Exterior problems in the half-space for the Laplace operator in weighted Sobolev spaces

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The purpose of this work is to solve exterior problems in the half-space for the Laplace operator. We give existence and unicity results in weighted $L^p$ theory with $1 < p < \infty$. This paper extends the studies done in [C. Amrouche, V. Girault, J. Giroire, Dirichlet and Neumann exterior problems for the $n$-dimensional Laplace operator, an approach in weighted Sobolev spaces, J. Math. Pures Appl. 76 (1) (1997) 55–81] with Dirichlet and Neumann conditions. © 2008 Elsevier Inc. All rights reserved.

1. Introduction and preliminaries

Many problems in fluid dynamics, such as flows past obstacles, around corners or through pipes or apertures, are first conceptualized by Stokes or Navier–Stokes equations in unbounded domains. Our aim is to solve such systems in a particular unbounded domain for which any result is known. This domain, that we call exterior domain in the half-space, is the complement in the upper half-space of a compact region $\omega_0$. We can see this geometry as an extension of the “classical” exterior domain, i.e. the complement of $\omega_0$ in the whole space. In a forthcoming paper, we study a Stokes system on such a domain but prior to that, it can be interesting to give results for the Laplace’s equation. Thus, in this work, we want to solve the exterior Laplace’s problem in the half-space.

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First, let us recall some elements for the Laplace’s equation in a classical exterior domain. Several families of spaces are used for this operator, like the completion of \( D(\omega_0) \) for the norm of the gradient in \( L^p(\omega_0) \) (where \( \omega_0 \) is the complement of \( \omega_0 \) in \( \mathbb{R}^n \)), which has the inconvenience that, when \( p \geq n \), some very treacherous Cauchy sequences exist in \( D(\omega_0) \) that do not converge to distributions, a behaviour carefully described in 1954 by Deny and Lions (cf. [9]). Another family of spaces is the subspace in \( L^p(\omega_0) \) of functions whose gradients belong to \( L^p(\omega_0) \), subspace which have an imprecision at infinity inherent to the \( L^p \) norm.

Another approach is to set problems in weighted Sobolev spaces where the growth or decay of functions at infinity are expressed by means of weights. These spaces have several advantages: they satisfy an optimal weighted Poincaré-type inequality; they allow us to describe the behaviour of functions and not just of their gradient, which is vital from the mathematical and the numerical point of view.

Without being exhaustive, we can recall works of several authors who have contributed to the solution of Laplace’s equation in a classical exterior domain by means of weighted Sobolev spaces: see Cantor [8], Giroire [10], Giroire and Nedelec [11], Nedelec [18], Nedelec and Planchard [19], Hsiao and Wendland [13], Leroux [14] and [15], McOwen [16] and Amrouche, Girault and Giroire [5].

In this paper, we choose to set our problems in weighted Sobolev spaces and we remind that here, our originality, with respect to results previously quoted, is to extend the resolution of the exterior Laplace’s problem in the whole space to the exterior problem in the half-space. From this extension, comes an additional difficulty due to the nature of the boundary. Indeed, as it contains here, our originality, with respect to results previously quoted, is to extend the resolution of the problem of functions whose gradients belong to \( L^p(\omega_0) \), subspace which have an imprecision at infinity inherent to the \( L^p \) norm.

Moreover, we deal with problems which have Dirichlet or Neumann conditions on the bounded boundary but also on the unbounded boundary \( \mathbb{R}^{n-1} \).

We define \( \omega_0 \) a compact and non-empty subset of \( \mathbb{R}^n_+ \) \((n \geq 2)\), \( \Gamma_0 \) its boundary and we denote by \( \Omega \) the complement of \( \omega_0 \) in \( \mathbb{R}^n_+ \). We want to solve the four following problems:

\[
(P_D) \quad - \Delta u = f \quad \text{in} \; \Omega, \quad u = g_0 \quad \text{on} \; \Gamma_0, \quad u = g_1 \quad \text{on} \; \mathbb{R}^{n-1}, \\
(P_N) \quad - \Delta u = f \quad \text{in} \; \Omega, \quad \frac{\partial u}{\partial n} = g_0 \quad \text{on} \; \Gamma_0, \quad \frac{\partial u}{\partial n} = g_1 \quad \text{on} \; \mathbb{R}^{n-1}, \\
(P_{M_1}) \quad - \Delta u = f \quad \text{in} \; \Omega, \quad u = g_0 \quad \text{on} \; \Gamma_0, \quad \frac{\partial u}{\partial n} = g_1 \quad \text{on} \; \mathbb{R}^{n-1}, \\
(P_{M_2}) \quad - \Delta u = f \quad \text{in} \; \Omega, \quad \frac{\partial u}{\partial n} = g_0 \quad \text{on} \; \Gamma_0, \quad u = g_1 \quad \text{on} \; \mathbb{R}^{n-1}.
\]

We supposed that \( \Omega \) is connected and that it is of class \( C^{1,1} \), even if, for some values of the exponent \( p \), it can be less regular.

Each section of this paper is devoted to the study of one of the four problems. We will call \((P_{M_1})\) and \((P_{M_2})\) the first and the second mixed problem. The main results of this work are Theorems 2.2, 3.3, 4.3 and 5.4.

We complete this introduction with a short review of the weighted Sobolev spaces and their trace spaces. For any integer \( q \) we denote by \( P_q \) the space of polynomials in \( n \) variables, of degree less than or equal to \( q \), with the convention that \( P_q \) is reduced to \( \{0\} \) when \( q \) is negative.

Let \( \mathbf{x} = (x_1, \ldots, x_n) \) be a typical point of \( \mathbb{R}^n \), \( \mathbf{x}' = (x_1, \ldots, x_{n-1}) \) and let \( r = |\mathbf{x}| = (x_1^2 + \cdots + x_n^2)^{1/2} \) denote its distance to the origin.

We define \( \omega'_0 \) the symmetric region of \( \omega_0 \) with respect to \( \mathbb{R}^{n-1} \), \( \Gamma'_0 \) the boundary of \( \omega'_0 \), \( \Omega' \) the symmetric region of \( \Omega \), \( \tilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1} \) and \( \tilde{\Gamma}_0 = \Gamma_0 \cup \Gamma'_0 \).
We define the following functions $u^*$ and $u_*$. For $(\mathbf{x}', x_n) \in \mathbb{R}^n$ and $u$ any function, we set:

$$u^*(\mathbf{x}', x_n) = \begin{cases} 
  u(\mathbf{x}', x_n) & \text{if } x_n \geq 0, \\
  -u(\mathbf{x}', -x_n) & \text{if } x_n < 0.
\end{cases}$$

and

$$u_*(\mathbf{x}', x_n) = \begin{cases} 
  u(\mathbf{x}', x_n) & \text{if } x_n \geq 0, \\
  u(\mathbf{x}', -x_n) & \text{if } x_n < 0.
\end{cases}$$

For any real number $p \in ]1, +\infty[$, we denote by $p'$ the dual exponent of $p$:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$  

We shall use two basic weights:

$$\rho = (1 + r^2)^{1/2} \quad \text{and} \quad \lg \rho = \ln(2 + r^2).$$

As usual, $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support, $\mathcal{D}'(\Omega)$ its dual space, called the space of distributions and $\mathcal{D}(\mathbb{R}^n)$ the space of restrictions to $\Omega$ of functions in $\mathcal{D}(\mathbb{R}^n)$.

Then, we define:

$$W^{1,p}_0(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \ u \frac{\omega_1}{\omega_1} \in L^p(\Omega), \ \nabla u \in L^p(\Omega) \right\}$$

and

$$W^{2,p}_1(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \ u \frac{\omega_1}{\omega_1} \in L^p(\Omega), \ \nabla u \in L^p(\Omega), \ \rho \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega), \ i, j = 1, \ldots, n \right\},$$

where $\omega_1$ is defined by

$$\omega_1 = \begin{cases} 
  \rho & \text{if } n \neq p, \\
  \rho \lg \rho & \text{if } n = p.
\end{cases}$$

They are reflexive Banach spaces equipped, respectively, with natural norms:

$$\|u\|_{W^{1,p}_0(\Omega)} = \left( \|\omega_1^{-1}u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p \right)^{1/p}$$

and

$$\|u\|_{W^{2,p}_1(\Omega)} = \left( \|\omega_1^{-1}u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p + \sum_{1 \leq i, j \leq n} \left( \rho \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{L^p(\Omega)}^p \right)^{1/p}.$$

We also define semi-norms:

$$|u|_{W^{1,p}_0(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$$
and

\[ |u|_{W_1^{1,p}(\Omega)} = \left( \sum_{1 \leq i, j \leq n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(\Omega)}^p \right)^{1/p}. \]

We set the following spaces:

\[ W_0^{1,p}(\Omega) = \frac{D(\Omega)}{P[1-n/p]}, \quad \text{and} \quad W_1^{2,p}(\Omega) = \frac{D(\Omega)}{P[1-n/p]}, \]

and we easily check that

\[ W_0^{1,p}(\Omega) = \{ u \in W_1^{1,p}(\Omega), \ u = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1} \} \]

and that

\[ W_1^{2,p}(\Omega) = \{ u \in W_1^{2,p}(\Omega), \ u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1} \}, \]

where the sense of traces of these functions are given below. The weights defined previously are chosen so that the space \( D(\Omega) \) is dense in \( W_1^{1,p}(\Omega) \) and in \( W_1^{2,p}(\Omega) \) and so that the following Poincaré-type inequalities hold:

**Theorem 1.1.**

(i) The semi-norm \( \cdot \) defined on \( W_0^{1,p}(\Omega)/P[1-n/p] \) is a norm equivalent to the quotient norm.

(ii) The semi-norm \( \cdot \) defined on \( W_0^{1,p}(\Omega) \) is a norm equivalent to the full norm \( \| \cdot \|_{W_0^{1,p}(\Omega)} \).

**Proof.** We extend the problem in \( \tilde{\Omega} \) and we use results of [5] in an exterior domain. Here, we prove only the case (i), the case (ii) is similar.

Let \( u \) be in \( W_0^{1,p}(\Omega) \) and \( u_* \in W_0^{1,p}(\tilde{\Omega}) \) its extension in \( \tilde{\Omega} \). We have

\[ \| u_* \|_{W_0^{1,p}(\tilde{\Omega})} \leq C \| u \|_{W_0^{1,p}(\Omega)}. \]

Moreover, by [5]

\[ \inf_{k \in P[1-n/p]} \| u_* + k \|_{W_0^{1,p}(\tilde{\Omega})} \leq C \| u_* \|_{W_0^{1,p}(\tilde{\Omega})}. \]

Finally, since for all \( k \in P[1-n/p] \),

\[ \| u + k \|_{W_0^{1,p}(\Omega)} \leq \| u_* + k \|_{W_0^{1,p}(\tilde{\Omega})}, \]

we have

\[ \inf_{k \in P[1-n/p]} \| u + k \|_{W_0^{1,p}(\Omega)} \leq C \| u_* \|_{W_0^{1,p}(\tilde{\Omega})} \leq C \| u \|_{W_0^{1,p}(\Omega)}. \]

We have similar inequalities for the space \( W_1^{2,p}(\Omega) \). We denote by \( W_0^{-1,p}(\Omega) \) (respectively \( W_0^{-2,p}(\Omega) \)) the dual space of \( W_0^{1,p}(\Omega) \) (respectively of \( W_1^{2,p}(\Omega) \). They are spaces of distributions.
Then, we define too, for \( \ell \in \mathbb{R} \), the space \( W^{0,p}_{\ell}(\Omega) \) by
\[
W^{0,p}_{\ell}(\Omega) = \{ u \in D'(\Omega), \ \rho^\ell u \in L^p(\Omega) \}.
\]
We have, if \( n \neq p \), the continuous injections
\[
W^{1,p}_0(\Omega) \subset W^{0,p}_{-1}(\Omega) \quad \text{and} \quad W^{0,p}_{1}(\Omega) \subset W^{-1,p}_0(\Omega).
\]

Now, we want to define the traces of functions of \( W^{1,p}_0(\Omega) \) and \( W^{2,p}_1(\Omega) \). These traces have a component on \( \Gamma_0 \) and another component on \( \mathbb{R}^{n-1} \). For the traces on \( \Gamma_0 \), we refer to Adams [1] or Nečas [17] for the definition of the two spaces \( W^{-1,p}_{1}(\Gamma_0) \) and \( W^{2-1,p}_{1}(\Gamma_0) \) and for the usual trace theorems. For the traces of functions on \( \mathbb{R}^{n-1} \), we refer to Amrouche and Nečasová [6] for general definitions and here, we define the three following spaces:
\[
W^{1-1,p}_{0}(\mathbb{R}^{n-1}) = \left\{ u \in D'(\mathbb{R}^{n-1}), \ \omega^{1-1+p} u \in L^p(\mathbb{R}^{n-1}), \ \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p-2}} \, dx \, dy < \infty \right\},
\]
where
\[
\omega_2 = \begin{cases} \rho' & \text{if } n \neq p, \\ \rho' \log \rho' & \text{if } n = p, \end{cases}
\]
with \( \rho' = (1 + |x'|^2)^{1/2} \) and \( \log \rho' = \ln(2 + |x'|^2) \). It is a reflexive Banach space equipped with its natural norm
\[
\left( \| \omega^{1-1+p} u \|^p_{L^p(\mathbb{R}^{n-1})} + \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p-2}} \, dx \, dy \right)^{1/p}.
\]
For \( x' \in \mathbb{R}^{n-1} \), we set
\[
\gamma_0 u(x') = u(x', 0),
\]
and we have the following trace lemma (see [6]):

**Lemma 1.2.** The mapping
\[
\gamma_0 : D(\mathbb{R}^n_+) \to D(\mathbb{R}^{n-1}), \quad u \mapsto \gamma_0 u
\]
can be extended by continuity to a linear and continuous mapping still denoted by \( \gamma_0 \) from \( W^{1,p}_0(\mathbb{R}^n_+) \) to \( W^{1-1,p}_{0}(\mathbb{R}^{n-1}) \). Moreover, \( \gamma_0 \) is onto and
\[
\text{Ker} \gamma = W^{1,p}_0(\mathbb{R}^n_+) = D(\Omega) \| \| W^{1,p}_{0}(\mathbb{R}^n_+).
In other words, for any \( g_0 \) in \( W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \), there exists \( u \in W_0^{1,p}(\mathbb{R}^n) \) such that \( \gamma_0 u = g_0 \) and we have the estimate

\[
\|u\|_{W_0^{1,p}(\mathbb{R}^n)} \leq C\|g_0\|_{W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1})}.
\]

Then, we define

\[
W_1^{1-rac{1}{p}}(\mathbb{R}^{n-1}) = \{ u \in W_1^{0,p}(\mathbb{R}^{n-1}), \, \rho' u \in W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \}
\]

and

\[
W_1^{2-rac{1}{p}}(\mathbb{R}^{n-1}) = \{ u \in W_1^{1,p}(\mathbb{R}^{n-1}), \, \rho' \nabla u \in W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \},
\]

where

\[
W_1^{1,p}(\mathbb{R}^{n-1}) = \{ u \in \mathcal{D}'(\mathbb{R}^{n-1}), \, \omega_2^{-1+rac{1}{p}} u \in L^p(\mathbb{R}^{n-1}), \, (\rho')^{-1} \nabla u \in L^p(\mathbb{R}^{n-1}) \}.
\]

Here again, we equip these spaces with their natural norm. For \( x' \in \mathbb{R}^{n-1} \), we set

\[
\gamma_1 u(x') = \frac{\partial u}{\partial n}(x', 0),
\]

and we have the following traces lemma (see [6]):

**Lemma 1.3.** The mapping

\[
\gamma : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}(\mathbb{R}^{n-1}),
\]

\[
u \mapsto (\gamma_0 u, \gamma_1 u)
\]

can be extended by continuity to a linear and continuous mapping still denoted by \( \gamma \) from \( W_1^{2,p}(\mathbb{R}^n) \) to \( W_1^{2-rac{1}{p}}(\mathbb{R}^{n-1}) \times W_1^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \). Moreover, \( \gamma \) is onto and

\[
\text{Ker} \, \gamma = W_1^{2,p}(\mathbb{R}_+) = \overline{\mathcal{D}(\Omega)}^\|.\|_{W_1^{2,p}(\mathbb{R}^n)}.
\]

In other words, for any \((g_0, g_1)\) in \( W_1^{2-rac{1}{p}}(\mathbb{R}^{n-1}) \times W_1^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \), there exists \( u \in W_1^{2,p}(\mathbb{R}^n) \) such that \( \gamma u = (g_0, g_1) \) and we have the estimate

\[
\|u\|_{W_1^{2,p}(\mathbb{R}^n)} \leq C(\|g_0\|_{W_1^{1-rac{1}{p}}(\mathbb{R}^{n-1})} + \|g_1\|_{W_1^{1-rac{1}{p}}(\mathbb{R}^{n-1})}).
\]

Finally, we denote, for \( p \in ]1, \infty[ \), the duality pairing \( W_1^{1-rac{1}{p}}(\Omega), W_1^{1-rac{1}{p}}(\Omega)' \), with \( \Omega = \Gamma_0 \) or \( \Gamma_0 \) and \((\cdot, \cdot)_{\mathbb{R}^{n-1}}\), the pairing \( W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1}), W_0^{1-rac{1}{p}}(\mathbb{R}^{n-1}) \).

We remind that in all this article, we suppose that \( \Omega \) is of class \( C^{1,1} \).

We will denote by \( C \) a positive and real constant which may vary from line to line.
2. The problem of Dirichlet

In this section, we want to solve the following problem of Dirichlet:

\[
(P_D) \begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = g_0 & \text{on } \Gamma_0, \\
u = g_1 & \text{on } \mathbb{R}^{n-1}.
\end{cases}
\]

First, we characterize the following kernel:

\[
\mathcal{D}^0_p(\Omega) = \{ z \in W^{1,p}_0(\Omega), \Delta z = 0 \text{ in } \Omega, z = 0 \text{ on } \Gamma_0, z = 0 \text{ on } \mathbb{R}^{n-1}\}.
\]

**Proposition 2.1.** For any \( p > 1 \), \( \mathcal{D}^0_p(\Omega) = \{0\} \).

**Proof.** Let \( z \) be in \( \mathcal{D}^0_p(\Omega) \), we define, for almost all \((x', x_n) \in \tilde{\Omega}\) the function \( z^* \in \tilde{W}^{1,p}_0(\tilde{\Omega}) \). For any \( \varphi \in D(\tilde{\Omega}) \), we have

\[
\langle \Delta z^*, \varphi \rangle_{D'(\tilde{\Omega}), D(\tilde{\Omega})} = \langle z^*, \Delta \varphi \rangle_{D'(\tilde{\Omega}), D(\tilde{\Omega})}
= \int_{\tilde{\Omega}} z(x', x_n) \Delta \varphi(x', x_n) \, dx - \int_{\tilde{\Omega}'} z(x', -x_n) \Delta \varphi(x', x_n) \, dx.
\]

Moreover

\[
\int_{\tilde{\Omega}} z(x', x_n) \Delta \varphi(x', x_n) \, dx = -\left( \frac{\partial z}{\partial n}, \varphi \right)_{\mathbb{R}^{n-1}}.
\]

Setting \( \psi(x', x_n) = \varphi(x', -x_n) \), we have \( \psi \in D(\tilde{\Omega}) \) and

\[
\int_{\tilde{\Omega}'} z(x', -x_n) \Delta \varphi(x', x_n) \, dx = \int_{\tilde{\Omega}} z(x', x_n) \Delta \psi(x', x_n) \, dx
= -\left( \frac{\partial z}{\partial n}, \varphi \right)_{\mathbb{R}^{n-1}}.
\]

Thus, we deduce that \( \langle \Delta z^*, \varphi \rangle_{D'(\tilde{\Omega}), D(\tilde{\Omega})} = 0 \), i.e. \( \Delta z^* = 0 \) in \( \tilde{\Omega} \). So, the function \( z^* \) is in the space \( \mathcal{A}^0_p(\tilde{\Omega}) \) defined by

\[
\mathcal{A}^0_p(\tilde{\Omega}) = \{ v \in W^{1,p}_0(\tilde{\Omega}), \Delta v = 0 \text{ in } \tilde{\Omega}, v = 0 \text{ on } \tilde{\Gamma}_0 \}.
\]

Now, we use the characterization of \( \mathcal{A}^0_p(\tilde{\Omega}) \) (see [5]). For this, we set \( \mu_0 \) the function defined by

\[
\mu_0 = U \ast \left( \frac{1}{|\tilde{\Gamma}_0|} \delta_{\tilde{\Gamma}_0} \right),
\]

where \( U = \frac{1}{2\pi} \ln(r) \) is the fundamental solution of the Laplace's equation in \( \mathbb{R}^2 \) and \( \delta_{\tilde{\Gamma}_0} \) is defined by

\[
\forall \varphi \in D(\mathbb{R}^2), \quad \langle \delta_{\tilde{\Gamma}_0}, \varphi \rangle = \int_{\tilde{\Gamma}_0} \varphi \, d\sigma.
\]
(i) If \( p < n \) or \( p = n = 2 \), then \( A_0^p(\Omega) = \{0\} \) and \( z^* = 0 \) in \( \tilde{\Omega} \), i.e. \( z = 0 \) in \( \Omega \) and \( D_0^p(\Omega) = \{0\} \).

(ii) If \( p \geq n \geq 3 \), then we have \( z^* = c(\lambda - 1) \), where \( c \) is a real constant and \( \lambda \) is the unique solution in \( W_0^{1,p}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega}) \) of the problem

\[
\Delta \lambda = 0 \quad \text{in} \quad \tilde{\Omega}, \quad \lambda = 1 \quad \text{on} \quad \Gamma_0.
\]

Thus, on \( \mathbb{R}^{n-1} \), \( z^* = z = c(\lambda - 1) = 0 \). This implies that \( c = 0 \) because otherwise, \( \lambda \) will be equal to 1 on \( \mathbb{R}^{n-1} \), that is not possible because \( 1 \not\in W_0^{1,2}(\mathbb{R}^{n-1}) \). Finally, we deduce that \( z = 0 \), i.e. \( D_0^p(\Omega) = \{0\} \).

(iii) If \( p > n = 2 \), then we have \( z^* = c(\mu - \mu_0) \), where \( c \) is a real constant and the function \( \mu \) is the unique solution in \( W_0^{1,p}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega}) \) of the problem

\[
\Delta \mu = 0 \quad \text{in} \quad \tilde{\Omega}, \quad \mu = \mu_0 \quad \text{on} \quad \Gamma_0.
\]

Thus, on \( \mathbb{R} \), \( z = c(\mu - \mu_0) = 0 \). This implies again that \( c = 0 \) because otherwise \( \mu \) will be equal to \( \mu_0 \) on \( \mathbb{R} \), that is not possible because \( \mu_0 \notin W_0^{1,2}(\mathbb{R}) \). Indeed, let \( x = (x',0) \) be in \( \mathbb{R} \), since

\[
\mu_0(x) = \frac{1}{2\pi|\Gamma_0|} \int_{\Gamma_0} \ln(|y-x|) d\sigma_y,
\]

then \( \mu_0(x') \geq C \ln|x'| \) if \( |x'| > \alpha \) with \( \alpha \) enough big and

\[
\int_{|x'|>\alpha} \frac{|\mu_0(x',0)|^2}{|x'| \log^2(2+|x'|)} dx' \geq C \int_{|x'|>\alpha} \frac{dx'}{|x'|} = +\infty
\]

that is contradictory with \( \mu_0 \in W_0^{1,2}(\mathbb{R}) \). Thus \( c = 0 \) and we deduce that \( z = 0 \), i.e. \( D_0^p(\Omega) = \{0\} \). \( \square \)

**Theorem 2.2.** For any \( p > 1 \), \( f \in W_0^{-1,p}(\Omega) \), \( g_0 \in W_1^{-1,p}(\Gamma_0) \) and \( g_1 \in W_0^{1,\frac{1}{2}-p}(\mathbb{R}^{n-1}) \), there exists a unique \( u \in W_0^{1,p}(\Omega) \) solution of \((P_D)\). Moreover, \( u \) satisfies

\[
\|u\|_{W_0^{1,p}(\Omega)} \leq C \left( \|f\|_{W_0^{-1,p}(\Omega)} + \|g_0\|_{W_1^{-\frac{1}{2}-p}(\Gamma_0)} + \|g_1\|_{W_0^{1,\frac{1}{2}-p}(\mathbb{R}^{n-1})} \right),
\]

where \( C \) is a real positive constant which depends only on \( p \) and \( \omega_0 \).

**Proof.** (i) We begin to show that solving \((P_D)\) amounts to solve a problem with homogeneous boundary conditions. We know there exists \( u_{g_1} \in W_0^{1,p}(\mathbb{R}^{n-1}) \) such that \( u_{g_1} = g_1 \) on \( \mathbb{R}^{n-1} \) and

\[
\|u_{g_1}\|_{W_0^{1,p}(\mathbb{R}^{n-1})} \leq C \|g_1\|_{W_0^{1,\frac{1}{2}-p}(\mathbb{R}^{n-1})}.
\]

We set \( u_1 = u_{g_1}|_{\Omega} \). Then \( u_1 \in W_0^{1,p}(\Omega) \) and the trace \( \eta \) of \( u_1 \) on \( \Gamma_0 \) is in \( W_1^{-\frac{1}{2}-p}(\Gamma_0) \). Setting \( z = u - u_1 \), the problem \((P)\) is equivalent to the problem:

\[
(P_1) \quad -\Delta z = f + \Delta u_1 \quad \text{in} \quad \Omega, \quad z = g_0 - \eta \quad \text{on} \quad \Gamma_0, \quad z = 0 \quad \text{on} \quad \mathbb{R}^{n-1}.
\]

We set \( g = g_0 - \eta \), and let \( R > 0 \) be such that \( \omega_0 \subset B_R \subset \mathbb{R}^{n+} \) (where \( B_R \) is an open ball of radius \( R \)). The function \( h_0 \) defined by

\[
h_0 = g \quad \text{on} \quad \Gamma_0, \quad h_0 = 0 \quad \text{on} \quad \partial B_R,
\]
is in $W^{1-\frac{1}{p}}(\Omega)$. We know there exists $u_{h_0} \in W^{1,1}(\Omega_R)$, where $\Omega_R = \Omega \cap B_R$, such that $u_{h_0} = h_0$ on $\Gamma_0 \cup \partial B_R$ and satisfying the estimate

$$\|u_{h_0}\|_{W^{1,1}(\Omega_R)} \leq C \|h_0\|_{W^{1-\frac{1}{p}}(\Omega_0 \cup \partial B_R)}.$$  

We set

$$u_0 = u_{h_0} \text{ in } \Omega_R, \quad u_0 = 0 \text{ in } \Omega \setminus \Omega_R.$$  

We have $u_0 \in W^{1,1}(\Omega)$, $u_0 = g$ on $\Gamma_0$, $u_0 = 0$ on $\mathbb{R}^{n-1}$ and $u_0$ satisfies

$$\|u_0\|_{W^{1,1}(\Omega)} \leq C \left(\|g_0\|_{W^{1,1}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p}}(\mathbb{R}^{n-1})}\right).$$  

Finally, setting $v = z - u_0$, the problem ($P_1$) is equivalent to the following problem ($P'$):

$$(P') \quad -\Delta v = h \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1}.$$  

where $h = f + \Delta u_1 + \Delta u_0 \in W^{-1,1}(\Omega)$.

(ii) Now, we want to return to a problem setted in the open region $\tilde{\Omega}$, problem that we know solving. Let $\psi$ be in $\tilde{W}_0^1(\tilde{\Omega})$, we set for almost all $(\mathbf{x}', x_n) \in \Omega$,

$$\pi \psi(\mathbf{x}', x_n) = \psi(\mathbf{x}', x_n) - \psi(\mathbf{x}', -x_n).$$  

It is obvious that $\pi \psi \in W^1(\tilde{\Omega})$ and, for any $\psi \in W^1(\tilde{\Omega})$, we define the operator $h_\pi$ by

$$\langle h_\pi, \psi \rangle := \langle h, \pi \psi \rangle_{W^1(\tilde{\Omega}) \times W^1(\tilde{\Omega})}.$$  

We notice that $h_\pi$ is in $W^{-1,1}(\tilde{\Omega})$ and satisfies

$$\|h_\pi\|_{W^{-1,1}(\tilde{\Omega})} \leq 2\|h\|_{W^{-1,1}(\Omega)}.$$  

Now, we suppose that $p \geq 2$. By [5], we know there exists $w \in W^1(\tilde{\Omega})$ solution of

$$-\Delta w = h_\pi \text{ in } \tilde{\Omega}, \quad w = 0 \text{ on } \tilde{\Gamma}_0,$$

satisfying the estimate

$$\|w\|_{W^1(\tilde{\Omega})} \leq C \|h_\pi\|_{W^{-1,1}(\tilde{\Omega})}.$$  

The function $v = \frac{1}{2} \pi w$ belongs to $W^1(\Omega)$ and we have

$$\|v\|_{W^1(\Omega)} \leq 2\|w\|_{W^1(\tilde{\Omega})}.$$  

Now, let us show that $-\Delta v = h$ in $\Omega$, i.e. $v$ solution of ($P'$). Let $\psi$ be in $D(\Omega)$, then

$$2\langle \Delta v, \psi \rangle_{D(\Omega)} = 2\langle v, \Delta \psi \rangle_{D(\Omega)} = \int_{\tilde{\Omega}} [w(\mathbf{x}', x_n) - w(\mathbf{x}', -x_n)] \Delta \psi d\mathbf{x}.$$
Moreover, setting $\psi(x', x_n) = \varphi(x', -x_n)$, then $\psi \in \mathcal{D}(\Omega')$ and we have the relations

$$
\int_{\Omega} w(x', x_n) \Delta \psi \, dx = \langle \Delta w, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}
$$

and

$$
\int_{\Omega} w(x', -x_n) \Delta \psi \, dx = \int_{\Omega'} w(x', x_n) \Delta \psi \, dx = \langle \Delta w, \psi \rangle_{\mathcal{D}'(\Omega'), \mathcal{D}(\Omega')}
$$

Setting $\tilde{\psi}$ and $\tilde{\varphi}$ the extensions by 0 in $\tilde{\Omega}$ of $\varphi$ and $\psi$ respectively, we deduce that:

$$
2 \langle \Delta v, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle \Delta w, \tilde{\varphi} - \tilde{\psi} \rangle_{\mathcal{D}'(\tilde{\Omega}), \mathcal{D}(\tilde{\Omega})}
$$

$$
= - \langle h, \tilde{\varphi} - \tilde{\psi} \rangle_{\mathcal{D}'(\tilde{\Omega}), \mathcal{D}(\tilde{\Omega})}
$$

$$
= - \langle h, \pi \tilde{\varphi} - \pi \tilde{\psi} \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}
$$

$$
= -2 \langle h, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}
$$

i.e. $-\Delta v = h$ in $\Omega$. So, we have checked that, if $p \geq 2$, the operator

$$
\Delta : W^{1,p}_0(\Omega) \mapsto W^{-1,p}_0(\Omega)
$$

is an isomorphism, and, by duality, the operator

$$
\Delta : W^{1,p'}_0(\Omega) \mapsto W^{-1,p'}_0(\Omega)
$$

is an isomorphism too. So, if $p < 2$, the problem $(\mathcal{P}')$ has also a unique solution $v \in W^{1,p}_0(\Omega)$. Thus, the problem $(\mathcal{P}_D)$ has a unique solution for $1 < p < \infty$. Finally, by (2)–(5) and (6), we have the estimate (1). $\Box$

### 3. The problem of Neumann

We remind that in this section and in the following ones, $\Omega$ is supposed to be of class $C^{1,1}$. In this section, we want to solve the following problem:

$$
(\mathcal{P}_N) \quad \begin{cases}
-\Delta u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = g_0 & \text{on } \Gamma_0, \\
\frac{\partial u}{\partial n} = g_1 & \text{on } \mathbb{R}^{n-1}.
\end{cases}
$$

First, we characterize the following kernel:

$$
\mathcal{N}_0^p(\Omega) = \left\{ z \in W^{1,p}_0(\Omega), \Delta z = 0 \text{ in } \Omega, \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_0, \frac{\partial z}{\partial n} = 0 \text{ on } \mathbb{R}^{n-1} \right\}.
$$

**Proposition 3.1.** For any $p > 1$, $\mathcal{N}_0^p(\Omega) = \mathcal{P}_{[1-n/p]}$. 
The following theorem allows us to obtain strong solutions of the problem \((P_N)\).

**Theorem 3.2.** For each \( p > 1 \) such that \( \frac{n}{p} \neq 1 \) and for any \( f \in W^{0,p}_1(\Omega) \), \( g_0 \in W^{1-\frac{1}{p},p}_1(\Gamma_0) \) and \( g_1 \in W^{1-\frac{1}{p},p}_1(\mathbb{R}^{n-1}) \) satisfying, if \( p < \frac{n}{n-1} \), the following compatibility condition:

\[
\int_{\Omega} f \, dx + \int_{\Gamma_0} g_0 \, d\sigma + \int_{\mathbb{R}^{n-1}} g_1 \, dx' = 0, \tag{7}
\]

the problem \((P_N)\) has a unique solution \( u \in W^{2,p}_1(\Omega)/P_{1-n/p} \). Moreover, \( u \) satisfies

\[
\|u\|_{W^{2,p}_1(\Omega)/P_{1-n/p}} \leq C(\|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}_1(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}_1(\mathbb{R}^{n-1})}), \tag{8}
\]

where \( C \) is a real positive constant which depends only on \( p \) and \( \omega_0 \).

**Proof.** First, we notice that, thanks to the hypothesis on the data, any integral of (7) has a meaning when \( p < \frac{n}{n-1} \), the last one being finished because \( W^{1-\frac{1}{p},p}_1(\mathbb{R}^{n-1}) \subset W^{0,p}_1(\mathbb{R}^{n-1}) \subset L^1(\mathbb{R}^{n-1}) \). We know there exists a function \( u_{g_1} \in W^{2,p}_1(\mathbb{R}^{n-1}) \) such that \( \frac{\partial u_{g_1}}{\partial n} = g_1 \) and \( u_{g_1} = 0 \) on \( \mathbb{R}^{n-1} \) satisfying

\[
\|u_{g_1}\|_{W^{2,p}_1(\mathbb{R}^{n-1})} \leq C\|g_1\|_{W^{1-\frac{1}{p},p}_1(\mathbb{R}^{n-1})}. \tag{9}
\]

We set \( u_1 \) the restriction of \( u_{g_1} \) to \( \Omega \) and \( \eta \) the normal derivative of \( u_1 \) on \( \Gamma_0 \). Finally, we set \( g = g_0 - \eta \in W^{1-\frac{1}{p},p}_1(\Gamma_0) \) and \( h = f + \Delta u_1 \in W^{0,p}_1(\Omega) \). Then, setting \( v = u - u_1 \in W^{2,p}_1(\Omega) \), the problem \((P_N)\) is equivalent to the following problem \((P')\):

\[
(P') \begin{cases} 
-\Delta v = h & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = g & \text{on } \Gamma_0, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \mathbb{R}^{n-1}.
\end{cases}
\]

We construct the two functions \( h_\ast \in W^{0,p}_1(\widetilde{\Omega}) \) and \( g_\ast \in W^{1-\frac{1}{p},p}_1(\widetilde{\Gamma}_0) \) which satisfy, if \( p < \frac{n}{n-1} \) and by (7), the equality \( \int_{\widetilde{\Omega}} h_\ast \, dx + \int_{\widetilde{\Gamma}_0} g_\ast \, d\sigma = 0 \). By [5], there exists a function \( w \in W^{2,p}_1(\widetilde{\Omega}) \), unique up to an element of \( P_{1-n/p} \), solution of

\[
-\Delta w = h_\ast \quad \text{in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial n} = g_\ast \quad \text{on } \widetilde{\Gamma}_0,
\]

satisfying

\[
\|w\|_{W^{2,p}_1(\widetilde{\Omega})/P_{1-n/p}} \leq C(\|h\|_{W^{0,p}_1(\widetilde{\Omega})} + \|g\|_{W^{1-\frac{1}{p},p}_1(\widetilde{\Gamma}_0)}),
\]

The following theorem allows us to obtain strong solutions of the problem \((P_N)\).
Now, let \( w_0 \in W^{2,p}_1(\tilde{\Omega}) \) be a solution of the above problem. For almost all \((x',x_n) \in \tilde{\Omega},\) we set 
\[ v_0(x',x_n) = w_0(x',-x_n). \]
As \( h_* \) is even with respect to \( x_n, \) we easily check that we have \(-\Delta v_0 = h_* \) in \( \tilde{\Omega} \). Moreover, by the definition of the normal derivative on \( \tilde{\Gamma}_0, \) we notice that we have, for almost all \((x',x_n) \in \tilde{\Gamma}_0:\)
\[
\frac{\partial v_0}{\partial n}(x',x_n) = \frac{\partial w_0}{\partial n}(x',-x_n).
\]
As \( g_* \) is even with respect to \( x_n, \) we easily show that we have \( \frac{\partial v_0}{\partial n} = g_* \) on \( \tilde{\Gamma}_0. \) So \( v_0 \in W^{2,p}_1(\tilde{\Omega}) \) is solution of the same problem that \( w_0 \) satisfies. Thus, the difference \( v_0 - w_0 \) is equal to a constant \( c \) which is necessary nil. So \( w_0(x',x_n) = w_0(x',-x_n) \) and thus \( \frac{\partial w_0}{\partial n} = 0 \) on \( \mathbb{R}^{n-1}. \) The restriction \( v \) of \( w_0 \) to \( \Omega \) being in \( W^{2,p}_1(\Omega), \) is solution of \( (P') \) and satisfies
\[
\| v \|_{W^{2,p}_1(\Omega)/\mathcal{F}_1} \leq C(\| h \|_{W^{0,p}_1(\Omega)} + \| g \|_{W^{-1,p}(\tilde{\Gamma}_0)}) \quad .
\]
Finally, from this inequality and (9), comes the estimate (8). \( \square \)

Now, we search weak solutions of the problem \( (P_N): \)

**Theorem 3.3.** For each \( p > 1 \) such that \( \frac{n}{p} \neq 1 \) and for any \( f \in W^{0,p}_1(\Omega), \) \( g_0 \in W^{-1,p}(-\tilde{\Gamma}_0)\) and \( g_1 \in W^{-1,p}_0(\mathbb{R}^{n-1})\) satisfying, if \( p < \frac{n}{n-1}, \) the following condition of compatibility:
\[
\int_{\Omega} f \, dx + \langle g_0, 1 \rangle_{\Gamma_0} + \langle g_1, 1 \rangle_{\mathbb{R}^{n-1}} = 0, \quad (10)
\]
the problem \( (P_N) \) has a unique solution \( u \in W^{1,p}_0(\Omega)/\mathcal{F}_1 \). Moreover, \( u \) satisfies
\[
\| u \|_{W^{1,p}_0(\Omega)/\mathcal{F}_1} \leq C(\| f \|_{W^{0,p}_1(\Omega)} + \| g_0 \|_{W^{-1,p}_1(\tilde{\Gamma}_0)} + \| g_1 \|_{W^{-1,p}_0(\mathbb{R}^{n-1})}), \quad (11)
\]
where \( C \) is a real positive constant which depends only on \( p \) and \( \omega_0. \)

**Proof.** (i) First, we suppose \( \frac{p}{n} > 1. \)

Theorem 3.2 assures the existence of a function \( s \in W^{2,p}_1(\Omega) \subset W^{1,p}_1(\Omega) \) solution of the problem
\[
-\Delta s = f \quad \text{in } \Omega, \quad \frac{\partial s}{\partial n} = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial n} = 0 \quad \text{on } \mathbb{R}^{n-1},
\]
and satisfying
\[
\| s \|_{W^{1,p}_0(\Omega)/\mathcal{F}_1} \leq C(\| f \|_{W^{2,p}_1(\Omega)} + \| g_0 \|_{W^{-1,p}_0(\tilde{\Gamma}_0)}), \quad (12)
\]
Then, by [2], there exists a function \( z \in W^{1,p}_0(\mathbb{R}^{n+}) \) solution of
\[
\Delta z = 0 \quad \text{in } \mathbb{R}^{n+}, \quad \frac{\partial z}{\partial n} = g_1 \quad \text{on } \mathbb{R}^{n-1},
\]
 satisfying the estimate
\[
\| z \|_{W^{1,p}_0(\mathbb{R}^{n+})} \leq C \| g_1 \|_{W^{-1,p}_0(\mathbb{R}^{n-1})}. \quad (13)
\]
We denote again by $z$ the restriction of $z$ to $\Omega$. It is obvious that the normal derivative $\eta$ of $z$ on $\Gamma_0$ is in $W^{-\frac{1}{p}, p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p}, p}(\Gamma_0)$ and we want to solve the following problem:

$$
(P') \quad \Delta v = 0 \quad \text{in} \ \Omega, \quad \frac{\partial v}{\partial n} = g \quad \text{on} \ \Gamma_0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1}.
$$

Let $\mu$ be in $W^{1-\frac{1}{p}, p'}(\tilde{\Gamma}_0)$. For almost all $(\mathbf{x}', x_n) \in \Gamma_0$, we set

$$
\pi \mu(\mathbf{x}', x_n) = \mu(\mathbf{x}', x_n) + \mu(\mathbf{x}', -x_n).
$$

We notice that $\pi \mu \in W^{1-\frac{1}{p}, p'}(\Gamma_0)$ and we define

$$
(g, \mu) := \langle g, \pi \mu \rangle_{\Gamma_0}.
$$

It is obvious that $g_\pi \in W^{-\frac{1}{p}, p}(\tilde{\Gamma}_0)$ and that $g$ is the restriction of $g_\pi$ to $\Gamma_0$. Moreover, we easily check that $g_\pi$ is even with respect to $x_n$, i.e.

$$
\langle g_\pi, \hat{\xi} \rangle_{\tilde{\Gamma}_0} = \langle g_\pi, \mu \rangle_{\tilde{\Gamma}_0},
$$

where $\hat{\xi}(\mathbf{x}', x_n) = \mu(\mathbf{x}', -x_n)$ with $(\mathbf{x}', x_n) \in \tilde{\Gamma}_0$. By [5], there exists a function $w \in W^{1, p}_0(\tilde{\Omega})$, unique up to an element of $\mathcal{P}_{1-n/p}$ solution of the following problem:

$$
\Delta w = 0 \quad \text{in} \ \tilde{\Omega}, \quad \frac{\partial w}{\partial n} = g_\pi \quad \text{on} \ \tilde{\Gamma}_0,
$$

and satisfying

$$
\|w\|_{W^{1, p}_0(\tilde{\Omega})/\mathcal{P}_{1-n/p}} \leq C \|g_\pi\|_{W^{-\frac{1}{p}, p}(\tilde{\Gamma}_0)} \leq C \|g\|_{W^{-\frac{1}{p}, p}(\Gamma_0)}.
$$

Let $w_0$ be a solution of the problem and we set for almost all $(\mathbf{x}', x_n) \in \tilde{\Omega}$:

$$
v_0(\mathbf{x}', x_n) = w_0(\mathbf{x}', -x_n).
$$

The function $v_0$ is in $W^{1, p}_0(\tilde{\Omega})$ and since $\Delta w_0 = 0$ on $\tilde{\Omega}$, we easily check that $\Delta v_0$ is nil too. Thus, $\frac{\partial v_0}{\partial n}$ has a meaning in $W^{-\frac{1}{p}, p}(\tilde{\Gamma}_0)$. Now, we want to show that $\frac{\partial v_0}{\partial n} = g_\pi$ on $\tilde{\Gamma}_0$. Let $\mu$ be in $W^{1-\frac{1}{p}, p'}(\tilde{\Gamma}_0)$. We know there exists $\varphi \in W^{1, p'}_0(\tilde{\Omega})$ such that $\varphi = \mu$ on $\tilde{\Gamma}_0$ and $\|\varphi\|_{W^{1, p'}_0(\tilde{\Omega})} \leq C \|\mu\|_{W^{1-\frac{1}{p}, p'}(\tilde{\Gamma}_0)}$. We have

$$
\left\{ \frac{\partial v_0}{\partial n}, \mu \right\}_{\tilde{\Gamma}_0} = \int_{\tilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \, dx.
$$

For almost all $(\mathbf{x}', x_n) \in \tilde{\Omega}$, we set $\psi(\mathbf{x}', x_n) = \varphi(\mathbf{x}', -x_n)$. The function $\psi$ is in $W^{1, p'}_0(\tilde{\Omega})$ and we set $\xi \in W^{1-\frac{1}{p}, p'}(\tilde{\Gamma}_0)$ the trace of $\psi$ on $\tilde{\Gamma}_0$. We notice that $\xi(\mathbf{x}', x_n) = \mu(\mathbf{x}', -x_n)$. Moreover, we show that

$$
\int_{\tilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \, dx = \int_{\tilde{\Omega}} \nabla w_0 \cdot \nabla \psi \, dx.
Thus,
\[
\left\langle \frac{\partial v_0}{\partial n}, \mu \right\rangle = \left\langle \frac{\partial w_0}{\partial n}, \xi \right\rangle = \langle g_\pi, \xi \rangle_{\Gamma_0} = \langle g_\pi, \mu \rangle_{\Gamma_0}.
\]

So \( \frac{\partial v_0}{\partial n} = g_\pi \) on \( \Gamma_0 \) and \( v_0 \) is solution of the same problem that \( w_0 \) satisfies, which implies that \( v_0 - w_0 \) is a constant, constant which is necessary nil. The restriction of \( w_0 \) to \( \Omega \), that we denote by \( v \), being in \( W_0^{1,p}(\Omega) \), is solution of the problem \((P')\) and we have the estimate
\[
\|v\|_{W_0^{1,p}(\Omega)/P(1-n/p)} \leq C \|g\|_{W^{-1,\frac{1}{p}}(\Gamma_0)}.
\]

Finally, the function \( u = z + s + v \in W_0^{1,p}(\Omega) \) is solution of the problem \((P_N)\) and by (12), (13) and (14), we have (11).

(ii) Now, we suppose that \( \frac{n}{p} < 1 \).

Let \( \alpha \) be in \( W_1^{0,p}(\Omega) \), \( \beta \) in \( W_1^{-\frac{1}{p},p}(\Gamma_0) \) and \( \gamma \) in \( W_1^{-\frac{1}{p},p}(\mathbb{R}^{n-1}) \) such that
\[
\int_\Omega \alpha \, dx = \int_{\Gamma_0} \beta \, d\sigma = \int_{\mathbb{R}^{n-1}} \gamma \, dx' = 1.
\]

Here, we notice that we have \( W_1^{-\frac{1}{p},p}(\Gamma_0) \subset W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}) \) and since \( \frac{n}{p} \neq 1 \), \( W_1^{-\frac{1}{p},p}(\mathbb{R}^{n-1}) \subset W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}) \). We set
\[
F = \left( \int_\Omega f \, dx \right) \alpha, \quad G_0 = (g_0, 1)_{\Gamma_0} \beta \quad \text{and} \quad G_1 = (g_1, 1)_{\mathbb{R}^{n-1}} \gamma.
\]

Thanks to Theorem 3.2, we know there exists \( r \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega) \) solution of the problem
\[
\Delta r = f - F \quad \text{in} \ \Omega, \quad \frac{\partial r}{\partial n} = 0 \quad \text{on} \ \Gamma_0, \quad \frac{\partial r}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1},
\]
satisfying by (8) (since \( \frac{n}{p} > 1 \))
\[
\|r\|_{W_0^{2,p}(\Omega)} \leq \|r\|_{W_1^{2,p}(\Omega)} \leq C \|f - F\|_{W_0^{0,p}(\Omega)}.
\]

We notice, by the Hölder’s inequality and because \( \frac{n}{p} < 1 \), that
\[
\|F\|_{W_0^{0,p}(\Omega)} \leq C \|f\|_{L^1(\Omega)} \leq C \|f\|_{W_1^{0,p}(\Omega)}.
\]

So, we have the estimate
\[
\|r\|_{W_0^{1,p}(\Omega)} \leq C \|f\|_{W_0^{0,p}(\Omega)}.
\]

Now, by [2], since \( (G_1 - g_1, 1)_{\mathbb{R}^{n-1}} = 0 \), there exists a function \( z \in W_0^{1,p}(\mathbb{R}^{n-1}) \) solution of
\[
\Delta z = 0 \quad \text{in} \ \mathbb{R}^{n-1}, \quad \frac{\partial z}{\partial n} = g_1 - G_1 \quad \text{on} \ \mathbb{R}^{n-1},
\]
satisfying
\[ \|z\|_{W_0^{1,p}(\Omega)} \leq C \|g_1 - G_1\|_{W_0^{1,p}(\mathbb{R}^{n-1})} \leq C \|g_1\|_{W_0^{1,p}(\mathbb{R}^{n-1})}. \] (16)

We denote again by \( z \) the restriction of \( z \) to \( \Omega \). It is obvious that the normal derivative \( \eta \) of \( z \) on \( \Gamma_0 \) is in \( W^{-\frac{1}{p},p}(\Gamma_0) \) and satisfies the following equality:
\[ \langle \eta, 1 \rangle_{\Gamma_0} = 0. \]

We set \( g = g_0 - G_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0) \), and we apply the same reasoning as in the point (i) to show there exists \( v \in W_0^{1,p}(\Omega) \) solution of the problem
\[ \Delta v = 0 \quad \text{in} \ \Omega, \quad \frac{\partial v}{\partial n} = g \quad \text{on} \ \Gamma_0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1}. \]
satisfying
\[ \|v\|_{W_0^{1,p}(\Omega)} \leq C \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)} \leq C \left( \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})} \right). \] (17)

We notice that the compatibility condition on \( g_\pi \) is satisfied because \( \langle g_\pi, 1 \rangle_{\Gamma_0} = 2\langle g, 1 \rangle_{\Gamma_0} = 0 \). Finally, noticing that \( F \in W_0^{1,p}(\Omega) \), \( G_0 \in W_1^{-\frac{1}{p},p}(\Gamma_0) \), \( G_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \) and that the condition (10) is satisfied, by Theorem 3.2, there exists a function \( s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega) \) solution of the problem
\[ \Delta s = F \quad \text{in} \ \Omega, \quad \frac{\partial s}{\partial n} = G_0 \quad \text{on} \ \Gamma_0, \quad \frac{\partial s}{\partial n} = G_1 \quad \text{on} \ \mathbb{R}^{n-1}, \]
and satisfying the following estimate:
\[ \|s\|_{W_0^{1,p}(\Omega)} \leq C \left( \|F\|_{W_0^{1,p}(\Omega)} + \|G_0\|_{W_1^{-\frac{1}{p},p}(\Gamma_0)} + \|G_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})} \right). \] (18)

Finally, the function \( u = r + z + v + s \in W_0^{1,p}(\Omega) \) is solution of the problem \( (P_N) \) and the estimate (11) is given by (15)–(17) and (18). \( \square \)

**Remark.** We notice that, when the data are more regular, the weak solution is also more regular; in fact, it is the solution of Theorem 3.2.

### 4. The first mixed problem

In this section, we want to solve the following problem:

\[
(P_{M_1}) \begin{cases}
-\Delta u = f & \text{in} \ \Omega, \\
u = g_0 & \text{on} \ \Gamma_0, \\
\frac{\partial u}{\partial n} = g_1 & \text{on} \ \mathbb{R}^{n-1}.
\end{cases}
\]

First, we characterize the following kernel:
\[ E_0^p(\Omega) = \left\{ z \in W_0^{1,p}(\Omega), \ \Delta z = 0 \ \text{in} \ \Omega, \ z = 0 \ \text{on} \ \Gamma_0, \ \frac{\partial z}{\partial n} = 0 \ \text{on} \ \mathbb{R}^{n-1} \right\}. \]
We have the following result (we refer to the proof of Proposition 2.1 for the definition of $\mu_0$):

**Proposition 4.1.**

(i) If $p < n$ or $p = n = 2$, then $\mathcal{E}_0^p(\Omega) = \{0\}$.

(ii) If $p > n \geqslant 3$, then $\mathcal{E}_0^p(\Omega) = \{c(\lambda - 1), \ c \in \mathbb{R}\}$ where $\lambda$ is the unique solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the following problem $(P_1)$:

\[
(\mathcal{P}_1) \quad \Delta \lambda = 0 \quad \text{in} \ \Omega, \quad \lambda = 1 \quad \text{on} \ \Gamma_0, \quad \frac{\partial \lambda}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1}.
\]

(iii) If $p > n = 2$, then $\mathcal{E}_0^p(\Omega) = \{c(\mu - \mu_0), \ c \in \mathbb{R}\}$ where $\mu$ is the unique solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the following problem $(P_2)$:

\[
(\mathcal{P}_2) \quad \Delta \mu = 0 \quad \text{in} \ \Omega, \quad \mu = \mu_0 \quad \text{on} \ \Gamma_0, \quad \frac{\partial \mu}{\partial n} = 0 \quad \text{on} \ \mathbb{R}.
\]

**Proof.** Let $z$ be in $\mathcal{E}_0^p(\Omega)$. We define, for almost all $(\mathbf{x}', x_n) \in \tilde{\Omega}$ the function $z_\ast \in W_0^{1,p}(\tilde{\Omega})$, $z_\ast = 0$ on $\tilde{\Gamma}_0$ and we check, like done in the proof of Proposition 2.1 that $\Delta z_\ast = 0$ in $\tilde{\Omega}$. So the function $z_\ast$ is in the space

\[\mathcal{A}_0^p(\tilde{\Omega}) = \{z \in W_0^{1,p}(\tilde{\Omega}), \ \Delta z = 0 \text{ in } \tilde{\Omega}, \ z = 0 \text{ on } \tilde{\Gamma}_0\}.
\]

Now, we use the characterization of $\mathcal{A}_0^p(\tilde{\Omega})$ (see [5]).

(i) If $p < n$ or if $p = n = 2$, then $\mathcal{A}_0^p(\tilde{\Omega}) = \{0\}$ which implies that $z_\ast = 0$ in $\tilde{\Omega}$ and so $z = 0$ in $\Omega$, i.e. $\mathcal{E}_0^p(\Omega) = \{0\}$.

(ii) If $p > n \geqslant 3$, then $z_\ast = c(\tilde{\lambda} - 1)$, where $c$ is a real constant and $\tilde{\lambda}$ is the unique solution in $W_0^{1,p}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega})$ of the problem

\[\Delta \tilde{\lambda} = 0 \quad \text{in} \ \tilde{\Omega}, \quad \tilde{\lambda} = 1 \quad \text{on} \ \tilde{\Gamma}_0.
\]

Now, we set, for almost all $(\mathbf{x}', x_n) \in \tilde{\Omega}$, $\beta(\mathbf{x}', x_n) = \tilde{\lambda}(\mathbf{x}', -x_n)$. We easily check that $\beta$, belonging to $W_0^{1,p}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega})$, is solution of the same problem that $\tilde{\lambda}$ satisfies, but this solution is unique, so we deduce that $\beta = \tilde{\lambda}$ and so on $\mathbb{R}^{n-1}$, $\frac{\partial \beta}{\partial n} = 0$. Thus, setting $\lambda$ the restriction of $\tilde{\lambda}$ to $\Omega$, $\lambda \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is solution of the problem $(\mathcal{P}_1)$. Moreover, this solution is unique. Indeed, if $\theta$ is another solution, $\theta_\ast$ is solution of the same problem that $\tilde{\lambda}$ satisfies in $\tilde{\Omega}$, so $\theta_\ast = \tilde{\lambda}$ in $\tilde{\Omega}$ and $\theta = \tilde{\lambda}$ in $\Omega$.

(iii) If $p > n = 2$, so, we have $z_\ast = c(\tilde{\mu} - \mu_0)$, where $c$ is a real constant and $\tilde{\mu}$ the unique solution in $W_0^{1,p}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega})$ of the problem

\[\Delta \tilde{\mu} = 0 \quad \text{in} \ \tilde{\Omega}, \quad \tilde{\mu} = \mu_0 \quad \text{on} \ \tilde{\Gamma}_0.
\]

But, we notice that $\mu_0$ can also be written

\[\mu_0(x) = \frac{1}{2\pi |\tilde{\Gamma}_0|} \int_{\tilde{\Gamma}_0} \ln(|y - x|) \, d\sigma_y.
\]

As $\tilde{\Gamma}_0$ is symmetric with respect to $\mathbb{R}^{n-1}$, we deduce that $\mu_0$ is symmetric too, and so $\frac{\partial \mu_0}{\partial n} = 0$ on $\mathbb{R}^{n-1}$. Now, for $(\mathbf{x}', x_n) \in \tilde{\Omega}$, we set $\xi(\mathbf{x}', x_n) = \tilde{\mu}(\mathbf{x}', -x_n)$. We check that $\xi$, belonging to
Let $f$ be in $W_0^{0,p}(\Omega)$, $g_0$ in $W^{1-\frac{1}{p}}(\Gamma_0)$ and $g_1$ in $W^{-1\frac{1}{p}}(\mathbb{R}^{n-1})$. We remind that we search $u \in W_0^{1,p}(\Omega)$ solution of the problem $(P_M_1)$. We suppose that such a solution $u \in W_0^{1,p}(\Omega)$ exists. Then, for any $v \in W_0^{1,p}(\Omega)$, we have

$$\int_{\Omega} -v \Delta u \, dx = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \left( \left( \frac{\partial u}{\partial n} \right)_{\Gamma_0}, v \right)_{\mathbb{R}^{n-1}}.$$ \hspace{1cm} \text{(19)}$$

In particular, for any $\varphi \in \mathcal{E}_0^p(\Omega)$:

$$\int_{\Omega} f \varphi \, dx = \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - (g_1, \varphi)_{\mathbb{R}^{n-1}}.$$ \hspace{1cm} \text{(20)}$$

We have too

$$0 = \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} \nabla \varphi \cdot \nabla u \, dx - \left( \left( \frac{\partial \varphi}{\partial n} \right)_{\Gamma_0}, g_0 \right)_{\mathbb{R}^{n-1}}.$$ \hspace{1cm} \text{(21)}$$

We deduce from this that if $u \in W_0^{1,p}(\Omega)$ is solution of the problem $(P_M_1)$, the data must satisfy the following compatibility condition:

$$\forall \varphi \in \mathcal{E}_0^p(\Omega), \quad \int_{\Omega} f \varphi \, dx = \left( \left( \frac{\partial \varphi}{\partial n} \right)_{\Gamma_0}, g_0 \right)_{\mathbb{R}^{n-1}} - (g_1, \varphi)_{\mathbb{R}^{n-1}}.$$ \hspace{1cm} \text{(22)}$$

Now, we are going to search strong solutions for the problem $(P_M_1)$.

**Theorem 4.2.** For any $p > \frac{n}{n-1}$, $f \in W_0^{0,p}(\Omega)$, $g_0 \in W^{2-\frac{1}{p}}(\Gamma_0)$ and $g_1 \in W^{-1\frac{1}{p}}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_0^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)$ solution of $(P_M_1)$. Moreover, $u$ satisfies

$$\|u\|_{W_0^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)} \leq C \left( \|f\|_{W_0^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p}}(\Gamma_0)} + \|g_1\|_{W^{-1\frac{1}{p}}(\mathbb{R}^{n-1})} \right),$$ \hspace{1cm} \text{(23)}$$

where $C$ is a real positive constant which depends only on $p$ and $\omega_0$.

**Proof.** We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}^n)$ such that $u_{g_1} = 0$ and $\frac{\partial u_{g_1}}{\partial n} = g_1$ on $\mathbb{R}^{n-1}$ satisfying the estimate

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}^n)} \leq C \|g_1\|_{W^{-1\frac{1}{p}}(\mathbb{R}^{n-1})}.$$ \hspace{1cm} \text{(24)}$$
We set $u_1$ the restriction of $u_{\tilde{g}_1}$ to $\Omega$ and $\eta$ the trace of $u_1$ on $\Gamma_0$. Then, we set $g = g_0 - \eta \in W_0^{2-\frac{1}{p}}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_0^{1,p}(\Omega)$.

Now, we must find $v \in W_1^{2,p}(\Omega)$ solution of the following problem $(P')$:

$$(P') \quad -\Delta v = h \quad \text{in} \ \Omega, \quad v = g \quad \text{on} \ \Gamma_0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1}.$$ 

For this, we define the functions $h_* \in W_0^{1,p}(\tilde{\Omega})$ and $g_* \in W_2^{2-\frac{1}{p}}(\tilde{\Gamma}_0)$. By [5], there exists a function $w \in W_1^{2,p}(\tilde{\Omega})$, unique up to an element of $A_0^p(\tilde{\Omega})$, solution of

$$-\Delta w = h_* \quad \text{in} \ \tilde{\Omega}, \quad w = g_* \quad \text{on} \ \tilde{\Gamma}_0,$$

and satisfying the estimate

$$\|w\|_{W_1^{2,p}(\tilde{\Omega})} \leq C (\|h\|_{W_0^{1,p}(\Omega)} + \|g\|_{W_0^{2-\frac{1}{p}}(\Gamma_0)}).$$

Let $w_0$ be a solution of this problem and for almost all $(\xi', x_n) \in \tilde{\Omega}$, we set:

$$v_0(\xi', x_n) = w_0(\xi', -x_n).$$

Thanks to the symmetry of $h_*, g_*, \tilde{\Omega}$ and $\tilde{\Gamma}_0$ with respect to $\mathbb{R}^{n-1}$, we easily show that $v_0$ is solution of the same problem that $w_0$. Thus $v_0 = w_0 + k$ where $k \in A_0^p(\tilde{\Omega})$. Moreover, we show that $\frac{\partial k}{\partial n} = 0$ on $\mathbb{R}^{n-1}$ and we deduce that $\frac{\partial w_0}{\partial n} = 0$ on $\mathbb{R}^{n-1}$, so, the function $v$, restriction of $w_0$ to $\Omega$, is in $W_1^{2,p}(\Omega)$, is solution of $(P')$ and satisfies

$$\|v\|_{W_1^{2,p}(\Omega)} \leq C (\|f\|_{W_0^{1,p}(\Omega)} + \|g_0\|_{W_0^{2-\frac{1}{p}}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p}}(\mathbb{R}^{n-1})}).$$

Finally, $u = v + u_1 \in W_1^{2,p}(\Omega)$ is solution of $(P_{M_1})$ and (20) comes from (21) and (22). 

Now we search weak solutions of the problem $(P_{M_1})$. For this, in the following theorem, we shall introduce a lemma between points (i) and (ii). This lemma, proved thanks to the point (i), allows us to obtain an “inf-sup” condition, fundamental condition for the resolution of the point (ii).

**Theorem 4.3.** For each $p > 1$ such that $\frac{n}{p} \neq 1$ and for any $f \in W_0^{1,p}(\Omega)$, $g_0 \in W_0^{1-\frac{1}{p}}(\Gamma_0)$ and $g_1 \in W_0^{1-\frac{1}{p}}(\mathbb{R}^{n-1})$, satisfying if $p < \frac{n}{n-1}$, the compatibility condition (19), there exists a unique $u \in W_0^{1,p}(\Omega) / E_0^p(\Omega)$ solution of $(P_{M_1})$. Moreover, $u$ satisfies

$$\|u\|_{W_0^{1,p}(\Omega) / E_0^p(\Omega)} \leq C (\|f\|_{W_0^{1,p}(\Omega)} + \|g_0\|_{W_0^{1-\frac{1}{p}}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p}}(\mathbb{R}^{n-1})}),$$

where $C$ is a real positive constant which depends only on $p$ and $\omega_0$.

**Proof.** (i) First, we suppose $\frac{n}{p} > 1$, i.e. $p > \frac{n}{n-1}$.

Thanks to the previous theorem, we begin to show that there exists a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \quad \text{in} \ \Omega, \quad s = 0 \quad \text{on} \ \Gamma_0, \quad \frac{\partial s}{\partial n} = 0 \quad \text{on} \ \mathbb{R}^{n-1}.$$

and satisfying the estimate
\[ \|s\|_{W_0^{1,p}(\Omega)} \leq \|s\|_{W_1^{2,p}(\Omega)} \leq C \|f\|_{W_0^{1,p}(\Omega)}. \] (24)

Moreover, by [2], there exists a function \( z \in W_0^{1,p}(\mathbb{R}^n_+) \) solution of
\[ \Delta z = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial n} = g_1 \text{ on } \mathbb{R}^{n-1}, \]
and satisfying
\[ \|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \|g_1\|_{W_0^{1-p,p}(\mathbb{R}^{n-1})}. \] (25)

We denote again by \( z \) the restriction of \( z \) to \( \Omega \), so \( z \in W_1^{1,p}(\Omega) \) and satisfying
\[ \Delta z = 0 \text{ in } \Omega, \quad \frac{\partial z}{\partial n} = 0 \text{ on } \mathbb{R}^{n-1}, \]
and checking the estimate
\[ \|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \|g_1\|_{W_0^{1-p,p}(\mathbb{R}^{n-1})}. \] (26)

Finally, the function \( u = s + z + v \in W_0^{1,p}(\Omega) \) is solution of the problem \((PM_1)\) and the estimate \((23)\) comes from \((24), (25)\) and \((26)\).

Now, we set
\[ V_p = \{ v \in W_0^{1,p}(\Omega), \ v = 0 \text{ on } \Gamma_0 \}, \]
and we introduce the following lemma to solve the point (ii) of the theorem:

**Lemma 4.4.** Let \( p \) be such that \( p > \frac{n}{n-1} \). There exists a real constant \( \beta > 0 \) such that
\[
\inf_{w \in V_p} \sup_{v \in V_p, v \neq 0} \frac{\int_{\Omega} \nabla v \cdot \nabla w \, dx}{\|\nabla v\|_{L^p(\Omega)} \|\nabla w\|_{L^p(\Omega)}} \geq \beta,
\]
and the operators \( B \) from \( V_p / \text{Ker } B \) to \( (V_p')' \) and \( B' \) from \( V_p' \) to \( (V_p)' \bot \text{Ker } B \) defined by
\[ \forall v \in V_p, \forall w \in V_p', \quad \langle B v, w \rangle = \langle v, B' w \rangle = \int_{\Omega} \nabla v \cdot \nabla w \, dx \]
are isomorphisms.

**Proof.** We must firstly show an equivalent proposition to Proposition 3.2 of [3], i.e. for any \( g \in L^p(\Omega) \), there exist \( z \in \tilde{H}_p(\Omega) \) and \( \varphi \in V_p \), such that
\[
\frac{1}{C} \|f\|_{W_0^{1,p}(\Omega)} \leq \|z\|_{W_1^{1,p}(\Omega)} \leq C \|f\|_{W_0^{1,p}(\Omega)}.
\]
\[ g = \nabla \varphi + z, \]
\[ \| \nabla \varphi \|_{L^p(\Omega)} \leq C \| g \|_{L^p(\Omega)}, \]
where \( C > 0 \) is a real constant which depends only on \( \Omega \) and \( p \) and
\[ H_p(\Omega) = \{ z \in L^p(\Omega), \ \text{div} \ z = 0 \ in \ \Omega, \ z \cdot n = 0 \ on \ \mathbb{R}^{n-1} \}. \]

The proof takes its inspiration from the proof of [3]. First, setting \( \tilde{g} = g \) in \( \Omega \), \( \tilde{g} = 0 \) in \( \omega_0 \), \( \tilde{g} = 0 \) in \( \mathbb{R}^n \), we remind (see [4]) that there exists \( v \in W_{1,p}^0(\mathbb{R}^n) \), unique if \( p < n \) and unique up to an additive constant otherwise, solution of
\[ \Delta v = \text{div} \ \tilde{g} \ in \ \mathbb{R}^n \]
and satisfying
\[ \| \nabla v \|_{L^p(\mathbb{R}^n)} \leq C \| \text{div} \ \tilde{g} \|_{W^{-1,p}_0(\mathbb{R}^n)} \leq C \| g \|_{L^p(\Omega)}. \]

We denote again by \( v \) the restriction of \( v \) to \( \Omega \). We notice that, by [7], \( (g - \nabla v) \cdot n \in W_{0}^{-1,p}(\mathbb{R}^{n-1}) \)
because \( \text{div}(g - \nabla v) = 0 \ in \ W_{1,p}^0(\Omega) \) and
\[ \| (g - \nabla v) \cdot n \|_{W_{0}^{-1,p}(\mathbb{R}^{n-1})} \leq C \| g - \nabla v \|_{L^p(\Omega)}. \]

Moreover, by the point (i), there exists a unique \( w \in W_{0}^{1,p}(\Omega) \) solution of
\[ \Delta w = 0 \ in \ \Omega, \quad w = -v \ on \ \Gamma_0, \quad \frac{\partial w}{\partial n} = (g - \nabla v) \cdot n \ on \ \mathbb{R}^{n-1}, \]
such that
\[ \| w \|_{W_{0}^{1,p}(\Omega)} \leq C \left( \| v \|_{W_{0}^{1,\frac{n}{n-1}}(\Omega)} + \| (g - \nabla v) \cdot n \|_{W_{0}^{-1,p}(\mathbb{R}^{n-1})} \right) \]
\[ \leq C \left( \| v \|_{W_{0}^{1,1}(\Omega)} + \| g - \nabla v \|_{L^p(\Omega)} \right) \leq C \| g \|_{L^p(\Omega)}. \]

Then, setting \( \varphi = v + w \) and \( z = g - \nabla \varphi \), we have the searched result, and, like done in [3], the “inf-sup” condition. The second part of the lemma comes from the Babuška-Brezzi’s theorem (see [3] for example). \( \square \)

(ii) We suppose \( \frac{n}{p} < 1 \), i.e. \( p < \frac{n}{n-1} \).

Thanks to Section 2, we know there exists a unique \( z \in W_{0}^{1,p}(\Omega) \) solution of the problem
\[ \Delta z = 0 \ in \ \Omega, \quad z = g_0 \ on \ \Gamma_0, \quad z = 0 \ on \ \mathbb{R}^{n-1}, \]
and satisfying the estimate
\[ \| z \|_{W_{0}^{1,p}(\Omega)} \leq C \| g_0 \|_{W_{0}^{1,\frac{n}{n-1}}(\Gamma_0)}. \]
Since $\Delta z = 0 \in W^{0,p}_1(\Omega)$, $\eta = \frac{\partial z}{\partial n}$ has a meaning in $W^{-1,p}_0(\mathbb{R}^{n-1})$. We set $g = g_1 - \eta \in W^{-1,p}_0(\mathbb{R}^{n-1})$ and we want to solve the following problem $(P')$:

$$(P') \quad -\Delta v = f \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial v}{\partial n} = g \quad \text{on} \quad \mathbb{R}^{n-1}.$$ For this, for any $w \in V_{p'}$ we define the operator:

$$Tw = \int_{\Omega} f w \, dx + (g, w)_{\mathbb{R}^{n-1}}.$$ We easily check that $T \in (V_{p'})'$. We define the following problem $(FV)$: find $v \in V_p$ such that for any $w \in V_{p'}$, we have

$$\int_{\Omega} \nabla v \cdot \nabla w \, dx = Tw.$$ We notice that if $v \in W^{1,p}_0(\Omega)$ is solution of $(P')$, it is also solution of $(FV)$. Conversely, let $v \in V_p$ be a solution of $(FV)$ and let $\varphi$ be in $D(\Omega) \subset V_{p'}$. So

$$(\Delta v, \varphi)_{D'(\Omega), D(\Omega)} = -\int_{\Omega} \nabla v \cdot \nabla \varphi \, dx = -T \varphi = -(f, \varphi)_{D'(\Omega), D(\Omega)},$$ i.e. $-\Delta v = f$ in $\Omega$. The function $\Delta v \in W^{1,p}_1(\Omega)$, so $\frac{\partial v}{\partial n}$ has a meaning in $W^{-1,p}_0(\mathbb{R}^{n-1})$. Now, we want to show that we have $\frac{\partial v}{\partial n} = g$ on $\mathbb{R}^{n-1}$. We know that, for any $\mu \in W^{1,p}_1(\mathbb{R}^{n-1})$, there exists $u_1 \in W^{1,p}_1(\mathbb{R}^{n-1})$ such that

$$u_1 = \mu \quad \text{and} \quad \frac{\partial u_1}{\partial n} = 0 \quad \text{on} \quad \mathbb{R}^{n-1},$$ with $\|u_1\|_{W^{1,p}_1(\mathbb{R}^{n-1})} \leq C \|\mu\|_{W^{2-\frac{1}{p'},p'}_1(\mathbb{R}^{n-1})}$. We denote again by $u_1 \in W^{1,p}_1(\Omega)$ the restriction of $u_1$ to $\Omega$ and $\xi \in W^{2-\frac{1}{p'},p'}_1(\Gamma_0)$ the trace of $u_1$ on $\Gamma_0$. There exists $u_0 \in W^{2,p'}_2(\Omega_R)$, where $R > 0$ is such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$ and $\Omega_R = \Omega \cap B_R$, satisfying

$$u_0 = \xi \quad \text{and} \quad \frac{\partial u_0}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \quad u_0 = \frac{\partial u_0}{\partial n} = 0 \quad \text{on} \quad \partial B_R$$ and

$$\|u_0\|_{W^{2,p'}_2(\Omega_R)} \leq C \|\xi\|_{W^{2-\frac{1}{p'},p'}_1(\Gamma_0)}.$$ We set $\tilde{u}_0$ the extension of $u_0$ by 0 outside $B_R$. We have $\tilde{u}_0 \in W^{2,p'}_1(\Omega)$ and

$$\tilde{u}_0 = \xi \quad \text{and} \quad \frac{\partial \tilde{u}_0}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \quad \tilde{u}_0 = \frac{\partial \tilde{u}_0}{\partial n} = 0 \quad \text{on} \quad \mathbb{R}^{n-1},$$
with $\|\tilde{u}_0\|_{W^{2,p'}_1(\Omega)} \leq C \|u_1\|_{W^{2,p'}_1(\Omega)}$. We set $u = u_1 - \tilde{u}_0 \in W^{2,p'}_1(\Omega)$, then $u$ satisfies

$$u = 0 \quad \text{on } \Gamma_0, \quad u = \mu \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}^{n-1}$$

and

$$\|u\|_{W^{2,p'}_1(\Omega)} \leq C \|\mu\|_{W^{2,p'}_1(\mathbb{R}^{n-1})}.$$ 

Thus, noticing that $u \in V_{p'}$ and $\mu \in W^{1-\frac{1}{p},p'}_0(\mathbb{R}^{n-1})$ because, for any value of $n$ and $p'$, $W^{2,p'}_1(\Omega) \subset W^{1,p'}_0(\Omega)$, we have

$$\left\langle \frac{\partial v}{\partial n}, \mu \right\rangle_{\mathbb{R}^{n-1}} = \left\langle \frac{\partial v}{\partial n}, u \right\rangle_{\Gamma_0 \cup \mathbb{R}^{n-1}} = \int_\Omega u \Delta v \, dx + \int_\Omega \nabla v \cdot \nabla u \, dx$$

$$= -\int_\Omega f u \, dx + Tu$$

$$= \left\langle g, \mu \right\rangle_{\mathbb{R}^{n-1}}.$$

i.e. $\frac{\partial v}{\partial n} = g$ on $\mathbb{R}^{n-1}$. So, problems $(P')$ and $(FV)$ are equivalents. Moreover, since $p < \frac{n}{n-1}$, $p' > \frac{n}{n-1}$ and we apply the previous lemma noticing that we have $\text{Ker } B = E_{p'}^0(\Omega)$. We deduce that $B'$ is an isomorphism from $V_p$ to $(V_{p'})' \perp E_{p'}^0(\Omega)$. 

(28)

Moreover $T \in (V_{p'})' \perp E_{p'}^0(\Omega)$. Indeed, for any $\varphi \in E_{p'}^0(\Omega)$, we have

$$T \varphi = \int_\Omega f \varphi \, dx + \left\langle g_1, \varphi \right\rangle_{\mathbb{R}^{n-1}} - \left\langle \eta, \varphi \right\rangle_{\mathbb{R}^{n-1}}$$

and

$$\left\langle \eta, \varphi \right\rangle_{\mathbb{R}^{n-1}} = \left\langle \frac{\partial z}{\partial n}, \varphi \right\rangle_{\Gamma_0 \cup \mathbb{R}^{n-1}} = \int_\Omega \nabla z \cdot \nabla \varphi \, dx = \left\langle \frac{\partial \varphi}{\partial n}, g_0 \right\rangle_{\Gamma_0},$$

which implies, by the condition (19), that $T \varphi = 0$. This allows us to deduce, by (28), that there exists a unique $v \in V_p$ such that $B'v = T$, i.e. solution of $(FV)$ and consequently of $(P')$ and we have the following estimate:

$$\|v\|_{W^{1,p}_0(\Omega)} \leq C \left( \|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p'}_0(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p'}_0(\mathbb{R}^{n-1})} \right).$$

(29)

Finally, $u = z + v \in W^{1,p}_0(\Omega)$ is solution of $(P_{M_1})$ and we have the estimate (23) by (27) and (29).

**Remark.** We notice that when $p > \frac{n}{n-1}$ and when the data are more regular, the weak solution is more regular too; it is in fact the solution of Theorem 4.2.
5. The second mixed problem

In this section, we want to solve the following problem:

\[(\mathcal{P}_{M_2}) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g_0 & \text{on } \Gamma_0, \\ u = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases} \]

First, we characterize the following kernel:

\[\mathcal{F}_0^p(\Omega) = \left\{ z \in W^{1,p}_0(\Omega), \Delta z = 0 \text{ in } \Omega, \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_0, z = 0 \text{ on } \mathbb{R}^{n-1} \right\}. \]

**Proposition 5.1.** For any \( p > 1 \), \( \mathcal{F}_0^p(\Omega) = \{0\} \).

**Proof.** Let \( z \) be in \( \mathcal{F}_0^p(\Omega) \). We define, for almost all \( (x', x_n) \in \tilde{\Omega} \) the function \( z^* \in W^{1,p}_0(\tilde{\Omega}) \). Then \( \frac{\partial z^*}{\partial n} = 0 \) on \( \tilde{\Gamma}_0 \) and we check, like done in the proof of Proposition 2.1 that \( \Delta z^* = 0 \) in \( \tilde{\Omega} \). The function \( z^* \) is in the space \( \{ z \in W^{1,p}_0(\tilde{\Omega}), \Delta z = 0 \} \) and \( \frac{\partial z^*}{\partial n} = 0 \) on \( \tilde{\Gamma}_0 \) which is equal to \( \mathcal{P}_{[1-n/p]} \) (see [5]). Thus, if \( p < n \), \( z^* = 0 \) in \( \tilde{\Omega} \) and \( z = 0 \) in \( \Omega \) and if \( p \geq n \), \( z^* \) is a constant in \( \tilde{\Omega} \) so \( z \) is constant in \( \Omega \), but \( z = 0 \) on \( \mathbb{R}^{n-1} \), so \( z = 0 \) in \( \Omega \) and \( \mathcal{F}_0^p(\Omega) = \{0\} \). \( \square \)

The following theorem allows us to obtain strong solutions of the problem \( (\mathcal{P}_{M_2}) \).

**Theorem 5.2.** For any \( p > \frac{n}{n-1} \), \( f \in W^{0,p}_1(\Omega) \), \( g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0) \) and \( g_1 \in W^{2-\frac{1}{p},p}(\mathbb{R}^{n-1}) \), there exists a unique \( u \in W^{2,p}_1(\Omega) \) solution of \( (\mathcal{P}_{M_2}) \). Moreover, \( u \) satisfies

\[ \|u\|_{W^{2,p}_1(\Omega)} \leq C \left( \|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})} \right), \]

where \( C \) is a real positive constant which depends only on \( p \) and \( \omega_0 \).

**Proof.** We know there exists a function \( u_{g_1} \in W^{2,p}_1(\mathbb{R}^{n+}) \) such that \( u_{g_1} = g_1 \) and \( \frac{\partial u_{g_1}}{\partial n} = 0 \) on \( \mathbb{R}^{n-1} \), satisfying the estimate

\[ \|u_{g_1}\|_{W^{2,p}_1(\mathbb{R}^{n+})} \leq C \|g_1\|_{W^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \]

We set \( u_1 \) the restriction of \( u_{g_1} \) to \( \Omega \) and \( \eta \) the normal derivative of \( u_1 \) on \( \Gamma_0 \). Then, we set \( g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0) \) and \( h = f + \Delta u_1 \in W^{0,p}_1(\Omega) \). Now, we want to find \( v \in W^{2,p}_1(\Omega) \) solution of the following problem \( (\mathcal{P}') \):

\[ (\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad \frac{\partial v}{\partial n} = g \quad \text{on } \Gamma_0, \quad v = 0 \quad \text{on } \mathbb{R}^{n-1}. \]

We define the functions \( h^* \in W^{0,p}_1(\tilde{\Omega}) \) and \( g^* \in W^{1-\frac{1}{p},p}(\tilde{\Gamma}_0) \) and, by [5], there exists a function \( w \in W^{2,p}_1(\tilde{\Omega}) \), unique up to an element of \( \mathcal{P}_{[1-n/p]} \), solution of

\[ -\Delta w = h^* \text{ in } \tilde{\Omega}, \quad \frac{\partial w}{\partial n} = g^* \text{ on } \tilde{\Gamma}_0. \]
and satisfying the estimate
\[ \|w\|_{W^{2,p}_1(\Omega)} \leq C \left( \|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{-1,p}_1(I_0)} + \|g_1\|_{W^{-1,p}_1(\mathbb{R}^{n-1})} \right). \]

Let \( w_0 \) be a solution of this problem and, for almost all \((x',x_0) \in \tilde{\Omega}\), we set
\[ v_0(x',x_0) = -w_0(x',x_0). \]

We easily check that \( v_0 \) is solution of the same problem that \( w_0 \) satisfies. Thus \( v_0 - w_0 \in P_{1-n/p}[\Omega] \).

(i) We suppose that \( \frac{n}{p} > 1 \). In this case, \( v_0 = w_0 \) in \( \tilde{\Omega} \) and we deduce that \( w_0 = 0 \) on \( \mathbb{R}^{n-1} \). So, the function \( v \in W^{2,p}_1(\Omega) \), restriction of \( w_0 \) to \( \Omega \) is a solution of \((P')\).

(ii) We suppose that \( \frac{n}{p} \leq 1 \). In this case, \( v_0 = w_0 + \alpha \) in \( \tilde{\Omega} \), where \( \alpha \) is a real constant, and, setting \( c = -\frac{1}{\alpha} \), we deduce that \( w_0 = c \) on \( \mathbb{R}^{n-1} \). The function \( v = w_0|_\Omega - c \) is an element of \( W^{2,p}_1(\Omega) \) and \( v \) is solution of \((P')\).

Moreover, \( v \), solution of \((P')\), satisfies the estimate
\[ \|v\|_{W^{2,p}_1(\Omega)} \leq C \left( \|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{-1,p}_1(I_0)} + \|g_1\|_{W^{-1,p}_1(\mathbb{R}^{n-1})} \right). \] (32)

Finally, the function \( u = v + u_1 \in W^{2,p}_1(\Omega) \) is solution of \((P_{M_2})\) and the estimate (30) comes from (31) and (32).

Now, we search weak solutions of the problem \((P_{M_2})\). We set
\[ W_p = \{ v \in W^{1,p}_0(\Omega), \ v = 0 \text{ on } \mathbb{R}^{n-1} \}, \]
and we firstly give the following lemma that we demonstrate like to Lemma 4.4 reversing only \( \Gamma_0 \) and \( \mathbb{R}^{n-1} \) (and so, using in its proof the result of the point (i) of the following theorem):

**Lemma 5.3.** Let \( p \) be such that \( p > \frac{n}{p-1} \). There exists a real constant \( \beta > 0 \) such that
\[ \inf_{\substack{w \in W_{p'} \\setminus \{0\} \\setminus \text{Ker } B \\cap (W_{p'})' \\setminus \text{Ker } B \\cap (W_{p'})' \\setminus \text{Ker } B \\cap (W_{p'})' \\setminus \text{Ker } B \\cap (W_{p'})' \\setminus \text{Ker } B \\cap (W_{p'})'}} \sup_{v \in W_p} \frac{\int_\Omega \nabla v \cdot \nabla w \, dx}{\|\nabla v\|_{L^p(\Omega)} \|\nabla w\|_{L^p(\Omega)}} \geq \beta, \]
and the operators \( B \) from \( W_p/\text{Ker } B \) to \((W_{p'})'\) and \( B' \) from \( W_{p'} \) to \((W_p)' \) are isomorphisms.

**Theorem 5.4.** For each \( p > 1 \) such that \( \frac{n}{p} \neq 1 \) and for any \( f \in W^{0,p}_1(\Omega) \), \( g_0 \in W^{-1,p}_1(I_0) \) and \( g_1 \in W^{1-\frac{1}{p},p}_0(\mathbb{R}^{n-1}) \), there exists a unique \( u \in W^{1,p}_0(\Omega) \) solution of \((P_{M_2})\). Moreover, \( u \) satisfies
\[ \|u\|_{W^{1,p}_0(\Omega)} \leq C \left( \|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{-1,p}_1(I_0)} + \|g_1\|_{W^{-1,p}_1(\mathbb{R}^{n-1})} \right), \] (33)
where \( C \) is a real positive constant which depends only on \( p \) and \( \omega_0 \).
Proof. (i) We suppose $\frac{n}{p} > 1$, i.e. $p > \frac{n}{n-1}$.

First, we apply Theorem 5.2 to have the existence of $s \in W^{2,p}_1(\Omega) \subset W^{1,p}_0(\Omega)$ solution of the problem

$$-\Delta s = f \quad \text{in } \Omega, \quad \frac{\partial s}{\partial n} = 0 \quad \text{on } \Gamma_0, \quad s = 0 \quad \text{on } \mathbb{R}^{n-1},$$

and satisfying

$$\|s\|_{W^{1,p}_0(\Omega)} \leq \|s\|_{W^{2,p}_1(\Omega)} \leq C \|f\|_{W^{0,p}_0(\Omega)}. \quad (34)$$

Then, by [6], there exists a function $z \in W^{1,p}_0(\mathbb{R}^n_{+})$ solution of

$$\Delta z = 0 \quad \text{in } \mathbb{R}^n_{+}, \quad z = g_1 \quad \text{on } \mathbb{R}^{n-1},$$

satisfying

$$\|z\|_{W^{1,p}_0(\mathbb{R}^n_{+})} \leq C \|g_1\|_{W^{-\frac{1}{p}}_0(\mathbb{R}^{n-1})}. \quad (35)$$

We denote again by $z$ the restriction of $z$ to $\Omega$. It is obvious that the normal derivative $\eta$ of $z$ on $\Gamma_0$ is in $W^{-\frac{1}{p}}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p}}(\Gamma_0)$ and we want to solve the following problem $(P')$:

$$(P') \quad \Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} = g \quad \text{on } \Gamma_0, \quad v = 0 \quad \text{on } \mathbb{R}^{n-1}.$$ 

Let $\mu$ be in $W^{1-\frac{1}{p}}(\Gamma_0)$. For almost all $(x', x_n) \in \Gamma_0$, we set

$$\pi \mu(x', x_n) = \mu(x', x_n) - \mu(x', -x_n).$$

We notice that $\pi \mu \in W^{1-\frac{1}{p}}(\Gamma_0)$, and we define

$$\langle g_{\pi}, \mu \rangle := \langle g, \pi \mu \rangle_{\Gamma_0}.$$ 

It is obvious that $g_{\pi} \in W^{-\frac{1}{p}}(\Gamma_0)$ and that $g$ is the restriction of $g_{\pi}$ to $\Gamma_0$. Moreover, we easily check that

$$\langle g_{\pi}, \xi \rangle_{\Gamma_0} = -\langle g_{\pi}, \mu \rangle_{\Gamma_0},$$

where $\xi(x', x_n) = \mu(x', -x_n)$ with $(x', x_n) \in \Gamma_0$. By [5], there exists a function $w \in W^{1,p}_0(\tilde{\Omega})$, unique up to an element of $P_{1-n/p}$ solution of the following problem:

$$\Delta w = 0 \quad \text{in } \tilde{\Omega}, \quad \frac{\partial w}{\partial n} = g_{\pi} \quad \text{on } \tilde{\Gamma}_0,$$

and satisfying

$$\|w\|_{W^{1,p}_0(\tilde{\Omega})/P_{1-n/p}} \leq C \|g_{\pi}\|_{W^{-\frac{1}{p}}(\tilde{\Gamma}_0)}.$$
Let \( w_0 \) be a solution of this problem. We set for almost all \((\mathbf{x}', x_0) \in \tilde{\Omega}\):

\[
v_0(\mathbf{x}', x_0) = -w_0(\mathbf{x}', -x_0).\]

The function \( v_0 \) is in \( W^{1,p}_0(\tilde{\Omega}) \) and since \( \Delta w_0 \) is nil in \( \tilde{\Omega} \), we easily check that \( \Delta v_0 \) is nil too. Thus \( \frac{\partial v_0}{\partial n} \) has a meaning in \( W^{-\frac{1}{p}}(\tilde{\Gamma}_0) \) and we show, like done in the proof of Theorem 3.3 that \( \frac{\partial v_0}{\partial n} = g_0 \) on \( \tilde{\Gamma}_0 \). So, the function \( v_0 \) is solution of the same problem that \( w_0 \) satisfies, which implies that \( v_0 - w_0 \in P_{1-n/p} \). We conclude like done in the proof of the previous theorem to show the existence of the solution \( v \in W^{1,p}_0(\Omega) \) of the problem \((P')\) satisfying

\[
\|v\|_{W^{1,p}_0(\Omega)} \leq C \|g\|_{W^{-\frac{1}{p}}(\tilde{\Gamma}_0)}.
\]

Finally, the function \( u = z + s + v \in W^{1,p}_0(\Omega) \) is solution of the problem \((P_N)\) and the estimate \((33)\) comes from \((34), (35)\) and \((36)\).

(ii) We suppose \( \frac{n}{p} < 1 \), i.e. \( p < \frac{n}{n-1} \).

Thanks to Section 2, we know there exists a unique \( z \in W^{1,p}_0(\Omega) \) solution of the problem

\[
\Delta z = 0 \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \Gamma_0, \quad z = g_1 \quad \text{on} \quad \mathbb{R}^{n-1},
\]
satisfying the estimate

\[
\|z\|_{W^{1,p}_0(\Omega)} \leq C \|g_1\|_{W^{-\frac{1}{p}}(\mathbb{R}^{n-1})}.
\]

Since \( \Delta z = 0 \in L^p(\Omega), \eta = \frac{\partial z}{\partial n} \) has a meaning in \( W^{-\frac{1}{p}}(\tilde{\Gamma}_0) \). We set \( g = g_0 - \eta \) and we want to find \( v \in W^{1,p}_0(\Omega) \) solution of the following problem \((P')\):

\[
(P') \quad -\Delta v = f \quad \text{in} \quad \Omega, \quad \frac{\partial v}{\partial n} = g \quad \text{on} \quad \Gamma_0, \quad v = 0 \quad \text{on} \quad \mathbb{R}^{n-1}.
\]

For this, we follow the same idea as the proof of the point (ii) in Theorem 4.4 reversing only \( \Gamma_0 \) and \( \mathbb{R}^{n-1} \) and noticing that, for \( \mu \in W^{2,\frac{1}{p}}(\tilde{\Gamma}_0) \), we know easily finding \( s \in W^{2,\frac{1}{p}}(\Omega) \) such that

\[
s = \mu \quad \text{and} \quad \frac{\partial s}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \quad s = \frac{\partial s}{\partial n} = 0 \quad \text{on} \quad \mathbb{R}^{n-1},
\]
satisfying

\[
\|s\|_{W^{2,\frac{1}{p}}(\Omega)} \leq C \|\mu\|_{W^{2,\frac{1}{p}}(\tilde{\Gamma}_0)}
\]

and that \( \text{Ker} \ B' = F_{\Gamma_0}^0(\Omega) = \{0\} \). We have also the following estimate:

\[
\|v\|_{W^{1,p}_0(\Omega)} \leq C (\|f\|_{W^{0,p}_1(\Omega)} + \|g_0\|_{W^{-\frac{1}{p}}(\tilde{\Gamma}_0)} + \|g_1\|_{W^{1,\frac{1}{p}}(\mathbb{R}^{n-1})}) \quad (38)
\]

Finally, \( u = z + v \in W^{1,p}_0(\Omega) \) is solution of \((P_{M_2})\) and we have the estimate \((33)\) by \((37)\) and \((38)\). \( \square \)

**Remark.** We notice that when \( p > \frac{n}{n-1} \) and when the data are more regular, the weak solution is more regular too; it is in fact the solution of Theorem 5.2.
References