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Nonsmooth Ducks and Regular Perturbations of Rivers, I

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1. INTRODUCTION

The *duck* phenomenon concerns certain "overstable" solutions of nonstandard slow–fast dynamical systems (especially ordinary differential equations), mostly in a context of bifurcation; in this case the values of the bifurcation parameter for which such solutions exist are called *duck values*, the overstable solutions being called *duck solutions*. The classical domains closely related to this theory are singular perturbations and matched asymptotic expansions (cf. [3]). This article presents a new method to obtain shadow-expansions for duck parameters and duck solutions.

Until recently, the methods used to prove the existence of and calculate such expansions were based upon the differentiability of the duck branches appearing in the successive scales given by appropriate nonstandard transformations (so-called *microscopes*).

The first papers on the subject (such as [1, 6, 8, 13]) assumed a $\mathscr{C}^\infty$ equation with a Morse point bifurcation, which has been shown to imply the existence of infinite expansions. In our paper [11] (with V. Gautheron) we assumed only finite differentiability for a slow–fast differential equation of the type

$$\varepsilon \dot{y} = F(x, y) + a, \tag{1}$$

F being standard and having a nondegenerate saddle point. We stated up to what order the algebraic method yields ε-expansions of the duck-values of a and Taylor expansions for the terms of the ε-expansion of the duck solutions. We showed how a computation of those expansions can be obtained by using only polynomial operations (once given a Taylor expan-

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sion of the "duck branch"), and gave an appropriate algorithm. In the current article, such expansions will be called *algebraic expansions*, as opposed to the *transcendental expansions* that we present here. In the previous paper we gave examples of equations for which the algebraic method fails after a few steps, and in one case we proved that no further ε-expansion exists.

A new idea, introduced by M. Diener [9] to deal with critical points other than Morse points, consists of applying an appropriate linear magnification around the critical point, in order to obtain a near-standard equation. The slow solutions of the initial equation which come infinitely close to the critical point were shown to appear in the new scale as near-standard solutions whose shadows have polynomial growth at infinity, i.e., *rivers* of the corresponding standard equation. Duck solutions can exist only if some rivers at $-\infty$ and at $+\infty$ are connected, which gives a condition on the parameter involved. Those ideas clarify the close connection between the theory of ducks (about the local behaviour of solutions of slow–fast equations) and the theory of rivers (about the behavior at infinity of solutions of standard equations—cf. [10, 12], for example).

Our contribution is to show the existence of expansions in (often fractional) powers of ε for duck solutions and duck parameters, even in many "nonsmooth" and "non-Morse" cases. To do so, we combine M. Diener's techniques with a study of regular perturbations of a standard equation with river solutions; we state some results about the expandability of some solutions of the perturbed equation which stay close to a given river of the unperturbed equation; we also give induction formulas to compute these expansions (analogous to those given for algebraic expansions in [13, 2]). Even when the classical method applies, ours yields an algorithm to compute those expansions which is more efficient than former ones (such as we gave in [11]).

We must point out that, independently, Delcroix used a partial version of our Theorem 2, in order to study solutions of a slow–fast equation in the halo of a fold point of the slow curve. In that context he put forth the idea that "rivers are the good special functions" in the sense that they are the universal terms in which "interesting" solutions can be expanded [5].

Because it is well adapted and widely used in differential equations, the version of nonstandard analysis we adopt here is the formulation of Nelson—the so-called *internal set theory* (IST). For a general reference, see [7], for example. The reader unfamiliar with that formalism will find below some indications which, hopefully, can help him to understand the results and proofs of this paper.

I thank Professors R. E. O'Malley, M. Diener, and F. Weissler for their useful suggestions about the exposition of that paper, and for correcting my English.

1.1. *Notations*

As usual in IST, we shall denote $x \simeq y$ for "x infinitely close to y," $x \lessapprox y$ for "$x < y$ or $x \simeq y$," $x \lnapprox y$ for "$x < y$ and not $x \simeq y$," and similarly $x \gtrapprox y$ and $x \gnapprox y$. We use the symbol £ to denote a limited value, ϕ for an infinitesimal one (roughly speaking, if $\varepsilon \simeq 0$ the notations £ε and $\phi\varepsilon$ are the nonstandard equivalent of $O(\varepsilon)$ and $o(\varepsilon)$); we use @ for an apprecia-ble (i.e., limited but not infinitesimal) value and $°x$ for the standard part of x (when x is limited).

If A is a point or a subset of a metric space, we denote the halo of A, i.e., the external set[1] of all points infinitely close to A, by hal(A), so $x \simeq 0$ is equivalent to $x \in \text{hal}(0)$.

When F and G are real-valued functions, we write $F \sim G$ for "$F(X)/G(X) \to 1$" (as X tends to some finite or infinite value), $F \ll G$ for "$F(X)/G(X) \to 0$" and $F \gg G$ for "$G(X)/F(X) \to 0$." By "F is a function of polynomial growth" (at ∞, for instance) we mean that there exists some integer p such that $F(X) \ll \|X\|^p$ as $X \to \infty$ (if X is multidimensional, we shall make precise the nature of the limit).

1.2. *How to Understand This Paper in Classical Terms*

First we recall some basic definitions about nonstandard differential equations. Consider an ODE

$$\varepsilon y' = f(x, y, \varepsilon, a), \tag{1}$$

where f is a standard function (at least $\mathscr{C}^1$) and ε and a are nonstandard real parameters, with $\varepsilon \simeq 0$, $\varepsilon > 0$, and $a \simeq a_0$ (a_0 a standard real). Denote by Γ the curve $f(x, y, 0, a_0) = 0$. If the point (x, y) is not infinitely close to Γ, the field determined by (1) is "almost vertical" with infinitely large module, and somewhere within the halo of Γ (i.e., infinitely close to it) the field is limited. That is why (1) is called a *slow–fast* equation and Γ is the *slow curve* of (1).

Suppose that for $x \in I$, a standard interval of $\mathbb{R}$, a branch of Γ is the graph of $y = \varphi_0(x)$. That section of branch will be called *attractive* (resp. *repelling*) if there exists a standard neighborhood V of that section, such that for $(x_0, y_0) \in V$ the solution $y = \psi(x)$ of (1) with initial condition (x_0, y_0) satisfies $(\forall x \in I) x \gnapprox x_0 \Rightarrow \psi(x) \simeq \varphi_0(x)$ (resp. $(\forall x \in I) x \lnapprox x_0 \Rightarrow \psi(x) \simeq \varphi_0(x)$).

[1] The term *external set* emphasizes the fact that this type of "set" does not obey some rules of set theory; for instance, in $\mathbb{R}$, hal(0) is bounded but has no l.u.b.

This notion has a classical translation: first, consider

$$\varepsilon y' = f(x, y, \varepsilon) \tag{2}$$

(without the parameter a); its slow curve is $\Gamma : f(x, y, 0) = 0$. The assertion "*for all positive* $\varepsilon \simeq 0$ *the branch* $\gamma : y = \varphi_0(x)$ of Γ *is attractive*" may be translated as: "there exist a neighborhood V of γ in $\mathbb{R}^2$ and a neighbourhood U of 0 in $\mathbb{R}$, such that if $(x_0, y_0) \in V$ and, for any positive $\varepsilon \in U$, $y = \psi_\varepsilon(x)$ is the solution of (1) with initial condition (x_0, y_0), then when $\varepsilon \to 0^+$, $\psi_\varepsilon \to \varphi_0$ uniformly on every compact in $\{x \in I/x > x_0\}$." Now, introducing the parameter a, one may translate the proposition "*for all positive* $\varepsilon \simeq 0$ *and all* $a \simeq a_0$, γ *is attractive*" by replacing "when $\varepsilon \to 0$" with "when $(\varepsilon, a) \to (0^+, a_0)$" in the above sentence.

More generally, the behaviour of solutions for $\varepsilon \simeq 0$ must be interpreted in classical terms as a limit behaviour when $\varepsilon \to 0$, with the topology of convergence on every compact for the solutions.

Suppose the slow curve has a Morse point for $a \simeq a_0$; a *duck branch* of this slow curve is a branch attracting on the left side of the Morse point, repelling on the right side. A *duck solution* is one which remains infinitely close to the duck branch, at least on some standard open interval containing the Morse point—such a solution is "overstable" in the sense that, except for a in a very narrow domain of values, all the solutions coming from the attractive part of the branch are immediately repelled when crossing the halo of the Morse point; a *duck value* for the parameter (or duck-parameter) is a value of a for which there exist some duck solutions. It has been shown (cf. [1, 8]) that if the equation is at least $\mathscr{C}^2$, then the duck values are contained in an exponentially short interval (relative to ε), so when they are expandable in powers of ε the expansion is unique.

The existence of a duck parameter with expansion $\sum a_n \varepsilon^n$ must be understood as the existence of a function $a(\varepsilon)$, with the same asymptotic expansion, such that when $\varepsilon \to 0^+$ some solutions of $\varepsilon y' = f(x, y, a(\varepsilon))$ have a limit behaviour as described above.

We hope that this can help the reader unfamiliar with ANS to understand most results of the present paper.

2. OBSTRUCTIONS TO THE ε-EXPANSIONS OF DUCKS

2.1. *Limitations of the Algebraic Method*

When applying the algebraic method, one deals with successive slow−fast equations with Morse points on the slow curve, where from the second transformation one of the branches is vertical and the other is a duck branch. If this duck branch has a nonvertical tangent at the Morse point, one can determine the unique possible standard part of duck parameters for the equation appearing at that stage, which corresponds to a new term for the ε-expansion of the duck values of a. That value is obtained, for example, by showing that duck solutions are impossible if, in a standard neighbourhood of the Morse point, the slope of the field remains appreciably different from that of the duck branch (cf. [11, 6]). The existence of a *finite slope* for the duck branch at each scale is thus essential for that method.

We now present two typical cases in which a stage in the construction is reached where the duck branch of the slow curve does not have a finite slope at the Morse point.

Jumping Duck (cf. [11]). The equation

$$\varepsilon\dot{y} = y^2 - x^2(1 + |x|)^2 + a,$$

through a first microscope (around the curve $y = x(1 + |x|)$), yields $\varepsilon\dot{Y} = 2x[-\operatorname{sgn}(x) + Y(1 + |x|)] + \varepsilon Y^2 + \alpha$, where $\alpha \simeq 0$; the duck branch of the equation $y = \operatorname{sgn}(x)/(1 + |x|)$ is discontinuous at the origin. This jump may be considered as an infinite slope, so the conclusion of Proposition 2 of [11] is not surprising: the parameter α is not of the form $\varepsilon£$, so no further ε-expansion can occur. We shall see that an expansion can be continued with powers of $\sqrt{\varepsilon}$.

Angular Duck. This example is perhaps more puzzling. Consider the equation

$$\varepsilon\dot{y} = \left(y - \frac{|x^3|}{3}\right)\left(y + x - \frac{|x^3|}{3}\right) + a. \tag{3}$$

If $a \simeq 0$, it has a slow curve with a Morse point at the origin, where the duck branch $y = |x^3|/3$ has a horizontal tangent. So if $\bar{a}$ is a duck value of the parameter, then $\bar{a} = \varepsilon\phi$ (i.e., $a_1 = 0$ in the expansion of $\bar{a}$). A first microscope $\varepsilon y_1 = y - |x^3|/3$, $\varepsilon A = a$ gives

$$\varepsilon\dot{y}_1 = x(y_1 - |x|) + \varepsilon y_1^2 + A \tag{4}$$

with $A \simeq 0$. So, in a standard neighbourhood of the origin, the duck solutions of (3) may be written $y = |x^3|/3 + \varepsilon|x| + \varepsilon\phi$. In order to carry on the expansion, one must determine the standard part of A/ε (if it exists); unfortunately, the usual geometric argument given above allows us to show that this standard part exists, but not how to compute it!

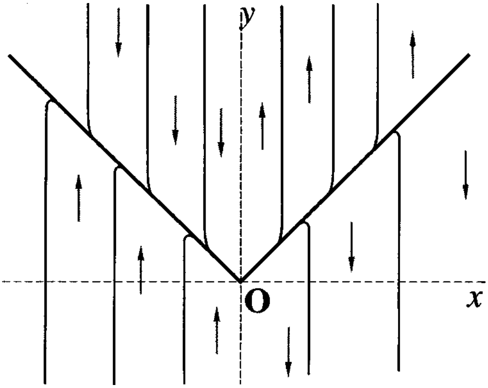

FIG. 1. Angular duck.

PROPOSITION 1. *If $\bar{A}$ is a duck value of A for Eq.* (4), *then*

$$^{\circ}\left(\frac{\bar{A}}{\varepsilon}\right) \in [-1,1].$$

Proof. The field of Eq. (4) is as in Fig. 1. The argument is the same as in Theorem 1 of [11]; were it not the case, the existence of duck solutions would be impossible, for that solution would have to cross the branch in a "wrong" direction.[2] Q.E.D.

However, that standard part is certainly *uniquely determined* (we recall that two duck values $\bar{a}$ and $\tilde{a}$ must satisfy $|\bar{a} - \tilde{a}| < e^{-K/\varepsilon}$, with $K \gtrneqq 0$). So a further expansion exists, for at least one more term (i.e., there exists a standard a_2 such that $\bar{a} = \varepsilon^2(a_2 + \phi)$), but we must find a new method to compute it.

2.2. *One Further Term*

The method we shall use here to compute the value of a_2 in the above example is that of the so-called *regularizing microscope* (that we shall abbreviate RM). It has been introduced by Diener in [9] (cf. below Section 6.2, the Union Jack equation). The idea is to make a linear change of variable $\varepsilon^p X = x$, $\varepsilon^q Y = y$, p and q being chosen (according to the nature of the critical point; cf. [9]) to obtain a nontrivial near-standard

[2] In [6], F. Diener uses a different reasoning, first showing that $|\bar{A}/\varepsilon|$ is limited, then applying a microscope; that argument, when applied to (4), leads exactly to the same conclusion as ours!

equation; in the new scale, the original duck solutions follow the halo of river solutions of the standard part of the new equation.

For Eq. (4) we must choose $p = q = \frac{1}{2}$, which yields

$$\dot{Y} = X(Y - |X|) + \varepsilon Y^2 + A/\varepsilon. \tag{5}$$

Since by Proposition 1 we know that A/ε is limited, the standard part of (5) is

$$\dot{Y} = X(Y - |X|) + B, \qquad \text{where } B = {}^\circ(A/\varepsilon). \tag{$\bar{5}$}$$

PROPOSITION 2. *If $\bar{a}$ is a duck value for* (3), *then $\bar{a} = \phi\varepsilon^2$.*

Proof. This is a special case of Theorem 1 below. Q.E.D.

Remark. One can check that, if $B = 0$, the equation of the "double river" is

$$F(X) = |X| + \int_{|X|}^{\infty} e^{(X^2 - t^2)/2}\, dt.$$

That $a_2 = 0$ is due to the symmetry of the equation.

The following elementary standard lemma describes the river solutions of a linear equation.

LEMMA 1. *Let (E) be the linear equation*

$$\dot{Y} = C(X)Y + D(X), \tag{E}$$

where C and D are $\mathscr{C}^1$. If $XC(X) \to \infty$ as $X \to \infty$ and D has polynomial growth at ∞, then (E) has a unique solution $Y = H(X)$ with polynomial growth at ∞. More precisely, if $D(X) \ll X^p$ then for all $\sigma > p + 1$ one has $H(X) \ll X^\sigma$ and

$$H(X) = -\int_X^{\infty} D(u) e^{W(X) - W(u)}\, du,$$

where W is a primitive of C.

Proof. Existence: let $\sigma > p + 1$ (one may assume $\sigma > 0$) and $\tau > 0$ such that $p + 1 < \tau < \sigma$. For X large enough, $|D(X)/C(X)| = |D(X)/W'(X)| < X^\tau$. Defining H as above, it is a solution of (E) and, for X large enough,

$$|H(X)| < e^{W(X)} \int_X^{\infty} u^\tau W'(u) e^{-W(u)}\, du.$$

Integrating by parts,

$$\int_X^\infty u^\tau W'(u)e^{-W(u)}\,du = X^\tau e^{-W(X)} + \tau\int_X^\infty u^{\tau-1}e^{-W(u)}\,du.$$

For large X, the second term of the sum is bounded by $\frac{1}{2}\int_X^\infty u^\tau W'(u)$ $e^{-W(u)}\,du$ (if X is large enough, $2\tau < uW'(u)$ for all $u > X$), whence $|H(X)| < 2X^\tau$ and $H(X)/X^\sigma \to 0$.

Uniqueness: at most one solution of (E) can have polynomial growth at ∞, since the difference between two solutions has the form $Ke^{W(X)}$, which by the hypothesis on C grows faster than any power of X (for every n, $e^{-W(X)}/X^n$ has a negative derivative for X large enough, so it is bounded). Q.E.D.

Remarks. Naturally, with analogous conditions at $-\infty$, (E) has a unique solution with polynomial growth at $-\infty$, namely

$$K(X) = \int_{-\infty}^X D(u)e^{W(X)-W(u)}\,du.$$

Such solutions will be called *river solutions* (or simply *rivers*) of the linear equation (even if they do not always satisfy the more restrictive definition of *macroscopic rivers*, that we shall recall in Section 3).

From a nonstandard point of view, changing the scale, Lemma 1 has the following consequences (the proof is straightforward):

LEMMA 2. *Let* (E) *be a standard linear ODE, satisfying Lemma* 1, *with* $\rho > p + 1$. *Then for all nonlimited* ω, *the macroscope* $(\omega x = X,\ \omega^\rho y = Y)$ *transforms* (E) *into a slow-fast equation*

$$\dot{y} = \gamma(x)y + \delta(x), \tag{e}$$

where $\delta(x) \simeq 0$ *for limited* x, *and* $\gamma(x) \simeq +\infty$ *for* $x \gtrsim\!\!\!\!\!\neq 0$—*so that for* $x \gtrsim\!\!\!\!\!\neq 0$ *the x-axis is the unique slow curve, which is regularly repelling,*[3] *and the image of the unique river solutions of* (E) *is a slow solution of* (e), *following the axis* $x = 0$.

The following theorem allows us to calculate one further term in the expansion of the duck parameters, in a category of cases containing Equation (4); note that, even if the hypotheses seem a bit restrictive, it is often easy to obtain them by a simple change of variables; the hypothesis of a vertical tangent for the second branch $g(x, y) = 0$ of the slow curve can be

[3]For a slow−fast equation $\dot{y} = p(x, y)$, a part Γ of a slow curve is said to be *regularly repelling* (resp. *regularly attracting*) if $(\partial p/\partial y)(x, y) \simeq +\infty$ (resp. $(\partial p/\partial y)(x, y) \simeq -\infty$) for all $(x, y) \in \mathrm{hal}(\Gamma)$. In both cases, Γ is said to be *regular* (cf. [4]).

obtained by a mere linear transformation and is always satisfied when the "angularity" appears after the application of one or more microscope(s).

THEOREM 1. *Let (e) $\varepsilon\dot{y} = F(x, y)$ be a slow–fast differential equation, where for limited x, y, $F(x, y) = F_0(x, y) + \varepsilon(F_1(x, y) + \phi)$, with standard continuous F_0 and F_1. Suppose that in a standard neighbourhood of the origin, there exist standard f and g such that $F_0(x, y) = f(x, y)g(x, y)$, satisfying:*

- *$f(0,0) = g(0,0) = 0$; g is $\mathscr{C}^1$ and $g'_x(0,0) \neq 0$, $g'_y(0,0) = 0$; $f(x, y)$ is $\mathscr{C}^1$ for all $x \neq 0$;*
- *$f(x, y) = 0$ is locally equivalent to $y = \varphi(x)$, φ possessing a left semi-slope p_- and a right semi-slope p_+ at the origin.*

If (e) *has duck solutions following the slow branch $f(x, y) = 0$, then necessarily*

$$F_1(0,0) \simeq \tfrac{1}{2}(p_- + p_+).$$

Remarks. • If, instead, the tangent to the branch $g(x, y) = 0$ at the origin has slope r, the conclusion becomes

$$F_1(0,0) = \frac{r(p_- + p_+) - 2p_-p_+}{2r - (p_- + p_+)}.$$

• If $F(x, y) = G(x, y, b)$, and $\bar{b} \simeq 0$ is a duck value for b, then $F(x, y, \bar{b}) = F_0(x, y, b_0) + \varepsilon(F_1(x, y, b_1) + \phi)$, where $\bar{b} = b_0 + \varepsilon(b_1 + \phi)$; then the theorem gives an implicit equation for b_1—of course, b_0 must satisfy $F_0(0, 0, b_0) = 0$.

Proof. We use the notations $q = g'_x(0,0)$, $s = f'_y(0,0)$, $t_1 = f'_{x-}(0,0)$, $t_2 = f'_{x+}(0,0)$, and $\lambda = F_1(0,0)$—so that $p_- = -t_1/s$ and $p_+ = -t_2/s$. Applying the RM ($x = \sqrt{\varepsilon}X$, $y = \sqrt{\varepsilon}Y$) yields an equation whose standard part $(\bar{E})$ is

$$\dot{Y} = qX(sY + t_1X) + \lambda \qquad \text{for } X \leq 0$$

$$\dot{Y} = qX(sY + t_2X) + \lambda \qquad \text{for } X \geq 0.$$

From Lemma 1, the unique river solution of $(\bar{E})$ at $-\infty$ (resp. $+\infty$) is

$$Y = e^{qsX^2/2}\int_{-\infty}^{X}\left(qt_1u^2 + \lambda\right)e^{-qsu^2/2}\,du$$

$$\left(\text{resp. } Y = -\,e^{qsX^2/2}\int_{X}^{\infty}\left(qt_2u^2 + \lambda\right)e^{-qsu^2/2}\,du\right).$$

Moreover, one can show, as in [9], that any slow solution of (e) for $x < 0$ (resp. $x > 0$) must, by the RM, stay in the halo of the river at $-\infty$ (resp. $+\infty$) of (E) (see also Lemma 2 of the present paper). So if a duck solution of (e) exists, its image in (E) must follow both rivers, which therefore must be connected at 0. An easy calculation shows that it implies $\lambda = -(t_1 + t_2)/2s$. Q.E.D.

A direct application of that theorem gives a partial solution to the problem of the "angular duck."

COROLLARY 1. *Suppose the equation* $\varepsilon\dot{y} = f(x, y)g(x, y) + a$ (*with standard* f *and* g) *satisfies the following hypotheses on a standard neighbourhood* V *of* $(0, 0)$:

- g *is* $\mathscr{C}^3$ *on* V *and* f *is* $\mathscr{C}^2$ *at the origin and* $\mathscr{C}^3$ *on* $V \setminus \{(0,0)\}$;
- *at* $(0, 0)$, $f = g = 0$, $f'_y > 0$ *and* $f'_x g'_y - f'_y g'_x < 0$;
- *when some duck solutions* $y = \varphi(x)$ *exist, their first order* ε*-expansion is* $\varphi = \varphi_0 + \varepsilon(\varphi_1 + \phi)$, *with* φ_1 *possessing semi-derivatives* α_l *and* α_r *at* $x = 0$.

Then the duck parameters have the ε*-expansion*

$$\bar{a} = \varepsilon\varphi'_0(0) + \varepsilon^2\left(\frac{\alpha_l + \alpha_r}{2} + \phi\right).$$

Remark. At the scale corresponding to Eq. (3), this method gives some information about the behaviour of the duck solutions in hal(0); instead of y remaining at a distance £ε from the slow curve $y = |x|$ throughout a standard neighborhood of $x = 0$ (even for $x \simeq 0$), as would be the case if the slow curve of (4) were smooth, y actually takes values of the form $|x| + @\sqrt{\varepsilon}$ for some $x \simeq 0$. Accordingly, after one more microscope (of magnification rate $1/\varepsilon$), one obtains a new slow curve with a pole at 0 (cf. below Section 5.2, Flying duck).

3. PERSISTENT SOLUTIONS ALONG STANDARD RIVERS

3.1. *Transcendental Expansion*: *Heuristic Result*

Returning to the "angular duck," let us consider Eq. (5), which is a regular perturbation of the standard linear equation ($\bar{5}$), taking the form

$$\dot{Y} = X(Y - |X|) + \varepsilon Y^2 + \alpha,$$

where $a \simeq 0$ (now we know that $\bar{a} = \phi\varepsilon^2$).

Diener's theorem shows that in that scale the duck solutions of the initial equation follow the halo of the unique river $Y = F(X)$ (attracting at $-\infty$, repelling at $+\infty$) of $(\bar{5})$. To obtain more precise information about those solutions, it seems natural to apply a change of scale, with magnification rate $1/\varepsilon$ around the river (imitating the "algebraic" method, with the river playing the role of the slow curve). The change of variables $\varepsilon Z = Y - F(X)$ in (5) gives

$$\dot{Z} = XZ + F(X)^2 + 2\varepsilon ZF(X) + \varepsilon^2 Z^2 + \frac{\alpha}{\varepsilon}. \tag{6}$$

One observes that *if* α/ε *is limited* (i.e., if $\bar{a}$ has an expansion up to order 3) one obtains a new near-standard equation, with linear standard part

$$\dot{Z} = XZ + F(X)^2 + b \tag{$\bar{6}$}$$

(where $b = {}^{\circ}(\alpha/\varepsilon)$). According to the well-known "short shadow lemma" (cf. [7]) every solution of (6) starting from a limited point remains infinitely close to a standard solution of $(\bar{6})$ while x remains limited. From Lemma 1, $(\bar{6})$ has a unique attracting river at $-\infty$ and a unique repelling one at $+\infty$. In the new scale, the images of some solutions of (3) follow the halo of each of these rivers, and if one chooses a value for b such that these rivers are connected, those solutions will follow the "double river" for every limited value of X, just as the images of the duck solutions at the preceding scale. Such a choice, however, is only possible (as in the proof of Theorem 1) for

$$b = -\frac{1}{\sqrt{2\pi}} \int_0^{\infty} F(t)^2 e^{-t^2/2}\, dt.$$

It is natural to conjecture that at that scale, the images of the duck solutions of (3) (and (4)) still follow the halo of the river for all limited X, which imposes the above value of b for the coefficient of ε^3 in the expansion of $\bar{a}$, if such a solution exists. Heuristically the latter process can be iterated without obstacle, giving a new connecting value a_n and a new river solution F_n for the standard part of the equation appearing at each new scale.

If our method is valid, at the level of the initial scale the expansion of the duck parameters must be $\sum a_n \varepsilon^n$, and the expansion of the duck solutions must be $\sum \varepsilon^n F_n$. However, that expansion is of a new type, for the coefficients are obtained, not by algebraical operations as in the "smooth" case, but by successive quadratures of transcendental functions (more precise formulas will be provided later). In order to justify such a method, we still must prove that in (6) the infinitesimal α is of the form £ε, and that the images of the duck solutions through the RM are visible at the new scale (i.e., are functions with limited values for limited x) and follow the halo of the rivers. Moreover, we must prove that those facts remain true at each new scale.

We shall prove these facts and give existence conditions for such expansions and recurrence formulas to compute them.

3.2. *Regular perturbations of standard rivers*

Starting from a problem of ducks in a slow-fast equation (i.e., a *singular* perturbation) we have been led to a question concerning an infinitesimal (*regular*) perturbation of a standard equation with river solutions. From that point of view, the problems are the following:

1. which solutions of the perturbed equation do the following: (a) follow the halo of a river of the standard equation for limited x; (b) keep that property after successive microscopes around the rivers?

2. if the initial standard equation possesses a parameter and if, for some value of that parameter, one observes a connection between a river at $-\infty$ and a river at $+\infty$, are there values of the parameter for which that property remains true at those successive scales and do they have ε-expansions?

RIVERS: THEORETICAL BACKGROUND. The standard equations that we deal with may contain transcendental functions, so, rather than using the theory of F. and M. Diener and F. Blais, adapted for rational equations, we shall work in the context of the more general theory of *macroscopic rivers*, due to I. P. Van den Berg [14]. We rapidly recall the main points of that theory which are used in the present paper:

Let ω, ξ be two internal positive real numbers with ω unlimited; we shall denote the macroscope $(X = \omega x,\ Y = \xi y)$ by $\mathscr{M}_{\omega,\xi}$ (X and Y are the old variables, x and y the new ones); if G is a function, with only nonzero values for X large enough, we shall use $\mathscr{M}_{\omega,G}$ for $\mathscr{M}_{\omega,G(\omega)}$.

If F is a real-valued function, we shall note $F_{\omega,\xi}$ for the image of F by $\mathscr{M}_{\omega,\xi}$ (and similarly $F_{\omega,G}$). A function G is called *macroscopically observable* (in short, m.o.) if for all $\omega \simeq \infty$, $G_{\omega,G}$ is S-continuous for all $x \gtrapprox 0$, with nonzero shadow; it has been proved by Van den Berg that if such a function G is standard, then G and $1/G$ have polynomial growth and $G_{\omega,G}$ has appreciable values for $x \gtrapprox 0$. If (E) $\dot{Y} = F(X,Y)$ is a standard equation and G is standard and macroscopically observable, G will be called *macroscopically slow* (resp. *macroscopically regular*—in short (m.r.) with respect to (E) if for all $\omega \simeq \infty$ the image of (E) by $\mathscr{M}_{\omega,G}$ is a slow−fast equation for which the graph of ${}^{\circ}G_{\omega,G}$ is a part (resp. a *regular* part—cf. footnote in Lemma 2)) of the slow curve.

Finally, a standard solution of (E) which is macroscopically slow (resp. regular) with respect to (E) will be called a *macroscopic river* (resp. *macroscopically regular river*) of (E). Such a river will be called *attracting* (resp. *repelling*) if the corresponding slow curve (by the associated macroscope) is attracting (resp. repelling). So the rivers are standard solutions of (E), with polynomial growth, and in the regular repelling case at $+\infty$ (resp. the regular attracting case at $-\infty$) they are locally unique and asymptotically they are exponential repellers (resp. attractors).

The main result of [14] is an existence theorem for rivers (here referred to as Van den Berg's theorem): *if there exists a standard function G, macroscopically regular for (E), then (E) has a macroscopically regular river H with $H \sim G$.*

In fact, that result may be expressed in standard form, for if G is standard the properties of being m.o. or m.r. have standard characterizations:

- G is m.o. at ∞ if and only if, for every function H such that $H(X) \sim X$ as $X \to \infty$, one has $G(H(X)) \sim G(X)$ (that property being called *asymptotic continuity*);

- G is m.r. with respect to (E) $\dot{Y} = F(X,Y)$ if and only if the five following conditions are satisfied (noting $V_G(X) = F'_Y(X, G(X))$)[4]:

1. G is asymptotically continuous;
2. $F(X,G(X)) \ll G(X)V_G(X)$ as $X \to \infty$;
3. $V_G(X) \gg 1/X$ as $X \to \infty$;
4. if $H(X) \sim X$ as $X \to \infty$, there exist A and B $(0 < A < B)$ s.t. for X large enough $AV_G(X) \le V_G(H(X)) \le BV_G(X)$ (that property of V_G is called *order continuity* in [14]);
5. if $H(X) \sim G(X)$ as $X \to \infty$, there exist positive A and B s.t. for X large enough, $AF'_Y(G(X)) \le F'_Y(X, H(X)) \le BF'_Y(X, G(X))$.

In the following, we shall refer to this set of properties as the *contraction conditions* (for G, with respect to (E)).

PERTURBATION OF A REPELLING RIVER. We shall use the two following results, the first of which is a "perturbed" version of Proposition 11 of [14][5] and has a similar proof.

LEMMA 3. *Let $(\overline{E})$ $\dot{Y} = F(X,Y)$ be a $\mathscr{C}^1$ standard equation having a m.r. repelling river G at $+\infty$, and (E) $\dot{Y} = F(X,Y) + \delta(X,Y)$ a $\mathscr{C}^1$*

[4] At $+\infty$, V_G is positive in the repelling case, negative in the attracting one, and the reverse at $-\infty$.

[5] See also [9, Theorem 8].

perturbation of $(\bar{E})$, *such that* $\delta(X,Y) \simeq 0$ *for all limited* X, Y. *Suppose there exists* $\omega_0 \simeq +\infty$ *such that for all nonlimited* $\omega \leq \omega_0$, ${}^\circ G_{\omega,G}$ *is for all* $x \gtrsim 0$ *a regular repelling part of the slow curve of the image of* $(\bar{E})$ *by* $\mathscr{M}_{\omega,G}$. *Then there exists a solution* Φ *of* (E) *such that, for all nonlimited* $\omega \leq \omega_0$ *one has* $\Phi(\omega) = (1+\phi)G(\omega)$. *Moreover, for all limited* X, $\Phi(X) \simeq G(X)$.

When the river is unique and repelling, one gets a similar conclusion without the assumption of regularity; in particular, it is applicable to linear equations satisfying Lemma 1.

LEMMA 4. *Let* $(\bar{H})\ \dot{Y} = P(X,Y)$ *be a standard equation, possessing a unique solution* $Y = F(X)$ *with polynomial growth at* $+\infty$, *and* $(H)\ \dot{Y} = P(X,Y) + A(X,Y)$ *an infinitesimal perturbation of* (H). *Suppose there exists* $\omega_0 \simeq +\infty$ *and* $\rho \gtrsim 0$ *such that for all nonlimited* $\omega \leq \omega_0$, *the image* $(h)_\omega$ *of* (H) *under the macroscope* $\mathscr{N}_{\omega,\rho}$: $\omega x = X$, $\omega^\rho y = Y$ *has the x-axis as a unique slow curve for* $x \gtrsim 0$, *with* $\dot{y} \simeq +\infty$ *for* $y \gtrsim 0$ *and* $\dot{y} \simeq -\infty$ *for* $y \lesssim 0$. *Then any slow solution of* $(h)_\omega$ *for* $\omega \leq \omega_0$ *is by* $\mathscr{N}_{\omega,\rho}$ *the image of a near standard solution* Φ *of* (H), *such that* ${}^\circ\Phi = F$.

3.3. *The Main Theorem*

DEFINITIONS. If $(\bar{E})\ \dot{Y} = F(X,Y)$ is a standard ODE and ε is an infinitesimal, we shall call an equation of the form

$$\dot{Y} = F(X,Y) + \varepsilon A(X,Y), \tag{E}$$

with A being a near-standard function whose shadow is not identically zero, a *regular perturbation of* $(\bar{E})$ *of order* ε. One has then $A(X,Y) = {}^\circ A(X,Y) + R(X,Y)$, where for all limited X, Y, $R(X,Y) \simeq 0$; $\varepsilon {}^\circ A$ will be called the *principal part* of the perturbation, and R its *ε-remainder*.

We shall call $\mathscr{G}$ (resp. $\mathscr{G}^2$) the principal galaxy[6] of $\mathbb{R}$ (resp. $\mathbb{R}^2$); for $\omega \simeq \infty$, and $\mathscr{G}_\omega$ (resp. $\mathscr{G}^2_\omega$) the external set $\{X/X = \omega £\}$ (resp. $\{(X,Y)/X = \omega £,\ Y = \omega £\}$).

Let $\mathscr{A}$ be an unbounded interval of $\mathbb{R}$ (resp. an unbounded domain of $\mathbb{R}^2$); an internal function $\varphi(X)$ (resp. $f(X,Y)$) has *standard polynomial growth* on $\mathscr{A}$ if there exist positive standard M, p such that $\forall X \in \mathscr{A}$, $|\varphi(X)| \leq M(1+|X|)^p$ (resp. $\forall (X,Y) \in \mathscr{A} |f(X,Y)| \leq M(1+|X|+|Y|)^p$). If φ (resp. f) is standard we just say that it has polynomial growth on $\mathscr{A}$.

Finally, let Φ be a standard solution of $(\bar{E})$; a solution φ of (E) will be called *ε-persistent along* Φ at $+\infty$ (resp. $-\infty$) if there exists an $r \gtrsim 0$ such that $\varphi(X) = (1+\phi)\Phi(X)$ on $\mathscr{G}_{\varepsilon^{-r}}+$ (resp. $\mathscr{G}^-_{\varepsilon^{-r}}$).

Remark. By the short shadow lemma, any solution φ of (E) passing through a point of $\mathscr{G}^2 \cap \mathrm{Graph}(\Phi)$ satisfies $\varphi(X) \simeq \Phi(X)$ for all limited

[6] That is, the external set of its limited point.

X; a persistent solution remains "close" to Φ on a much larger domain, i.e., up to an appreciable power of $1/\varepsilon$.

The following result, which plays a central part in this paper, gives conditions for existence of persistent solutions and shows their expandability up to order 1 at least—the expansion being computed by the method described in the preceding section.

THEOREM 2. *Let* $(\overline{\mathrm{E}})$ $\dot{Y} = P(X,Y)$ *be a standard continuous ODE, with* F *a m.r. repelling river at* $+\infty$ (*resp. a m.r. attracting river at* $-\infty$) *and let* (E) $\dot{Y} = P(X,Y) + \varepsilon A(X,Y)$ *be a continuous regular perturbation of* $(\overline{\mathrm{E}})$, *of order* ε, *with* $A = A_0 + \eta R$ (A_0 *standard*, $\eta \simeq 0$, R *near-standard*).

Suppose that for X (*resp.* $-X$) *large enough*:

- P, $\partial P/\partial Y$, $\partial^2 P/\partial Y^2$, A_0, *and* $\partial A_0/\partial Y$ *are defined and have polynomial growth on* $\mathbb{R}^+ \times \mathbb{R}$ (resp. $\mathbb{R}^- \times \mathbb{R}$);
- *there exists a* $r \gtrsim 0$ *such that* R *has standard polynomial growth on* $\mathscr{G}^2_{\varepsilon^{-r}} \cap \mathbb{R}^+ \times \mathbb{R}$ (resp. $\mathscr{G}^2_{\varepsilon^{-r}} \cap \mathbb{R}^- \times \mathbb{R}$).

Then (E) *has* ε-*persistent solutions along* F, *which may be written*

$$\Phi(X) = F(X) + \varepsilon\big(F_1(X) + \phi\big)$$

for limited X, *where* F_1 *is the unique river at* $+\infty$ (resp. $-\infty$) *of the equation obtained from* (E) *through the microscope* $\varepsilon Z = Y - F(X)$.

Proof. We shall only treat the case of a repelling river at $+\infty$. To begin with, let us show the existence of persistent solutions (bear in mind that, being m.r., F satisfies the contraction conditions with respect to (E)). For any $\omega \simeq +\infty$, let $(\mathrm{e})_\omega$ and $(\overline{\mathrm{e}})_\omega$ be the images of (E) and $(\overline{\mathrm{E}})$ through the macroscope $\mathscr{M}_{\omega, F}$. The case of $(\overline{\mathrm{e}})_\omega$ has been treated in [14]: for all $\omega \simeq \infty$, it is a slow–fast equation of the form $\alpha \dot{y} = p_\omega(x,y)$, where $\alpha = 1/(\omega V_F(\omega)) \simeq 0$ and p_ω is near-standard; the graph of $^\circ F_\omega$ (which is continuous, as F is m.o.) is a regularly repelling slow curve, for one has for all $x \gtrsim 0$, $\partial p_\omega/\partial y \gtrsim 0$ over a standard zone around the slow curve.

We claim that $(\mathrm{e})_\omega$ has the same properties, provided ω is not too large; indeed, $(\mathrm{e})_\omega$ may be written

$$\alpha \dot{y} = p_\omega(x,y) + \varepsilon \frac{A(\omega x, F(\omega)y) + \eta R(\omega x, F(\omega)y)}{F(\omega)V_F(\omega)}.$$

As F and $1/F$ have polynomial growth, let r be a positive standard number such that $X^{-r} < |F(X)| < X^r$ for all $X \simeq \infty$. Then if $\omega < \omega_1 = \min(\varepsilon^{-\tau}, \varepsilon^{-\tau/r})$ for all limited x, y ($x \gtrsim 0$), one has $(\omega x, F(\omega y) \in \mathscr{G}^2_{\varepsilon^{-r}}$, so there exist standard p and K such that, for such ω, x, y, both

$A(\omega x, F(\omega)y)$ and $R(\omega x, F(\omega)y)$ are bounded by $K(1 + |\omega x| + |F(\omega)y|)^p$, whence $|A(\omega x, F(\omega)y) + \eta R(\omega x, F(\omega)y)| = £\,\omega^{(1+r)p}$. On the other hand, by the third contraction condition $1/\omega V_F(\omega) \simeq 0$, one has also

$$\frac{A(\omega x, F(\omega)y) + \eta R(\omega x, F(\omega)y)}{F(\omega)V_F(\omega)} = \phi\omega^{(p+1)(r+1)}.$$

If we choose $\omega_0 < \omega_1$ and $\omega_0 < \omega_2 = \varepsilon^{-1/(p+1)(r+1)}$, then for all $\omega \leq \omega_0$, $(\mathrm{e})_\omega$ is a slow–fast equation for which the graph of ${}^\circ F_\omega$ is a repelling slow curve for $x \gtrapprox 0$.

There remains to show that this slow curve is regular; if we call $q_\omega(x, y)$ the right-hand member of $(\mathrm{e})_\omega$, one must make sure that $(\partial q_\omega/\partial y)(x, y) \gtrapprox 0$; one checks that, if q, L are standard positive numbers such that $|(\partial A_0/\partial Y)(X, Y)|$ and $|(\partial B/\partial Y)(X, Y)|$ are bounded by $L(1 + |X| + |Y|)^q$ on $\mathscr{G}^2_{\varepsilon^{-r}}$, then the condition $\omega_0 < \omega_3 = \min(\omega_1, \omega_2, \varepsilon^{-1/[1+q(1+r)]})$ implies that for all $\omega \leq \omega_0$, x, y limited, $x \gtrapprox 0$, one has $(\partial q_\omega/\partial y)(x, y) \simeq \partial p_\omega/\partial y$ and the latter is known to be positive appreciable. Now by Lemma 3 there exists a Φ solution of (E) such that $\Phi(X) = (1 + \phi)F(X)$ for all $X \in \mathbb{R}^+ \cup \mathscr{G}_{\omega_0}$. Finally, the different conditions imposed on ω_0 allow us to choose it of the form $\varepsilon^{-@}$, which completes the proof of assertion 1.

Now for assertion 2. Start with a lemma describing the equation obtained from (E) through the microscope $\mathscr{L}_\varepsilon$: $\varepsilon Z = Y - F(X)$.

LEMMA 5. *The image of* (*E*) *through* $\mathscr{L}_\varepsilon$ *is a near-standard equation* (E_1), *whose standard part is the linear equation,*

$$\dot{Z} = ZV_F(X) + A_0(X, F(X)), \qquad (\overline{\mathrm{E}_1})$$

satisfying Lemma 1—*so that* $(\overline{\mathrm{E}_1})$ *has a unique solution with polynomial growth at* $+\infty$.

Proof. By application of $\mathscr{L}_\varepsilon$, taking to account that F is a solution of $(\overline{\mathrm{E}})$, one obtains Eq. (E_1) which can be written $\dot{Z} = ZV_F(X) + A_0(X, F(X)) + \lambda(X, Z)$, where $\lambda(X, Z) \simeq 0$ for limited X, Z. Because $F(X)$ has polynomial growth at ∞, so does $A_0(X, F(X))$, and in view of the contraction conditions $XV_F(X) \to +\infty$ as $X \to +\infty$, Lemma 1 also applies to $(\overline{\mathrm{E}_1})$. Q.E.D.

Now let a persistent solution Φ of (E) along F be given, and let $s_1 \gtrapprox 0$ be such that $\Phi(X) = (1 + \phi)F(X)$ for all nonlimited $X \leq \omega_0 = (1/\varepsilon)^{s_1}$—we want to show that Φ has a first-order expansion. For $\omega \simeq \infty$ and ρ a standard real number, we shall denote by $\mathscr{N}_{\omega,\rho}$ the macroscope $(\omega x = X,\ \omega^\rho z = Z)$, $(\mathrm{e}_1)_{\omega,\rho}$ and $(\overline{\mathrm{e}_1})_{\omega,\rho}$ the images of (E_1) and $(\overline{\mathrm{E}_1})$ through $\mathscr{N}_{\omega,\rho}$, and $\mathscr{O}_{\omega,\rho}$ the transformation $\mathscr{N}_{\omega,\rho} \circ \mathscr{L}_\varepsilon \circ \mathscr{M}^{-1}_{\omega,F}$ (cf. Fig. 2). The images of Φ

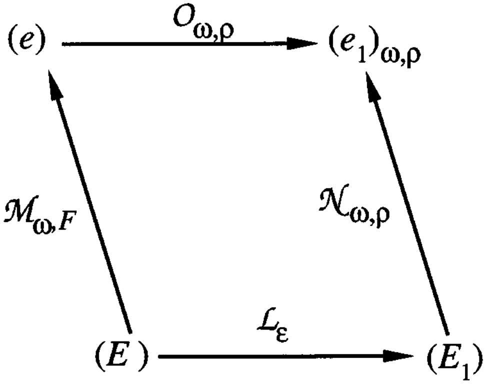

FIGURE 2

by $\mathscr{M}_{\omega, F}$ and $\mathscr{L}_\varepsilon$ will be called φ and Φ_1, respectively, and the image of Φ_1 by $\mathscr{N}_{\omega, \rho}$ (or of φ by $\mathscr{O}_{\omega, \rho}$) will be called φ_1. The proof of Lemma 6 is based upon simultaneous consideration of the four scales in that diagram (cf. Fig. 2).

LEMMA 6. *One can choose standard* $\rho > 0$ *and* $\sigma > 0$ *such that for all nonlimited* $\omega \leq \omega_4 = \varepsilon^{-\sigma}$:

(i) $(\mathrm{e})_\omega$ *is a slow–fast equation, for which the graph of* $°F_\omega$ *is a regular repelling slow curve for* $x \gtrapprox 0$, φ *being a slow solution for all* $x \gtrapprox 0$;

(ii) $(\mathrm{e}_1)_{\omega, \rho}$ *admits the axis* $y = 0$ *as a unique repelling slow curve for* $x \gtrapprox 0$; *for all appreciable* y *and* $x \gtrapprox 0$; *the value of* $\dot{y}$ *is nonlimited, with the same sign as* y;

(iii) *the transformation* $\mathscr{O}_{\omega, \rho}$, *which leaves* x *unchanged, is a* magnification *in the vertical direction* (*i.e., if* (x, z_1) *and* (x, z_2) *are the images of* (x, y_1) *and* (x, y_2), *then* $|y_1 - y_2|/|z_1 - z_2| \simeq 0$), *whose rate is a limited power of* $1/\varepsilon$.

Proof. Condition (i) is obtained by imposing $\omega_4 < \omega_3$ (so that the graph of $°F_\omega$ is a repelling slow curve) and $\omega_4 = \phi\varepsilon^{-s_1}$ (so that φ is a slow solution).

Now for condition (ii). In view of Lemma 1 and Lemma 5, F_1 is the unique river of $(\overline{\mathrm{e}_1})$ at $+\infty$, and if $\rho \geq p(1 + r) + 2$ (r being defined as in the first part of the proof), then by Lemma 2, Eq. $(\overline{\mathrm{e}_1})_{\omega, \rho}$ will have the x-axis as a unique slow curve for $x \gtrapprox 0$ which will be regularly repelling. It remains to show that, by a suitable choice of ω, the same will be true for the perturbed equation $(e_1)_{\omega,\rho}$. Equation (E_1) may be written

$$\dot{Z} = Z\frac{\partial P}{\partial Y}(X, F(X)) + A_0(X, F(X)) + \varepsilon[\Delta(X, Z) + \eta R(X, F(X) + \varepsilon Z)],$$

where $\Delta(X, Z) = \frac{1}{2}\partial^2 P/\partial Y^2(X, F(X) + \theta\varepsilon Z) + \partial A_0/\partial Y(X, F(X) + \theta_1\varepsilon Z)$, with $\theta, \theta_1 \in (0, 1)$. The hypotheses imply the existence of standard m and M such that $|\Delta(X, Z)| < M(1 + |X| + |Z|)^m$. Applying $\mathscr{N}_{\omega,\rho}$ to (E_1) and dividing both members by $\omega^\rho V_F(\omega)$ gives an expression for $(e_1)_{\omega,\rho}$,

$$\frac{1}{\omega V_F(\omega)}\dot{z} = z\frac{V_F(\omega x)}{V_F(\omega)} + \frac{1}{\omega^\rho V_F(\omega)}A_0(\omega x, F(\omega x)) + \frac{\varepsilon}{\omega^\rho V_F(\omega)}[\Delta(\omega x, \omega^\rho z) + R(\omega x, F(\omega x) + \varepsilon\omega^\rho z)].$$

By the third contraction condition, $1/(\omega V_F(\omega)) \simeq 0$; from the fifth one of those conditions one easily infers (as in [14]) that $V_F(\omega x)/V_F(\omega) \gtrsim 0$ for all $x \gtrsim 0$. Further, for appreciable x and z one easily gets

$$\left|\frac{A_0(\omega x, F(\omega x))}{\omega^\rho V_F(\omega)}\right| = \frac{£\,\omega^{p(1+r)}}{\omega^{p(1+r+2)}V_F(\omega)} = \frac{£}{\omega^2 V_F(\omega)} \simeq 0,$$

$$\left|\frac{\varepsilon\Delta(\omega x, \omega^\rho z)}{\omega^\rho V_F(\omega)}\right| = \frac{£\,\varepsilon\omega^{m+\rho(m-1)+1}}{\omega V_F(\omega)} = \phi\varepsilon\omega^{m+\rho(m-1)+1};$$

this last term is infinitesimal if one imposes $\omega \le \varepsilon^{-\sigma}$, with $\sigma < 1/[m + \rho(m-1)+1]$. Moreover, $\sigma < 1/\rho$ implies $F(\omega x) + \varepsilon\omega^p z \simeq F(\omega x)$ for limited z, so for appreciable x and z we get

$$\left|\frac{\varepsilon\eta}{\omega^\rho V_F(\omega)}R(\omega x, F(\omega x + \varepsilon\omega^\rho z)\right| = \phi\varepsilon\omega^{p(1+r)},$$

which is infinitesimal if $\sigma < 1/[p(1+r)]$. If σ and ρ satisfy those conditions, $(e_1)_{\omega,\rho}$ may be written

$$\alpha\dot{z} = \lambda(x)z + \mu(x, z),$$

where $\alpha \simeq 0$, $\lambda(x) \gtrsim 0$ for $x \gtrsim 0$ and $\mu(x, z) \simeq 0$ for appreciable x and z, which is enough to satisfy condition (ii).

Finally, imposing $\sigma < 1/(\rho + r)$ also satisfies condition (iii), for if (x, z_1) and (x, z_2) are the images of (x, y_1) and (x, y_2) by $\mathscr{O}_{\omega,\rho}$ one obtains

$|y_1 - y_2| = |z_1 - z_2|\varepsilon\omega^{\rho}/F(\omega)$, with $\omega^{\rho}/F(\omega) < \omega^{\rho+r}$; as $F(\omega) < \omega^{r}$ and $\omega < \varepsilon^{-\sigma}$, the magnification rate is a limited power of $1/\varepsilon$, which ends the proof. Q.E.D.

Now we complete the proof of the theorem. Let some $\omega \leq \omega_4$ and ρ be given as in the above lemma; let ψ_1 be a slow solution of $(\mathrm{e}_1)_{\omega,\rho}$, following the halo of the slow curve, say for $0 \lessapprox x \lessapprox 2$. Lemma 4 shows that ψ_1 is, by $\mathscr{N}_{\omega,\rho}$, the image of a solution Ψ_1 of (E_1) with polynomial growth—so that ${}^{\circ}\Psi_1$ is the unique river F_1 of $(\overline{\mathrm{E}_1})$ at $+\infty$; therefore Ψ_1 is, by $\mathscr{L}_{\varepsilon}$, the image of a solution Ψ of (E) whose shadow is the river F, and the image of Ψ by $\mathscr{M}_{\omega,F}$ is a solution ψ of $(\mathrm{e})_{\omega}$. The construction implies that ψ_1 is the image of ψ by $\mathscr{O}_{\omega,\rho}$. The values of ψ_1 remain limited for $0 \lessapprox x \lessapprox 2$, and particularly for $x = 1$; we remark that by $\mathscr{O}_{\omega,\rho}$, $(1,1) \to (1,0)$ and $(1,1)$ belongs to the slow curve of $(\mathrm{e})_{\omega}$; so, by condition (iii) of Lemma 6, $\psi(1)$ belongs to the halo of the slow curve, and, due to its repellingness, so do the values of $\psi(x)$ for $0 \lessapprox x \lessapprox 1$—thus ψ is a slow solution of $(\mathrm{e})_{\omega}$ and Ψ is a persistent solution of (E) by Lemma 3. By construction, for limited X,

$$\Psi(X) = F(X) + \varepsilon\Psi_1(X) = F(X) + \varepsilon\left[F_1(X) + \phi\right],$$

so Ψ has the required expansion.

To complete the proof it remains to show that Φ, the initially given persistent solution of (E), has the same expansion. By condition (i) of Lemma 6, the image φ of Φ by $\mathscr{M}_{\omega,F}$ is a slow solution for all $x \gtrapprox 0$; so, as the slow curve is regularly repelling, the slow solutions φ and ψ satisfy $|\varphi(x) - \psi(x)| < e^{-k/\varepsilon}$, for $0 \lessapprox x \lessapprox 1$, with $k \gtrapprox 0$ (this is a well-known result; see [4], for example). The magnification rate of $\mathscr{O}_{\omega,\rho}$ being a limited power of $1/\varepsilon$, one has also $\varphi_1(x) \simeq \psi_1(x)$ for $0 \lessapprox x \lessapprox 1$, so φ_1 is a slow solution of $(\mathrm{e}_1)_{\omega,\rho}$; so by the same argument as for Ψ_1, $\Phi_1(X) \simeq F_1(X)$ for all limited X, whence the announced expansion for Φ. Q.E.D.

When the unperturbed equation is linear, one does not need the complicated contraction conditions, because of the uniqueness of rivers; replacing $\mathscr{M}_{\omega,F}$ by a macroscope of the type $\mathscr{N}_{\omega,\lambda}$: $\omega x = X$, $\omega^{\lambda} y = Y$, and chosing λ so that $X^{\lambda-1}/F(X) \to 0$ as $X \to +\infty$, one proves the following.

PROPOSITION 3. *The conclusion of Theorem* 2 *holds if one replaces the assumptions on* $(\overline{\mathrm{E}})$ (*the hypotheses on the perturbation remaining unchanged*) *by*

$$\dot{Y} = C(X)Y + D(X), \qquad (\overline{\mathrm{E}})$$

where C *and* D *have polynomial growth at* $+\infty$ (*resp.* $-\infty$), *and* $C(X) \gg 1/X$ *as* $X \to +\infty$ (*resp.* $X \to -\infty$).

This work is continued in Part II.

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