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Vector F-implicit complementarity problems in Banach spaces☆

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Abstract

In this work, a new class of vector F-implicit complementarity problems and vector F-implicit variational inequality problems are introduced and studied, and the equivalence between of them is presented under certain assumptions in Banach spaces. We also derive some new existence theorems of solutions for the vector F-implicit complementarity problems and the vector F-implicit variational inequality problems by using the FKKM theorem under some suitable assumptions without monotonicity.

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1. Introduction

Vector variational inequality was first introduced and studied by Giannessi [4] in the setting of finite-dimensional Euclidean spaces. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criteria consideration. Throughout the development over the last twenty years, existence theorems of solutions of vector variational inequalities in various situations have been studied by many authors (see, for example, [1,5–10] and the references therein). At the same time, the

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vector variational inequality has found many applications in vector optimization, approximate vector optimization, vector equilibria, vector traffic equilibria and other areas (see [5,8]).

Let X be a real Banach space with dual space X^*, and $\langle t, x\rangle$ denote the value of the linear continuous function $t \in X^*$ at x. Let K be a closed convex cone of X. In 2001, Yin et al. [10] introduced a class of F-complementarity problems (F-CP), which consists of finding $x \in K$ such that

$$\langle Tx, x\rangle + F(x) = 0, \qquad \langle Tx, y\rangle + F(y) \geq 0$$

for all $y \in K$, where $T : K \to X^*$ and $F : K \to (-\infty, +\infty)$, and they proved that ($F$-CP) is equivalent to the following generalized variational inequality problem (GVIP): find $x \in K$ such that

$$\langle Tx, y - x\rangle + F(y) - F(x) \geq 0, \quad \forall y \in K,$$

when K is a nonempty closed convex cone and F is a positively homogeneous and convex function. They also proved the existence of solutions for (F-CP) under some assumptions with F-pseudomonotonicity.

Recently, by using the combination of demicontinuity and pseudomonotonicity, Fang and Huang [3] introduced and studied a new class of vector F-complementarity problems with demipseudomonotone mappings in Banach spaces. They also presented the solvability of this class of vector F-complementarity problems with demipseudomonotone mappings and finite-dimensional continuous mappings in reflexive Banach spaces.

Very recently, Huang and Li [6] introduced and studied a new class of F-implicit complementarity problems and F-implicit variational inequality problems in Banach spaces. The equivalence between the F-implicit complementarity problem and F-implicit variational inequality problem was presented, and some new existence theorems of solutions for F-implicit complementarity problems and F-implicit variational inequality problems were also proved.

The main purpose of this work is to generalize some results of [6] for the vector case. We introduce a new class of vector F-implicit complementarity problems and vector F-implicit variational inequality problems in Banach spaces, and prove the equivalence between of them under certain assumptions. Furthermore, we derive some new existence theorems of solutions for the vector F-implicit complementarity problems and the vector F-implicit variational inequality problems by using the FKKM theorem [2] under some suitable assumptions without any monotonicity.

2. Preliminaries

Let Y be a real Banach space. A nonempty subset P of Y is said to be a convex cone if: (i) $P + P = P$; (ii) $\lambda P \subseteq P$ for all $\lambda > 0$. P is called a pointed cone if P is a convex cone and $P \cap \{-P\} = \{0\}$. An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a closed convex cone $P \subseteq Y$ with an apex at the origin, in the form of

$$x \geq y \quad \Leftrightarrow \quad x - y \in P$$

and

$$x \not\geq y \quad \Leftrightarrow \quad x - y \notin P.$$

If the interior of P, say int P, is nonempty, then a weak ordering in X is also defined by

$$y < x \Leftrightarrow x - y \in \operatorname{int} P, \quad \forall x, y \in Y,$$

and

$$y \not< x \Leftrightarrow x - y \notin \operatorname{int} P, \quad \forall x, y \in Y.$$

Let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex set, (Y, P) be an ordered Banach space induced by a pointed closed convex cone P. Denote by $L(X, Y)$ the space of all the continuous linear mappings from X into Y.

We first recall some definitions and lemmas which are needed in the main results of this work.

Definition 2.1. Let K be a nonempty subset of topological vector space X. A point-to-set mapping $T : K \to 2^X$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, \dots, x_n\}$ of K,

$$\operatorname{conv}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i),$$

where conv denotes the convex hull.

Definition 2.2. A mapping $F : K \to Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$.

Lemma 2.1 ([2]). *Let K be a nonempty subset of Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM-mapping, such that for any $y \in K$, $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$, i.e., $\cap_{y\in K} G(y) \neq \emptyset$.*

Lemma 2.2. *Let (Y, P) be an ordered Banach space induced by a pointed closed convex cone P. Then, $x \geq 0$ and $y \geq 0$ imply that $x + y \geq 0$, $\forall x, y \in Y$.*

Proof. Since P is a convex cone, $x \geq 0 \Leftrightarrow x \in P$ and $y \geq 0 \Leftrightarrow y \in P$, we know that $x+y \in P+P = P$ and so $x + y \geq 0$. This completes the proof. □

3. Vector F-implicit complementarity problems and vector variational inequality problems

Throughout this section, let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex set, and (Y, P) be an ordered Banach space induced by a pointed closed convex cone P. Denote by $L(X, Y)$ the space of all the continuous linear mappings from X into Y, and $\langle t, x\rangle$ the value of the linear continuous mapping $t \in L(X, Y)$ at x. Let $f : K \to L(X, Y)$, $g : K \to K$ and $F : K \to Y$. In this section, we consider the following vector F-implicit complementarity problem (VF-ICP): find $x^* \in K$ such that

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle + F(y) \geq 0, \quad \forall y \in K.$$

Examples of (VF-ICP):

(1) If g is an identity mapping on K, then (VF-ICP) collapses to the vector F-complementary problem (for short VF-CP) of finding $x^* \in K$ such that

$$\langle f(x^*), x^*\rangle + F(x^*) = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle + F(y) \geq 0, \quad \forall y \in K.$$

(2) If $F = 0$, then (VF-CP) collapses to the vector complementary problem (for short VCP) of finding $x^* \in K$ such that

$$\langle f(x^*), x^*\rangle = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle \geq 0, \quad \forall y \in K,$$

which has been studied by Chen and Yang [1], and Yang [9].

(3) If $L(X, Y) = X^*$ and $F : K \to R$, then (VF-ICP) collapses to the F-implicit complementary problem (for short F-ICP) of finding $x^* \in K$ such that

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle + F(y) \geq 0, \quad \forall y \in K,$$

which is considered by Huang and Li [6].

(4) If g is an identity mapping on K, then (F-ICP) collapses to the F-complementary problem (for short F-CP) of finding $x^* \in K$ such that

$$\langle f(x^*), x^*\rangle + F(x^*) = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle + F(y) \geq 0, \quad \forall y \in K,$$

which has been studied by Yin et al. [10].

(5) If $F = 0$, then (F-ICP) reduces to the implicit complementary problem (for short ICP) of finding $x^* \in K$ such that

$$\langle f(x^*), g(x^*)\rangle = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle \geq 0, \quad \forall y \in K.$$

which has been studied by Isac [7,8].

(6) If g is an identity mapping on K and $F = 0$, then (F-ICP) reduces to the complementary problem (for short CP) of finding $x^* \in K$ such that

$$\langle f(x^*), x^*\rangle = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle \geq 0, \quad \forall y \in K.$$

which has been studied by many authors, see [8]. If $X = X^* = R^n$, then (CP) becomes the classical complementarity problem.

We also introduce the following vector F-implicit variational inequality problem (for short VF-IVIP): find $x^* \in K$ such that

$$\langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*)) \geq 0, \quad \forall y \in K.$$

We first establish the equivalence between (VF-ICP) and (VF-IVIP).

Theorem 3.1. (i) *If x^* solves* (VF-ICP), *then x^* solves* (VF-IVIP)*;* (ii) *if $F : K \to Y$ is positively homogeneous and x^* solves* (VF-IVIP), *then x^* solves* (VF-ICP).

Proof. (i) Let x^* be a solution of (VF-ICP). Then, $x^* \in K$ such that

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) = 0 \qquad \text{and} \qquad \langle f(x^*), y\rangle + F(y) \geq 0, \quad \forall y \in K.$$

It follows that

$$\begin{aligned}&\langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*))\\ &\quad = [\langle f(x^*), y\rangle + F(y)] - [\langle f(x^*), g(x^*)\rangle + F(g(x^*))]\\ &\quad = \langle f(x^*), y\rangle + F(y)\\ &\quad \geq 0\end{aligned}$$

for all $y \in K$. Thus, x^* is a solution of (VF-IVIP).

(ii) Let x^* be a solution of (VF-IVIP). Then, $x^* \in K$ such that

$$\langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*)) \geq 0, \quad \forall y \in K. \tag{3.1}$$

Since $F : K \to Y$ is a positively homogeneous mapping, and K is a convex cone, letting $y = 2g(x^*)$ and $y = \frac{1}{2}g(x^*)$ in (3.1), we have

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) \geq 0, \qquad \text{and} \qquad \langle f(x^*), g(x^*)\rangle + F(g(x^*)) \leq 0,$$

that is,

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) \in P \cap \{-P\}.$$

Since P is a pointed cone,

$$\langle f(x^*), g(x^*)\rangle + F(g(x^*)) = 0.$$

By using this equality and (3.1), we obtain

$$\begin{aligned}
&\langle f(x^*), y\rangle + F(y)\\
&\quad = [\langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*))] + [\langle f(x^*), g(x^*)\rangle + F(g(x^*))]\\
&\quad = \langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*))\\
&\quad \geq 0
\end{aligned}$$

for all $y \in K$, which shows that x^* solves (VF-ICP). This completes the proof. □

If g is an identity mapping on K, then we have the following:

Corollary 3.1. (i) *If x^* solves (* VF-CP*), then x^* solves* (VF-VIP)*;* (ii) *if $F : K \to Y$ is positively homogeneous and x^* solves* (VF-VIP)*, then x^* solves (* VF-CP*).*

Theorem 3.2. *Assume that:*

(a) *$f : K \to L(X, Y)$, $g : K \to K$ and $F : K \to Y$ are continuous;*
(b) *there exists a mapping $h : K \times K \to Y$ such that*
  (i) *$h(x, x) \geq 0, \forall x \in K$;*
  (ii) *$\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) - h(x, y) \geq 0, \forall x, y \in K$;*
  (iii) *the set $\{y \in K : h(x, y) \ngeq 0\}$ is convex, $\forall x \in K$;*
(c) *there exists a nonempty, compact, convex subset C of K, such that $\forall x \in K\backslash C$, $\exists y \in C$ such that*

$$\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) \ngeq 0.$$

Then, (VF-IVIP) *has a solution. Furthermore, the solution set of* (VF-IVIP) *is closed.*

Proof. Define

$$G(y) = \{x \in C \mid \langle f(x), y - g(x)\rangle + F(y) - F(g(x)) \geq 0\}, \quad \forall y \in K.$$

From assumption (a), we have that for any $y \in K$, $G(y)$ is closed in C. Since every element $x^* \in \cap_{y\in K} G(y)$ is a solution of (VF-IVIP), we have to show that $\cap_{y\in K} G(y) \neq \emptyset$. Since C is compact, it is sufficient to prove that the family $\{G(y)\}_{y\in K}$ has the finite intersection property. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of K and set $B = \overline{\mathrm{conv}}(C \cup \{y_1, \ldots, y_n\})$. Then B is a compact and convex subset of K.

We define two point-to-set mappings $F_1, F_2 : B \to 2^B$ as follows:

$$F_1(y) = \{x \in B \mid \langle f(x), y - g(x)\rangle + F(y) - F(g(x)) \geq 0\}, \quad \forall y \in B$$

and

$$F_2(y) = \{x \in B \mid h(x, y) \geq 0\}, \quad \forall y \in B.$$

From assumptions (i) and (ii) of (b), we have

$$h(y, y) \geq 0$$

and

$$\langle f(y), y - g(y)\rangle + F(y) - F(g(y)) - h(y, y) \geq 0.$$

Now Lemma 2.2 implies

$$\langle f(y), y - g(y)\rangle + F(y) - F(g(y)) \geq 0$$

and so $F_1(y)$ is nonempty. Similarly, we can prove that for any $y \in K$, $F_1(y)$ is closed. Since $F_1(y)$ is a closed subset of a compact set B, we know that $F_1(y)$ is compact. Next, we prove that F_2 is a KKM-mapping. Suppose that there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of B and $\lambda_i \geq 0 (i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j).$$

Then

$$h(u, u_j) \ngeq 0, \quad j = 1, 2, \ldots, n.$$

From assumption (b)(iii), we have

$$h(u, u) \ngeq 0,$$

which contradicts assumption (b)(i). Hence F_2 is a KKM-mapping. From assumption (b)(ii), we have $F_2(y) \subseteq F_1(y), \forall y \in B$. In fact, $x \in F_2(y)$ implies that $h(x, y) \geq 0$, and by assumption (b)(ii), we have

$$\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) - h(x, y) \geq 0.$$

It follows from Lemma 2.2 that

$$\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) \geq 0,$$

i.e., $x \in F_1(y)$. Thus, F_1 is also a KKM-mapping. From Lemma 2.1, there exists $x^* \in B$ such that $x^* \in F_1(y)$ for all $y \in B$. Therefore, there exists $x^* \in B$ such that $\langle f(x^*), y - g(x^*)\rangle + F(y) - F(g(x^*)) \geq 0$ for all $y \in B$. By assumption (c), we get $x^* \in C$ and moreover $x^* \in G(y_i), i = 1, 2, \ldots, n$. Hence $\{G(y)\}_{y\in K}$ has the finite intersection property.

Since $f : K \to X^*, g : K \to K$ and $F : K \to Y$ are continuous, it is easy to see that the solution set of (VF-IVIP) is closed. This completes the proof of Theorem 3.2. □

Example 3.1. Let $X = Y = R^2$, $K = P = R^2_+ = [0, \infty) \times [0, \infty)$, $C = [0, 1] \times [0, 1]$. Let

$$g(x) = \left(\frac{x_2}{2}, \frac{x_1}{2}\right), \qquad F(x) = (x_1, 0), \qquad f(x) \equiv f$$

and $\langle f(x), z\rangle = f(z) = (z_1 + z_2, 0)$ for any $x, z \in K$, with $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Then,

$$\langle f(x), y - g(x)\rangle = \left((y_1 + y_2) - \frac{x_1 + x_2}{2}, 0\right)$$

for any $x, y \in K$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If we set

$$h(x, y) = \left((2y_1 + y_2) - \left(\frac{x_1}{2} + x_2\right), 0\right)$$

for any $x, y \in K$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then all assumptions in Theorem 3.2 hold. It is easy to see that $(0, 0) \in K$ is a unique solution of (VF-IVIP).

If g is an identity mapping on K, then from Theorem 3.2, we obtain an existence theorem for (F-VIP) as follows.

Corollary 3.2. *Assume that*

(a) $f : K \to L(X, Y)$, $g : K \to K$ *and* $F : K \to Y$ *are continuous;*
(b) *there exists a mapping* $h : K \times K \to Y$ *such that*
  (i) $h(x, x) \geq 0, \forall x \in K$;
  (ii) $\langle f(x), y - x\rangle + F(y) - F(x) - h(x, y) \geq 0, \forall x, y \in K$;
  (iii) *the set* $\{y \in K : h(x, y) \not\geq 0\}$ *is convex,* $\forall x \in K$;
(c) *there exists a nonempty, compact, convex subset* C *of* K, *such that* $\forall x \in K \backslash C$, $\exists y \in C$ *such that*

$$\langle f(x), y - x\rangle + F(y) - F(x) \not\geq 0.$$

Then, (VF-VIP) *has a solution. Furthermore, the solution set of* (VF-VIP) *is closed.*

Theorem 3.3. *Assume that* $f : K \to L(X, Y)$ *and* $g : K \to K$ *are continuous, and* $F : K \to Y$ *is positively homogeneous and continuous. If assumptions* (b) *and* (c) *in* Theorem 3.2 *hold, then* (VF-ICP) *has a solution. Furthermore, the solution set of* (VF-ICP) *is closed.*

Proof. It follows directly from Theorems 3.1 and 3.2 that the conclusion holds. This completes the proof. □

Example 3.2. Let $X = Y = R^2$, $K = P = R^2_+ = [0, \infty) \times [0, \infty)$ and $C = [0, 1] \times [0, 1]$. Let

$$g(x) = \left(x_1 + \frac{x_2}{2}, \frac{x_2}{2}\right), \qquad F(x) = \left(-\frac{x_1 + x_2}{2}, 0\right), \qquad f(x) \equiv f$$

and $\langle f(x), z\rangle = f(z) = (z_1 + z_2, 0)$ for any $x, z \in K$, with $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Then,

$$\langle f(x), y - g(x)\rangle = ((y_1 + y_2) - (x_1 + x_2), 0)$$

for any $x, y \in K$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If we set

$$h(x, y) = \left(\frac{(y_1 + y_2) - (x_1 + x_2)}{2}, 0\right)$$

for any $x, y \in K$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then all assumptions in Theorem 3.3 hold. It is easy to see that $(0, 0) \in K$ is a unique solution of (VF-ICP).

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