Locally linearly dependent operators and reflexivity of operator spaces

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Abstract

We obtain two results on the existence of large subspaces of operators of small rank in locally linearly dependent spaces of operators. As a consequence we obtain an upper bound for the rank of operators belonging to a minimal locally linearly dependent space of operators. It has been known that the only obstruction to the reflexivity of a finite-dimensional operator space comes from the operators with small ranks. Our results improve known bounds on the minimal rank that guarantees the reflexivity.

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1. Introduction

Let U and V be vector spaces over a field F. Linear operators T1, . . . , Tn : U → V are locally linearly dependent if T1u, . . . , Tnu are linearly dependent for every u ∈ U. Locally linearly dependent operators need not be linearly dependent. To see this take any linear space V of dimension n − 1. Then every n-tuple of operators from U into V is locally linearly dependent.

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The study of such operators was motivated by some problems in algebra and operator theory (see [1,2,7]). The basic result proved in [3,9] states that for every $n$-tuple of locally linearly dependent operators $T_1, \ldots, T_n : U \to V$ there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that $S = \alpha_1 T_1 + \cdots + \alpha_n T_n$ satisfies $\text{rank } S \leq n - 1$. This estimate is sharp.

The problem of understanding the structure of locally linearly dependent operators is closely related to the apparently very difficult classification problem for maximal vector spaces of $n \times n$ matrices with zero determinant that was treated by Fillmore et al. [6]. A detailed explanation of the connection between these two problems can be found in [3].

When studying locally linearly dependent operators $T_1, \ldots, T_n$ we can always assume that we have a nontrivial case, that is, $T_1, \ldots, T_n$ are linearly independent. Then we can denote by $S$ the $n$-dimensional linear span of these operators. The assumption of local linear dependence then reads as follows: for every $u \in U$ there is a nonzero $S \in S$ with $Su = 0$, or equivalently, $\dim S = \dim \{ Su : S \in S \} \leq n - 1$ for every $u \in U$.

This concept has been further generalized in [9]. We denote by $\mathcal{L}(U, V)$ the space of all linear operators from $U$ into $V$. Let $n$ and $c$ be positive integers with $c \leq n - 1$. An $n$-dimensional subspace $\mathcal{F} \subset \mathcal{L}(U, V)$ is $c$-locally linearly dependent if $\dim \mathcal{F}u \leq n - c$ for every $u \in U$. When $c = 1$ we simply say that $\mathcal{F}$ is locally linearly dependent. In [9] the basic result on locally linearly dependent operators was improved by showing that every $c$-locally linearly dependent space $\mathcal{F}$ contains a nonzero operator $S$ of rank at most $n - c$. We will improve this result by showing that such a space $\mathcal{F}$ contains a large space of operators whose ranges are contained in some $(n - c)$-dimensional subspace.

The rest of this note is based on the following simple observation. When considering locally linearly dependent spaces of operators $\mathcal{F} \subset \mathcal{L}(U, V)$ there is no loss of generality in assuming that they are minimal, that is, $\{0\} = T \subset S$ is a locally linearly dependent subspace, then $\mathcal{F} = S$.

The basic result on locally linearly dependent operators can be improved for sufficiently large fields in the following way [9]. If $\mathbb{F}$ is a field with at least $n + 2$ elements and $\mathcal{F} \subset \mathcal{L}(U, V)$ an $n$-dimensional locally linearly dependent space of operators then either all nonzero operators in $\mathcal{F}$ are of rank $n - 1$, or there exists a nonzero $S \in \mathcal{F}$ with rank $S \leq n - 2$. We will further improve this result for minimal locally linearly dependent spaces of operators by showing that such spaces contain a chain of large subspaces of operators of small rank.

All the study of locally linearly dependent spaces of operators so far was concentrated on the problem of finding a nonzero operator (or a subspace of operators) of a rank as small as possible. Our result shows that if such a space is minimal then we can get also an estimate on the maximal possible rank.

As an application we will improve some known results on the reflexivity of finite-dimensional operator spaces. Let $\mathcal{H} \subset \mathcal{L}(U, V)$ be a finite-dimensional subspace. Set $n = \dim \mathcal{H}$. We say that $T \in \mathcal{L}(U, V)$ locally belongs to $\mathcal{H}$ if $Tx \in \mathcal{H}$ for
every \( x \in U \). The space \( \mathcal{H} \) is called (algebraically) reflexive if every operator that locally belongs to \( \mathcal{H} \) is automatically a member of \( \mathcal{H} \). Such spaces were treated in [4,5,7,8]. Larson proved that if \( \mathcal{H} \) contains no nonzero finite rank operators then it is reflexive. Ding improved this result by showing that the obstruction to reflexivity comes from operators with small ranks. His results were further improved by Li and Pan who showed that if \( \mathbb{F} = \mathbb{C} \) and every nonzero operator in \( \mathcal{H} \) has rank at least \( 2n - 1 \) then \( \mathcal{H} \) is reflexive. They also showed that the weaker bound \( 2n \) works for all fields that are sufficiently large. We will conclude the paper by showing that our results yield an improvement of these bounds.

2. Results

We start with an improvement of the above mentioned result on \( c \)-locally linearly dependent operators.

**Theorem 2.1.** Let \( U \) and \( V \) be vector spaces over a field \( \mathbb{F} \) and let \( n, c \) be positive integers with \( c \leq n - 1 \). Assume also that \( |\mathbb{F}| \geq n - c + 1 \). If \( \mathcal{H} \subset L(U, V) \) is a \( c \)-locally linearly dependent subspace of dimension \( n \) then there exist \( (n - c) \)-dimensional subspaces \( W \subset V \) and \( \mathcal{R} \subset \mathcal{H} \) such that \( SU \subset W \) for every \( S \in \mathcal{R} \).

**Proof.** We will first prove the special case that \( V \) is finite-dimensional. The general case will follow easily.

So, assume that \( V \) is \( m \)-dimensional. There is nothing to prove if \( m \leq n - c \). So, assume that \( m > n - c \). By induction we may also assume that

\[
\dim \mathcal{H} u = n - c
\]

for some \( u \in U \). Thus, we can find operators \( S_1, \ldots, S_{n-c} \in \mathcal{H} \) such that the vectors \( S_1 u, \ldots, S_{n-c} u \) are linearly independent. If \( S \) is any operator from \( \mathcal{H} \) then \( Su \) belongs to the linear span of \( S_1 u, \ldots, S_{n-c} u \), and therefore, there exist scalars \( \alpha_1, \ldots, \alpha_{n-c} \) such that \( (S - \alpha_1 S_1 - \cdots - \alpha_{n-c} S_{n-c}) u = 0 \). Consequently, we can extend the set of linearly independent operators \( \{S_1, \ldots, S_{n-c}\} \) to the basis \( \{S_1, \ldots, S_{n-c}, S_{n-c+1}, \ldots, S_m\} \) of \( \mathcal{H} \) with the property that \( S_j u = 0 \) whenever \( k > n - c \). We also choose a basis \( v_1 u, \ldots, v_{m-n-c}, v_{n-c+1}, \ldots, v_{m} \) of the space \( V \).

Next, we will define a linear map \( \phi : U \to M_{m \times n}(\mathbb{F}) \). For every \( w \in U \) we define \( \phi(w) \) to be the matrix whose entries in the \( j \)th column are the coordinates of \( S_j w \) with respect to the chosen basis of \( V \). The assumption that \( \mathcal{H} \) is \( c \)-locally linearly dependent can be equivalently reformulated as the condition that \( \text{rank} \phi(w) \leq n - c \) for every \( w \in U \). Clearly,

\[
\phi(u) = \begin{bmatrix}
I_{n-c} & 0_{n-c,c} \\
0_{m-n+c,n-c} & 0_{m-n+c,c}
\end{bmatrix},
\]

where \( I_k \) is the \( k \times k \) identity matrix and \( 0_{k,l} \) is the \( k \times l \) zero matrix.
If $w$ is any vector from $U$, then rank($\alpha\phi(u) + \phi(w)$) $\leq n - c$ for every scalar $\alpha$. We will prove that the $(k, j)$-entry of $\phi(w)$ is zero whenever $k > n - c$ and $j > n - c$. Assume that $(\phi(w))_{kj} \neq 0$ for some pair of indexes $k$ and $j$. The determinant of a submatrix of $\alpha\phi(u) + \phi(w)$ lying in rows 1, 2, ..., $n - c$, $k$ and columns 1, 2, ..., $n - c$, $j$ is a polynomial in $\alpha$ of degree at most $n - c$. In fact, this polynomial has degree $n - c$ since the coefficient at $\alpha^{n-c}$ is $(\phi(w))_{kj} \neq 0$. Our assumption on cardinality of $\mathbb{F}$ implies that there is a scalar $a_0$ such that this polynomial is nonzero at $a_0$ contradicting the fact that rank($\alpha_o\phi(u) + \phi(w)$) $\leq n - c$.

So, for every $w \in U$, the $(k, j)$-entry of $\phi(w)$ is zero whenever $k > n - c$ and $j > n - c$. Hence, every linear combination of $S_{n-c+1}, ..., S_n$ maps $w$ into the linear span of $S_1u, ..., S_{n-c}u$. This completes the proof when dim $V < \infty$.

Assume now that dim $V = \infty$ and let $P \in \mathcal{L}(V)$ be any idempotent of finite rank whose range contains $W$, the linear span of the vectors $S_1u, ..., S_{n-c}u$. The space of operators $P\mathcal{S} = \{PS : S \in \mathcal{S}\}$ is $c$-locally linearly dependent space of operators of dimension at most $n$ mapping $U$ into $PV$. As before we see that $PS_{n-c+1}U \subset W, ..., PS_nU \subset W$. This is true for any finite rank idempotent $P$ whose range contains $W$, and therefore, $S_{n-c+1}U \subset W, ..., S_nU \subset W$, as desired. □

We continue by studying minimal locally linearly dependent spaces of operators.

**Theorem 2.2.** Let $n$ be a positive integer $\geq 2$ and let $\mathbb{F}$ be a field with at least $n + 2$ elements. Suppose that $U$ and $V$ are vector spaces over $\mathbb{F}$ and $\mathcal{S} \subset \mathcal{L}(U, V)$ is an $n$-dimensional minimal locally linearly dependent space of operators. Then either

1. all nonzero operators in $\mathcal{S}$ are of rank $n - 1$, or
2. there exist a positive integer $k, k < n - 1$, a sequence of subspaces $W_1 \subset \cdots \subset W_k \subset V$ with $\dim W_1 \leq n - 2$, $\dim W_2 \leq (n - 2) + (n - 3)$, ..., $\dim W_k \leq (n - 2) + (n - 3) + \cdots + (n - k - 1)$, and a sequence of subspaces $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_k \subset \mathcal{S}$ with $\dim \mathcal{S}_j = j$, $j = 1, ..., k$, such that $SU \subset W_j$ whenever $S \in \mathcal{S}_j$, and the range of every $S \in \mathcal{S}$ is contained in a sum of $W_k$ and a subspace of dimension at most $n - k - 1$.

**Proof.** If $r$, the minimal rank of nonzero elements from $\mathcal{S}$, is $n - 1$, then by [9, Theorem 2.4] we have rank $S = n - 1$ for every nonzero $S \in \mathcal{S}$, and we are done.

So, assume that $r \leq n - 2$ and choose $S \in \mathcal{S}$ with rank $S = r$. Denote the range of $S$ by $W_1$ and let $\mathcal{F}$ be the subspace of all operators from $\mathcal{S}$ whose range is contained in $W_1$ (the range of every nonzero operator from $\mathcal{F}$ is, of course, equal to $W_1$). Set dim $\mathcal{F} = p$ and $W_j = W_{1j}, 1 \leq j \leq p$. Let $\mathcal{F}_1$ be the linear span of $S$ and choose a sequence of subspaces $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_p = \mathcal{F}$ such that dim $\mathcal{F}_j = j$, $j = 1, ..., p$. Note that $p < n - 1$ since otherwise $\mathcal{F}_{n-1}$ would be locally linearly dependent.

We start the next step of the proof by choosing an idempotent $P_1 \subset \mathcal{L}(V)$ whose kernel is $W_1$. We also choose a direct summand $\mathcal{F}_1$ of $\mathcal{F}_p$ in $\mathcal{S}$. Our aim is to show
that the \((n - p)\)-dimensional space of operators \(P_1 \mathcal{F}_1 = \{ P_1 S : S \in \mathcal{F}_1 \} \) is locally linearly dependent. Assume to the contrary that there exists \( x \in U \) and \( S_{p+1}, \ldots, S_{n} \in \mathcal{F}_1 \) such that \( P_1 S_{p+1}x, \ldots, P_1 S_{n}x \) are linearly independent. Because \( \mathcal{F} \) is a minimal locally linearly dependent space of operators we can find \( y \in U \) and \( S_1, \ldots, S_p \in \mathcal{F}_p \) such that the vectors \( S_1 y, \ldots, S_p y \) are linearly independent. Applying [3, Lemma 2.1] we see that there are at most \((n - p)\) nonzero scalars \( \alpha \) such that \( P_1 S_{p+1}(x + \alpha y), \ldots, P_1 S_{n}(x + \alpha y) \) are linearly dependent. Similarly, there are at most \( p \) nonzero scalars \( \alpha \) such that \( S_1(\alpha^{-1}x + y), \ldots, S_p(\alpha^{-1}x + y) \) are linearly dependent. It follows that there exists a nonzero scalar \( \beta \) such that both sets of vectors \( P_1 S_{p+1}(x + \beta y), \ldots, P_1 S_{n}(x + \beta y) \) and \( S_1(x + \beta y), \ldots, S_p(x + \beta y) \) are linearly independent. Let

\[
\lambda_1 S_1(x + \beta y) + \cdots + \lambda_p S_p(x + \beta y) + \lambda_{p+1} S_{p+1}(x + \beta y) + \cdots + \lambda_n S_n(x + \beta y) = 0
\]

for some scalars \( \lambda_1, \ldots, \lambda_n \). Applying the operator \( P_1 \) to the both sides of the above equation and using the fact that the vectors \( P_1 S_{p+1}(x + \beta y), \ldots, P_1 S_{n}(x + \beta y) \) are linearly independent we conclude that \( \lambda_{p+1} = \cdots = \lambda_n = 0 \). It follows that all \( \lambda_i \)'s have to be zero, contradicting the fact that the space \( \mathcal{F} \) is locally linearly dependent.

Now, we have two possibilities. If all operators in \( P_1 \mathcal{F}_1 \) have rank at most \( n - p - 1 \) then set \( k = p \) and we are done. Otherwise choose a nonzero \( R \in P_1 \mathcal{F}_1 \) with minimal rank and denote by \( W_{p+1} \) the direct sum of \( W_1 \) and the range of \( R \). Denote by \( \mathscr{R} \) the subspace of all operators in \( \mathcal{F} \) whose range is contained in \( W_{p+1} \). Set \( \dim \mathscr{R} = p + q \) and choose a sequence of subspaces \( \mathcal{F}_{p+1} \subset \cdots \subset \mathcal{F}_{p+q} = \mathscr{R} \) and \( \dim \mathcal{F}_j = j, j = p + 1, \ldots, p + q \). Put also \( W_{p+1} = \cdots = W_{p+q} \).

We continue by choosing an idempotent \( P_2 \subset \mathcal{L}(V) \) whose kernel is \( W_{p+1} \). We also choose a direct summand \( \mathcal{F}_2 \) of \( \mathcal{F}_{p+q} \) in \( \mathcal{F} \). As before we see that the \((n - p - q)\)-dimensional space of operators \( P_2 \mathcal{F}_2 \) is locally linearly dependent. In particular, \( \mathcal{F}_2 \) is at least two-dimensional. Repeating the same procedure one can now complete the proof. \( \square \)

**Corollary 2.3.** Let \( n \) be a positive integer \( \geq 2 \) and let \( \mathbb{F} \) be a field with at least \( n + 2 \) elements. Suppose that \( U \) and \( V \) are vector spaces over \( \mathbb{F} \) and \( \mathcal{F} \subset \mathcal{L}(U, V) \) is an \( n \)-dimensional minimal locally linearly dependent space of operators. Then \( \dim S \leq \frac{(n-1)(n-2)}{2} + 1 \) for every \( S \in \mathcal{F} \). This estimate is sharp for \( n \in \{2, 3, 4\} \).

**Proof.** By the previous theorem, the maximal possible rank of an operator belonging to an \( n \)-dimensional minimal locally linearly dependent space of operators is at most \((n - 2) + (n - 3) + \cdots + 1 + 1 = \frac{(n-1)(n-2)}{2} + 1 \).

Let us now show that this estimate is sharp for \( n \leq 4 \). In the case \( n = 2 \) the whole space of operators \( \mathcal{L}(\mathbb{F}^2, \mathbb{F}) \) is a minimal locally linearly dependent space. So, in this case our result on maximal possible rank is sharp.

When \( n = 3 \) define \( \mathcal{F} \subset \mathcal{L}(\mathbb{F}^2, \mathbb{F}^2) \) to be the linear span of operators

\[
T_1(a, b) = (a, 0), \quad (a, b) \in \mathbb{F}^2,
\]
Because the target space of operators $T_1, T_2, T_3$ is two-dimensional, $\mathcal{S}$ is locally linearly dependent. And since rank $T_3 = 2$ we have to show that $\mathcal{S}$ is minimal in order to see that our estimate is sharp when $n = 3$. Assume to the contrary that $\mathcal{S}$ is not minimal. Then there exists a two-dimensional locally linearly dependent subspace $\mathcal{T} \subset \mathcal{S}$. Such a space must be minimal, and therefore, it is a two-dimensional space of operators all of whose nonzero members have rank one. Obviously, rank($\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$) $\leq 1$ if and only if $\alpha_1 \alpha_2 = \alpha_2^3$. But there is no two-dimensional subspace $\mathcal{Y} \subset \mathcal{F}$ with the property that $\alpha_1 \alpha_2 = \alpha_2^3$ whenever $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{Y}$. This contradiction shows that $\mathcal{S}$ is indeed minimal.

In the case $n = 4$ we use an idea from [6]. Define $\mathcal{S} \subset \mathcal{L}(\mathbb{F}^4, \mathbb{F}^4)$ to be the linear span of operators

$T_1(x, y, z, w) = (x, z, 0, 0)$, \quad $(x, y, z, w) \in \mathbb{F}^4$,

$T_2(x, y, z, w) = (0, -y, x, 0)$, \quad $(x, y, z, w) \in \mathbb{F}^4$,

$T_3(x, y, z, w) = (w, 0, 0, z)$, \quad $(x, y, z, w) \in \mathbb{F}^4$,

and

$T_4(x, y, z, w) = (y, w, z, -x)$, \quad $(x, y, z, w) \in \mathbb{F}^4$.

Let us first prove that $\mathcal{S}$ is locally linearly dependent. We have to show that $T_1(x, y, z, w), T_2(x, y, z, w), T_3(x, y, z, w)$, and $T_4(x, y, z, w)$ are linearly dependent for every $(x, y, z, w) \in \mathbb{F}^4$. If $x \neq 0$ then

$$-\frac{wx + yz}{x}T_1(x, y, z, w) - \frac{z^2}{x}T_2(x, y, z, w) + xT_3(x, y, z, w) + zT_4(x, y, z, w) = 0.$$ 

So, assume that $x = 0$ and $z \neq 0$. Then

$$yT_1(x, y, z, w) + zT_2(x, y, z, w) = 0.$$ 

Finally, if both $x$ and $z$ are zero we have $T_1(x, y, z, w) = 0$.

Clearly, rank $T_4 = 4$. So, in order to complete the proof we have to show that $\mathcal{S}$ is minimal among locally linearly dependent spaces of operators. First note that in the standard basis of $\mathbb{F}^4$ the following matrix corresponds to the operator $\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \alpha_4 T_4$:

$$\begin{pmatrix}
\alpha_1 & \alpha_4 & 0 & \alpha_3 \\
0 & -\alpha_2 & \alpha_1 & \alpha_4 \\
\alpha_2 & 0 & \alpha_4 & 0 \\
-\alpha_4 & 0 & \alpha_3 & 0
\end{pmatrix}.$$
The rank of this matrix is the same as the rank of
\[
\begin{bmatrix}
\alpha_1 & 0 & \alpha_4 & \alpha_3 \\
0 & \alpha_1 & -\alpha_2 & \alpha_4 \\
\alpha_2 & \alpha_4 & 0 & 0 \\
-\alpha_4 & \alpha_3 & 0 & 0
\end{bmatrix}.
\]
Obviously, if such a matrix has rank \( \leq 2 \) then \( \alpha_2\alpha_3 + \alpha_4^2 = 0 \), and if its rank is at most one than it has to be zero. If \( \mathcal{S} \) was not minimal then we would have either a three-dimensional subspace \( \mathcal{S}_1 \subset \mathcal{S} \) whose nonzero members would have rank at most two, or a two-dimensional subspace \( \mathcal{S}_2 \subset \mathcal{S} \) all of whose nonzero members would have rank one. It is easy to see that none of these possibilities can occur. This completes the proof. □

**Remark.** If we want to have a quadratic estimate on the maximal rank in a minimal locally linearly dependent space of operators which is also sharp when \( n = 2, 3, 4 \), then the estimate \( (n-1)(n-2)/2 + 1 \) given in Corollary 2.3 is the only possibility.

**Corollary 2.4.** Let \( n \) be a positive integer and let \( F \) be a field with at least \( n+2 \) elements. Suppose that \( U \) and \( V \) are vector spaces over \( F \) and \( \mathcal{S} \subset \mathcal{L}(U, V) \) is an \( n \)-dimensional locally linearly dependent space of operators. Assume also that \( \mathcal{S} \) contains an operator of rank at least \( (n-1)(n-2)/2 + 2 \). Then either \( \mathcal{S} \) contains a nonzero operator of rank at most \( n-3 \), or \( \mathcal{S} \) contains an \( (n-1) \)-dimensional subspace whose nonzero members are all of rank \( n-2 \).

**Proof.** By Corollary 2.3, \( \mathcal{S} \) is not minimal. Hence, \( \mathcal{S} \) contains an \( (n-1) \)-dimensional locally linearly dependent subspace \( \mathcal{S}_1 \). The desired conclusion follows now directly from [9, Theorem 2.4]. □

Li and Pan [8] proved that if \( \mathcal{R} \) is a nonreflexive \( n \)-dimensional operator space over the complex field then \( \mathcal{R} \) contains a nonzero operator of rank at most \( 2n-2 \). We can improve this result as follows.

**Corollary 2.5.** Let \( F \) be a field with at least \( n+3 \) elements. Assume that \( U \) and \( V \) are vector spaces over \( F \) and \( \mathcal{R} \subset \mathcal{L}(U, V) \) an \( n \)-dimensional nonreflexive space. Then either we can find a nonzero \( R \in \mathcal{R} \) with rank \( \leq 2n-3 \), or all nonzero members of \( \mathcal{R} \) have rank \( 2n-2 \).

**Proof.** We can find \( T \in \mathcal{L}(U, V) \) that locally belongs to \( \mathcal{R} \) but is not a member of \( \mathcal{R} \). Then the space \( \mathcal{R}' = \text{span}(T) \oplus \mathcal{R} \) has dimension \( n+1 \). It is locally linearly dependent. We choose any minimal locally linearly dependent subspace \( \mathcal{R}'' \) of \( \mathcal{R}' \). Set \( p = \dim \mathcal{R}'' \). Clearly, \( 2 \leq p \leq n+1 \), and consequently, \( \mathcal{R}'' \) contains at least one nonzero member of \( \mathcal{R} \). Now we apply Theorem 2.2. We first assume that \( \mathcal{R}'' \) satisfies the first condition from Theorem 2.2. Then we can find a nonzero member of \( \mathcal{R} \) of
rank $p - 1$. If $n \geq 3$ then $p - 1 \leq n \leq 2n - 3$ and we are done. A nonreflexive space cannot be one-dimensional. So, it remains to consider the case that $n = 2$. If $p = 2$ then there is a nonzero member of $\mathcal{H}''$ of rank 1, $2n - 3$. If $p = 3$ then all members of $\mathcal{H}$ are of rank 2, $2n - 2$. We continue by considering the case that $\mathcal{H}''$ satisfies the second condition from Theorem 2.2 with $k = 1$. Then all operators from $\mathcal{H}''$ have rank at most $(p - 2) + (p - 2) \leq 2n - 2$. Here, the equality can occur only when $p = n + 1$, or equivalently, $\mathcal{H}'' = \mathcal{H}$. In this case all members of $\mathcal{H}$ have rank at most $2n - 2$. And finally, if $\mathcal{H}''$ satisfies the second condition from Theorem 2.2 with $k > 1$, then there exists a two-dimensional subspace $\mathcal{H}''' \subset \mathcal{H}''$ all of whose members are of rank at most $2p - 5 \leq 2n - 3$. This completes the proof. □

The obtained bound is sharp when $n = 2$. To see this define $\mathcal{H} \subset L(\mathbb{R}^2, \mathbb{R}^2)$ to be the linear span of operators

$$R_1(a, b) = (a, b), \quad (a, b) \in \mathbb{R}^2,$$

and

$$R_2(a, b) = (b, -a), \quad (a, b) \in \mathbb{R}^2.$$

It is easy to see that every nonzero member of $\mathcal{H}$ is invertible. Thus, rank $R = 2n - 2$ for every nonzero $R \in \mathcal{H}$. Moreover, every $S \in L(\mathbb{R}^2, \mathbb{R}^2)$ locally belongs to $\mathcal{H}$. Indeed, $R_1(a, b)$ and $R_2(a, b)$ are linearly independent unless $(a, b) = 0$. So, $\mathcal{H}$ is nonreflexive.

References