On certain quotients of the Green rings of dihedral 2-groups

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Abstract

In this paper we study the properties of Green rings of dihedral 2-groups, and in particular certain quotients of these Green rings introduced by Benson and Carlson. It is shown that these quotients can be realised as group rings over \( \mathbb{Z} \). The properties of the corresponding groups are investigated: they are shown to be abelian, torsion-free and infinitely generated. We also show how taking products of elements of these groups is related to the structure of the Auslander–Reiten quivers for dihedral 2-groups.

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1. Introduction

When studying representations of groups, it is important to understand how the structures of the indecomposable modules are related. In particular, the tensor product of two group algebra modules is again a group algebra module; tensor products are an important tool throughout representation theory. The Green ring \( A(kG) \) of a group algebra \( kG \) was introduced by J.A. Green as a tool for studying direct sums and tensor products of group algebra modules in cases where the characteristic of the field divides the order of the group \([10]\). The isomorphism classes of indecomposable modules are given the structure of a ring by taking direct sums and tensor products of modules to give the addition and multiplication, respectively, in the ring. Studying this ring and its properties provides a method for addressing the question of how tensor products of indecomposable group algebra modules behave.

The Green ring of a cyclic \( p \)-group is well understood. In particular we have the following theorem of J. A. Green.

**Theorem 1.1** ([10, Theorem 2]). Let \( k \) be a field of characteristic \( p \), and let \( G \) be a cyclic group of order a power of \( p \). Then \( A(kG) \) is semisimple.

This theorem is the main result of [10]; this paper is the origin of the study of Green rings as algebraic objects, explaining why they have since been given this name.

The Green ring \( A(kV_4) \) of the Klein four group, with \( k \) a field of characteristic 2, is also well understood: the decompositions into direct sums of indecomposable modules of all possible tensor products of indecomposable \( kV_4 \)-modules have been calculated by Conlon [9]—see Section 2.5.
In contrast with the case of the cyclic $p$-groups and the Klein four group, relatively little is known about the structure of the Green ring $A(kD_{4q})$, with $k$ a field of characteristic 2 and $q$ a power of 2 with $q \geq 2$. In particular, there is no description of how tensor products of $kD_{4q}$-modules decompose, as there is for $A(kV_4)$. Research into the structure of $A(kD_{4q})$ has focussed on the question of whether it is semisimple, or equivalently whether $A(kD_{4q})$ has any nilpotent elements. It was shown by Zemanek [14] that, unlike $A(kC_p)$, the Green ring $A(kD_{4q})$ will have nilpotent elements. Methods for constructing such elements can be found in [15] and in the survey articles [3,4]. The existence of nilpotent elements of order greater than 2 was shown in [11]; it is not known whether there are any nilpotent elements of order greater than 3.

In this paper, we investigate the structure of the Green ring $A(kD_{4q})$ of the dihedral 2-group of order $4q$. In particular, we study the properties of a certain quotient of $A(kD_{4q})$, which we shall call the Benson–Carlson quotient, as this proves more tractable than tackling $A(kD_{4q})$ directly. We show (Theorem 3.2) that these quotients can be realised as group rings over $\mathbb{Z}$. We therefore move to looking at the properties of the groups providing these realisations (Theorems 3.2, 3.4 and 3.5). Finally, we consider the problem of calculating products of elements in these groups. Although we are not able to give an explicit description of how to calculate such products, we demonstrate a relationship between these products and the structure of the Auslander–Reiten quiver for $kD_{4q}$. This relationship is presented in Theorem 4.3, which shows how taking products in these groups induces graph isomorphisms between components of the Auslander–Reiten quiver for $kD_{4q}$. This result—and the approach to studying the Green rings of dihedral 2-groups which it suggests—is related to that used by Christine Bessenrodt in classifying the endotrivial modules for dihedral 2-groups [7].

2. Preliminaries

2.1. Green rings

Let $kG$ be a group algebra. Take $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all elements $g \in G$ and extend linearly to $kG$. This makes $kG$ into a bialgebra. Let $M$ and $N$ be left $kG$-modules. Putting $g(m \otimes n) = (g \otimes g)(m \otimes n) = gm \otimes gn$ for $g \in G$ makes $M \otimes N$ into a $kG$-module in the usual way.

Let $kG$ be a group algebra and let $M$ and $N$ be left $kG$-modules. Then the action of $kG$ on $\text{Hom}_k(M, N)$ is given by $(g\phi)(m) = g(\phi(g^{-1}m))$ for $g \in G$. This action allows us to define the dual module $M^* = \text{Hom}_k(M, k)$. Note that we are able to view the dual module $M^*$ as a left $A$-module.

**Definition 2.1.** Let $kG$ be a group algebra. Let $A(kG)$ be the ring generated by all isomorphism classes of $kG$-modules, with the only relations being $[M] + [N] = [M \oplus N]$ and with multiplication defined by $[M] \circ [N] = [M \otimes N]$. This is the Green ring (also sometimes called the representation ring) of $kG$.

The addition defined on $A(kG)$ is both associative and commutative; multiplication is also associative. Multiplication in the Green ring is also commutative since we have $M \otimes N \simeq N \otimes M$ as $kG$-modules.

The Krull–Schmidt Theorem tells us that each element of $A(kG)$ can be written uniquely as a sum of isomorphism classes of indecomposable $kG$-modules. So, the additive group of $A(kG)$ is a free abelian group with generators the isomorphism classes of indecomposable $kG$-modules.

2.2. The Benson–Carlson quotient

**Proposition 2.2.** Let $k$ be an algebraically closed field of characteristic $p$. Let $kG$ be a group algebra. Let $A(kG; p)$ be the span in $A(kG)$ of all $kG$-modules $M$ such that all direct summands of $M$ have dimension divisible by $p$. Then $A(kG; p)$ is an ideal of $A(kG)$.

This proposition is a version of [6, Lemma 2.5].

As a consequence of Proposition 2.2, we will be able to study quotients of the form $A(kG)/A(kG; p)$. We shall call these the Benson–Carlson quotients, as they were first introduced and studied by Benson and Carlson [6].

2.3. Dihedral 2-groups and their indecomposable modules

Let $k$ be a field of characteristic 2. Let
\[ k D_{4q} = \{ x, y : x^{2q} = y^2 = 1, y^{-1}xy = x^{-1} \}, \]

with \( q \geq 2 \) and \( q \) a power of 2, be the dihedral group of order \( 4q \). If we put \( X \mapsto 1 + y \) and \( Y \mapsto 1 + xy \), we have \( k D_{4q} \) isomorphic to

\[ \Lambda = \frac{k(X, Y)}{\langle X^2, Y^2, (XY)^q - (YX)^q \rangle}. \]

The indecomposable \( k D_{4q} \)-modules were classified by Ringel [12]. There is a single projective indecomposable module, namely \( \Lambda \). The non-projective indecomposable modules can be described as belonging to two classes, the string modules and the band modules. (This terminology was not used by Ringel, but comes from the fact that \( \Lambda \) is a special biserial algebra as defined in [13]; the non-projective indecomposable \( \Lambda \)-modules are thus string modules and band modules as defined for string algebras.)

**The string modules:**

Let \( W \) be the set of words in the letters \( X, Y, X^{-1} \) and \( Y^{-1} \) such that

(i) \( X \) and \( X^{-1} \) are always followed by \( Y \) or \( Y^{-1} \), and vice versa.

(ii) no word in \( W \) contains any of \( (X^{-1}Y)^q, (Y^{-1}X)^q, (XY)^q, (YX)^q \).

Also include in \( W \) the word 1 of length 0. For a word \( w = a_1a_2 \cdots a_n \) in \( W \), define \( w^{-1} := a_n^{-1} \cdots a_2^{-1}a_1^{-1} \), where we set \( (X^{-1})^{-1} = X \) and \( (Y^{-1})^{-1} = Y \) and \( 1^{-1} = 1 \). Then we define an equivalence relation \( \sim \) on \( W \) by setting \( w \sim w^{-1} \) for all \( w \in W \).

The indecomposable string modules for \( \Lambda \) are in one-to-one correspondence with equivalence classes of \( W \) under \( \sim \). Given a word \( w = a_1a_2 \cdots a_n \), we have a natural basis \( v_1, v_2, \ldots, v_{n+1} \) for the corresponding module \( M(w) \). The actions of \( X \) and \( Y \) with respect to this basis are defined by

\[
\begin{align*}
Xv_i &= v_{i-1} & \text{if } a_{i-1} = X \\
Yv_i &= v_{i-1} & \text{if } a_{i-1} = Y \\
Xv_i &= v_{i+1} & \text{if } a_i = X^{-1} \\
Yv_i &= v_{i+1} & \text{if } a_i = Y^{-1} \\
Xv_i &= 0 & \text{otherwise} \\
Yv_i &= 0 & \text{otherwise}
\end{align*}
\]

**The band modules:**

Details of the structure of band modules for \( D_{4q} \) will not be required for the work that follows; it will be sufficient to note the following two facts:

(i) All band modules are even-dimensional.

(ii) All band modules satisfy \( \text{im } X \cap \text{ker } Y \subseteq \text{im } Y \) and \( \text{im } Y \cap \text{ker } X \subseteq \text{im } X \).

### 2.4. Almost split sequences

In the work that follows we will make use of information about the components of the Auslander–Reiten quiver for \( k D_{4q} \), which contain string modules. Here we use the results and notation of [2, Appendix] (see also [5, Section 4.17]).

Define two functions \( R_q : W \rightarrow W \) and \( L_q : W \rightarrow W \) as follows. If \( w \in W \) ends in \( XY^{-1}(X^{-1}Y^{-1})q^{-1} \) or \( YX^{-1}(Y^{-1}X^{-1})q^{-1} \) then \( wR_q \) is obtained from \( w \) by cancelling that part; otherwise define \( wR_q = wX^{-1}(XY)^qY^{-1} \) or \( wR_q = wY^{-1}(XY)^qX^{-1} \), whichever is a word. Similarly, if \( w \in W \) begins with \( (XY)^qXY^{-1} \) or \( (YX)^qYX^{-1} \) then \( wL_q \) is obtained by cancelling this part; otherwise define \( wL_q = X^{-1}(XY)^qY^{-1}Yw \) or \( wL_q = Y^{-1}(X^{-1}Y^{-1})q^{-1}Xw \), whichever is a word. Note that \( R_q \) and \( L_q \) depend on the actual word \( w \in W \) and not just on the equivalence class under \( \sim \). It is clear that \( R_q \) and \( L_q \) commute. Ringel gives the following construction for taking even powers of \( \Omega \):

**Theorem 2.3** (Ringel [12]). Let \( M(w) \) be the indecomposable string \( k D_{4q} \)-module corresponding to a word \( w \in W \). Then \( \Omega^2 \langle M(w) \rangle \simeq M(wR_qL_q) \).

Furthermore, the almost split sequence terminating in \( M(w) \) is

\[
0 \rightarrow M(wR_qL_q) \rightarrow M(wL_q) \oplus M(wR_q) \rightarrow M(w) \rightarrow 0,
\]

unless \( w \) or \( w^{-1} \) is \( (XY)^qXY^{-1}(X^{-1}Y^{-1})q^{-1} \), in which case it is

\[
0 \rightarrow M(wR_qL_q) \rightarrow M(wL_q) \oplus M(wR_q) \oplus \Lambda^A \rightarrow M(w) \rightarrow 0.
\]
The Auslander–Reiten quiver components for $kD_{4q}$ involving string modules are: two 1-tubes

$M(u_1) \xrightarrow{} M(u_1 R_q) \xrightarrow{} M(u_1 R_q^2) \xrightarrow{} \cdots$  

and

$M(u_2) \xleftarrow{} M(u_2 R_q) \xleftarrow{} M(u_2 R_q^2) \xleftarrow{} \cdots$  

where $u_1 = (XY)^{q-1}X$ and $u_2 = (YX)^{q-1}Y$, and an infinite set of components of type $ZA_\infty$, one of which has the projective module $\Lambda$ attached (see [2, Appendix] for pictures of these quiver components). Note that the two 1-tubes described above involve only modules of even dimension, so all odd-dimensional $kD_{4q}$-modules lie in some Auslander–Reiten quiver component of type $ZA_\infty$.

The components of the Auslander–Reiten quiver containing band modules will not be used in the work that follows, though it is known that all band modules lie in 1-tubes (see again [2, Appendix] and [5, Section 4.17]).

### 2.5. Klein four group modules and their products

In our study of the Green ring $A(kD_{4q})$ of the dihedral group of order $4q$ and the associated Benson–Carlson quotient, useful information will be gained by looking at restrictions of $kD_{4q}$-modules to a subgroup isomorphic to the Klein four group and at tensor products of such restricted modules.

Let $k$ be a field of characteristic 2 and let $V_4 = \langle g, h \rangle$ be the Klein four group. Putting $X = 1 + g$ and $Y = 1 + h$ we see that

$$kV_4 \simeq k[X, Y] / (X^2, Y^2).$$

The indecomposable $kV_4$-modules were classified by Kronecker. The families of indecomposable modules are as follows:

- $\Omega^n(k)$, for all $n \in \mathbb{Z}$.
- $C_n(\pi)$, for all $n \in \mathbb{N}$ and all irreducible polynomials $\pi = T^m + u_{m-1}T^{m-1} + \cdots + u_1 T + u_0$ over $k$, where the actions of $X$ and $Y$ on the module $C_n(\pi)$ are given by

$$X \mapsto \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & C_Y \\ 0 & 0 \end{pmatrix},$$

where $C_Y$ is an $mn \times mn$ matrix consisting of $m \times m$ blocks

$$C_Y = \begin{pmatrix} M & 0 \\ N & \ddots \\ 0 & N & M \end{pmatrix},$$

with

$$M = \begin{pmatrix} 0 & 1 \\ u_0 & u_1 & \cdots & u_{m-2} & u_{m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

- $C_n(\infty)$, for all $n \in \mathbb{N}$, where the actions of $X$ and $Y$ on $C_n(\infty)$ are given by

$$X \mapsto \begin{pmatrix} 0 & C_X \\ 0 & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$
where $C_X$ is the $n \times n$ matrix

$$C_X = \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $D = kV_4$, which is both the only projective indecomposable module and the only injective indecomposable module for our algebra.

This notation for the indecomposable modules of type $C_n(\pi)$ and $C_n(\infty)$ is that used in [9].

The multiplications of any pair of modules in the Green ring $A(kV_4)$ have been calculated by Conlon [9]; the results are recorded in the tables shown in Fig. 1.

### 3. Realisation of the Benson–Carlson quotient as a group algebra

Let $k$ be a field of characteristic 2, and let $D_{4q}$ with $q \geq 2$ be the dihedral group of order $4q$. We have seen that

$$kD_{4q} \simeq \frac{k\langle X, Y \rangle}{(X^2, Y^2, (XY)^q - (YX)^q)}.$$ 

The co-multiplication on $kD_{4q}$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X + X \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y + Y \otimes Y$.

**Lemma 3.1.** Let $M, N$ be two odd-dimensional indecomposable $kD_{4q}$-modules. The product $M \otimes N$ of $kD_{4q}$-modules decomposes as a direct sum of indecomposables as one odd-dimensional string module plus band modules and projective summands.

**Proof.** The classification of indecomposable $kD_{4q}$-modules in [12] tells us that $M$ and $N$ must be string modules corresponding to strings in $X$ and $Y$ of even length. Let $A_X$ be the subalgebra of $kD_{4q}$ generated by $X$. There are precisely two indecomposable $A_X$-modules, the trivial module $k$ where $X$ acts as 0 and the projective module $A_X$. We have $M|_{A_X} = k \oplus$ projectives and $N|_{A_X} = k \oplus$ projectives. Hence $(M \otimes N)|_{A_X} = k \oplus$ projectives, since taking the tensor product of modules commutes with restriction to $A_X$.

Similarly, let $A_Y$ be the subalgebra of $kD_{4q}$ generated by $Y$. By symmetry, the two indecomposable $A_Y$-modules have the same structure as the two indecomposable $A_X$-modules. As in the case of $A_X$, we then have $(M \otimes N)|_{A_Y} = k \oplus$ projectives.
Let \( N \) be an odd-dimensional indecomposable \( k D_4 \)-module. It is clear that every odd-dimensional summand (which must be a string module) will contribute a copy of \( k \) when restricted to \( A \). Hence \( M \otimes N \) has at most one odd-dimensional indecomposable summand; the fact that \( \dim(M \otimes N) \) is odd tells us that there must be at least one such summand.

Furthermore, any even-dimensional string module given by a string which begins and ends with \( Y^{\pm 1} \) will contribute two copies of \( k \) when restricted to \( A \). Hence \( M \otimes N \) cannot have any summands of this form. Similarly, if we consider restrictions to \( A \), we see that \( M \otimes N \) cannot have any summands which are even-dimensional string modules given by strings beginning and ending with \( X^{\pm 1} \). Any string module of even dimension must be of one of these two forms. So, we see that \( M \otimes N \) cannot have any summands which are even-dimensional string modules. \( \square \)

**Theorem 3.2.** The cosets of the odd-dimensional indecomposable \( kD_4 \)-modules form an abelian group under the tensor product multiplication in

\[
\frac{A(kD_4)}{A(kD_4; 2)}.
\]

To prove Theorem 3.2, we will need the following result of Benson and Carlson:

**Theorem 3.3** ([6, Theorem 2.1]). Let \( k \) be an algebraically closed field of characteristic \( p \). Let \( G \) be a finite group. Let \( M \) and \( N \) be indecomposable \( kG \)-modules. Then \( M \otimes N \) has the trivial module \( k \) as a direct summand if and only if the following two conditions are satisfied:

(i) \( M \simeq N^* \)

(ii) \( p \nmid \dim N \).

Moreover, if \( k \) is a component of \( N^* \otimes N \) then it has multiplicity one.

**Proof of Theorem 3.2.** Let \( M \) and \( N \) be odd-dimensional indecomposable \( kD_4 \)-modules, and let \([M]\) and \([N]\) be the corresponding cosets in \( A(kD_4)/A(kD_4; 2) \). By Lemma 3.1 we have \([M] \otimes [N] = [U]\) for some odd-dimensional indecomposable \( kD_4 \)-module \( U \). It is clear that our multiplication will be associative and commutative, since this follows from the corresponding properties of \( A(kD_4) \). It is also clear that the coset of the trivial module \( k \) (on which both \( X \) and \( Y \) act as 0) will be a multiplicative identity. Given an odd-dimensional indecomposable module \( M \), Theorem 3.3 tells us that \( M \otimes M^* \) has a summand isomorphic to \( k \). By Lemma 3.1, \( M \otimes M^* \) has precisely one odd-dimensional summand. Hence this summand must be \( k \) and we have \([M] \otimes [M^*] = [k]\) in \( A(kD_4)/A(kD_4; 2) \), telling us that \([M^*]\) is the inverse for \([M]\). \( \square \)

We now introduce notation for this group: write \( \Gamma(kD_4) \) for the group given by Theorem 3.2.

This result tells us that the Benson–Carlson quotient we are considering is in fact the group ring over \( \mathbb{Z} \) of the group given by Theorem 3.2. That is,

\[
\frac{A(kD_4)}{A(kD_4; 2)} \simeq \mathbb{Z}\Gamma(kD_4).
\]

We now study the properties of the group \( \Gamma(kD_4) \). For \([M]\) an element of \( \Gamma(kD_4) \), let \( M \) denote the unique odd-dimensional indecomposable summand of \( M \).

**Theorem 3.4.** The group \( \Gamma(kD_4) \) is torsion-free.

**Proof.** Let \( N \) be an indecomposable \( kD_4 \)-module, and suppose \([N]^{\otimes r} = [k]\) in \( A(kD_4)/A(kD_4; 2) \), for some \( r \geq 1 \). Then, by Theorem 3.3, we must have \( N^{\otimes (r-1)} \simeq N^* \). We will show that this is not possible unless \( N \simeq k \).

Let \( N \) be an odd-dimensional indecomposable \( kD_4 \)-module with \( N \not\simeq k \), that is, a string module given by some string \( w = u_1u_2 \cdots u_{2n} \) of even length. If \( u_1 = X^\pm 1 \) then \( u_{2n} = Y^\pm 1 \) (and vice versa). Assume without loss of generality that \( u_1 = X^\varepsilon \) with \( \varepsilon \in \{\pm 1\} \). We claim that \( N^{\otimes t} \) is also given by a word beginning with \( X^t \), for all \( t \geq 1 \). This would be sufficient to show that \( N^{\otimes (r-1)} \not\simeq N^* \), since the dual module \( N^* \) is given by the string \( u_1^{-1}u_2^{-1} \cdots u_{2n}^{-1} \).

If the claim is correct then this is different from the string corresponding to \( N^{\otimes r} \). The only string equivalent to \( u_1^{-1}u_2^{-1} \cdots u_{2n}^{-1} \) under Ringel’s equivalence relation on strings is \( u_{2n}u_{2n-1} \cdots u_1 \). This is also different from the string...
corresponding to $N^*\otimes t$, since $u_{2n} = Y^{\pm 1}$. So, the strings corresponding to $N^*$ and $N^*\otimes t$ are not equivalent, which tells us that $N^*\otimes t \not\simeq N^*$, as required.

So, it remains to show that if $N$ is given by a string beginning with $X^e$ then $N^*\otimes t$ is also given by a string beginning with $X^e$ for all $t \geq 1$. We first take the case $e = 1$. We will give a proof by induction on $t$, so suppose $N$ and $N^*\otimes (t-1)$ are both given by words beginning with $X$. In this case there must exist an element $b \in N$ such that $b \in \text{im} X \cap X \cap \ker Y$ and $b \not\in \text{im} Y$. Similarly, there must exist an element $c \in N^*\otimes (t-1)$ with the same properties. Then we have an element $b \otimes c \in N^*\otimes t$ with $b \otimes c \in \text{im} X \cap X \cap \ker Y$ and $b \otimes c \not\in \text{im} Y$.

Suppose, for a contradiction, that $N^*\otimes t$ is given by a word beginning with $X^{-1}$. Inspection of the natural basis for a string module shows that such a module satisfies $\text{im} X \cap \ker Y \subseteq \text{im} Y$. Lemma 3.1 tells us that $N^*\otimes t$ decomposes as $N^*\otimes t$ plus band modules, plus projective summands. All band modules and projective modules for $kD_{4q}$ also satisfy $\text{im} X \cap \ker Y \subseteq \text{im} Y$. Hence $N^*\otimes t$ satisfies $\text{im} X \cap \ker Y \subseteq \text{im} Y$. The existence of $b \otimes c$ gives the required contradiction.

Now take the case $e = -1$, so $N$ is given by a string beginning with $X^{-1}$. Then $N^*$ is given by a string beginning with $X$. So, by the case above, $(N^*)^* \simeq (N^*\otimes t)^*$ is given by a word beginning with $X$. Hence $N^*\otimes t$ is given by a word beginning with $X^{-1}$. This completes the proof. □

Theorem 3.5. The group $\Gamma(kD_{4q})$ is not finitely generated.

Proof. Suppose, for a contradiction, that $\Gamma(kD_{4q})$ is finitely generated by elements $[M_1], [M_2], \ldots, [M_n]$. Then for all odd-dimensional indecomposable $kD_{4q}$-modules $U$, we must have $[U] = [M_1^{\otimes r_1} \otimes M_2^{\otimes r_2} \otimes \cdots \otimes M_n^{\otimes r_n}]$ in $\Gamma(kD_{4q})$.

Recall that $kD_{4q} = k\langle X, Y \rangle / (X^2, Y^2, (XY)^q = (YX)^q)$. Let

$$A = (XY)^q + (XY)^q - 1 + X + (XY)^q - 1 + \sum_{i=0}^{(\log_2 q)-1} ((XY)^q - 2^i + (YX)^q - 2^i)$$

and $B = X$. Then $AB = BA$ and $A^2 = B^2 = 0$. So, $A$ and $B$ generate a subalgebra $A'$ isomorphic to $kV_4$. Note that $A = 1 + z$, where $z$ is the central involution in the group $D_{4q}$, so $A'$ is indeed the group algebra of a subgroup of $D_{4q}$ isomorphic to $V_4$. Consider the restrictions $M_i|_{A'}$. We have

$$M_i|_{A'} = C \oplus \bigoplus_{s \in \mathbb{Z}} \beta_{i,s} \Omega^s(k) \oplus \gamma_1 D$$

where $C$ is some direct sum of non-projective even-dimensional indecomposable $kV_4$-modules, with $\beta_{i,s}, \gamma_1 \in \mathbb{N}_0$ and only finitely many of the coefficients $\beta_{i,s}$ non-zero.

Now, given an odd-dimensional indecomposable $kD_{4q}$-module $U$, we have supposed $U \simeq M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n}$. Equivalently, $U$ is a direct summand of $M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n}$. Then $U|_{A'}$ is a direct summand of $(M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n})|_{A'}$, which is isomorphic to $(M_1|_{A'})^{\otimes r_1} \otimes \cdots \otimes (M_n|_{A'})^{\otimes r_n}$. Now, by inspection of the multiplication tables for $kV_4$-modules in Section 2.5, we see that a $kV_4$-module of type $C_t(T)$ for $t \in \mathbb{N}$ can only occur as a summand of $(M_1|_{A'})^{\otimes r_1} \otimes \cdots \otimes (M_n|_{A'})^{\otimes r_n}$ if $C_t(T)$ occurs with non-zero coefficient in the restriction $M_i|_{A'}$ for some $i$. Only finitely many $C_t(T)$ occur in these restrictions. So, we see that only finitely many $C_t(T)$ can occur with non-zero coefficient in restrictions $U|_{A'}$ of odd-dimensional indecomposable $kD_{4q}$-modules $U$.

To give the required contradiction, we now present a family of $kD_{4q}$-modules whose restrictions to $A'$ involve infinitely many different $C_t(T)$. Let $U_r$ be the odd-dimensional string module given by the string $XY(X^{-1}Y)^{r-1}Y^{-1}Y^{-1}X^{-1}$, for $r \geq 1$. These satisfy $U_r|_{A'} \simeq C_{q+2+(r-1)2^{q-1}}$. This gives the required infinite list of $C_t(T)$ appearing in restrictions to $A'$. This tells us that $\Gamma(kD_{4q})$ cannot be finitely generated. □

4. Tensor products and Auslander–Reiten theory

In this section we shall look at how taking tensor products of odd-dimensional $kD_{4q}$-modules is related to the Auslander–Reiten quiver.

The key tool in this section will be the following theorem:
Theorem 4.1. Let $kG$ be a group algebra and let $M$ be a finitely generated indecomposable $kG$-module such that $\text{char}(k)$ does not divide $\dim M$. If
\[ 0 \rightarrow \tau(k) \rightarrow E \rightarrow k \rightarrow 0 \] is an almost split sequence, then
\[ 0 \rightarrow M \otimes \tau(k) \rightarrow M \otimes E \rightarrow M \rightarrow 0 \] is an almost split sequence modulo injectives.

Two formulations of this result can be found in [1, Theorem 3.6] and [6, Proposition 2.15].

Let $L$ be the unique indecomposable $kD_{4q}$-module such that the sequence
\[ 0 \rightarrow L \rightarrow k^\uparrow D_{4q} \rightarrow k \rightarrow 0 \] is exact. For example, in the case $q = 2$ we have $L = M(XY)$.

We can use Theorem 4.1 to prove the following lemma describing a particular tensor product involving this module. First recall Benson’s description of the Auslander–Reiten sequences of odd-dimensional $kD_{4q}$-modules outlined in Section 2.4.

Lemma 4.2. Let $M$ be an odd-dimensional $kD_{4q}$-module. We may assume that $M = M(w)$ where $w$ is a word beginning with $X^e$ for some $e \in \{ \pm 1 \}$. Then the tensor product $M(w) \otimes \Omega(L) \equiv M(wR_q)$ modulo projectives.

Proof. It can be seen that the almost split sequence terminating in $k$ is
\[ 0 \rightarrow \Omega^2(k) \rightarrow \Omega(L) \oplus \Omega(L^*) \rightarrow k \rightarrow 0. \]
So, by Theorem 4.1,
\[ 0 \rightarrow M(w) \otimes \Omega^2(k) \rightarrow M(w) \otimes \Omega(L) \oplus M(w) \otimes \Omega(L^*) \rightarrow M(w) \rightarrow 0 \] is almost split modulo projectives. But, by Benson’s construction, the almost split sequence terminating in $M(w)$ is
\[ 0 \rightarrow M(wR_q L_q) \rightarrow M(wR_q) \oplus M(wL_q) \rightarrow M(w) \rightarrow 0. \]
Since the almost split sequence terminating in a particular module is unique up to isomorphism, we must therefore have either $M(w) \otimes \Omega(L) \equiv M(wR_q)$ or $M(w) \otimes \Omega(L) \equiv M(wL_q)$ modulo projectives.

We now look at restrictions to $H = k\langle A, X \rangle \simeq kV_4$. The module $L$ was used by Carlson and Thévenaz in their classification of the endotrivial $kD_{4q}$-modules, where it is shown that $(\Omega^a_{D_{4q}}(L^{\otimes b}))|_H = \Omega_{V_4}^{a+b}(k)$ modulo projectives [8, Theorem 5.4]. Suppose $M|_H = C \oplus \Omega^t(k) \oplus$ projectives, where $C$ is a direct sum of non-projective even-dimensional indecomposable $kV_4$-modules. Then $(M(w) \otimes \Omega(L))|_H = M(w)|_H \otimes \Omega(L)|_H = M(w)|_H \otimes (\Omega^t(k) \oplus mD) = C \oplus \Omega^{t+2}(k) \oplus$ projectives, by our tables of multiplications for $kV_4$-modules. So, it is enough to show that $M(wL_q)|_H \not\cong C \oplus \Omega^{t+2}(k) \oplus$ projectives.

Suppose, for a contradiction, that $M(wL_q)|_H \cong C \oplus \Omega^{t+2}(k) \oplus$ projectives. Note that $\dim M(wL_q) = \dim M(w) \pm 2q$ (according to whether $wL_q$ is longer or shorter than $w$). Furthermore, $\dim \Omega^{t+2}(k) \cong \Omega^{t+2}(k) \oplus$ projectives (according to whether $|t+2|$ is greater or less than $|t|$). So, by comparing dimensions, our assumption tells us that the number of projective summands of $M(w)|_H$ and $M(wL_q)|_H$ must differ either by $\pm (q - 2)/2$ or by $\pm (q + 2)/2$. We will show that this is not the case.

First, consider the case where $wL_q = X^{-1}(Y^{-1}X^{-1})^{q-1}Yw$. Let $v_1, v_3, \ldots, v_{q-1}$ be the first $q/2$ odd-numbered basis elements of the corresponding natural basis for the string module $M(wL_q)$. Then for $i \in \{1, 3, \ldots, q-1\}$ we have $AXv_i \neq 0$, so $v_i$ must be the top of a projective $kH$-module, which must then be a direct summand of $M(wL_q)|_H$. But any element $v$ which is the top of a projective summand of $M(w)|_H$ will clearly also be the top of a projective summand of $M(wL_q)|_H$. So, in this case $M(wL_q)|_H$ has $q/2$ more projective summands than $M(w)|_H$.

Similarly, if $w$ begins with $(XY)^{-q-1}XY^{-1}$ and $wL_q$ is obtained from $w$ by cancelling this part, then $M(w)|_H$ has $q/2$ more projective summands than $M(wL_q)|_H$. \(\square\)

We now use this to show how products of elements in the group $\Gamma(kD_{4q})$ are related to the structure of the Auslander–Reiten quiver.
Theorem 4.3. Let $M = M(u)$ and $N$ be odd-dimensional indecomposable $kD_{4q}$-modules. Suppose $M(u) \otimes N = M(w)$. We may assume without loss of generality that $u$ and $w$ both begin with $X^\varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Then for all $i, j \in \mathbb{Z}$ we have $M(uR_q^jL_q^{-i}) \otimes N \simeq M(wR_q^iL_q^j)$.

Remark. Recall that the elements $M(uR_q^jL_q^{-i})$ for $i, j \in \mathbb{Z}$ are the vertices of the Auslander–Reiten quiver component containing $M$, with their position in the quiver relative to $M$ determined by the indices $i$ and $j$. So, this result tells us that the component on $M \otimes N$ can be obtained by taking the component of $M$ and replacing all vertices with their image under the map $U \mapsto U \otimes N$. So, tensoring with $N$ induces a graph isomorphism from the component of $M(u)$ to the component of $M(w)$. Alternatively, Theorem 4.3 says that, if we know $M \otimes N$ then $U \otimes N$ is determined for all $U$ in the same Auslander–Reiten quiver component as $U$, since $U \otimes N$ is determined by its position relative to $M \otimes N$ in the corresponding component.

This result is similar in structure to Theorem 2.3 of [7], in which Bessenrodt shows that, for any finite group $G$, tensoring with an endotrivial $kG$-module induces, modulo projectives, an isomorphism between components of the Auslander–Reiten quiver for $kG$.

Proof of Theorem 4.3. First we look at modules in the same column of the Auslander–Reiten quiver as $M(u)$; that is, we show that

\[ \frac{M(uR_q^jL_q^{-i}) \otimes N}{M(wR_q^iL_q^j)} \quad \text{for all } i \in \mathbb{Z}. \] (2)

We prove this for $i \geq 0$ by induction on $i$; the case $i \leq 0$ is analogous using induction on $-i$. The base case $i = 0$ holds by definition. Assume

\[ \frac{M(uR_q^{-1}L_q^{-i}) \otimes N}{M(wR_q^{-1}L_q^{-i})}. \]

Now

\[ \frac{M(uR_q^jL_q^{-i}) \otimes N}{\Omega^{-2}(k) \otimes M(uR_q^{i+1}L_q^{-1}) \otimes N} \]

by Theorem 2.3. Then, by two applications of Lemma 4.2, this is

\[ \frac{\Omega^{-2}(k) \otimes \Omega^{2}(L \otimes 2) \otimes M(uR_q^{i-1}L_q^{-i}) \otimes N}{\Omega^{-2}(k) \otimes M(wR_q^{i+1}L_q^{-i})}. \]

which equals

\[ \Omega^{-2}(k) \otimes \Omega^{2}(L \otimes 2) \otimes M(wR_q^{i-1}L_q^{-i}) \]

by the inductive hypothesis. Then, applying Lemma 4.2 again, this is

\[ \Omega^{-2}(k) \otimes M(wR_q^{i+1}L_q^{-i}). \]

Finally, Theorem 2.3 tells us that this is $M(wR_q^iL_q^j)$ as required.

Now we look at modules in the preceding column of the Auslander–Reiten quiver; that is, we show that

\[ \frac{M(uR_q^{i+1}L_q^{-i}) \otimes N}{M(wR_q^{i+1}L_q^{-i}) \quad \text{for all } i \in \mathbb{Z}. \] (3)

For $i = 0$, we have $M(uR_q) \otimes N = \Omega(L) \otimes M(u) \otimes N$ by Lemma 4.2. By definition this is $\Omega(L) \otimes M(w)$, which equals $M(wR_q)$ by Lemma 4.2. Using this as the base case for an inductive argument exactly analogous to that presented above then completes the proof of (3).

We can now use (2) and (3) to look at the general case $M(uR_q^jL_q^j)$. This will be done in two cases, according to whether $i + j$ is even or odd. First suppose $i + j$ is even. Then we can re-write $M(uR_q^jL_q^j) \otimes N$ as

\[ \frac{M(uR_q^{(i-j)/2}L_q^{(i-j)/2}R_q^{i+j/2}L_q^{i+j/2}) \otimes N}{\Omega(i+j)(k) \otimes M(uR_q^{(i-j)/2}L_q^{i-j/2}) \otimes N}, \]

which is

\[ \frac{\Omega(i+j)(k) \otimes M(uR_q^{(i-j)/2}L_q^{i-j/2}) \otimes N}{M(uR_q^{(i-j)/2}L_q^{(i-j)/2}) \otimes N}. \]
by Theorem 2.3. Then by (2) this is
\[ \Omega^{(i+j)}(k) \otimes M(w R_q^{(i-j)/2} L_q^{(j-i)/2}) \]
and applying Theorem 2.3 again this is \( M(w R_q^i L_q^j \otimes N) \) as required. Similarly, if \( i + j \) is odd, we can use Theorem 2.3 to take out a factor of \( \Omega^{(i+j-1)}(k) \) from \( M(u R_q^i L_q^j) \), apply (3) and then multiply \( \Omega^{(i+j-1)}(k) \) back in to get the required result. \( \square \)

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