THE GRADED LIE ALGEBRAS OF AN ALGEBRA

BY

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(Communicated at the meeting of June 24, 1967)

1. OUTLINE

In a recent note [11] the author showed the existence of a natural graded Lie (GLA) structure on the cohomology $H^*(B, A/B)$ when $A$ is an (associative, commutative or Lie) algebra and $B$ a subalgebra. The construction depends on a GLA structure on the standard complex of multilinear maps (with possible additional conditions) of $A$ into itself, which is little known in algebra, and finds its origin in differential geometry [2], in the context of the Lie algebra of vector fields on a manifold and its structure as a module over the functions.

The main purpose of this paper is the development of the properties of this GLA, and the preparation of machinery to use $H^*(B, A/B)$ in the study of deformations of $B$ as a subalgebra of $A$. A rigidity theorem in this context was recently obtained by RICHARDSON [12].

As we deal, in principle, with three cases; that of associative algebras, that of commutative associative algebras (called just "commutative" for short) and that of Lie algebras, a certain amount of repetition can be expected. (The reader who is familiar with this material can immediately proceed to section 5.) A substantial amount of repetition can be avoided, however, by the observation that the common element in all these cases is a GLA with ("comp") product $[, \, ]^\circ$ which acts on its own underlying graded vector space (the action of $f$ on $g$ is denoted $g \circ f$ and is called the composition product) in a manner which in general is not the adjoint representation. Furthermore, an element $\mu$ is chosen, with $\mu \circ \mu = 0$. Given these structures, a coboundary map can be defined by $\delta f = -[\mu, f]^\circ$, and another GLA ("cup") product

$$[f, g]^\cup = (-1)^{mn+m+1}\{\mu \circ f \circ g - \mu \circ (f \circ g)\}$$

is obtained, for which the initially given action consists of derivations. The semi-direct product of the two GLA's contains a subalgebra in which the product can be given by

$$[f, g] = [f, g]^\cup + (-1)^n g \circ \delta f + (-1)^{mn+m+1} f \circ \delta g.$$

1) Research partially supported by N.S.F. Grant GP 4503.
It is this product alone which induces the product in $H^*(B, A/B)$. It also was this product which, in the differential-geometric case, produced a vector form (differential form whose values are tangent vectors) from two such by a process involving differentiation but no auxiliary structures (e.g. a connection).

The sections 2, 3 and 4 deduce the basic identities for the three types of algebras under consideration. The material on associative algebras is a summary and condensation of work of Gerstenhaber [3]; that on commutative algebras is deduced from some comments by Harrison [4], elaborated on in [8], and that on Lie algebras can mostly already be found (in a somewhat different form) in Frölicher-Nijenhuis [2], and improved on under the influence of Kodaira-Spencer [5] and of Nickerson [6]. More recent references for the Lie case are joint works [9, 10] with Richardson. Section 5 constructs the GLA's which are the subject of this paper. Section 6 discusses the cohomology of algebras, subalgebras and quotients as mentioned in the beginning, in a rather general setting.

The base field in all the considerations may be of prime characteristic. If the characteristic is 2, some changes may be necessary, for which we refer to [7, 11]. It is also a simple exercise to allow the base field to be replaced by a commutative ring, with the usual unitary properties.

2. THE CASE OF AN ASSOCIATIVE ALGEBRA

Let $V$ be a vector space over a field $k$. The graded system

$$\{\text{Hom}_k(\otimes^{n+1} V, V)\}_{n \geq -1}$$

of vector spaces can be given a GLA structure through the introduction of the composition product: let $f, g$ be of respective degrees $n, m$; then $f \circ g$, of degree $n+m$, is given by

$$\begin{align*}
(f \circ g)(x_0, \ldots, x_{n+m}) = \\
= \sum_{i=0}^{n} (-1)^{imf(x_0, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m}), x_{i+m+1}, \ldots, x_{n+m})}.
\end{align*}$$

The composition product (2.1) is not associative, in general, but satisfies, for $h$ of degree $p$, the commutative-associative law

$$(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{mp}\{(f \circ h) \circ g - f \circ (h \circ g)\}.\quad (2.2)$$

The validity of (2.2) is easily seen after the observation that the left side can be re-written:

$$\begin{align*}
\{\{f \circ g \circ h - f \circ (g \circ h)\}(x_0, \ldots, x_{n+m+p}) = \\
= \sum_1 (-1)^{im+fp} f(x_0, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m}), x_{i+m+1}, \ldots, x_{f-1}, \\
= \sum (-1)^{(f-p)m+ip} g(x_f, \ldots, x_{f+p}) \}
\end{align*}$$

(2.3)
where \( L' \) extends over \( 0 \leq i < n \) and \( i + m < j < n + m \), while \( \Sigma_2 \) extends over \( 0 \leq i < n \) and \( i + p < j < n + p \).

The GLA product is now defined by

\[
[f, g]^{\circ} = g \circ f - (-1)^{mn} f \circ g.
\]

The Jacobi identity for \([ , ]^{\circ}\) follows by writing (2.2) as

\[
(-1)^{np}(f \circ g) \circ h - (-1)^{(m+n)p}(f \circ h) \circ g = (-1)^{np} f \circ [h, g]^{\circ},
\]
cyclically permuting \( f, g, h \) and adding. Other properties of GLA’s follow easily. See [7] for a short discussion of GLA’s.

If \( V \) has the structure of an associative algebra, then a multiplication map \( \mu \in \text{Hom}_k(V \otimes V, V) \) is given, which satisfies the associativity condition

\[
\mu(\mu(x, y), z) - \mu(x, \mu(y, z)) = 0.
\]

This condition is equivalent to \( \mu \circ \mu = 0 \), and if \( \text{char } k \neq 2 \), is also equivalent to \( [\mu, \mu]^{\circ} = 0 \).

The system \( \{\text{Hom}_k(\otimes^{n+1} V, V)\}_{n \geq 1} \) is, apart from its grading, the standard cochain complex for the Hochschild cohomology of the associative algebra \( A = (V, \mu) \) with coefficients in \( A \). The Hochschild coboundary map is given by

\[
\delta f = - [\mu, f]^{\circ}.
\]

The product \( \mu \) gives rise to a structure of graded associative algebra to the system \( \{\text{Hom}_k(\otimes^n V, V)\}_{n \geq 0} \) (note the change in grading!) by the product \( f \cup g \) of \( f, g \) of degrees \( n, m \) given by

\[
(f \cup g)(x_1, \ldots, x_{n+m}) = \mu(f(x_1, \ldots, x_n), g(x_{n+1}, \ldots, x_{n+m})).
\]

This product gives rise to a GLA structure \([ , ]^U\) given by

\[
[f, g]^U = f \cup g - (-1)^{mn} g \cup f.
\]

Note that \([ , ]^{\circ}\) and \([ , ]^U\) are different products, based on different gradings, and that further, \([ , ]^{\circ}\) depends only on the vector space structure of \( V \), while \([ , ]^U\) depends on the choice of an associative multiplication on \( V \).

The product \([ , ]^U\) is expressible in composition products by replacing \( f, g, h \) in (2.3) by \( f, g, h \) where \( f \in \text{Hom}_k(\otimes^{n+1} V, V) \) and \( g \in \text{Hom}_k(\otimes^{m+1} V, V) \):

\[
[f, g]^U = (-1)^{mn+m+1} \{(\mu \circ f) \circ g - \mu \circ (f \circ g)\}.
\]

The \( \mu \)'s can be eliminated by the use of \( \delta \)'s through (2.4, 5):

\[
[f, g]^U = (-1)^{mn+m+1} \{(-1)^n \delta f \circ g - (-1)^{m+n} \delta (f \circ g) + (-1)^{m+n} f \circ \delta g\}.
\]

The right side does not look symmetric in \( f \) and \( g \), though the left side is. We therefore write the formula also the other way, with a simultaneous
change in grading: if \( f \in \text{Hom}_k(\otimes^n V, V) \) and \( g \in \text{Hom}_k(\otimes^m V, V) \), then
\[
[f, g]^U = \delta g \overline{\sigma} f - (-1)^n g \overline{\sigma} \delta f + (-1)^n \delta(g \overline{\sigma} f).
\]
The material in this section is a summary of work by Gerstenhaber [3].

3. The case of a commutative algebra

To construct the cochain complex suggested by Harrison [4] we use the shuffle product defined on \( \otimes V = \oplus_{n \geq 1} \otimes^n V \). Let
\[
u = u_1 \otimes \ldots \otimes u_m, \quad v = u_{m+1} \otimes \ldots \otimes u_{m+n},
\]
where \( u_i \in V \) for \( 1 \leq i \leq m + n \). Then the shuffle product \( u \star v \) is defined by (cf. [1])
\[
u \star v = \sum_s g_s \sigma u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(m+n)},
\]
where \( \Sigma \) extends over those permutations \( \sigma \) of \( \{1, \ldots, m+n\} \) which do not change the relative orders of \( \{1, \ldots, m\} \) and of \( \{m+1, \ldots, m+n\} \).

Let \( T \) denote the space \( \otimes V \) with bilinear product \( \star \) defined by (3.1), then \( T \) is a graded-commutative and associative algebra. The ideal \( T \star T \) is the set of all shuffles.

The graded system \( \text{Har}(V) = \otimes_{n \geq 0} \text{Har}^n(V) \) has as its summand \( \text{Har}^n(V) \) the subset of all those elements \( \nu \) of \( \text{Hom}_k(\otimes^n V, V) \) which vanish on all shuffles of degree \( n \). Note that \( \text{Har}^1(V) = \text{Hom}_k(V, V) \), and that \( \text{Har}^2(V) \) consists of the symmetric bilinear maps of \( V \) into \( V \). In particular, a commutative algebra structure on \( V \) is an element \( \mu \in \text{Har}^2(V) \).

Proposition 3.1. Let \( f, \ g \in \text{Har}(V) \); then \( f \circ g \in \text{Har}(V) \).

The proof is tedious but reasonably straightforward, and is left to the reader.

The immediate consequence of Proposition 3.1 is that \( \text{Har}(V) \), with product \( \circ \) is a system in which all GLA operations of Section 2 are meaningful, since they are based on \( \circ \) only. (This does not apply to \( \cup \), which is not expressible through \( \circ \). This is one reason why \( \cup \) will not be used again.) With the above exception, the statements of Section 2 remain valid if everywhere \( \text{Hom}_k(\otimes^n V, V) \) is replaced by \( \text{Har}^n(V) \), and "Hochschild" by "Harrison".

4. The case of a Lie algebra

Let \( V \) be a vector space over a field \( k \). The graded system
\[
\{\text{Hom}_k(A^{n+1} V, V)\}_{n \geq -1}
\]
of vector spaces can be given a GLA structure through the introduction
of a composition product \(^1\): let \(f, g\) be of respective degrees \(n, m\); then \(f \circ g\), of degree \(n + m\), is given by

\[
(f \circ g)(x_0, \ldots, x_{n+m}) = \sum s g \sigma f(g(x_{\sigma(0)}), \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n)})
\]

where the summation extends over those permutations \(\sigma\) of \(\{0, \ldots, m+n\}\) for which \(\sigma(0) < \ldots < \sigma(m)\) and \(\sigma(m+1) < \ldots < \sigma(m+n)\). The composition product is not associative, in general, but satisfies, for \(h\) of degree \(p\), the commutative-associative law

\[
(f \circ g) \circ h = f \circ (g \circ h) = (-1)^{mp} \{f \circ h) \circ g - f \circ (h \circ g)\}.
\]

The validity of (4.2) is easily seen after the observation that the left side can be re-written:

\[
(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{mp} \{f \circ h) \circ g - f \circ (h \circ g)\}
\]

where \(\Sigma\) extends over those permutations \(\sigma\) with \(\sigma(0) < \ldots < \sigma(m)\) and \(\sigma(m+1) < \ldots < \sigma(m+p+1)\) and \(\sigma(m+p+2) < \ldots < \sigma(m+n+p)\).

The GLA product is now defined by

\[
[f, g] = \circ \circ g - (-1)^{mn} \circ \circ f.
\]

The Jacobi identity for \([, ]^{\circ}\) follows as sketched in section 2.

If \(V\) has the structure of a Lie algebra, then a multiplication map \(\mu \in \text{Hom}_k(V \wedge V, V)\) is given, which satisfies the Jacobi identity

\[
\mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y) = 0.
\]

This condition is equivalent to \(\mu \circ \mu = 0\), and, if \(\text{char } k \neq 2\), is also equivalent to \([\mu, \mu]^{\circ} = 0\).

The system \(\{\text{Hom}_k(A^{n+1}V, V)\}_{n \geq -1}\) is, apart from its grading, the standard cochain complex for the Chevalley-Eilenberg cohomology of the Lie algebra \(A = (V, \mu)\) with coefficients in \(A\). The Chevalley-Eilenberg coboundary map \(\delta\) is given by

\[
\delta f = -[\mu, f]^{\circ}.
\]

The product \(\mu\) gives rise to a GLA structure \([, ]^u\) on the system \(\{\text{Hom}_k(A^nV, V)\}_{n \geq 0}\) (note the change in grading!) given by

\[
[f, g]^u(x_1, \ldots, x_{n+m}) = \sum s g \sigma \mu(f(x_{\sigma(1)}), \ldots, x_{\sigma(n)}), g(x_{\sigma(n+1)}, \ldots, x_{\sigma(n+m)})
\]

\(^1\) Traditionally, the composition product here was called the hook product, and denoted \(\natural\). It seemed simpler here to use the same term and the same notation in all the three cases.
where $\Sigma$ extends over the permutations $\sigma$ for which $\sigma(1) < \ldots < \sigma(n)$ and $\sigma(n+1) < \ldots < \sigma(n+m)$.

The product $[,]^U$ is expressible in composition products by replacing $f, g, h$ in (4.3) by $\mu, f, g$, where $f \in \text{Hom}_k(A^{n+1}V, V)$ and $g \in \text{Hom}_k(m+1V, V)$:

$$[f, g]^U = (-1)^{mn+m+1} \{ (\mu \circ f) \circ g - \mu \circ (f \circ g) \}.$$  

The $\mu$'s can be eliminated by the use of $\delta$'s through (4.5):

$$[f, g]^U = (-1)^{mn+m+1} \{ (-1)^n \delta f \circ g - (-1)^{m+n} \delta (f \circ g) + (-1)^{m+n} f \delta g \}.$$  

The right side is not symmetric in $f$ and $g$, though the left side is. We therefore write the formula also the other way, with a simultaneous change in grading: if $f \in \text{Hom}_k(A^nV, V)$ and $g \in \text{Hom}_k(A^mV, V)$, then

$$(4.7) \quad [f, g]^U = \delta g \circ f - (-1)^n g \circ \delta f + (-1)^n \delta (g \circ f).$$

The material, thus far, in this section is a summary of work by Frölicher-Nijenhuis [2], Kodaira-Spencer [5] and Nickerson [6]. Portions also appear in [9, 10].

The strong similarity between the cases of associative algebras and Lie algebras is made more explicit by the following considerations, in which, temporarily, for reasons of clarity, we revert to the notation $\circ$ for the composition (hook) product in $\text{Hom}_k(AV, V)$. Define for $f \in \text{Hom}_k(\otimes^n V, V)$ the element $\mathcal{G}f \in \text{Hom}_k(A^nV, V)$ by

$$(\mathcal{G}f) (x_1, \ldots, x_n) = \sum \sigma \sigma f (x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $\Sigma$ extends over all elements $\sigma$ of the permutation group. Then we have the following, whose proof we leave to the reader.

**Proposition 4.1.** The map $\mathcal{G}: \text{Hom}_k(\otimes V, V) \rightarrow \text{Hom}_k(AV, V)$ is a homomorphism with respect to the composition products: $\mathcal{G}(f \circ g) = \mathcal{G}f \circ \mathcal{G}g$.

This implies that $\mathcal{G}$ is a homomorphism with respect to $[,]^\circ$, and if $\mu \in \text{Hom}_k(V \otimes V, V)$ is an associative algebra structure ($\mu \circ \mu = 0$), then $\nu = \mathcal{G}\mu \in \text{Hom}_k(V \wedge V, V)$ is a Lie algebra structure ($\nu \wedge \nu = 0$), and $\mathcal{G}$ is also a homomorphism for the corresponding products $[,]^\circ$. As $\mathcal{G}$ commutes with $\delta$, homomorphisms of the cohomologies and their products are induced.

5. **The GLA's of an Algebra**

In this section we deal with a graded system $E = \otimes_{n \geq 0} E^n$ of vector spaces over a field $k$. $n$ is the degree of the elements of $E^n$; $n-1$ is the reduced degree. A bilinear product $\circ$ is defined on $E$ with respect to the reduced grading, which satisfies the commutative-associative law: if $f, g, h$ have reduced degrees $n, m, p$, respectively, then

$$(5.1) \quad (f \circ g) \circ h - f \circ (g \circ h) = (-1)^{mp} \{ (f \circ h) \circ g - f \circ (h \circ g) \}.$$
The product $[,]^o$ given by

$$[f, g]^o = g \circ f - (-1)^{mn} f \circ g$$

defines a GLA structure on $E$ with respect to the reduced grading. We shall call it the comp structure. Let $\mu \in E^2$ satisfy $\mu \circ \mu = 0$. Then the operator $\delta$ defined by

$$\delta f = -[\mu, f]^o$$

gives rise to a cohomology theory $H^*(E, \delta)$. As $[\mu, ]^o$ is a derivation with respect to $[,]^o$, a comp structure is induced on $H^*(E, \delta)$.

A second GLA structure, the cup structure $[,]^u$, is given, with respect to the normal grading, for $f, g$ of degrees $n, m$ by

$$[f, g]^u = (-1)^{mn+n} \{ (\mu \circ f) \circ g - \mu \circ (f \circ g) \} = \delta g \circ f - (-1)^n g \circ \delta f + (-1)^n \delta (g \circ f).$$

The anti-commutativity of $[f, g]^u$ is immediate from (5.1). $\delta$ is a derivation with respect to the cup structure; the induced GLA structure on $H^*(E, \delta)$ is, however abelian, as is obvious from (5.4). Also, $\circ h$ is a derivation:

**Proposition 5.1.** Let $h \in E^p$. Then the mapping $i_h: E \to E$ defined by $i_h f = f \circ h$ is a derivation with respect to $[,]^u$ of degree $p-1$.

**Proof.** A straightforward application of (5.1) to (5.4).

**Proposition 5.2.** Let $\text{Der}(E)$ be the space of all derivations on $E$ with respect to $[,]^u$. Then $\text{Der}(E)$ is a GLA under formation of commutators. The map $h \to i_h$ sends $E$ into $\text{Der}(E)$, and is a homomorphism with respect to the comp structure $[,]^o$ on $E$ and the reduced grading.

**Proof.** For the first statement, see [7]. The second statement is equivalent to

$$(f \circ g) \circ h - (-1)^{(m-1)(p-1)} (f \circ h) \circ g = f \circ [h, g]^o,$$

for $g \in E^m, h \in E^p$; and this formula is a trivial re-write of (5.1) by (5.2).

Propositions 5.1 and 5.2 imply the following, cf. [7].

**Theorem 1.** The graded system $E = \bigotimes_{n \geq -1} E^n$, with $E^n = E^n \otimes E^{n+1}$, has a GLA structure (semi-direct product) $[,]^*$. If each element $(f, g)$ of $E^n$ is represented as $f + i_g$, then the product is given by

$$[f, g]^* = [f, g]^u, \quad [i_f, g]^* = g \circ f, \quad [i_f, i_g]^* = i[f, g]^u.$$

Next, we establish a third GLA structure on $E$. It is the aim of our work.
Theorem 2. The map \( g : E \to \mathcal{E} \) given by \( g f = f + (-1)^n i_M \) for \( f \in E_n \) is an injection, and its image is a subalgebra of \( \mathcal{E} \).

Proof. Let \( f \in E_n, g \in E_m \); then \([g f, g g]^* = F + (-1)^n i_M \) for some \( F, G \in E \). To show: \( G = \delta F \); so \( F + (-1)^m i_M = g F \). We have:

\[
[g f, g g]^* = [f, g]^u + (-1)^n [i_M, g]^* + (-1)^m [f, i_T] + (-1)^{n+m} i_M [i_M, i_M]^o.
\]

In what follows we use (5.4) as a rule for finding \( \delta(g \circ f) \), and observe (5.2).

\[
\delta F = \delta([f, g]^u) + (-1)^n g \circ f + (-1)^{n+m+1} f \circ g = \delta f + (-1)^n g \circ f + g \circ f \circ g
\]

A re-formulation of Theorem 2 is the following

Theorem 2'. The space \( E \) with product given by

\[
[f, g] = [f, g]^u + (-1)^n g \circ f + (-1)^{n+m+1} f \circ g
\]

for \( f \in E_n, g \in E_m, \) is a GLA.

The space \( \mathcal{E} \) is, as a vector space, the direct sum of \( qE \) and \( i_E \). The Jacobi identity can therefore be expressed with respect to this decomposition, and so can the product in \( \mathcal{E} \).

Theorem 3. Let \( f \in E_n, g \in E_m, h \in E_p \). Then

\[
[i_h, g f]^* = g(f \circ h) + (-1)^n i_{[h, f]} + [i_h, i_T]^* = i_{[h, f]}^o, \quad [\delta h, g f]^* = g[h, f].
\]

give the product in \( \mathcal{E} \), and the Jacobi identity gives in particular

\[
[f \circ h, g] + (-1)^{p-1} [f, g \circ h] - [f, g] \circ h = (-1)^{m(n-1)} f \circ [h, g] + (-1)^{n-1} g \circ [h, f].
\]

Proof. We have

\[
[i_h, g f]^* = [i_h, f] + (-1)^n [i_h, i_T]^* = f \circ h + (-1)^n i_{[h, f]}^o = g(f \circ h) + (-1)^n i_{[h, f]} + (-1)^{n+p-1} i_{[h, f]}^o.
\]

The subscript of \( i \) can be reduced by (5.4):

\[
(-1)^n [h, \delta f]^o + (-1)^{n+p} \delta(f \circ h) = (-1)^n [h, \delta f]^o + (-1)^n ([h, f]^u - \delta f \circ h + (-1)^p f \circ \delta h) = (-1)^n ([h, f]^u + (-1)^p \circ \delta h + (-1)^{n+p+1} h \circ \delta f]. = (-1)^n [h, f].
\]
This proves the first formula of (5.7). The others are obvious. The formula (5.8) is deduced by considering the Jacobi identity on \( \mathcal{E} \), thus:
\[
(-1)^{n(p-1)}[[ef, eg]^*, i_b]^* + (-1)^{mn}[[eg, i_b]^*, ef]^* + (-1)^{(p-1)m}[[i_b, ef]^*, eg]^* = 0.
\]
Computation of the \( g \)-part of the left side by repeated application of (5.7) yields (5.8). For details, see the deduction of (5.22) in [2].

If \( E^1 \) contains an (''identity''' element \( I \) with the properties \( I \sim I = f \) and \( f \sim I = nf \) for \( f \in E^n \), then (5.7) implies
\[
[i_f, gI]^* = f, \quad [gf, gI]^* = 0.
\]
Hence, bracketing with \( gI \) is a coboundary operator in \( \mathcal{E} \). While, obviously, its cohomology vanishes, this is not the case any more for subalgebras of \( \mathcal{E} \) which are stable under this coboundary map. This idea was used by Kodaira and Spencer in [5].

6. Subalgebras and cohomology

In this section we assume that the graded system \( E \) not only has the properties stated in Section 5, but also that each \( E^n \) is some subspace of \( \text{Hom}_k(\otimes^n V, V) \) for some (fixed) vector space over a field \( k \). Furthermore, we assume that \( \bar{o} \) has the following properties, all of which are obviously satisfied in the cases discussed in Sections 2, 3, 4.

Let \( \mu \) be a product on \( V \); i.e. an element of \( E^2 \), and let \( \mu \bar{o} \mu = 0 \). Then \( A = (V, \mu) \) is called an algebra. Let \( W \) be a subspace such that \( \mu | W \rightarrow W \) has values in \( W \), while \( f | W = 0 \) shall mean that the same restriction vanishes. We assume that \( \bar{o} \) satisfies these properties for all subspaces \( W \) of \( V \).

a) If \( g|W \rightarrow W \) and \( f|W \rightarrow W \), then \( g \bar{o} f|W \rightarrow W \).
b) If \( g|W = 0 \) and \( f|W \rightarrow W \), then \( g \bar{o} f|W = 0 \).
c) If \( f|W = 0 \) and \( g \) arbitrary, then \( g \bar{o} f|W = 0 \).

Let \( \mu \) be a product on \( V \); i.e. an element of \( E^2 \), and let \( \mu \bar{o} \mu = 0 \). Then \( A = (V, \mu) \) is called an algebra. Let \( W \) be a subspace such that \( \mu | W \rightarrow W \). Then \( B = (W, \mu | W) \) is a subalgebra. We denote by \( \pi : A \rightarrow A/B \) the natural projection.

The space \( E \) is also denoted \( E(A, A) \). Furthermore, we set
\[
E(B, A) = \{ f | W | f \in E \}; \quad \text{then } E(B, A) \text{ is the quotient space of } E(A, A) \text{ by the subspace } \{ f \in E | f | W = 0 \}
\]
\[
E(B, B) = \{ f | W | f | W \rightarrow W \}; \quad \text{thus } E(B, B) \text{ is a subspace of } E(B, A).
\]
\[
E(B, A/B) = \{ \pi \circ f | f \in E(B, A) \}; \quad \text{thus } E(B, A/B) \text{ is the quotient space of } E(B, A) \text{ by the subspace } E(B, B).
\]

Then in the following cases, \( \bar{o} \) induces a product, again denoted \( \bar{o} \)
The proofs of these facts are routine exercises in kernels and restrictions based on (a), (b), (c).

It follows that operations $\delta$, $[,]^U$ and also $[,]$ are induced on $E(B, B)$. The corresponding cohomology is denoted $H^*(B, B)$; the induced GLA structures are abelian, by (5.4).

The operations $\delta$ and $[,]^U$ are also induced on $E(B, A)$; the induced GLA structure on the cohomology $H^*(B, A)$ is no longer necessarily abelian, as (5.4) does not hold in $E(B, A)$, though it was used in defining $[,]^U$ on $E(B, B)$ (by forming equivalence classes!).

The coboundary operator $\delta$ carries over to $E(B, A|B)$, but not the products $[,]^U$ and $[,]$. Yet, we show below that $[,]$ induces a GLA product on $H^*(B, A|B)$.

We note that the first and last members of (5.4) are equal when $f \in E(B, B)$ and $g \in E(B, A)$.

Proposition 6.1. Let $f \in E^n(B, A|B)$ and $g \in E^m(B, A|B)$, and let $\bar{f}$ and $\bar{g}$ be representatives of $f$, $g$ in $E(B, A)$.

1. If $\delta f = \delta g = 0$, then $[\bar{f}, \bar{g}]$ is a well-defined element of $E(B, A)$, and $\delta(\pi \circ [\bar{f}, \bar{g}]) = 0$.

2. If $\varphi \in E^n(B, B)$ and $\delta g = 0$, then $\pi \circ [\varphi, \bar{g}]$ is a well-defined element of $E(B, A|A)$, and is a coboundary.

3. If $\bar{f} = \delta h$, where $h \in E^{n-1}(B, A)$, and $\delta g = 0$, then $\pi \circ [\bar{f}, \bar{g}]$ is a co­boundary in $E(B, A|B)$.

Proof. 1. The hypothesis $\delta f = 0$ is equivalent to $\delta \bar{f} \in E(B, B)$. Then, by (6.1), $\bar{g} \circ \delta \bar{f}$ is well-defined. It was already remarked that $[\bar{f}, \bar{g}]^U$ is well-defined. Thus $[\bar{f}, \bar{g}]$ is a well-defined element of $E(B, A)$. Then $\delta [\bar{f}, \bar{g}]$ is well-defined, too. By the computation in the proof of Theorem 2 we have then

$$\delta [\bar{f}, \bar{g}] = [\delta \bar{f}, \delta \bar{g}]^\pi,$$

which belongs to $E(B, B)$. Hence, $\delta(\pi \circ [\bar{f}, \bar{g}]) = \pi \circ \delta [\bar{f}, \bar{g}] = 0$.

2. By (5.4) we have

$$[\varphi, \bar{g}] = [\varphi, \bar{g}]^U + (-1)^n \bar{g} \circ \delta \varphi + (-1)^{mn+m+1} \varphi \circ \delta \bar{g} =$$

$$= \delta \bar{g} \circ \varphi + (-1)^{n} \delta(\bar{g} \circ \varphi) + (-1)^{mn+m+1} \varphi \circ \delta \bar{g},$$

hence

$$\pi \circ [\varphi, \bar{g}] = \pi \circ (\delta \bar{g} \circ \varphi) + (-1)^n \delta(\pi \circ (\bar{g} \circ \varphi)) + (-1)^{mn+m+1} \pi \circ (\varphi \circ \delta \bar{g}).$$
The middle term is a coboundary; the other two vanish, as the expressions inside the parentheses belong to \( E(B, B) \).

3. First we use that \( \delta \) is a derivation with respect to \([ , ]_U\); then we use (5.4), which is applicable for \( \delta \in E(B, A) \) and \( \delta \tilde{g} \in E(B, B) \).

\[
[\delta h, \tilde{g}] = [\delta h, \tilde{g}]^U + (-1)^n \tilde{g} \circ \delta \delta h + (-1)^{mn+m+1} \delta h \circ \delta \tilde{g} = \\
\delta [h, \tilde{g}]^U + (-1)^n [h, \delta \tilde{g}]^U + (-1)^{mn+m+1} \delta h \circ \delta \tilde{g} = \\
\delta [h, \tilde{g}]^U + (-1)^n (n-1) (m+1)^{-1} \{ -(-1)^m+1 \delta \circ \delta \tilde{g} + \\
+ (-1)^m+1 \delta (h \circ \delta \tilde{g}) \} = \delta ([h, \tilde{g}]^U - (-1)^m h \circ \delta \tilde{g}^U) = 0 \\
\delta [h, \tilde{g}]^U + (-1)^n (n-1) (m+1)^{-1} \{ -(-1)^m+1 \delta \circ \delta \tilde{g} + \\
+ (-1)^m+1 \delta (h \circ \delta \tilde{g}) \} = 0.
\]

It follows that \( \pi \circ [\delta h, \tilde{g}] \) is a coboundary in \( E(B, A/B) \).

The exact sequence of maps \( 0 \to E(B, B) \xrightarrow{\imath_*} E(B, A) \xrightarrow{\pi_*} E(B, A/B) \to 0 \) gives rise to an exact triangle of maps

\[
(6.3)
\]

\[
\begin{array}{ccc}
H^\ast(B, B) & \xrightarrow{\iota_*} & H^\ast(B, A) \\
\Downarrow \psi & & \Downarrow \pi_* \\
H^\ast(B, A/B) & &
\end{array}
\]

in the usual manner, in which \( \iota_* \) and \( \pi_* \) preserve gradings, while \( \delta_* \) raises the grading by one. \( H^\ast(B, B) \) has two GLA products, \([ , ]^O\) and \([ , ]^U\); the latter, which is abelian, is also produced by \([ , ]\). Similarly, \( H^\ast(B, A) \) has a GLA product \([ , ]^U\), which is the same as \([ , ]\). It was just shown that \( H^\ast(B, A/B) \) has a GLA structure \([ , ]\). It is clear that \( \imath_* \) and \( \pi_* \) are homomorphisms for the structures \([ , ]\). Finally, (6.2) shows that \( \delta_* \) sends the structure \([ , ]\) on \( H^\ast(B, A/B) \) into the structure \([ , ]^O\) on \( H^\ast(B, B) \). Thus we have:

**Theorem 3.** Let \( A \) be an algebra, \( B \) a subalgebra. Then \( H^\ast(B, A/B) \) has a GLA structure \([ , ]\). The maps \( \iota_* \) and \( \pi_* \) in (6.3) are homomorphisms of the structures \([ , ]\). The map \( \delta_* \) is a homomorphism between the GLA structures \([ , ]\) on \( H^\ast(B, A/B) \) and the GLA structure \([ , ]^O\) of \( H^\ast(B, B) \) with respect to the reduced grading.

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**SUMMARY**

In a preceding article [11] the author has shown the existence of a natural graded Lie algebra (GLA) structure for the cohomology of an associative, commutative-assocative and of a Lie algebra \( B \) (in the sense of Hochschild, Harrison, resp. Chevalley and Eilenberg) with coefficients in the quotient \( A/B \), where \( A \) is an algebra containing \( B \). The GLA product on the standard cochain complex of \( A \) with coefficients in \( A \), which generates this cohomology product is an unusual one, known heretofore only in a differential-geometric context [2]. The details of the extension of this structure to the types of algebras at hand, and an axiomatization of sufficient conditions for its existence, are the subject of this paper.
BIBLIOGRAPHY