_j}. (If there is no minimally p-even cycle of modular class j in $G(A)$, set $\theta_j = 0$.) Further, since there is a minimally p-even cycle whose product is not 1, some $\theta_j \neq 0$. Let j_* denote the smallest value of j for which $\theta_j \neq 0$, and let α be a minimally p-even cycle with length l_α that satisfies l_α/h is in modular class j_*. Then $\wp(\alpha) = \eta^{\theta_{j_*}} \neq 1$. Suppose that there is a minimally p-even cycle β whose length l_β satisfies l_β/h is of modular class $k \neq j_*$ for some integer k with $0 < k < p$. Then $\wp(\beta) = \eta^{\theta_k}$. By Lemma 2,

$$\wp(\alpha)^{\frac{l_\beta}{g}} = \wp(\beta)^{\frac{l_\alpha}{g}}.$$

That is, $\theta_{j_*} l_\beta / g \equiv \theta_k l_\alpha / g \bmod p$. Note that since l_1 and l_2 are minimally p-even and since h divides $g = \gcd(l_1, l_2)$, both l_1/g and l_2/g must be p-odd. Applying the argument from Case I, $\theta_k \equiv \theta_{j_*} j_*^{-1} k$ for each integer k with $0 < k < p$ for which $G(A)$ contains a minimally p-even cycle whose length divided by h is in modular class k.

Let C_1 and C_2 be two circuits in $G(A)$ with the same length. For $\sigma = 1, 2$, let $n_{\sigma 0}$ count the number of p-even cycles in C_σ that are not minimally p-even, and for $0 < j < p$, let $n_{\sigma j}$ be the number of minimally p-even cycles in C_σ whose length divided by h is of modular class j. Since the two circuits have the same length, call it l, it follows that

$$l/h \equiv \sum_{j=1}^{p-1} jn_{1j} \equiv \sum_{j=1}^{p-1} jn_{2j} \mod p. \tag{3}$$

Since all cycles that are not minimally p-even have cycle products 1, the product of the circuit C_σ is given by

$$\wp(C_\sigma) = \eta^{\sum_{j=1}^{p-1} \theta_j n_{\sigma j}}.$$

The proof that $\wp(C_1) = \wp(C_2)$ proceeds exactly as the proof in Case I, using equivalence (3) rather than **(2)**.

In both Case I and in Case II, we observe that $\wp(C_1) = \wp(C_2)$ whenever C_1 and C_2 are circuits in $G(A)$ of the same length. Thus no power of A can have an ambiguous entry on the diagonal. Since A is irreducible, no power of A can have an ambiguous entry. $\square$

For a sign pattern, every cycle has its product in $\{+, -\}$, where $+ = 1 = \eta^0$, $- = -1 = \eta^1$, and $\eta = \exp(2\pi i/2)$. Thus sign patterns correspond to the case when $p = 2$. Hence:

Theorem 10. *Let A be an $n \times n$ irreducible sign pattern. Then A is powerful if and only if A^{n^2-2n+2} contains no ambiguous entry.*

Corollary 11. *Let A be an $n \times n$ irreducible ray pattern such that A^{n^2-2n+2} contains no ambiguous entry. If there is a cycle γ whose length is $l = gp$ where g is a positive integer and p is either 1 or a prime number, and if for every cycle $\gamma' \neq \gamma$ with length l', $\gcd(l, l') \in \{gp, g\}$, then A is powerful.*

Proof. Let $a \in \mathbb{C}$ such that $\wp(\gamma) = a^l$. If $a \neq 1$, replace A with $\bar{a}A$, so $\wp(\gamma) = 1$. Suppose that γ' is a cycle such that $\gcd(l, l') = g$. Then $\wp(\gamma)^{\frac{l'}{g}} = \wp(\gamma')^{\frac{l}{g}}$ becomes $1 = \wp(\gamma')^p$, and hence, $\wp(\gamma')$ is a pth root of unity when p is prime, and $\wp(\gamma') = 1$ when $p = 1$. Suppose that γ' is a cycle such that $\gcd(l, l') = gp$. Then $\wp(\gamma)^{\frac{l'}{gp}} = \wp(\gamma')^{\frac{l}{gp}}$ becomes $1 = \wp(\gamma')$. Thus every cycle product is either 1 or else a pth root of unity where p is a prime number. Apply Theorem 9 with $a = 1$. $\square$

A useful special case of the previous result:

Corollary 12. *Let A be an $n \times n$ irreducible ray pattern such that A^{n^2-2n+2} contains no ambiguous entry. If there is a cycle γ whose length is either 1 or a prime number, then A is powerful.*

What happens when there is no prime number p and no $a \in \mathbb{C}$ such that each cycle in $G(\bar{a}A)$ has product $\exp(2\pi it/p)$ for some integer t with $0 \leqslant t < p$? That is, what happens when we must choose p to be composite? The proofs given above strongly depend on the existence of inverses modulo p.

Conjecture 13. *Let A be an $n \times n$ irreducible ray pattern such that A^{n^2-2n+2} is unambiguous. Then A is powerful.*

4. The Wielandt graph

In this section, we show that there is an $n \times n$ irreducible matrix A, for $n \geqslant 3$, that can be viewed as either a sign pattern or a ray pattern, such that the first power of A with an ambiguous entry is the $\left(n^2 - 2n + 2\right)$th power. That is, $n^2 - 2n + 2$ cannot be replaced with a smaller power in Theorem 9 or in the conjecture (Fig. 1).

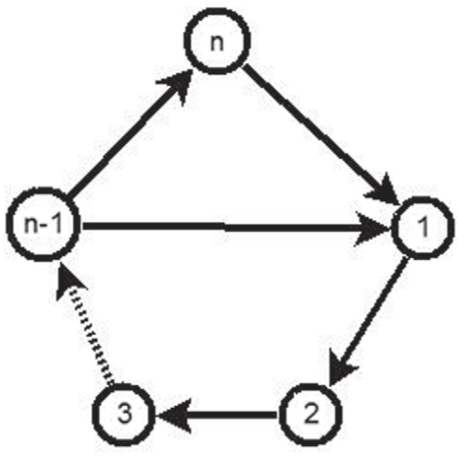

Fig. 1. The Wielandt graph.

The *Wielandt graph is the digraph* $W = (V, E)$ where $V = \{v_1, \dots, v_n\}$ and where

$$E = \{(v_i, v_{i+1}) | i = 1, \dots, n-1\} \cup \{(v_n, v_1)\} \cup \{(v_{n-1}, v_1)\}.$$

We consider the matrix $A = [a_{jk}]$ where

$$a_{jk} = \begin{cases} 1 = e^{i0} & \text{if k=j+1} \\ -1 = e^{i\pi} & \text{if } k = 1, \text{ and } \begin{cases} j = n & \text{if } n \text{ is even} \\ j = n-1 & \text{if } n \text{ is odd} \end{cases} \\ 1 = e^{i0} & \text{if } k = 1, \text{ and } \begin{cases} j = n & \text{if } n \text{ is odd} \\ j = n-1 & \text{if } n \text{ is even} \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $G(A) = W$, and A provides a weighting for the edges of W. The graph W has exactly two cycles: an n-cycle γ_1 and an $n-1$-cycle γ_2, where

$$\wp(\gamma_1) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

$$\wp(\gamma_2) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, A is irreducible whether viewed as a sign pattern or as a ray pattern. If C is a circuit, then C must be obtained by traversing γ_1 r times for some $r \geqslant 0$ and traversing γ_2 s times for some $s \geqslant 0$. Thus the length of C is $rn + s(n-1)$. If C_1 and C_2 are two distinct circuits of the same length, then $r_1 n + s_1(n-1) = r_2 n + s_2(n-1)$ with at least one of $r_1 \neq r_2$ and $s_1 \neq s_2$ holding. Further, if C_1 and C_2 are chosen so that there is no shorter pair of distinct circuits with a common length, then $\min(r_1, r_2) = 0$ and $\min(s_1, s_2) = 0$. Thus, without loss of generality, $r_1 n = s_1(n-1)$ with $r_1 s_1 \neq 0$. Since $\gcd(n, n-1) = 1$, the shortest pair occurs when $r_1 = n-1$ and $s_1 = n$. Thus for all j, $\left(A^k\right)_{jj}$ must be unambiguous for $k < n(n-1)$. Letting C_1 be the circuit obtained by traversing γ_1 $n-1$ times, $\wp(C_1) = \wp(\gamma_1)^{n-1}$. Letting C_2 be the circuit obtained by traversing γ_2 n times, $\wp(C_2) = \wp(\gamma_2)^n$. Note that $\wp(\gamma_1)^{n-1} = \wp(\gamma_1)$, and that $\wp(\gamma_2)^n = \wp(\gamma_2)$, so C_1 and C_2 are conflicting circuits of length $n(n-1)$. Consequently, the first occurrence of sharp in a diagonal entry of a power of A occurs for $A^{n(n-1)}$. Specifically, $\left(A^{n(n-1)}\right)_{n-1,n-1} = \#$. Since the two cycles share a common path of length $n-2$ from v_1 to v_{n-1}, it follows that $\left(A^{n(n-1)-n+2}\right)_{n-1,1} = \#$. Finally, observe that $n(n-1)-n+2 = n^2-2n+2$.

Suppose $(A^{\ell})_{jk} = \#$. Then there are two walks β_1 and β_2 from v_j to v_k with length ℓ such that $\wp(\beta_1) = -\wp(\beta_2)$. Extend β_1 and β_2 to circuits C_1 and C_2 by adding the same shortest path γ from v_k to v_j of length h. Unless $j=1$ and $k=n$, $h \leqslant n-2$. Note that C_1 and C_2 are distinct circuits in W with a common length, and hence their length must be at least $n(n-1)$. Unless $j=1$ and $k=n$, the common length of β_1 and β_2 must be at least $n(n-1)-h \geqslant n(n-1)-(n-2) = n^2-2n+2$. If $j=1$ and $k=n$, then $h=n-1$ and the circuits C_1 and C_2 must traverse γ_1 because they contain v_n. Since both circuits are distinct but have the same length, it means that at least one must also traverse γ_2, without loss of generality, C_1 does. Then $r_1n+s_1(n-1) = r_2n+s_2(n-1)$ with r_1, r_2 and s_1 positive. From the argument given above, r_1 and s_1 positive implies that the common length of these circuits must exceed $n(n-1)$. Then the common length of β_1 and β_2 must exceed $n(n-1)-(n-1) = n^2-2n+2$.

Summarizing,

Proposition 14. *Let k be the smallest positive integer such that if A is an $n \times n$ nonpowerful, irreducible sign (ray) pattern, then A^k must contain at least one ambiguous entry. Then $k \geqslant n^2-2n+2$.*

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