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Note

A note on: "Relaxation oscillators with exact limit cycles" ☆

Jaume Giné, Maite Grau *

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain

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Abstract

In this note we give a family of planar polynomial differential systems with a prescribed hyperbolic limit cycle. This family constitutes a corrected and wider version of an example given in the work [M.A. Abdelkader, Relaxation oscillators with exact limit cycles, J. Math. Anal. Appl. 218 (1998) 308–312]. The result given in this note may be used to construct models of Liénard differential equations exhibiting a desired limit cycle.

Keywords: Planar polynomial differential system; Limit cycle; Hyperbolicity; Liénard differential equation

1. Introduction and statement of the main result

Our purpose in this work is to give a family of planar polynomial differential systems of the form:

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

for which an explicit expression of a limit cycle, that is, an isolated periodic orbit, can be given. We assume that $P(x, y)$ and $Q(x, y)$ belong to the ring of real polynomials in two variables

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* Corresponding author.
E-mail addresses: gine@eps.udl.es (J. Giné), mtgrau@matematica.udl.es (M. Grau).

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$\mathbb{R}[x, y]$, and we will always assume that $P(x, y)$ and $Q(x, y)$ are coprime polynomials. We denote by $\mathrm{d} = \max\{\deg P, \deg Q\}$ and we say that d is the degree of system (1).

In the work [1] a family of planar polynomial differential systems like (1) is studied and the existence of an explicit limit cycle is pretended to be given. The author of [1] gives a family of systems of the form (1) with a prescribed invariant algebraic curve. This curve $f(x, y) = 0$ has an oval surrounding the origin of coordinates. However, in [1] there is no proof of the fact that the oval of $f(x, y) = 0$ is an isolated periodic orbit, that is, a limit cycle. It is stated as obvious. We have been able to weaken the hypothesis appearing in [1], getting a bigger family of planar polynomial systems, and we have been able to show that the oval of $f(x, y) = 0$ is a hyperbolic limit cycle.

Theorem 1. *We consider a polynomial $p(x)$ such that $p(x_e) = 0$ and $p(x_d) = 0$ for some $x_e < 0$ and $x_d > 0$, $p'(x_e) \neq 0$ and $p'(x_d) \neq 0$. We assume that $p(x) > 0$ for all x in the interval (x_e, x_d). We consider another polynomial $q(x)$ satisfying $p(x)q(x)^2 \neq 1$ for all $x \in (x_e, x_d)$ and $q'(x) \neq 0$ for all $x \in (x_e, x_d)$. Then, the algebraic curve given by $f(x, y) = 0$ with $f(x, y) := (y - p(x)q(x))^2 - p(x)$ has an oval in the band $x_e \leqslant x \leqslant x_d$ which is a hyperbolic limit cycle for the following system*:

$$\dot{x} = y,$$
$$\dot{y} = \left(\frac{3}{2}q(x)p'(x) + p(x)q'(x)\right)y - \frac{p'(x)}{2}\big(p(x)q(x)^2 - 1\big). \tag{2}$$

We note that a system of the form (2) can also be viewed as an autonomous Liénard differential equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3}$$

where $f(x)$ and $g(x)$ are the polynomials given by $f(x) := -3q(x)p'(x)/2 - p(x)q'(x)$ and $g(x) := (p'(x)/2)(p(x)q(x)^2 - 1)$. Therefore, Theorem 1 may be used to construct models of Liénard differential equations exhibiting a hyperbolic limit cycle.

One of the most famous Liénard differential equations is called van der Pol equation and it appears when studying the vacuum-tube circuits. This particular equation (3) has $f(x) = \mu(x^2 - 1)$ and $g(x) = 1$, with $\mu \in \mathbb{R}$, and it exhibits a unique hyperbolic limit cycle surrounding the origin. This limit cycle is shown to be non-algebraic in the work of Odani [3]. We are not considering van der Pol's equation since the systems described in Theorem 1 always exhibit an algebraic limit cycle. The systems given in (2) are examples of Liénard equations with an algebraic and hyperbolic limit cycle.

The same kind equations are studied in the work [1], but under other hypothesis for the polynomials $p(x)$ and $q(x)$, and the author of [1] pretends to state the existence of a limit cycle. The conditions for the polynomials $p(x)$ and $q(x)$ appearing in [1] are: $p(x)$ is an even polynomial, $p(0) > 0$, there exists a value $X > 0$ such that $p(X) = 0$, $p(x) < 0$ for all $x > X$, $q(x)$ is an odd polynomial and $q(0) = 0$. All these conditions are contained in the ones that we assume in Theorem 1. However, the authors noticed that in the work [1] the condition $p(x)q(x)^2 \neq 1$ for $x \in (-X, X)$ does not appear and it is not implied by the other hypothesis. As we will see in the proof of Theorem 1, condition $p(x)q(x)^2 \neq 1$ for $x \in (x_e, x_d)$ is necessary to have a limit cycle and it cannot be avoided.

We remark that each system of the family (2) has an algebraic limit cycle, which is an oval of the real algebraic curve $f(x, y) = 0$, and it may have other limit cycles which are not taken into consideration. These other limit cycles can be contained in $f(x, y) = 0$ or not. If they are contained in an invariant curve, we can treat them with the same methods described in this note. For instance, it can be shown that the system (2) with $p(x) = (1 - x^2)(4 - x^2)(9 - x^2)$ and $q(x) = x/100$ has 3 hyperbolic limit cycles all of them contained in the corresponding invariant algebraic curve $f(x, y) = 0$.

In order to prove Theorem 1, we need some results relating ovals of curves of the form $f(x, y) = 0$ with the fact that the oval of such a curve is a limit cycle. These preliminary results are stated in Section 2. Once we have stated these previous results, we prove Theorem 1 in Section 3.

2. Preliminary results

We are considering limit cycles which are contained in a real curve $f(x, y) = 0$, which does not need to be algebraic. This fact leads us to the definition of invariant of a system (1).

Definition 2. Let us consider an open set $\mathcal{U} \subseteq \mathbb{R}^2$ and a $\mathcal{C}^1(\mathcal{U})$ real function denoted by $f(x, y): \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$. We say that $f(x, y)$ is an *invariant* for a system (1) if

$$P(x, y)\frac{\partial f}{\partial x}(x, y) + Q(x, y)\frac{\partial f}{\partial y}(x, y) = k(x, y) f(x, y), \tag{4}$$

with $k(x, y)$ a polynomial of degree lower or equal than $\mathrm{d} - 1$, where d is the degree of the system. This polynomial $k(x, y)$ is called the *cofactor* of $f(x, y)$.

In case that $f(x, y)$ is a polynomial we say that $f(x, y) = 0$ is an *invariant algebraic curve* for system (1). We notice that if $f(x, y)$ is an invariant of system (1) and $f(x, y) = 0$ defines a curve in the real plane, then the function $P(x, y)(\partial f/\partial x) + Q(x, y)(\partial f/\partial y)$ equals zero on the points such that $f(x, y) = 0$. This fact implies that the real curve $f(x, y) = 0$ is formed by orbits of system (1). In particular if $f(x, y) = 0$ contains an oval without any singular point of system (1), this oval is a periodic orbit of system (1).

As well as invariant curves, the other objects taken into consideration in this paper are limit cycles. A *limit cycle* of system (1) is an isolated periodic orbit. Let γ be a limit cycle for system (1). We say that γ is *stable* if there exists a neighborhood such that all the orbits starting in it have γ as ω-limit set. We say that γ is *unstable* if there is a neighborhood such that all the orbits starting in it have γ as α-limit set. There might be limit cycles which are neither stable nor unstable. These limit cycles have a neighborhood such that in the interior of the limit cycle all the orbits have γ as ω-limit set and in the exterior of γ all the orbits have γ as α-limit set. Or the other way round: the orbits of the interior have γ as α-limit set and the orbits in the exterior have γ as ω-limit set. In this case, we say that γ is *semi-stable*. Any limit cycle γ of a system (1) is either stable, unstable or semi-stable as it is stated in [4].

A classical known result, given in the book of Perko [4], let us distinguish the hyperbolicity of a limit cycle. If we consider $\gamma(t)$ a periodic orbit of system (1) of period T, we may compute the finite value given by the following integral $\int_0^T \operatorname{div}(\gamma(t))\,dt$, where $\operatorname{div}(x, y) = (\partial P/\partial x) + (\partial Q/\partial y)$ is called the *divergence* of system (1). It can be shown that if $\int_0^T \operatorname{div}(\gamma(t))\,dt < 0$, then γ is a stable limit cycle, if $\int_0^T \operatorname{div}(\gamma(t))\,dt > 0$, then γ is a unstable limit cycle and if $\int_0^T \operatorname{div}(\gamma(t))\,dt = 0$, then γ may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles. When the quantity $\int_0^T \operatorname{div}(\gamma(t))\,dt$ is different from zero, we say that the limit cycle γ is *hyperbolic*. We notice that if $\int_0^T \operatorname{div}(\gamma(t))\,dt \neq 0$, then the periodic orbit γ is a limit cycle (either stable or unstable). We are going to use this property to ensure that a periodic orbit is a limit cycle, that is, that it does not belong to a continuous band of cycles.

We relate limit cycles with invariants in the following way. We assume that we have a periodic orbit γ of system (1) which is given in an implicit way, that is, there exists an invariant curve $f(x,y)=0$ such that $\gamma \subseteq \{(x,y) \mid f(x,y)=0\}$. In order to have a smooth curve $f(x,y)=0$ defining the periodic orbit, we will assume that $\nabla f(p) \neq 0$ for any $p \in \gamma$, that is, the gradient vector of $f(x,y)$ is different from zero in all the points of γ. Then we have the following result stated and proved in [2].

Theorem 3. *Let us consider a system* (1) *and $\gamma(t)$ a periodic orbit of period $T>0$. Assume that $f:\mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is an invariant curve with $\gamma \subseteq \{(x,y) \mid f(x,y)=0\}$ and let $k(x,y)$ be the cofactor of $f(x,y)$ as given in* (4)*. We assume that $\nabla f(p) \neq 0$ for any $p \in \gamma$. Then,*

$$\int_0^T k\big(\gamma(t)\big)\,dt = \int_0^T \operatorname{div}\big(\gamma(t)\big)\,dt. \tag{5}$$

Hence, we have an alternative way to compute the value $\int_0^T \operatorname{div}(\gamma(t))\,dt$. In the family of planar polynomial differential systems which we are considering, that is, the one described in Theorem 1, we will not be able to directly compute the value $\int_0^T \operatorname{div}(\gamma(t))\,dt$. This is due to the fact that we are not considering a fixed system with a concrete periodic orbit, but a family of systems each one with a different periodic orbit and, thus, the expression of the integrand is too general to be manipulated. By Theorem 3, we can also compute the value $\int_0^T k(\gamma(t))\,dt$ but this integral is as much difficult as the previous one. That is why we are going to use the fact that for any $w \in \mathbb{R}$:

$$\int_0^T \operatorname{div}\big(\gamma(t)\big)\,dt = \int_0^T \operatorname{div}\big(\gamma(t)\big)\,dt + w\left(\int_0^T \operatorname{div}\big(\gamma(t)\big)\,dt - \int_0^T k\big(\gamma(t)\big)\,dt\right).$$

The integrand in the right-hand side of this equality will be chosen strictly positive or negative in all the interval of integration for a suitable value of w. Therefore, the value of the integral will be different from zero. Using these steps, we will be able to prove that the oval of the invariant curve described in Theorem 1 is a limit cycle of the corresponding system.

3. Proof of Theorem 1

Proof. In order to prove this theorem, we first show that $f(x,y)=0$, where $f(x,y) := (y - p(x)q(x))^2 - p(x)$, is an invariant algebraic curve of system (2). Straightforward computations show that

$$\begin{aligned} &y\left(\frac{\partial f}{\partial x}\right) + \left[\left(\frac{3}{2}q(x)p'(x) + p(x)q'(x)\right)y - \frac{p'(x)}{2}\big(p(x)q(x)^2 - 1\big)\right]\left(\frac{\partial f}{\partial y}\right) \\ &\quad = q(x)p'(x)f(x,y), \end{aligned}$$

and, thus, we have that $f(x,y)=0$ is an invariant algebraic curve for system (2) with cofactor $k(x,y) := q(x)p'(x)$.

Since $p(x_e) = p(x_d) = 0$ for the values $x_e < 0$ and $x_d > 0$ and $p(x) > 0$ for $x \in (x_e, x_d)$, we deduce that $f(x, y) = 0$ has an oval in the band $x_e \leqslant x \leqslant x_d$ surrounding the origin of coordinates, which can be parameterized in two parts by:

$$x(\tau) = \tau, \qquad y_{\pm}(\tau) = p(\tau)q(\tau) \pm \sqrt{p(\tau)}, \tag{6}$$

with $\tau \in (x_e, x_d)$. We are going to prove that this oval does not contain any singular point of system (2), and then, we will have that it defines a periodic orbit of the system. The singular points of system (2) have coordinates of the form $(a, 0)$ where the value a is a root of the polynomial $p'(x)(p(x)q(x)^2 - 1)$. We have that $f(a, 0) = p(a)(p(a)q(a)^2 - 1)$ and $(p(a)q(a)^2 - 1)$ is different from zero in all the closed interval $a \in [x_e, x_d]$ by the hypothesis that $p(x)q(x)^2 \neq 1$ for all $x \in (x_e, x_d)$ and $p(x_e) = p(x_d) = 0$.

Here we notice that the assumption $p(x)q(x)^2 \neq 1$ for $x \in (x_e, x_d)$ is necessary for the oval of $f(x, y) = 0$ to be a limit cycle. In [1], this assumption is not given. We notice that an oval of an invariant algebraic curve of a system may contain singular points of the system, and in such a case, it is not even a periodic orbit.

Since x_e and x_d are simple zeroes of $p(x)$ and $p(x) > 0$ for all x in the interval (x_e, x_d) we have that there is no singular point of system (2) on the oval given by $f(x, y) = 0$ and parameterized by (6). From this fact and that $f(x, y) = 0$ is an invariant algebraic curve of the system, we deduce that this oval is a periodic orbit of system (1). We denote this periodic orbit by γ for the rest of the proof. We note that we do not know the parameterization of γ as explicit solution of system (2), that is, we do not know the periodic function $\gamma(t) := (\gamma_1(t), \gamma_2(t))$ such that $d\gamma_1(t)/dt = \gamma_2(t)$ and

$$\begin{aligned}\frac{d\gamma_2(t)}{dt} &= \left(\frac{3}{2} q\big(\gamma_1(t)\big) p'\big(\gamma_1(t)\big) + p\big(\gamma_1(t)\big) q'\big(\gamma_1(t)\big)\right)\gamma_2(t) \\ &\quad - \frac{p'(\gamma_1(t))}{2}\big(p\big(\gamma_1(t)\big) q\big(\gamma_1(t)\big)^2 - 1\big),\end{aligned}$$

for all $t \in \mathbb{R}$. We do neither know its period $T > 0$ but we have been able to show its existence by using the invariant algebraic curve $f(x, y) = 0$ and its properties in relation with system (2).

Finally, we need to prove that the periodic orbit γ is a hyperbolic limit cycle. To do so, we are going to show that the value of the integral $\int_0^T \operatorname{div}(\gamma(t))\,dt$ is different from zero. Since we do not know $\gamma(t)$ nor the period T, we use the parameterization of the oval γ given in (6). In order to get the correct sign of the integral $\int_0^T \operatorname{div}(\gamma(t))\,dt$, we need to know the sense of the flow over γ. We take the point of coordinates $(x_d, 0)$, which belongs to γ, and we have that the vector field defined by system (2) on that point is $(0, p'(x_d)/2)$ because $p(x_d) = 0$. Since $p(x_d) = 0$, $p'(x_d) \neq 0$ and $p(x) > 0$ in the interval (x_e, x_d), we deduce that $p'(x_d) < 0$. Hence, the sense of the flow over γ is clockwise. We can write the following equality, using the parameterization (6):

$$\begin{aligned}\int_0^T \operatorname{div}\big(\gamma(t)\big)\,dt &= \int_{x_e}^{x_d} \frac{\operatorname{div}(x(\tau), y_+(\tau))}{y_+(\tau)}\,d\tau + \int_{x_d}^{x_e} \frac{\operatorname{div}(x(\tau), y_-(\tau))}{y_-(\tau)}\,d\tau \\ &= \int_{x_e}^{x_d} \left(\frac{\operatorname{div}(x(\tau), y_+(\tau))}{y_+(\tau)} - \frac{\operatorname{div}(x(\tau), y_-(\tau))}{y_-(\tau)}\right) d\tau.\end{aligned}$$

We note that the divergence of system (2) is $\operatorname{div}(x,y) = 3q(x)p'(x)/2 + p(x)q'(x)$, and substituting this expression in the former equality, we get

$$\int_0^T \operatorname{div}\bigl(\gamma(t)\bigr)\,dt = \int_{x_e}^{x_d} \frac{-3q(\tau)p'(\tau) - 2p(\tau)q'(\tau)}{(p(\tau)q(\tau)^2 - 1)\sqrt{p(\tau)}}\,d\tau.$$

This integral is well defined because we are assuming that $(p(\tau)q(\tau)^2 - 1)p(\tau)$ is different from zero for $\tau \in (x_e, x_d)$. However, we are not able to distinguish if its value is positive, negative or zero. By using the same reasonings, we can write the following equality:

$$\int_0^T k\bigl(\gamma(t)\bigr)\,dt = \int_{x_e}^{x_d} \frac{-2q(\tau)p'(\tau)}{(p(\tau)q(\tau)^2 - 1)\sqrt{p(\tau)}}\,d\tau.$$

Using Theorem 3, we have that, for any value of $w \in \mathbb{R}$:

$$\begin{aligned}\int_0^T \operatorname{div}\bigl(\gamma(t)\bigr)\,dt &= \int_0^T \operatorname{div}\bigl(\gamma(t)\bigr)\,dt + w\left(\int_0^T \operatorname{div}\bigl(\gamma(t)\bigr)\,dt - \int_0^T k\bigl(\gamma(t)\bigr)\,dt\right)\\ &= \int_{x_e}^{x_d} \frac{-(w+3)q(\tau)p'(\tau) - 2(1+w)p(\tau)q'(\tau)}{(p(\tau)q(\tau)^2 - 1)\sqrt{p(\tau)}}\,d\tau.\end{aligned}$$

Taking $w = -3$, we get

$$\int_0^T \operatorname{div}\bigl(\gamma(t)\bigr)\,dt = \int_{x_e}^{x_d} \frac{4p(\tau)q'(\tau)}{(p(\tau)q(\tau)^2 - 1)\sqrt{p(\tau)}}\,d\tau. \tag{7}$$

The hypotheses of Theorem 1 on the sign of the polynomials $p(x)$ and $q(x)$ in the interval $x \in (x_e, x_d)$ are $p(x) > 0$, $p(x)q(x)^2 \neq 1$ and $q'(x) \neq 0$. Therefore the integrand of the right-hand side of (7) is strictly positive or negative in all the interval $\tau \in (x_e, x_d)$. We deduce that the value of the integral cannot be zero and, hence, the periodic orbit γ is a hyperbolic limit cycle as we wanted to show.

We would also like to characterize if this limit cycle is stable or unstable, so we are going to study the sign of the integrand in the right-hand side of (7). Since $p(x)q(x)^2 \neq 1$ for $x \in (x_e, x_d)$ and $p(x_d)q(x_d)^2 - 1 = -1$ (because $p(x_d) = 0$), we deduce that $p(x)q(x)^2 - 1 < 0$ for $x \in (x_e, x_d)$. Therefore, we have that the integrand in the right-hand side of (7) is strictly positive if $q'(\tau) < 0$ for all $\tau \in (x_e, x_d)$ and strictly negative if $q'(\tau) > 0$ for all $\tau \in (x_e, x_d)$. We can state that the hyperbolic limit cycle γ is stable if $q'(0) > 0$ and unstable if $q'(0) < 0$. □

We also note that in the work [1], the expression of an example of a more general limit cycle for a family of planar polynomial differential systems is pretended to be given. In fact, we are going to show that in the case that the oval of this example is a limit cycle, we are in the same family of systems as written in (2), that is, the one described in Theorem 1.

In [1] the following planar polynomial differential system is given as an example of a more general family of systems with an explicit limit cycle.

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \{p'(x)[(m+r)h(x)p(x)^{r-1} + (m+1)q(x)] + h'(x)p(x)^r + p(x)q'(x)\}y\\ &\quad - mp(x)p'(x)([h(x)p(x)^{r-1} + q(x)]^2 - p(x)^{2m-2}),\end{aligned} \tag{8}$$

where $p(x)$ is an even polynomial, $q(x)$ and $h(x)$ are odd polynomials, $m = n + 1/2$, n is an integer number with $n \geqslant 0$ and r is an integer number with $r \geqslant 2$. Moreover, it is assumed that $p(0) > 0$ and there exists a value $X > 0$ such that $p(X) = 0$, $p(x) < 0$ for all $x > X$ and $q(0) = 0$.

Some straightforward computations show that system (8) exhibits the invariant algebraic curve $f(x, y) = 0$ with $f(x, y) := (y - h(x)p(x)^r - q(x)p(x))^2 - p(x)^{2n+1}$ and with cofactor $k(x, y) := (2n+1)p'(x)[h(x)p(x)^{r-1} + q(x)]$. We have that $f(x, y) = 0$ has an oval in the band $-X \leqslant x \leqslant X$ which can be parameterized by:

$$x(\tau) = \tau, \qquad y(\tau) = h(\tau)p(\tau)^r + p(\tau)q(\tau) \pm p(\tau)^n\sqrt{p(\tau)}.$$

In order to show that this oval is a periodic orbit of system (8), we only need to show that it does not contain any singular point of the system. The singular points of system (8) in the band $|x| \leqslant X$ are of the form $(a, 0)$ where a is a root of the polynomial $p(x)p'(x)([h(x)p(x)^{r-1} + q(x)]^2 - p(x)^{2n-1})$, because $m = n + 1/2$. We notice that, unless $n = 0$, the points with coordinates $(-X, 0)$ and $(X, 0)$ are singular points of the system which are contained in the oval of $f(x, y) = 0$. Therefore, if $n > 0$, we have that the oval of $f(x, y) = 0$ cannot be a limit cycle. If $n = 0$, we can consider the polynomial $\tilde{q}(x) := q(x) + h(x)p(x)^{r-1}$ and we have that system (8) coincides with system (2) with polynomials $p(x)$ and $\tilde{q}(x)$. Therefore, this is not an example of a more general limit cycle.

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