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Stability and Bifurcation in Delay–Differential Equations with Two Delays

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The purpose of this paper is to study a class of differential–difference equations with two delays. First, we investigate the local stability of the zero solution of the equation by analyzing the corresponding characteristic equation of the linearized equation. General stability criteria involving the delays and the parameters are obtained. Second, by choosing one of the delays as a bifurcation parameter, we show that the equation exhibits the Hopf bifurcation. The stability of the bifurcating periodic solutions are determined by using the center manifold theorem and the normal form theory. Finally, as an example, we analyze a simple motor control equation with two delays. Our results improve some of the existing results on this equation. © 1999 Academic Press

1. INTRODUCTION

In the last two decades, great attention has been paid to equations with multiple delays, which have significant biological and physical background. Consider the following equation with two delays

$$\dot{x}(t) = f\big(x(t), x(t-\tau_1), x(t-\tau_2)\big), \tag{1.1}$$

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where τ_1, τ_2 are positive constants, $f(0,0,0) = 0$, and $f\colon R \times R \times R \to R$ is continuously differentiable. Let $-A_0$, $-A_1$, and $-A_2$ be the first derivatives of $f(u_1, u_2, u_3)$ with respect to u_1, u_2, and u_3 evaluated at $u_1 = u_2 = u_3 = 0$, respectively, i.e.,

$$-A_0 = \frac{\partial f}{\partial u_1}(0,0,0), \qquad -A_1 = \frac{\partial f}{\partial u_2}(0,0,0), \qquad -A_2 = \frac{\partial f}{\partial u_3}(0,0,0). \tag{1.2}$$

Then the linearized equation of (1.1) at the trivial solution is

$$\dot{x}(t) = -A_0 x(t) - A_1 x(t - \tau_1) - A_2 x(t - \tau_2). \tag{1.3}$$

Hale and Hunag [15] investigated the stability of Eq. (1.3) in the (τ_1, τ_2) plane for various intervals in A_0, A_1, and A_2 and determined the global geometry of the stable regions. For related work, we refer to Bellman and Cooke [3], Bélair, *et al*. [2], Hale [14], Hale and Tanaka [17], Mahaffy *et al*. [22], Marriot *et al*. [23], Mizuno and Ikeda [24], and Ruiz Claeyssen [26], among others.

Assume that

$$A_0 = 0, \qquad A_1 > 0, \qquad A_2 > 0. \tag{1.4}$$

Then Eq. (1.3) becomes

$$\dot{x}(t) = -A_1 x(t - \tau_1) - A_2 x(t - \tau_2). \tag{1.5}$$

Equation (1.5) is the linearized equation of some other equations with two delays in the form of (1.1). The first example is the logistic model with two delays (Braddock and van den Driessche [6]; Gopalsamy [13]):

$$\dot{N}(t) = RN(t)\left[1 - BN(t - \tau_1) - CN(t - \tau_2)\right], \tag{1.6}$$

where R, B, and C are positive constants. Equation (1.6) has a positive equilibrium $N^* = 1/(B + C)$. Let $N(t) = N^*(1 + n(t))$. Then Eq. (1.6) can be written as

$$\dot{n}(t) = -(1 + n(t))\left[A_1 n(t - \tau_1) + A_2 n(t - \tau_2)\right], \tag{1.7}$$

where $A_1 = RBN^*$, $A_2 = RCN^*$. Clearly, Eq. (1.5) is the linearized equation of (1.7) at $n = 0$. Braddock and van den Driessche [6] describe some linear stability regions for Eq. (1.7). They find that the two delay terms are equally important and observe stable limit cycles when τ_2/τ_1 is large. In modeling sexually transmitted disease, Cooke and Yorke [11] discuss Eq. (1.7) with $A_1 = -A_2$. They describe various stability properties of (1.7) and obtain some limit cycle solutions. Nussbaum [25] studies Eq. (1.7) with $\tau_1 = 1$. If $1 < \tau_2 < 2$, he proves the existence of a periodic solution of (1.7). When $\tau_2 > 2$, he shows that there may exist two positive solutions of period greater than τ_2. Stech [27] also considers Eq. (1.7) with $\tau_1 = 1$, $\tau_2 = 3$, and $A_1 + A_2 = 1$ and discusses the stable and unstable bifurcations.

The second example is a simple motor control equation (Bélair and Campbell [1]; Beuter *et al.* [4, 5]),

$$\dot{x}(t) = f_1(x(t-\tau_1)) + f_2(x(t-\tau_2)), \tag{1.8}$$

where $f_i(u) = -A_i \tan h(u)$, $i = 1, 2$, and A_1 and A_2 are positive constants. The linearized equation of (1.8) at the equilibrium $x = 0$ also takes the form of Eq. (1.5). Bélair and Campbell [1] analyze the linearized stability of (1.8) and study both single and double Hopf bifurcations.

The third example is the one considered in Ruiz Claeyssen [26]:

$$\dot{x}(t) = -A_1 x(t-\tau_1) - A_2 x(t-\tau_2) + x^3(t), \tag{1.9}$$

where $A_1 = A_2 = 1/2$. Ruiz Claeyssen studies the Hopf bifurcation in (1.9) and the stability of the bifurcating periodic solutions. Other examples can be found in Hale [14], Nussbaum [25], Stech [27], and the references cited therein.

The purpose of this paper is to study the two delay equation (1.1) under the assumption (1.4). First, we investigate the local stability of the zero solution of Eq. (1.1) by analyzing the corresponding characteristic equation of the linearized equation (1.5). General stability criteria involving the delays and the parameters are obtained. Second, by choosing one of the delays as a bifurcation parameter, we show that the two delay equation exhibits Hopf bifurcation. Then we discuss the properties of the bifurcating periodic solutions by using the center manifold theorem and the normal form theory. It is shown that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are orbitally stable under certain conditions. Finally, as an example, we analyze Eq. (1.8), the simple motor control equation. Our results improve some of the results obtained by Bélair and Campbell [1].

The following Rouché theorem on the continuity of the roots of an equation as a function of parameters will be needed throughout the paper in analyzing the characteristic equation of the linearized equation (1.5). For a proof, we refer to Dieudonné [12, p. 248].

ROUCHÉ'S THEOREM. *Let A be an open set in $\mathscr{C}$, the set of complex numbers, F a metric space, f a continuous complex valued function in $A \times F$, such that, for each $\alpha \in F$, $z \to f(z, \alpha)$ is analytic in A. Let B be an open set* *of A, whose closure $\overline{B}$ in $\mathscr{C}$ is compact and contained in A, and let $\alpha_0 \in F$ be such that no zero of $f(z, \alpha_0)$ is on the frontier of B. Then there exists a neighborhood W of α_0 in F such that*:

(i) *for any $\alpha \in W$, $f(z, \alpha)$ has no zeros on the frontier of B*;

(ii) *for any $\alpha \in W$, the sum of the orders of the zeros of $f(z, \alpha)$ belonging to B is independent of α.*

2. LOCAL STABILITY ANALYSIS

The characteristic equation of (1.5) is

$$z = -A_1 e^{-z\tau_1} - A_2 e^{-z\tau_2}. \tag{2.1}$$

Since, as observed by Braddock and van den Driessche [6], both delay terms are equally important, we do not scale the time to let one of the delays be equal to 1. Rather, as did Bélair and Campbell [1], we scale the variable so that one of the coefficients A_i will be equal to 1. Let

$$\lambda = \frac{z}{A_1}, \qquad A = \frac{A_2}{A_1}, \qquad r_1 = A_1\tau_1, \qquad r_2 = A_1\tau_2.$$

We obtain the normalized characteristic equation

$$\lambda = -e^{-\lambda r_1} - Ae^{-\lambda \tau_2}. \tag{2.2}$$

When $A = 0$, we can easily prove the following result.

LEMMA 2.1. *The transcendental equation*

$$\lambda = -e^{-\lambda r_1} \tag{2.3}$$

has purely imaginary roots if and only if $r_1 = 2j\pi + \frac{\pi}{2}$ $(j = 0, 1, 2, \ldots)$. Moreover, if $r_1 = 2j\pi + \frac{\pi}{2}$, Eq. (2.3) *has a pair of purely imaginary roots $\pm i$ which are simple.*

Denote $r_1^j = 2j\pi + \frac{\pi}{2}$ $(j = 0, 1, 2, \ldots)$ and let $\lambda_j(r_1)$ be the root of Eq. (2.3) satisfying $\operatorname{Re} \lambda_j(r_1^j) = 0$, $\operatorname{Im} \lambda_j(r_1^j) = 1$. Then we have

$$\left.\frac{d \operatorname{Re} \lambda_j(r_1)}{dr_1}\right|_{r_1 = r_1^j} = \frac{1}{1 + \left(2j\pi + \frac{\pi}{2}\right)^2}. \tag{2.4}$$

The proof of the following lemma can be found in Cooke and van den Driessche [10]; see also Cooke and Grossman [9].

LEMMA 2.2. *If* $r_1 \in [0, \frac{\pi}{2})$, *then all roots of Eq.* (2.3) *have strictly negative real parts. If* $r_1 \in (2j\pi + \frac{\pi}{2}, 2(j+1)\pi + \frac{\pi}{2}]$, *then Eq.* (2.3) *has exactly* $2j$ *roots with strictly positive real roots.*

By using Lemmas 2.1 and 2.2, we can prove the following lemma.

LEMMA 2.3. *For any* $r_1 > \frac{\pi}{2}$ *with* $r_1 \neq 2j\pi + \frac{\pi}{2}$ *and fixed* $r_2 > 0$, *there is a* $\delta > 0$, *such that when* $A = A_2/A_1 < \delta$ *Eq.* (2.2) *has at least one root with positive real part.*

Proof. Define

$$h(\lambda, A) = \lambda + e^{-\lambda r_1} + Ae^{-\lambda r_2}.$$

Then $h(\lambda, A)$ is an analytic function in λ and A. By Lemma 2.1, when $r_1 \neq 2j\pi + \frac{\pi}{2}$ the function $h(\lambda, 0)$ has no zeros on the boundary of Ω, where $\Omega = \{\lambda \mid \operatorname{Re} \lambda \geq 0, |\lambda| \leq 2\}$. Thus, Rouché's theorem implies that there exists a $\delta > 0$ such that, when $A < \delta$, $h(\lambda, A)$ and $h(\lambda, 0)$ have the same sum of the orders of zeros.

It follows from Lemma 2.2 that when $r_1 > \frac{\pi}{2}$ the sum of the orders of the zeros of $h(\lambda, 0)$ is at least 2. Thus, when $r_1 > \frac{\pi}{2}$, $r_1 \neq 2j\pi + \frac{\pi}{2}$, and $A < \delta$, the sum of the orders of the zeros of $h(\lambda, A)$ is also at least 2. This proves the lemma. ∎

LEMMA 2.4. *Suppose* $A \in (0, 1)$ *and* $r_1 \leq \frac{1}{1+A}$. *Then all roots of Eq.* (2.2) *have strictly negative real parts.*

Proof. Since all roots of Eq. (2.2) have negative real parts when $r_1 = 0$, if the conclusion fails, then there must be some $r_1 \in (0, \frac{1}{1+A}]$ such that Eq. (2.2) has purely imaginary roots $\pm i\omega$ ($\omega > 0$) satisfying

$$\begin{aligned} \cos \omega r_1 &= -A \cos \omega r_2 \\ \omega - \sin \omega r_1 &= A \sin \omega r_2. \end{aligned} \tag{2.5}$$

Adding up the squares of both equations, we have

$$\omega^2 - 2\omega \sin \omega r_1 + 1 = A^2,$$

that is,

$$g(\omega) \triangleq \frac{\omega^2 + 1 - A^2}{2\omega} = \sin \omega r_1. \tag{2.6}$$

Since $|\sin \omega r_1| \le 1$, it follows that $\omega \in [1 - A, 1 + A]$. On the other hand,

$$\begin{aligned} g(\omega) &= \frac{1}{2}\omega\left[1 + \frac{1 - A^2}{\omega^2}\right] \\ &\ge \frac{1}{2}\omega\left[1 + \frac{1 - A^2}{(1 + A)^2}\right] \\ &\ge \omega\frac{1}{1 + A} \\ &\ge \omega r_1 \\ &> \sin \omega r_1, \end{aligned}$$

a contradiction. Thus, all roots of Eq. (2.2) must have negative real parts. ∎

Applying Lemmas 2.3 and 2.4 to Eq. (1.5), we have the following results about the local stability of the zero solution of Eq. (1.1).

THEOREM 2.5. *For Eq.* (1.1) *under the assumption* (1.4), *we have*

(i) *for any* $\tau_1 > \pi/2A_1$ *with* $\tau_1 \ne (2j\pi + \pi/2)/A_1$ $(j = 1, 2, \dots)$ *and fixed* $\tau_2 > 0$, *there exists a* $\delta > 0$ *such that when* $A_2/A_1 < \delta$ *the zero solution of Eq.* (1.1) *is unstable*;

(ii) *when* $A_2 < A_1$ *and* $\tau_1 \le 1/(A_1 + A_2)$, *the zero solution of Eq.* (1.1) *is asymptotically stable*.

3. THE HOPF BIFURCATION

In this section, we shall study the Hopf bifurcation of Eq. (1.1) by choosing one of the delays as a bifurcation parameter. First, we would like to know when Eq. (2.2) has purely imaginary roots $\pm i\omega$ $(\omega > 0)$. Clearly, if $\pm i\omega$ are roots of Eq. (2.2), then (2.5) and hence (2.6) holds. We shall consider three cases: (a) $A = A_2/A_1 > 1$; (b) $A < 1$; and (c) $A = 1$.

3.1. $A > 1$

In this case, the function $g(\omega)$ defined by (2.6) has the following properties (see Fig. 3.1):

(1) $g(\omega)$ is strictly monotonically increasing and convex on $[0, +\infty)$ and $\lim_{\omega \to 0} g(\omega) = -\infty$, $\lim_{\omega \to +\infty} g(\omega) = +\infty$;

(2) $g(A + 1) = 1$, $g(A - 1) = -1$, and $g(\sqrt{A^2 - 1}) = 0$;

(3) $\omega - A \le g(\omega) \le \frac{\omega}{1 + A}$ if $\omega \in [A - 1, A + 1]$.

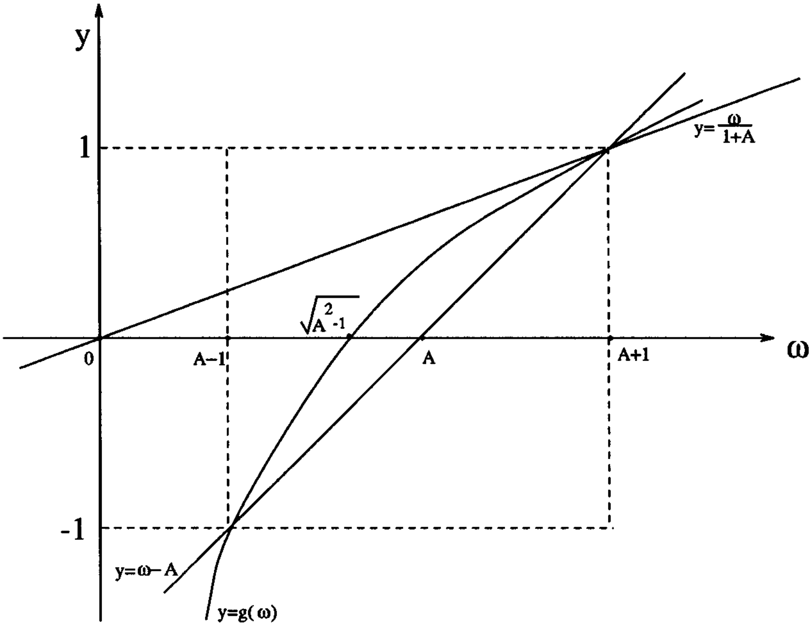

FIG. 3.1. The graph of $g(\omega)$ when $A > 1$.

Clearly, $g(\omega)$ intersects $\sin \omega r_1$ only in the rectangle bounded by $y = \pm 1$ and $\omega = A \pm 1$; that means, if Eq. (2.2) has purely imaginary roots $\pm\omega_0$, then $\omega_0 \in [A - 1, A + 1]$.

The above properties of $g(\omega)$ can be summarized into the following lemma.

LEMMA 3.1. *For $A > 1$, we have*

(i) *if $r_1 < \frac{5\pi}{2(A+1)}$, then Eq.* (2.6) *has a unique solution $\omega_0 \in [A - 1, A + 1]$;*

(ii) *if $r_1 \geq \frac{5\pi}{2(A+1)}$, then Eq.* (2.6) *has at least two solutions in $[A - 1, A + 1]$.*

LEMMA 3.2. *If $A > 1$, then for any $r_1 \geq 0$ all roots of the equation*

$$\lambda = -e^{-\lambda r_1} - A \tag{3.1}$$

have strictly negative parts.

Clearly, when $A > 1$ and $r_1 \geq 0$, Eq. (3.1) has neither purely imaginary roots nor roots with positive real part; the lemma thus follows.

For $r_1 < \frac{5\pi}{2(A+1)}$, since $A > 1$, it follows that

$$\cos \omega_0 r_1 = -A \cos \omega_0 r_2 \tag{3.2}$$

has a solution r_2^0, where ω_0 is defined in Lemma 3.1(i).

For $r_1 \geq \frac{5\pi}{2(A+1)}$, Lemma 3.1(ii) implies that Eq. (2.6) has at least two solutions, denoted by $\omega_1, \omega_2, \ldots, \omega_m$ $(m \geq 2)$. It follows from $A > 1$ that the equation

$$\cos \omega_j r_1 = -A \cos \omega_j r_2, \qquad j = 1, 2, \ldots, m \tag{3.3}$$

has a solution $r_2^{(j)}$. Set $\bar{r}_2 = \min\{r_2^{(1)}, \ldots, r_2^{(m)}\}$.

LEMMA 3.3. *Let r_2^0 and $\bar{r}_2$ be defined in* (3.2) *and* (3.3), *respectively*.

(i) *Suppose $r_1 < \frac{5\pi}{2(A+1)}$. If $r_2 \in [0, r_2^0)$, then all roots of Eq.* (2.2) *have strictly negative real parts; if $r_2 = r_2^0$, then Eq.* (2.2) *has a pair of purely imaginary roots and all other roots have strictly negative real parts.*

(ii) *Suppose $r_1 \geq \frac{5\pi}{2(A+1)}$. If $r_2 \in [0, \bar{r}_2)$, then all roots of Eq.* (2.2) *have strictly negative real parts; if $r_2 = \bar{r}_2$, then Eq.* (2.2) *has a pair of purely imaginary roots and all other roots have strictly negative real parts.*

Proof. We only prove the statement (i); statement (ii) can be proved similarly. By the definition of r_2^0, it follows that when $r_2 = r_2^0$ Eq. (2.2) has a unique pair of purely imaginary roots and when $r_2 < r_2^0$ Eq. (2.2) has no purely imaginary roots. On the other hand, if (2.2) has a root λ with positive real part, then we must have $|\lambda| < 2 + A$. Denote

$$\Omega_1 = \{\lambda \in \mathscr{C} | \operatorname{Re} \lambda \geq 0, |\lambda| \leq 2 + A\}.$$

Then all roots of Eq. (2.2) which have positive real parts lie in the interior of Ω_1.

By Lemma 3.2, when $r_2 = 0$ the sum of the orders of the roots of Eq. (2.2) is zero in Ω_1. Thus, Rouché's theorem implies that for $r_2 \in [0, r_2^0)$ Eq. (2.2) has no root in Ω_1. This completes the proof of Lemma 3.3. ∎

From Lemma 3.3, it seems that when $r_2 = r_2^0$ Eq. (1.1) may exhibit the Hopf bifurcation. To verify this, we need to consider the transversality condition.

LEMMA 3.4. *For any $r_1 > 0$, if $A > 1$ satisfies the condition*

$$\frac{\pi}{2r_1} < \sqrt{A^2 - 1} < \frac{3\pi}{2r_1}, \tag{3.4}$$

then there exists an $r_2^0 > 0$ *such that*

(i) *for* $r_2 \in [0, r_2^0)$ *all roots of Eq.* (2.2) *have strictly negative real parts and*

(ii) *for* $r_2 = r_2^0$ *Eq.* (2.2) *has a unique pair of purely imaginary roots* $\pm i\omega_0$ *and all other roots have strictly negative real parts, where* $\omega_0 r_2^0 < \frac{\pi}{2}$.

Proof. If $\pm i\omega$ are roots of Eq. (2.2), then by the property of $g(\omega)$ that $g(\sqrt{A^2 - 1}) = 0$ there exists an $\omega_0 \in (\sqrt{A^2 - 1}, \pi/r_1)$ such that

$$g(\omega_0) = \frac{\omega_0^2 + 1 - A^2}{2\omega_0} = \sin \omega_0 r_1.$$

It follows from (3.4) that $\omega_0 r_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and hence $\cos \omega_0 r_1 < 0$. Let

$$r_2^0 = \frac{1}{\omega_0} \arccos\left(-\frac{\cos \omega_0 r_1}{A}\right). \tag{3.5}$$

If $A > 1$ satisfies (3.4), then for $r_1 = r_2^0$ Eq. (2.5) has a solution ω_0; i.e., $\pm i\omega_0$ is the unique pair of purely imaginary roots of (2.2) when $r_2 = r_2^0$. By (3.5), we can see that $\omega_0 r_2^0 < \frac{\pi}{2}$.

If $r_2 = 0$, then Lemma 3.2 implies that all roots of Eq. (2.2) have strictly negative real parts and when $r_2 < r_2^0$ Eq. (2.2) has no purely imaginary roots. By using an argument similar to that in the proof of Lemma 3.3, we can show that if $r_2 \in [0, r_2^0)$, then all roots of Eq. (2.2) have strictly negative real parts and if $r_2 = r_2^0$, then Eq. (2.2) has a unique pair of purely imaginary roots and all other roots have strictly negative real parts.

Next, we show that $\pm i\omega_0$ are simple roots of Eq. (2.2). From the above analysis we know that $\omega_0 r_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\omega_0 r_2^0 \in (0, \frac{\pi}{2})$. Thus, $r_1 > r_2^0$. Set

$$h(\lambda) = \lambda + e^{-\lambda r_1} + Ae^{-\lambda r_2^0}.$$

We have

$$\frac{dh(\lambda)}{d\lambda} = 1 - r_1 e^{-\lambda r_1} - A r_2^0 e^{-\lambda r_2^0}$$

and

$$\frac{dh(i\omega_0)}{d\lambda} = 1 - r_1(\cos \omega_0 r_1 - i \sin \omega_0 r_1) - A r_2^0 \left(\cos \omega_0 r_2^0 - i \sin \omega_0 r_2^0\right).$$

Notice that $\cos \omega_0 r_1 = -A \cos \omega_0 r_2^0$, $\omega_0 r_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and $r_1 > r_2^0$; we have

$$\frac{d}{d\lambda} \operatorname{Re} h(i\omega_0) = 1 - \left(r_1 - r_2^0\right) \cos \omega_0 r_1 > 0,$$

that is, $dh(i\omega_0)/d\lambda \neq 0$. Hence, $\pm i\omega_0$ are simple roots of Eq. (2.2) when $r_2 = r_2^0$. ∎

Let $\lambda(r_2) = \alpha(r_2) + i\omega(r_2)$ be the root of Eq. (2.2) satisfying

$$\alpha\left(r_2^0\right) = 0, \qquad \omega\left(r_2^0\right) = \omega_0.$$

LEMMA 3.5. *Under the hypothesis of Lemma* 3.4, *we have*

$$\alpha'(r_2)|_{r_2=r_2^0} = \frac{\omega_0 A\left[\sin \omega_0 r_2^0 + \omega_0 r_1 \cos \omega_0 r_2^0\right]}{\left[1 + \left(r_2^0 - r_1\right)\cos \omega_0 r_1\right]^2 + \left[\omega_0 r_2^0 - \left(r_2^0 - r_1\right)\sin \omega_0 r_1\right]^2} > 0.$$

Proof. Differentiating with respect to r_2 on both sides of Eq. (2.2) gives

$$\frac{d\lambda(r_2)}{dr_2} = \frac{A\lambda e^{-\lambda r_2}}{1 - r_1 e^{-\lambda r_1} - Ar_2 e^{-\lambda r_2}}.$$

It follows from (2.5) that

$$\begin{aligned}\alpha'(r_2)|_{r_2=r_2^0} &= \frac{d}{dr_2}\operatorname{Re}\lambda(r_2)|_{r_2=r_2^0} \\ &= \frac{\omega_0(\omega_0 - \omega_0 r_1 \cos \omega_0 r_1 - \sin \omega_0 r_1)}{\left[1 + \left(r_2^0 - r_1\right)\cos \omega_0 r_1\right]^2 + \left[\omega_0 r_2^0 - \left(r_2^0 - r_1\right)\sin \omega_0 r_1\right]^2} \\ &= \frac{\omega_0 A\left[\sin \omega_0 r_2^0 + \omega_0 r_1 \cos \omega_0 r_2^0\right]}{\left[1 + (r_2 - r_1)\cos \omega_0 r_1\right]^2 + \left[\omega_0 r_2^0 - \left(r_2^0 - r_1\right)\sin \omega_0 r_1\right]^2} \\ &> 0,\end{aligned}$$

follows from the fact that $\omega_0 r_2^0 < \frac{\pi}{2}$. ∎

Applying Lemmas 3.4 and 3.5 to Eq. (1.1), we have

THEOREM 3.6. *For any* $\tau_1 > 0$, *if* $A_2 > A_2$, *and*

$$\frac{\pi}{2\tau_1} < \sqrt{A_2^2 - A_1^2} < \frac{3\pi}{2\tau_1}, \tag{3.6}$$

then there exists a $\tau_2^0 > 0$ *such that, for* $\tau_2 \in [0, \tau_2^0)$, *the zero solution of Eq.* (1.1) *is asymptotically stable. When* $\tau_2 = \tau_2^0$ *Eq.* (1.1) *exhibits the Hopf bifurcation, where* $\tau_2^0 = r_2^0/A_1$ *and* r_2^0 *is defined in* (3.5).

3.2. $A < 1$

In this case, the function $g(\omega)$ defined by (2.6) has the following properties (see Fig. 3.2):

(1) $g(\omega)$ attains its minimum value $\sqrt{1-A^2}$ when $\omega = \sqrt{1-A^2}$ and $g(1-A) = g(1+A) = 1$;

(2) $g(\omega)$ is a concave upward function and is strictly monotonically decreasing if $\omega \in (0, \sqrt{1-A^2})$ and strictly monotonically increasing if $\omega \in (\sqrt{1-A^2}, \infty)$. Moreover, $\lim_{\omega \to 0} g(\omega) = \lim_{\omega \to \infty} g(\omega) = \infty$;

(3) $g(\omega) > \frac{\omega}{2}$, $\omega \in (0, \infty)$.

If $\pm i\omega$ $(\omega > 0)$ are roots of Eq. (2.2), then ω must satisfy (2.6). From Fig. 3.2 we can see that solutions lie in $[1-A, 1+A]$. Also, from Fig. 3.2 we can see that, when $r_1 \geq 0$ is sufficiently small, $\sin r_1\omega$ and $g(\omega)$ do not intersect; when $r_1 \geq \frac{\pi}{2(1+A)}$, $\sin r_1\omega$ and $g(\omega)$ intersect at least twice. Set

$$r_1^0 = \min\{r_1 \mid \sin r_1\omega \text{ intersects } g(\omega)\}. \tag{3.7}$$

It follows that $r_1^0 > 0$, and, when $r_1 = r_1^0$, $\sin r_1\omega$ and $g(\omega)$ intersect exactly once; when $r_1 > r_1^0$, $\sin r_1\omega$ and $g(u)$ intersect at least twice. Clearly, for any $r_1 \geq r_1^0$, the equation $g(\omega) = \sin r_1\omega$ has finitely many solutions, denoted by $\omega_1, \omega_2, \ldots, \omega_m$. The first property of $g(\omega)$ implies

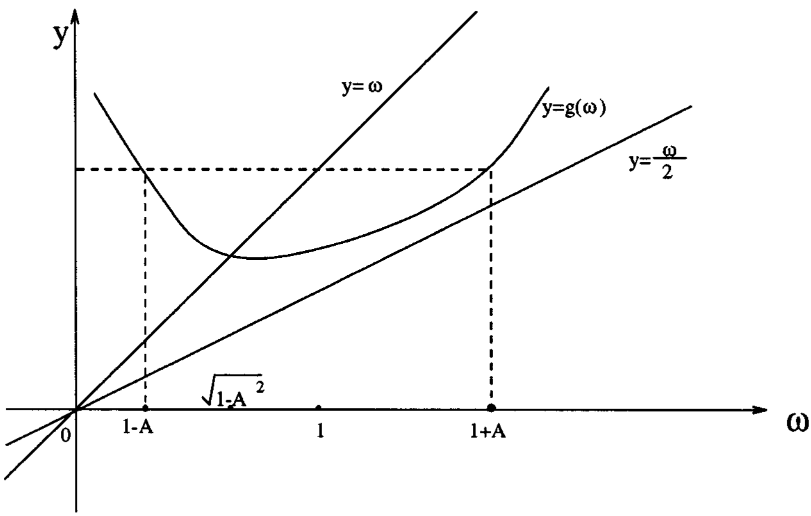

FIG. 3.2. The graph of $g(\omega)$ when $A < 1$.

that

$$g(\omega_i) = \sin r_1 \omega_i \geq \sqrt{1 - A^2}, \qquad i = 1, 2, \dots, m.$$

It then follows that

$$0 \leq \frac{|\cos r_1 \omega_i|}{A} = \frac{\sqrt{1 - \sin^2 r_1 \omega_i}}{A} \leq \frac{\sqrt{1 - (1 - A^2)}}{A} = 1.$$

Thus,

$$r_2^i = \frac{1}{\omega_i} \arccos\left(- \frac{\cos r_1 \omega}{A} \right) \tag{3.8}$$

is well defined and $r_2^i \omega_i \in [0, \pi)$. Denote

$$r_2^0 = \min\{r_2^1, r_2^2, \dots, r_2^m\}. \tag{3.9}$$

We have the following lemma.

LEMMA 3.7. *Let*

$$\bar{r}_1 = \frac{\arcsin\sqrt{1 - A^2}}{\sqrt{1 - A^2}}. \tag{3.10}$$

(i) *If* $r_1 \in [0, \bar{r}_1)$, *then all roots of the equation*

$$\lambda = -e^{-\lambda r_1} - A \tag{3.11}$$

have strictly negative real parts.

(ii) *If* $r_1 > \bar{r}_1$, *then at least one root of the equation* (3.11) *has positive real part.*

LEMMA 3.8. *Suppose* r_1^0, $\bar{r}_1$, *and* r_2^0 *are defined in* (3.7), (3.10), *and* (3.9), *respectively.*

(i) *If* $r_1 \in [0, r_1^0)$, *then all roots of Eq.* (2.2) *have strictly negative real parts.*

(ii) *If* $r_1 \in [r_1^0, \bar{r}_1)$, $r_2 \in [0, r_2^0)$, *then all roots of Eq.* (2.2) *have strictly negative real parts; if* $r_2 = r_2^0$, *then Eq.* (2.2) *has a unique pair of simply purely imaginary roots and all other roots have strictly negative real parts.*

Proof. (i) $\pm i\omega$ are roots of the equation (2.2) if and only if ω is a root of Eq. (2.6). By the definition of r_1^0, it follows that if $r_1 \in [0, r_1^0)$, then Eq. (2.6) has no solutions and thus Eq. (2.2) has no purely imaginary roots. If $r_1 = 0$, then Eq. (2.2) has no roots with positive real part for any $r_2 \geq 0$.

Therefore, Rouché's theorem implies that, for any $r_2 \geq 0$, if $r_1 \in [0, r_1^0)$, then all roots of Eq. (2.2) have negative real parts.

(ii) It follows from (3.8) and (3.9) that there exists a $j \in \{1, 2, \ldots, m\}$ such that

$$r_2^0 = \frac{1}{\omega_j} \arccos\left(-\frac{\cos r_1 \omega_j}{A}\right).$$

Denote $\omega_0 = \omega_j$. By Lemma 3.7, if $r_1 \in [r_1^0, \bar{r}_1)$ and $r_2 = 0$, then all roots of Eq. (2.2) have strictly negative real parts. By the definition of r_2^0, if $r_2 \in [0, r_2^0)$, then Eq. (2.2) has no purely imaginary roots. Rouché's theorem again implies that for any $r_2 \in [0, r_2^0)$ all roots of Eq. (2.2) have negative real parts.

The definition of r_2^0 also implies that, when $r_2 = r_2^0$, $\pm i\omega$ is a unique pair of purely imaginary roots of Eq. (2.2) and all other roots have strictly negative real parts. When $r_2^0 \omega_0 \in (0, \pi)$, we have $\sin r_1 \omega_0 > 0$. Denote $h(\lambda) = \lambda + e^{-\lambda r_1} + e^{-\lambda r_2^0}$. Using arguments similar to those in the proof of Lemma 3.4, we have

$$\frac{d}{d\lambda} \operatorname{Im} h(i\omega_0) = r_1 \sin r_1 \omega_0 + A r_2^0 \sin r_2^0 \omega_0 > 0,$$

that is, $dh(i\omega_0)/d\lambda \neq 0$. Thus, $\pm i\omega$ are simple roots of Eq. (2.2) when $r_2 = r_2^0$. ∎

For $r_1 \in [r_1^0, \bar{r}_1)$, let

$$\lambda(r_2) = \alpha(r_2) + i\omega(r_2)$$

be the solution of Eq. (2.2) satisfying

$$\alpha\left(r_2^0\right) = 0, \qquad \omega\left(r_2^0\right) = \omega_0.$$

Similar to the proof of Lemma 3.5, we can prove the following lemma.

LEMMA 3.9. *If* $u = r_2^0 \omega_0$ *is not a root of the equation* $\tan u = -u$ *on* $(\frac{\pi}{2}, \pi)$, *then*

$$\alpha'(r_2)|_{r_2 = r_2^0} \neq 0.$$

Now, we shall derive some conditions to ensure that $u = r_2^0 \omega_0$ is not a root of the equation $\tan u = -u$ on $(\frac{\pi}{2}, \pi)$.

LEMMA 3.10. *Suppose* $\bar{r}_1 > \frac{\pi}{2(1+A)}$. *If* $r_1 \in [\frac{\pi}{2(1+A)}, \bar{r}_1)$, *then*

$$\alpha'(r_2)|_{r_2 = r_2^0} > 0.$$

Proof. Since $r_1 \geq \frac{\pi}{2(1+A)}$, it follows that Eq. (2.6) has at least one solution ω_j satisfying $r_1 \omega_j \in [\frac{\pi}{2}, \pi)$. Thus, (3.8) and (3.9) imply that $r_2^0 \omega_0 \in (0, \frac{\pi}{2}]$. The conclusion follows from the same argument as in the proof of Lemma 3.5. ∎

Notice that in the above proof $r_2^0 \omega_0 \in (0, \frac{\pi}{2}]$; this certainly indicates that $r_2^0 \omega_0$ is not a solution of the equation $\tan u = -u$ on the interval $(\frac{\pi}{2}, \pi)$.

Applying the above lemmas to Eq. (1.1), we have the following.

THEOREM 3.11. *Assume that* r_1^0, r_2^0, *and* $\bar{r}_1$ *are defined by* (3.7), (3.9), *and* (3.10), *respectively. Denote* $\tau_1^0 = r_1^0/A_1$, $\tau_2^0 = r_2^0/A_1$, *and* $\bar{\tau}_1 = \bar{r}_1/A_1$.

(i) *If* $\tau_1 \in [0, \tau_1^0)$, *then the trivial solution of Eq.* (1.1) *is asymptotically stable.*

(ii) *Suppose* $\tau_1^0 < \bar{\tau}_1$. *If* $\tau_1 \in [\tau_1^0, \bar{\tau}_1)$ *and* $\tau_2 \in [0, \tau_2^0)$, *then the trivial solution of Eq.* (1.1) *is asymptotically stable*; *if* $u = A_1 \tau_2^0 \omega_0$ *is not a root of the equation* $\tan u = -u$ *on* $(\frac{\pi}{2}, \pi)$, *then* $\tau_2 = \tau_2^0$ *is the Hopf bifurcation point for Eq.* (1.1).

(iii) *Suppose* $\bar{\tau}_1 > \pi/2(A_1 + A_2)$. *If* $\tau_1 \in [\pi/2(A_1 + A_2), \bar{\tau}_1]$ *and* $\tau_2 = \tau_2^0$, *then* $\tau_2 = \tau_2^0$ *is the Hopf bifurcation point for Eq.* (1.1).

3.3. $A = 1$

In this case, Eq. (2.2) becomes

$$\lambda = -e^{-\lambda r_1} - e^{-\lambda r_2}. \tag{2.2a}$$

$\pm i\omega$ $(\omega > 0)$ are solutions of (2.2a) if and only if ω satisfies the following equations:

$$\begin{aligned} \omega - \sin r_1 \omega &= \sin r_2 \omega \\ \cos r_1 \omega &= \cos r_2 \omega. \end{aligned} \tag{2.5a}$$

Thus, the necessary condition for $\pm i\omega$ $(\omega > 0)$ to be solutions of (2.2a) is

$$\frac{\omega}{2} = \sin r_1 \omega. \tag{2.6a}$$

Obviously, all positive solutions of Equation (2.6a) lie on $(0, 2]$ and, for $\frac{1}{2} < r_1 \leq \frac{5\pi}{4}$, Eq. (2.6a) has exactly one positive solution; when $r_1 > \frac{5\pi}{4}$, it has at least two positive solutions (see Fig. 3.3).

For $r_1 > \frac{1}{2}$ denote the positive solutions of Eq. (2.6a) as

$$\omega_0 < \omega_1 < \cdots < \omega_m.$$

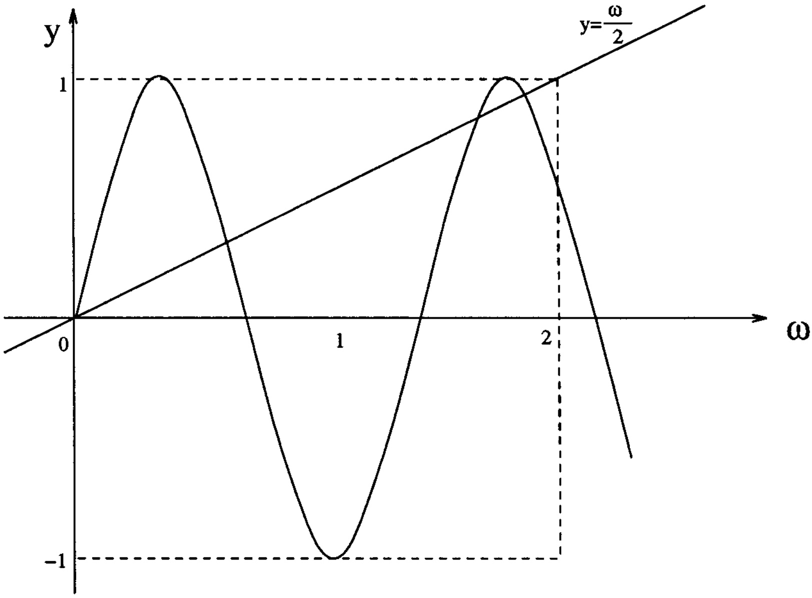

FIG. 3.3. The graph of $g(\omega)$ when $A = 1$.

For each ω_i, set

$$r_2^i = \frac{1}{\omega_i}\arccos(-\cos r_1\omega_i). \tag{3.12}$$

We can show that $r_2^i\omega_i \in (0, \pi]$,

$$r_2^0 = \min_{0\le i\le m}\{r_2^i\}, \tag{3.13}$$

and $r_2^0\omega_0 \in (0, \frac{\pi}{2}]$.

As argued in Sections 3.1 and 3.2, we have the following lemmas.

LEMMA 3.12. *All roots of the equation*

$$\lambda = -e^{-\lambda r_1} - 1$$

have strictly negative parts.

LEMMA 3.13. (i) *If* $r_1 \in [0, \frac{1}{2}]$, *then for any* $r_2 \ge 0$ *all roots of Eq.* (2.2a) *have strictly negative real parts.*

(ii) *For* $r_1 > \frac{1}{2}$, *there exists an* r_2^0 *defined by* (3.12) *such that if* $r_2 \in [0, r_2^0)$, *all roots of Eq.* (2.2a) *have strictly negative real parts*; *if* $r_2 = r_2^0$, *then Eq.* (2.2a) *has a unique pair of purely imaginary roots and all other roots have strictly negative real parts.*

LEMMA 3.14. *For* $r_1 \geq \frac{1}{2}$, *let* $\lambda(r_2) = \alpha(r_2) + i\omega(r_2)$ *be the solutions of Eq.* (2.2a) *satisfying* $\alpha(r_2^0) = 0$ *and* $\omega(r_2^0) = \omega_0$. *Then*

$$\left.\frac{d\alpha(r_2)}{dr_2}\right|_{r_2 = r_2^0} > 0.$$

Remark 3.15. The above analysis together with the implicit function theorem gives us the distribution of the roots of Eq. (2.2a) in the (r_1, r_2) plane (see Fig. 3.4). If (r_1, r_2) lies in the region bounded by the curve l and the r_1, r_2 axes, then all roots of Eq. (2.2a) have strictly negative real parts. If (r_1, r_2) lies on the curve l passing through the point $(\frac{\pi}{4}, \frac{\pi}{4})$, then Eq. (2.2a) has a unique pair of simply purely imaginary roots, all other roots have strictly negative real parts and the transversality condition is satisfied.

We should mention that the result of the case when $A_1 = A_2$ was also obtained by Ruiz Claeyssen [26] and Hale [14].

Applying Lemmas 3.13 and 3.14 to Eq. (1.1), we obtain the following theorem.

THEOREM 3.16. *Suppose* $A_1 = A_2$.

(i) *If* $\tau_1 \in [0, 1/(2A_1)]$, *then, for any* $\tau_2 \geq 0$, *the trivial solution of Eq.* (1.1) *is asymptotically stable.*

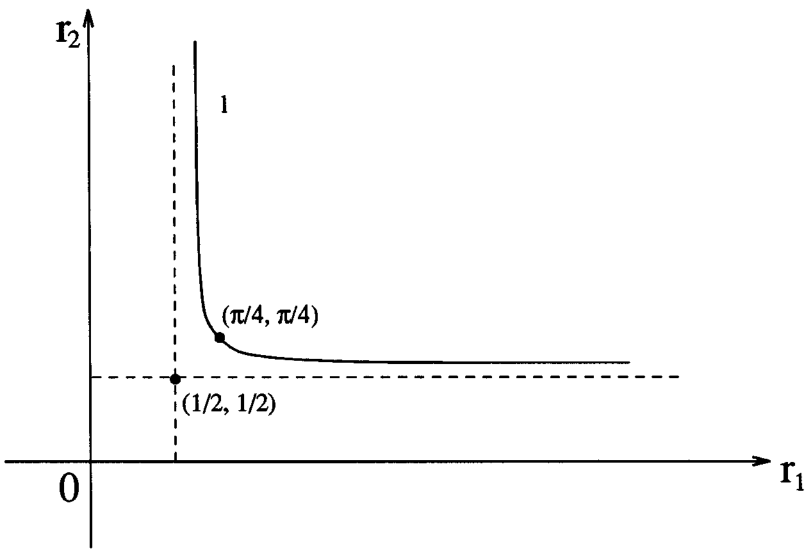

FIG. 3.4. The distribution of roots of (2.2) in the (r_1, r_2) plane.

(ii) *For* $\tau_1 > 1/(2A_1)$, *there exists a* $\tau_2^0 = r_2^0/A_1$ *such that if* $\tau_2 \in [0, \tau_2^0)$, *then the trivial solution of Eq.* (1.1) *is asymptotically stable*; *if* $\tau_2 = \tau_2^0$, *then Eq.* (1.1) *exhibits the Hopf bifurcation.*

4. STABILITY OF THE HOPF BIFURCATION

In this section, we shall use the normal form theory introduced in Hassard *et al.* [18] to study the stability of the bifurcating periodic solutions.

Without loss of generality, assume $\tau_1 > \tau_2^0$ and define the phase space as

$$C = C([-\tau_1, 0], R)$$

associated with the norm $|\phi| = \sup_{-\tau_1 \le \theta \le 0} |\phi(\theta)|$ for $\phi \in C$.

The expansion of Eq. (1.1) at the trivial solution is

$$\dot{x}(t) = -A_1 x(t-\tau_1) - A_2 x(t-\tau_2) + F(x(t), x(t-\tau_1), x(t-\tau_2)), \tag{4.1}$$

where

$$\begin{aligned} F(x(t), x(t-\tau_1), x(t-\tau_2)) &= \frac{1}{2}\big[a_{11}x^2(t) + a_{22}x^2(t-\tau_1) + a_{33}x^2(t-\tau_2) \\ &\quad + 2a_{12}x(t)x(t-\tau_1) + 2a_{13}x(t)x(t-\tau_2) \\ &\quad + 2a_{23}x(t-\tau_1)x(t-\tau_2)\big] \\ &\quad + \frac{1}{3!}\big[b_{111}x^3(t) + b_{222}x^3(t-\tau_1) + b_{333}x^3(t-\tau_2) \\ &\quad + 3b_{112}x^2(t)x(t-\tau_1) + 3b_{113}x^2(t)x(t-\tau_2) \\ &\quad + 3b_{122}x(t)x^2(t-\tau_1) + 3b_{133}x(t)x^2(t-\tau_2) \\ &\quad + 6b_{123}x(t)x(t-\tau_1)x(t-\tau_2) \\ &\quad + 3b_{223}x^2(t-\tau_1)x(t-\tau_2) \\ &\quad + 3b_{233}x(t-\tau_1)x^2(t-\tau_2)\big] + O(x^4) \end{aligned}$$

and

$$a_{ij} = \frac{\partial^2 f}{\partial u_i \, \partial u_j}(0,0,0), \qquad i,j = 1,2,3;$$

$$b_{ijk} = \frac{\partial^3 f}{\partial u_i \, \partial u_j \, \partial u_k}(0,0,0), \qquad i,j,k = 1,2,3.$$

Suppose that, for (A_1, A_2, τ_1), there exists a $\tau_2^0 > 0$ at which Eq. (4.1) exhibits the Hopf bifurcation. Denote $\tau_2 = \tau_2^0 + \mu$. In the following we shall regard μ as the bifurcation parameter. For $\phi \in C$, define

$$F(\mu, \phi) = F\big(\phi(0), \phi(-\tau_1), \phi(-\tau_2)\big).$$

By the Reisz representation theorem, for any $\phi \in C^1[-\tau_1, 0]$ we have

$$-A_1 x(t - \tau_1) - A_2 x(t - \tau_2) = \int_{-\tau_1}^{0} d\eta(\theta, \mu)\phi(\theta),$$

where

$$\eta(\theta, \mu) = \begin{cases} -A_2 \delta(\theta), & \theta \in (-\tau_2, 0], \\ A_1 \delta(\theta + \tau_1), & \theta \in [-\tau_1, -\tau_2]. \end{cases}$$

Set

$$L(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau_1, 0), \\ \displaystyle\int_{-\tau_1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-\tau_1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then Eq. (4.1) can be written as

$$\dot{x}_t = L(\mu) x_t + R(\mu) x_t. \tag{4.2}$$

For $\psi \in C^1[0, \tau_1]$, define

$$L^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, \tau_1], \\ \displaystyle\int_{-\tau_1}^{0} d\eta(t,0)\,\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C[-\tau_1, 0]$ and $\psi \in C[0, \tau_1]$, define the bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=-\tau_1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi.$$

Then L^* and $L = L(0)$ are adjoint operators.

By the results in Section 3, we assume that $\pm i\omega_0$ are eigenvalues of L; thus they are also eigenvalues of L^*. $q(\theta) = e^{i\omega_0\theta}$ is the eigenvector of L corresponding to $i\omega_0$; $q^*(s) = De^{i\omega_0 s}$ is the eigenvector of L^* corresponding to $-i\omega_0$. Moreover,

$$\langle q^*, q \rangle = 1, \qquad \langle q^*, \bar{q} \rangle = 0,$$

where $D = (1 - \tau_1 A_1 e^{i\omega_0\tau_1} - \tau_2^0 A_2 e^{i\omega_0\tau_2^0})^{-1}$.

Using the same notation as in Hassard *et al.* [18], we first compute the coordinates to describe the center manifold $\mathscr{C}_0$ at $\mu = 0$. Let x_t be the solution of Eq. (4.2) when $\mu = 0$. Define

$$z(t) = \langle q^*, x_t \rangle,$$

$$w(t, \theta) = x_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold $\mathscr{C}_0$ we have

$$w(t, \theta) = w(z(t), \bar{z}(t), \theta),$$

where

$$w(z, \bar{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + w_{30}\frac{z^3}{6} + \cdots.$$

z and $\bar{z}$ are local coordinates for the center manifold $\mathscr{C}_0$ in the direction of q^* and $\overline{q^*}$. Note that w is real if x_t is real. We consider only real solutions.

For solution $x_t \in \mathscr{C}_0$ of (4.1), since $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= i\omega_0 z(t) + \langle q^*(\theta), F(0, w + 2\,\mathrm{Re}\{z(t)q(\theta)\}) \rangle \\ &= i\omega_0 z(t) + \overline{q^*}(0) F(0, w(z, \bar{z}, 0) + 2\,\mathrm{Re}\{z(t)q(0)\}) \\ &\triangleq i\omega_0 z(t) + \overline{q^*}(0) F_0(z, \bar{z}). \end{aligned} \tag{4.3}$$

We rewrite this as

$$\dot{z} = i\omega_0 z(t) + g(z, \bar{z}),$$

where

$$g(z,\bar{z}) = \overline{q^*}(0)F(0,w(z,\bar{z},0) + 2\operatorname{Re}\{z(t)q(0)\})$$
$$= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{4.4}$$

By (4.2) and (4.3), we have

$$\dot{w} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$$
$$= \begin{cases} Lw - 2\operatorname{Re}\{\overline{q^*}(0)F_0q(\theta)\} & (\theta\in[-\tau_1,0)) \\ Lw - 2\operatorname{Re}\{\overline{q^*}(0)F_0q(\theta)\} + F_0 & (\theta = 0) \end{cases}$$
$$\triangleq Lw + H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta) = 2\operatorname{Re}\{g(z,\bar{z})q(\theta)\} + F(0,w + 2\operatorname{Re}\{z(t)q(\theta)\})$$
$$= H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{4.5}$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{aligned} (L - 2i\omega_0)\omega_{20}(\theta) &= -H_{20}(\theta) \\ Lw_{11}(\theta) &= -H_{11}(\theta) \\ (L + 2i\omega_0)\omega_{02}(\theta) &= -H_{02}(\theta) \\ &\cdots. \end{aligned} \tag{4.6}$$

Since $q^*(0) = D$, we have

$$\begin{aligned} g(z,\bar{z}) = \frac{\overline{D}}{2}\big[& a_{11}x^2(t) + a_{22}x^2(t-\tau_1) + a_{33}x^2(t-\tau_2^0) \\ & +2a_{12}x(t)x(t-\tau_1) + 2a_{13}x(t)x(t-\tau_2^0) \\ & +2a_{23}x(t-\tau_1)x(t-\tau_2^0)\big] \\ + \frac{\overline{D}}{3!}\big[& b_{111}x^3(t) + b_{222}x^3(t-\tau_1) + b_{333}x^3(t-\tau_2^0) \\ & +3b_{112}x^2(t)x(t-\tau_1) + 3b_{113}x^2(t)x(t-\tau_2^0) \\ & +3b_{122}x(t)x^2(t-\tau_1) + 3b_{133}x(t)x^2(t-\tau_2^0) \\ & +6b_{123}x(t)x(t-\tau_1)x(t-\tau_2^0) \\ & +3b_{223}x^2(t-\tau_1)x(t-\tau_2^0) \\ & +3b_{233}x(t-\tau_1)x^2(t-\tau_2^0)\big] + O(x^4). \end{aligned} \tag{4.7}$$

Notice that

$$
\begin{aligned}
x(t-\tau) &= w(t,-\tau) + z(t)q(-\tau) + \bar{z}(t)\bar{q}(-\tau) \\
&= w_{20}(-\tau)\frac{z^2}{2} + w_{11}(-\tau)z\bar{z} + w_{02}(-\tau)\frac{\bar{z}^2}{2} \\
&\quad + \cdots + e^{-i\omega_0\tau}z(t) + e^{i\omega_0\tau}\bar{z}(t),
\end{aligned}
$$

where $\tau = 0$, τ_1, or τ_2^0. Substituting it into (4.7) and comparing the coefficients with (4.4), we have

$$
\begin{aligned}
g_{20} &= \overline{D}M, \\
g_{11} &= \overline{D}B, \\
g_{02} &= \overline{DM}, \\
g_{21} &= \overline{D}\Big[a_{11}(2w_{11}(0) + w_{20}(0)) \\
&\quad + a_{22}\big(2w_{11}(-\tau_1)e^{-i\omega_0\tau_1} + w_{20}(-\tau_1)e^{i\omega_0\tau_1}\big) \\
&\quad + a_{33}\big(2w_{11}(-\tau_2^0)e^{-i\omega_0\tau_2^0} + w_{20}(-\tau_2^0)e^{i\omega_0\tau_2^0}\big) \\
&\quad + a_{12}\big(w_{20}(0)e^{i\omega_0\tau_1} + 2w_{11}(0)e^{-i\omega_0\tau_1} + 2w_{11}(-\tau_1) + w_{20}(-\tau_1)\big) \\
&\quad + a_{13}\big(w_{20}(0)e^{i\omega_0\tau_2^0} + 2w_{11}(0)e^{-i\omega_0\tau_2^0} + 2w_{11}(-\tau_2^0) + w_{20}(-\tau_2^0)\big) \\
&\quad + a_{23}\big(w_{20}(-\tau_1)e^{i\omega_0\tau_2^0} \\
&\qquad + 2w_{11}(-\tau_1)e^{-i\omega_0\tau_2^0} + 2w_{11}(-\tau_2^0)e^{-i\omega_0\tau_1} + w_{20}(-\tau_2^0)e^{i\omega_0\tau_1}\big) \\
&\quad + b_{111} + b_{222}e^{-i\omega_0\tau_1} + b_{333}e^{-i\omega_0\tau_2^0} + b_{112}(2e^{-i\omega_0\tau_1} + e^{i\omega_0\tau_1}) \\
&\quad + b_{113}(2e^{-i\omega_0\tau_2^0} + e^{i\omega_0\tau_2^0}) + b_{122}(e^{-2i\omega_0\tau_1} + 2) \\
&\quad + b_{133}(e^{-2i\omega_0\tau_2^0} + 2) \\
&\quad + 2b_{123}(e^{-i\omega_0(\tau_1-\tau_2^0)} + e^{i\omega_0(\tau_1-\tau_2^0)} + e^{-i\omega_0(\tau_1+\tau_2^0)}) \\
&\quad + b_{223}(e^{-i\omega_0(2\tau_1-\tau_2^0)} + 2e^{-i\omega_0\tau_2^0}) + b_{233}(e^{i\omega_0(\tau_1-2\tau_2^0)} + 2e^{-i\omega_0\tau_1})\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
M &= a_{11} + a_{22}e^{-2i\omega_0\tau_1} + a_{33}e^{-2i\omega_0\tau_2^0} + 2a_{12}e^{-i\omega_0\tau_1} \\
&\quad + 2a_{13}e^{-i\omega_0\tau_2^0} + 2a_{23}e^{-i\omega_0(\tau_1+\tau_2^0)}, \\
B &= a_{11} + a_{22} + a_{33} + a_{12}\,\mathrm{Re}\{e^{i\omega_0\tau_1}\} + 2a_{13}\,\mathrm{Re}\{e^{i\omega_0\tau_2^0}\} \\
&\quad + 2a_{23}\,\mathrm{Re}\{e^{i\omega_0(\tau_1-\tau_2^0)}\}.
\end{aligned}
$$

We still need to compute $w_{20}(\theta)$ and $w_{11}(\theta)$. For $\theta \in [-\tau_1, 0)$, we have

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}\{\overline{q^*}(0)F_0 q(\theta)\} \\ &= -gq(\theta) - \overline{gq}(\theta) \\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\right)e^{i\omega_0\theta} \\ &\quad -\left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \bar{g}_{21}\frac{\bar{z}^2 z}{2} + \cdots\right)e^{-i\omega_0\theta}. \end{aligned}$$

Comparing the coefficients with (4.5) gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}e^{i\omega_0\theta} - \bar{g}_{20}e^{-i\omega_0\theta} \\ &= -\overline{D}Me^{i\omega_0\theta} - DMe^{-i\omega_0\theta} \\ &= -2M\,\mathrm{Re}\{\overline{D}e^{i\omega_0\theta}\}, \\ H_{11}(\theta) &= -g_{11}e^{i\omega_0\theta} - \bar{g}_{11}e^{-i\omega_0\theta} \\ &= -\overline{D}Be^{i\omega_0\theta} - D\overline{B}e^{-i\omega_0\theta} \\ &= -2\,\mathrm{Re}\{\overline{D}Be^{i\omega_0\theta}\}. \end{aligned}$$

It follows from (4.6) that

$$\dot{w}_{20}(\theta) = 2i\omega_0\omega_{20}(\theta) + g_{20}e^{i\omega_0\theta} + \bar{g}_{02}e^{-i\omega_0\theta}. \tag{4.8}$$

Solving for w_{20}, we obtain

$$w_{20}(\theta) = -\frac{g_{20}}{i\omega_0}e^{i\omega_0\theta} - \frac{\bar{g}_{02}}{3i\omega_0}e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \tag{4.9}$$

and similarly

$$w_{11}(\theta) = \frac{g_{11}}{i\omega_0}e^{i\omega_0\theta} - \frac{1}{i\omega_0}\bar{g}_{11}e^{-i\omega_0\theta} + E_2, \tag{4.10}$$

where E_1 and E_2 can be determined by setting $\theta = 0$ in H. In fact, since

$$H(z,\bar{z},0) = -2\,\mathrm{Re}\{\overline{q^*}(0)F_0 q(0)\} + F_0,$$

we have

$$\begin{aligned} &H_{11}(0) = (1 - 2\,\mathrm{Re}\,D)B, \\ &H_{20}(0) = -g_{20} - \bar{g}_{02} + M = -\overline{D}M - DM + M = (1 - 2\,\mathrm{Re}\,D)M. \end{aligned} \tag{4.11}$$

It follows from the definition of L and (4.6) that

$$-A_1 w_{20}(-\tau_1) - A_2 w_{20}\left(-\tau_2^0\right) = 2i\omega_0 w_{20}(0) + (2\operatorname{Re} D - 1)M,$$
$$-A_1 w_{11}(-\tau_1) - A_2 w_{11}\left(-\tau_2^0\right) = (2\operatorname{Re} D - 1)B.$$

Substituting (4.9) and (4.10) into the above equations and noticing that $\pm i\omega_0$ are solutions of the equation

$$\lambda = -A_1 e^{-\lambda\tau_1} - A_2 e^{-\lambda(\tau_2^0+\mu)} \tag{4.12}$$

when $\mu = 0$, we obtain

$$E_1 = \frac{M}{N}, \qquad E_2 = \frac{B}{A_1 + A_2}, \tag{4.13}$$

where

$$N = 2i\omega_0 + A_1 e^{-2i\omega_0\tau_1} + A_2 e^{-2i\omega_0\tau_2^0}. \tag{4.14}$$

Based on the above analysis, we can see that each g_{ij} is determined by the parameters and delays in Eq. (1.1). Thus, we can compute the following quantities:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\alpha'(0)}, \\ T_2 &= -\frac{\operatorname{Im} c_1(0) + \mu_1 \operatorname{Im} \lambda_1'(\alpha_0)}{\omega_0}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}. \end{aligned} \tag{4.15}$$

We know that (Hassard *et al.* [18]) μ_2 determines the direction of the Hopf bifurcation [if $\mu_2 > 0$ (< 0), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau_2 > \tau_2^0$ $(< \tau_2^0)$]; β_2 determines the stability of the bifurcating periodic solutions [the bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0$ (> 0)]; and T_2 determines the period of the bifurcating periodic solutions [the period increases (decreases) if $T_2 > 0$ (< 0)]. In (4.15),

$$\lambda(\mu) = \alpha(\mu) + i\beta(\mu) \tag{4.16}$$

is a solution of Eq. (4.12) satisfying $\alpha(0) = 0$, $\omega(0) = \omega_0$. $\alpha'(0)$ and $\omega'(0)$ are the real and imaginary parts of $\lambda'(0)$, respectively.

Notice that the relation between $\lambda(\mu)$ in (4.16) and $\lambda(r_2)$ in Section 3 is that $\lambda(\mu) = A_1\lambda(r_2)$. Similarly, ω_0 in this section is the multiplication of A_1 and ω_0 in Section 3.

5. AN EXAMPLE

By using the results in Sections 2, 3, and 4, we can study the stability and bifurcation of the logistic equation (1.6), the simple motor control equation (1.8), and Eq. (1.9). As an example, we consider the following equation:

$$\begin{aligned}\dot{x}(t) &= -x(t-\tau_1) - Ax(t-\tau_2) + \tfrac{1}{3}\left[x^3(t-\tau_1) + Ax^3(t-\tau_2)\right]\\ &\quad + O\left(x^4(t-\tau_1), x^5(t-\tau_2)\right)\\ &\triangleq Lx_t + F(x_t) + O\left(x_t^5\right). \end{aligned} \tag{5.1}$$

Notice that Eq. (5.1) is a special case of Eq. (4.1) with $A_1 = 1$, $A_2 = A$, $a_{ij} = 0$ $(i, j = 1, 2, 3)$; $b_{111} = 0$, $b_{222} = 2$, $b_{333} = 2A$, $b_{ijk} = 0$ $(i, j, k = 1, 2, 3,\ i \neq j \neq k)$. Bélair and Campbell [1] study the single Hopf bifurcation of Eq. (5.1) and show that, for $\tau_1 < 1$, each branch of the Hopf bifurcation is everywhere supercritical. They also observe that, for $1 < \tau_1 < \frac{\pi}{2}$, the entire stability boundary is still supercritical; however, their theorem does not apply to this case. In the following, we shall apply the results in Section 4 to Eq. (5.1). Detailed and all possible parameter estimates will be given for the occurrence and stability of the Hopf bifurcation.

We can compute that

$$M = 0, \qquad B = 0, \qquad D = \left(1 - \tau_1 e^{i\omega_0\tau_1} - \tau_2^0 A e^{i\omega_0\tau_2^0}\right)^{-1},$$

and

$$g_{20} = g_{11} = g_{02} = 0, \qquad g_{21} = -2i\omega_0\overline{D}, \qquad c_1(0) = \frac{g_{21}}{2} = -i\omega_0\overline{D}.$$

Denote

$$\Delta = \left(1 - \tau_1 \cos\omega_0\tau_1 - \tau_2^0 A\cos\omega_0\tau_2^0\right)^2 + \left(\tau_1\sin\omega_0\tau_1 + \tau_2^0 A\sin\omega_0\tau_2^0\right)^2.$$

We have

$$c_1(0) = -\frac{\omega_0}{\Delta}\Big[\big(\tau_1 \sin \omega_0\tau_1 + \tau_2^0 A \sin \omega_0\tau_2^0\big) \\ +i\big(1 - \tau_1 \cos \omega_0\tau_1 - \tau_2^0 A \cos \omega_0\tau_2^0\big)\Big],$$

$$\mu_2 = \frac{\omega_0}{\Delta\,\alpha'(0)}\big(\tau_1 \sin \omega_0\tau_1 + \tau_2^0 A \sin \omega_0\tau_2^0\big),$$

$$\beta_2 = -\frac{2\,\omega_0}{\Delta}\big(\tau_1 \sin \omega_0\tau_1 + \tau_2^0 A \sin \omega_0\tau_2^0\big),$$

$$T_2 = \frac{1}{\Delta}\big(1 - \tau_1 \cos \omega_0\tau_1 - \tau_2^0 A \cos \omega_0\tau_2^0\big) \\ - \frac{\omega'(0)}{\Delta\,\alpha'(0)}\big(\tau_1 \sin \omega_0\tau_1 + \tau_2^0 A \sin \omega_0\tau_2^0\big).$$

By applying Theorem 3.6, (iii) of Theorem 3.11, (ii) of Theorem 3.16, and Lemma 3.5, we obtain the following bifurcation theorem for Eq. (5.1).

THEOREM 5.1. *If one of the following conditions is satisfied*:

(i) $A > 1$ *and* $\tau_1 > 0$ *satisfies* $\pi/2\tau_1 < \sqrt{A^2 - 1} < 3\pi/2\tau_1$;

(ii) $A < 1$ *and* $\bar{r}_1 > \frac{\pi}{2(1+A)}$ *such that* $\tau_1 \in [\frac{\pi}{2(1+A)}, \bar{r}_1)$, *where* $\bar{r}_1$ *is defined as in* (3.10);

(iii) $A = 1$ *and* $\tau_1 > \frac{1}{2}$;

then, at $\tau_2 = \tau_2^0$, *Eq.* (5.1) *undergoes the Hopf bifurcation*; *the Hopf bifurcation is supercritical* (*i.e., the bifurcating periodic solutions exist for* $\tau_2 > \tau_2^0$); *the bifurcating periodic solutions are orbitally asymptotically stable*; *the period of bifurcating periodic solutions is determined by*

$$T = \frac{2\pi}{\omega_0}\big(1 + T_2\,\varepsilon^2 + O(\varepsilon^4)\big),$$

where $\varepsilon = (\tau_2 - \tau_2^0)/\mu_2 + O((\tau_2 - \tau_2^0)^2)$.

6. DISCUSSION

Due to its complexity, the local and Hopf bifurcation analysis for scalar delay−differential equations with two delays is far from complete and many researchers have tried to fill in some ‘‘piece of the puzzle’’ of the two delay problem (Bélair and Campbell [1]).

In this paper, we have considered a class of two delay equations whose linearization at the zero solution takes the form

$$\dot{x}(t) = A_1 x(t - \tau_1) - A_2 x(t - \tau_2),$$

where A_1 and A_2 are positive. The logistic equation with two delays discussed by Braddock and van der Driessche [6] and Gopalsamy [13], the simple motor control equation studied by Bélair and Campbell [1] and Beuter *et al*. [4, 5], and some of the equations considered in Hale [14], Ruiz Claeyssen [26], Nussbaum [25], and Stech [27] are examples of such a class of equations.

By analyzing the corresponding characteristic equation, we have obtained some sufficient conditions on the stability and instability of the zero solution. Then we fixed the first delay τ_1 and increased the second delay τ_2 from zero to show that there exists a first critical value of τ_2 at which the zero solution loses its stability and the Hopf bifurcation occurs. The detailed local and Hopf bifurcation analysis was completed by classifying the parameters A_1 and A_2 into three possible cases: (a) $A_1 > A_2$, (b) $A_1 < A_2$, and (c) $A_1 = A_2$. The direction of the Hopf bifurcation and its stability for the perturbed equation were studied by using the normal form introduced by Hassard *et al*. [18]. As an application, we considered a simple moto control equation and extended the result of Bélair and Campbell [1].

Our results can be used to analyze some other two delay equations such as the logistic equation considered by Braddock and van den Driessche [6] and Gopalsamy [13].

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