Lane–Emden systems with negative exponents

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Received 15 September 2009; accepted 3 February 2010

Available online 12 February 2010

Communicated by J. Coron

Abstract

We study the elliptic system

\[
\begin{aligned}
-\Delta u &= u^{-p} v^{-q} \quad \text{in } \Omega, \\
-\Delta v &= u^{-r} v^{-s} \quad \text{in } \Omega, \\
        u &= v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) with a smooth boundary, \( p, s \geq 0 \) and \( q, r > 0 \). We investigate the existence, non-existence, and uniqueness of \( C^2(\Omega) \cap C(\overline{\Omega}) \) solutions in terms of \( p, q, r \) and \( s \). A necessary and sufficient condition for the \( C^1 \)-regularity of solutions up to the boundary is also obtained.

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Keywords: Lane–Emden equation; Elliptic system; Negative exponent; Boundary behavior

1. Introduction

In this paper we study the elliptic system

\[
\begin{aligned}
-\Delta u &= u^{-p} v^{-q}, \quad u > 0 \quad \text{in } \Omega, \\
-\Delta v &= u^{-r} v^{-s}, \quad v > 0 \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with $C^2$-boundary, $p, s \geq 0$ and $q, r > 0$. By solution of (1) we understand a pair $(u, v)$ with $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $u, v > 0$ in $\Omega$ and satisfies (1) pointwise.

The first motivation for the study of system (1) comes from the so-called Lane–Emden equation (see [5,8,15])

$$-\Delta u = u^p \quad \text{in } B_R(0), \ R > 0,$$

subject to Dirichlet boundary condition. In astrophysics, the exponent $p$ is called polytropic index and positive radially symmetric solutions of (2) are used to describe the structure of the polytropic stars (we refer the interested reader to the book by Chandrasekhar [2] for an account on the above equation as well as for various mathematical techniques to describe the behavior of the solution to Lane–Emden equation).

Systems of type (1) with $p, s \leq 0$ and $q, s < 0$ have received considerably attention in the last decade (see, e.g., [1,3,6,7,16,18–23] and the references therein). It has been shown that for such range of exponents system (1) has a rich mathematical structure. Various techniques such as moving plane method, Pohozaev-type identities, rescaling arguments have been developed and suitably adapted to deal with (1) in this case.

Recently, there has been some interest in systems of type (1) where not all the exponents are negative. In [10–12] the system (1) is considered under the hypothesis $p, r < 0 < q, s$. This corresponds to the singular Gierer–Meinhardt system arising in the molecular biology. In [9] the authors provide a nice sub and supersolution device that applies to general systems both in cooperative and non-cooperative setting. This method was then used to discuss singular counterpart of some well-known models such as Gierer–Meinhardt, Lotka–Voltera or predator-prey systems.

In this paper, we shall be concerned with system (1) in case $p, s \geq 0$ and $q, r > 0$. This corresponds to the prototype equation (2) in which the polytropic index $p$ is negative. For such range of exponents, the above mentioned methods do not apply; another difficulties in dealing with system (1) come from the non-cooperative character of our system and from the lack of a variational structure. In turn, our approach relies on the boundary behavior of solutions to (2) (with $p < 0$) or more generally, to singular elliptic problems of the type

$$\begin{cases}
-\Delta u = k(\delta(x))u^{-p}, & u > 0 \quad \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where

$$\delta(x) = \text{dist}(x, \partial \Omega), \ x \in \overline{\Omega},$$

and $k : (0, \infty) \to (0, \infty)$ is a decreasing function such that $\lim_{t \to 0} k(t) = \infty$.

The approach we adopt in this paper can be used to study more general systems in the form

$$\begin{cases}
-\mathcal{L}u = f(x, u, v), & u > 0 \quad \text{in } \Omega, \\
-\mathcal{L}v = g(x, u, v), & v > 0 \quad \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\mathcal{L}$ is a second order differential operator not necessarily in divergence form and

$$f(x, u, v) = k_1(x)u^{-p}v^{-q}, \quad g(x, u, v) = k_2(x)u^{-r}v^{-s},$$
or
\[
f(x, u, v) = k_{11}(x)u^{-p} + k_{12}(x)v^{-q}, \quad g(x, u, v) = k_{21}(x)u^{-r} + k_{22}(x)v^{-s},
\]

with \( k_i, k_{ij} : \Omega \to (0, \infty) \) \((i, j = 1, 2)\) continuous functions that behave like
\[
\delta(x)^{-q} \log^b \left( \frac{A}{\delta(x)} \right) \quad \text{near} \ \partial \Omega,
\]

for some \( A, a > 0 \) and \( b \in \mathbb{R} \).

Our first result concerning the study of (1) is the following.

**Theorem 1.1** (Non-existence). Let \( p, s \geq 0, q, r > 0 \) be such that one of the following conditions holds:

(i) \( r \min\{1, \frac{2-q}{1+p}\} \geq 2 \);
(ii) \( q \min\{1, \frac{2-r}{1+s}\} \geq 2 \);
(iii) \( p > \max\{1, r-1\}, 2r > (1-s)(1+p) \) and \( q(1+p-r) > (1+p)(1+s) \);
(iv) \( s > \max\{1, q-1\}, 2q > (1-p)(1+s) \) and \( r(1+s-q) > (1+p)(1+s) \).

Then the system (1) has no solutions.

Remark that condition (i) in Theorem 1.1 restricts the range of the exponent \( q \) to the interval \((0, 2)\) while in (iii) the exponent \( q \) can take any value greater than 2, provided we adjust the other three exponents \( p, r, s \) accordingly. The same remark applies for the exponent \( r \) from the above conditions (ii) and (iv).

The existence of solutions to (1) is obtained under the following assumption on the exponents \( p, q, r, s \):

\[
(1+p)(1+s) - qr > 0.
\]

We also introduce the quantities
\[
\alpha = p + q \min\left\{1, \frac{2-r}{1+s}\right\}, \quad \beta = r + s \min\left\{1, \frac{2-q}{1+p}\right\}.
\]

The above values of \( \alpha \) and \( \beta \) are related to the boundary behavior of the solution to the singular elliptic problem (3) as explained in Proposition 2.6 below. Our existence result is as follows.

**Theorem 1.2** (Existence). Let \( p, s \geq 0, q, r > 0 \) satisfy (5) and one of the following conditions:

(i) \( \alpha \leq 1 \) and \( r < 2 \);
(ii) \( \beta \leq 1 \) and \( q < 2 \);
(iii) \( p, s \geq 1 \) and \( q, r < 2 \).

Then, the system (1) has at least one solution.
The proof of the existence is based on the Schauder’s fixed point theorem in a suitable chosen closed convex subset of $C(\bar{\Omega}) \times C(\bar{\Omega})$ that contains all the functions having a certain rate of decay expressed in terms of the distance function $\delta(x)$ up to the boundary of $\Omega$.

From Theorem 1.1(i)–(ii) and Theorem 1.2(i)–(ii) we have the following necessary and sufficient conditions for the existence of solutions to (1):

**Corollary 1.3.** Let $p, s \geq 0, q, r > 0$ satisfy (5).

(i) Assume $p + q \leq 1$. Then system (1) has solutions if and only if $r < 2$;
(ii) Assume $r + s \leq 1$. Then system (1) has solutions if and only if $q < 2$.

A particular feature of system (1) is that it does no posses $C^2(\bar{\Omega})$ solutions. Indeed, due to the fact that $q, r < 0$ and to the homogeneous Dirichlet boundary condition imposed on $u$ and $v$ we have that $u^{-p}v^{-q}$ and $u^{-r}v^{-s}$ are unbounded around $\partial \Omega$, so there are no $C^2(\bar{\Omega})$ solutions of (1). In turn, $C^2(\Omega) \cap C^1(\bar{\Omega})$ may exist and our next result provides necessary and sufficient conditions in terms of $p, q, r$ and $s$ for the existence of such solutions.

**Theorem 1.4** ($C^1$-regularity). Let $p, s \geq 0, q, r > 0$ satisfy (5). Then:

(i) System (1) has a solution $(u, v)$ with $u \in C^1(\bar{\Omega})$ if and only if $\alpha < 1$ and $r < 2$;
(ii) System (1) has a solution $(u, v)$ with $v \in C^1(\bar{\Omega})$ if and only if $\beta < 1$ and $q < 2$;
(iii) System (1) has a solution $(u, v)$ with $u, v \in C^1(\bar{\Omega})$ if and only if $p + q < 1$ and $r + s < 1$.

Another feature of system (1) is that under some conditions on $p, q, r, s$ it has a unique solution (see Theorem 1.5 below). This is a striking difference between our setting and the case $p, s \leq 0$ and $q, r < 0$ largely investigated in the literature so far, where the uniqueness does not seem to occur. In our framework, the uniqueness is achieved from the boundary behavior of solution to (1) deduced from the study of the prototype model (3).

**Theorem 1.5** (Uniqueness). Let $p, s \geq 0, q, r > 0$ satisfy (5) and one of the following conditions:

(i) $p + q < 1$ and $r < 2$;
(ii) $r + s < 1$ and $q < 2$.

Then, the system (1) has a unique solution.

The rest of the paper is organized as follows. In Section 2 we obtain some useful properties related to the boundary behavior of the solution to (3). Sections 3–6 are devoted to the proofs of the above results.

2. Preliminary results

In this section we collect some old and new results concerning problems of type (3). Note that the method of sub and supersolutions is also valid in the singular framework as explained in [13, Theorem 1.2.3]. Our first result is a straightforward comparison principle between subsolutions and supersolutions for singular elliptic equations.
Proposition 2.1. Let \( p \geq 0 \) and \( \phi : \Omega \to (0, \infty) \) be a continuous function. If \( u \) is a subsolution and \( \bar{u} \) is a supersolution of

\[
\begin{aligned}
-\Delta u &= \phi(x)u^{-p}, \quad u > 0 \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

then \( u \leq \bar{u} \) in \( \Omega \).

**Proof.** If \( p = 0 \) the result follows directly from the maximum principle. Let now \( p > 0 \). Assume by contradiction that the set \( \omega := \{ x \in \Omega : \bar{u}(x) < u(x) \} \) is not empty and let \( w := u - \bar{u} \). Then, \( w \) achieves its maximum on \( \Omega \) at a point that belongs to \( \omega \). At that point, say \( x_0 \), we have

\[
0 \leq -\Delta w(x_0) \leq \phi(x_0)[u(x_0)^{-p} - \bar{u}(x_0)^{-p}] < 0,
\]

which is a contradiction. Therefore, \( \omega = \emptyset \), that is, \( u \leq \bar{u} \) in \( \Omega \). \( \square \)

Proposition 2.2. Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be such that \( u = 0 \) on \( \partial \Omega \) and

\[
0 \leq -\Delta u \leq c\delta(x)^{-a} \quad \text{in } \Omega,
\]

where \( 0 < a < 2 \) and \( c > 0 \). Then, \( u \in C^{0,\gamma}(\overline{\Omega}) \) for some \( 0 < \gamma < 1 \). Furthermore, if \( 0 < a < 1 \), then \( u \in C^{1,1-a}(\overline{\Omega}) \).

**Proof.** Let \( G \) denote Green’s function for the negative Laplace operator. Thus, for all \( x \in \Omega \) we have

\[
u(x) = -\int_\Omega G(x, y) \Delta u(y) \, dy.
\]

Let \( x_1, x_2 \in \Omega \). Then

\[
|u(x_1) - u(x_2)| \leq -\int_\Omega |G(x_1, y) - G(x_2, y)| \Delta u(y) \, dy
\]

\[
\leq c \int_\Omega |G(x_1, y) - G(x_2, y)| \delta(y)^{-a} \, dy.
\]

Next, using the method in [14, Theorem 1.1] we have

\[
|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\gamma \quad \text{for some } 0 < \gamma < 1.
\]

Hence \( u \in C^{0,\gamma}(\overline{\Omega}) \). Assume now \( 0 < a < 1 \). Then,

\[
\nabla u(x) = -\int_\Omega G_x(x, y) \Delta u(y) \, dy \quad \text{for all } x \in \Omega,
\]
and

\[ |∇u(x_1) - ∇u(x_2)| \leq -\int_{Ω} |G_x(x_1, y) - G_x(x_2, y)| Δu(y) \, dy \]

\[ \leq c \int_{Ω} |G_x(x_1, y) - G_x(x_2, y)| \delta(y)^{-a} \, dy. \]

The same technique as in [14, Theorem 1.1] yields

\[ |∇u(x_1) - ∇u(x_2)| \leq C|x_1 - x_2|^{1-a} \quad \text{for all } x_1, x_2 ∈ Ω. \]

Therefore \( u ∈ C^{1,1-a}(Ω) \).

**Proposition 2.3.** Let \((u, v)\) be a solution of system (1). Then, there exists a constant \( c > 0 \) such that

\[ u(x) ≥ c\delta(x) \quad \text{and} \quad v(x) ≥ c\delta(x) \quad \text{in } Ω. \] (6)

**Proof.** Let \( w \) be the solution of

\[
\begin{cases}
-Δw = 1, & \text{in } Ω, \\
w = 0 & \text{on } ∂Ω.
\end{cases}
\] (7)

Using the smoothness of \( ∂Ω \), we have \( w ∈ C^2(Ω) \) and by Hopf’s boundary point lemma (see [17]), there exists \( c_0 > 0 \) such that \( w(x) ≥ c_0\delta(x) \) in \( Ω \). Since \( -Δu ≥ C = -Δ(Cw) \) in \( Ω \), for some constant \( C > 0 \), by standard maximum principle we deduce \( u(x) ≥ Cw(x) ≥ c\delta(x) \) in \( Ω \) and similarly \( v(x) ≥ c\delta(x) \) in \( Ω \), where \( c > 0 \) is a positive constant. \( \square \)

Let \((λ_1, ϕ_1)\) be the first eigenvalue/eigenfunction of \(-Δ\) in \( Ω \). It is well known that \( λ_1 > 0 \) and \( ϕ_1 ∈ C^2(Ω) \) has constant sign in \( Ω \). Further, using the smoothness of \( Ω \) and normalizing \( ϕ_1 \) with a suitable constant, we can assume

\[ c_0\delta(x) ≤ ϕ_1(x) ≤ δ(x) \quad \text{in } Ω, \] (8)

for some \( 0 < c_0 < 1 \). By Hopf’s boundary point lemma we have \( \frac{∂ϕ_1}{∂n} < 0 \) on \( ∂Ω \), where \( n \) is the outer unit normal vector at \( ∂Ω \). Hence, there exists \( ω ⊂ Ω \) and \( c > 0 \) such that

\[ |∇ϕ_1| > c \quad \text{in } Ω \setminus ω. \] (9)

**Theorem 2.4.** Let \( p ≥ 0, A > \text{diam}(Ω) \) and \( k : (0, A) → (0, ∞) \) be a decreasing function such that

\[ \int_{0}^{A} tk(t) \, dt = ∞. \]
Then, the inequality

\[
\begin{aligned}
-\Delta u &\geq k(\delta(x))u^{-p}, & u > 0 & \text{in } \Omega, \\
u = 0 & & \text{on } \partial\Omega,
\end{aligned}
\]

has no solutions \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \).

**Proof.** Suppose by contradiction that there exists a solution \( u_0 \) of (10). For any 

\[ 0 < \varepsilon < A - \text{diam}(\Omega) \]

we consider the perturbed problem

\[
\begin{aligned}
-\Delta u &= k(\delta(x) + \varepsilon)(u + \varepsilon)^{-p}, & u > 0 & \text{in } \Omega, \\
u &= 0 & & \text{on } \partial\Omega.
\end{aligned}
\]

Then, \( \overline{u} = u_0 \) is a supersolution of (11). Also, if \( w \) is the solution of problem (7) it is easy to see that \( u = cw \) is a subsolution of (11) provided \( c > 0 \) is small enough. Further, by Proposition 2.1 it follows that \( u \leq \overline{u} \) in \( \Omega \). Thus, by the sub and supersolution method we deduce that problem (11) has a solution \( u_\varepsilon \in C^2(\overline{\Omega}) \) such that

\[
cw \leq u_\varepsilon \leq u_0 \quad \text{in } \Omega.
\]

Multiplying with \( \varphi_1 \) in (11) and then integrating over \( \Omega \) we find

\[
\lambda_1 \int_{\Omega} u_\varepsilon \varphi_1 \, dx = \int_{\Omega} k(\delta(x) + \varepsilon)(u_\varepsilon + \varepsilon)^{-p} \varphi_1 \, dx.
\]

Using (12) we obtain

\[
M := \lambda_1 \int_{\Omega} u_0 \varphi_1 \, dx \geq \lambda_1 \int_{\omega} u_\varepsilon \varphi_1 \, dx \geq \int_{\omega} k(\delta(x) + \varepsilon)(u_0 + \varepsilon)^{-p} \varphi_1 \, dx,
\]

for all \( \omega \subset \Omega \). Passing to the limit with \( \varepsilon \to 0 \) in the above inequality and using (8) we find

\[
M \geq \int_{\omega} k(\delta(x)) u_0^{-p} \varphi_1 \, dx \geq c_0 \|u_0\|_{-p}^{-p} \int_{\omega} k(\delta(x)) \delta(x) \, dx.
\]

Since \( \omega \subset \Omega \) was arbitrary, we deduce

\[
\int_{\Omega} k(\delta(x)) \delta(x) \, dx < \infty.
\]

Using the smoothness of \( \partial\Omega \), the above condition yields \( A t k(t) \, dt < \infty \), which contradicts our assumption on \( k \). Hence, (10) has no solutions. \( \Box \)
A direct consequence of Theorem 2.4 is the following result.

**Corollary 2.5.** Let \( p \geq 0 \) and \( q \geq 2 \). Then, there are no functions \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) such that

\[
\begin{cases}
-\Delta u \geq \delta(x)^{-q}u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

**Proposition 2.6.** Let \( p \geq 0 \) and \( 0 < q < 2 \). There exists \( c > 0 \) and \( A > \text{diam}(\Omega) \) such that any supersolution \( \bar{u} \) of

\[
-\Delta u = \delta(x)^{-q}u^{-p}, \quad u > 0 \text{ in } \Omega,
\]

satisfies:

(i) \( \bar{u}(x) \geq c\delta(x) \) in \( \Omega \), if \( p + q < 1 \);

(ii) \( \bar{u}(x) \geq c\delta(x) \log \frac{1}{\delta(x)} (\frac{A}{\delta(x)}) \) in \( \Omega \) if \( p + q = 1 \);

(iii) \( \bar{u}(x) \geq c\delta(x)^{\frac{2-q}{1+q}} \) in \( \Omega \), if \( p + q > 1 \).

A similar result holds for subsolutions of \( (13) \).

**Proof.** If \( p > 0 \) then the result follows from Theorem 3.5 in [4] (see also [13, Section 9]). If \( p = 0 \) we proceed as in [4, Theorem 3.5], namely, for \( m > 0 \) we show that the function

\[
u(x) = \begin{cases}
m\Phi_1(x) & \text{if } q < 1, \\
m\Phi_1(x) \log \left( \frac{A}{\Phi_1(x)} \right) & \text{if } q = 1, \ A > \text{diam}(\Omega), \\
m\Phi_1(x)^{2-q} & \text{if } q > 1,
\end{cases}
\]

satisfies \( -\Delta \nu \leq \delta(x)^{-q} \) in \( \Omega \). Thus, the estimates in Proposition 2.6 follow from (8) and the maximum principle. \( \Box \)

**Theorem 2.7.** Let \( 0 < a < 1 \), \( A > \text{diam}(\Omega) \), \( p \geq 0 \) and \( q > 0 \) be such that \( p + q = 1 \). Then, the problem

\[
\begin{cases}
-\Delta u = \delta(x)^{-q}A(\delta(x))^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

has a unique solution \( u \) which satisfies

\[
c_1\delta(x)^{\frac{1-q}{1+q}} \left( \frac{A}{\delta(x)} \right) \leq u(x) \leq c_2\delta(x)^{\frac{1-q}{1+q}} \left( \frac{A}{\delta(x)} \right) \text{ in } \Omega,
\]

for some \( c_1, c_2 > 0 \).
Proof. Let
\[ w(x) = \varphi_1(x) \log^b \left( \frac{A}{\varphi_1(x)} \right), \quad x \in \Omega, \]
where \( b = \frac{1-a}{1+p} \in (0, 1) \). A straightforward computation yields
\[
-\Delta w = \lambda_1 \varphi_1 \log^b \left( \frac{A}{\varphi_1(x)} \right) + b(1-b) |\nabla \varphi_1|^2 \varphi_1^{-1} \log^{b-2} \left( \frac{A}{\varphi_1(x)} \right) \quad \text{in} \ \Omega.
\]
Using (9) we can find \( C_1, C_2 > 0 \) such that
\[
C_1 \varphi_1^{-1} \log^{b-1} \left( \frac{A}{\varphi_1(x)} \right) \leq -\Delta w \leq C_2 \varphi_1^{-1} \log^{b-1} \left( \frac{A}{\varphi_1(x)} \right) \quad \text{in} \ \Omega,
\]
that is,
\[
C_1 \varphi_1^{-q} \log^{a} \left( \frac{A}{\varphi_1(x)} \right) w^{-p} \leq -\Delta w \leq C_2 \varphi_1^{-q} \log^{a} \left( \frac{A}{\varphi_1(x)} \right) w^{-p} \quad \text{in} \ \Omega.
\]
We now deduce that \( u = mw \) and \( \bar{u} = Mw \) are respectively subsolution and supersolution of (14) for suitable \( 0 < m < 1 < M \). Hence, the problem (14) has a solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) such that
\[
m \varphi_1^{\frac{1-a}{1+p}} \left( \frac{A}{\varphi_1(x)} \right) \leq u \leq M \log^{\frac{1-a}{1+p}} \left( \frac{A}{\varphi_1(x)} \right) \quad \text{in} \ \Omega. \tag{16}
\]
The uniqueness follows from Proposition 2.1 while the boundary behavior of \( u \) follows from (16) and (8). This finishes the proof. \( \square \)

**Corollary 2.8.** Let \( C > 0 \) and \( a, A, p, q \) be as in Theorem 2.7. Then, there exists \( c > 0 \) such that any solution \( u \) of
\[
-\Delta u \geq C \delta(x)^{-q} \log^{-a} \left( \frac{A}{\delta(x)} \right) u^{-p}, \quad u > 0 \quad \text{in} \ \Omega,
\]
\[
u = 0 \quad \text{on} \ \partial\Omega,
\]
satisfies
\[
u(x) \geq c \delta(x)^{\frac{1-a}{1+p}} \left( \frac{A}{\delta(x)} \right) \quad \text{in} \ \Omega.
\]
Proposition 2.9. Let $A > 3 \text{diam}(\Omega)$ and $C > 0$. There exists $c > 0$ such that any solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$
\begin{cases}
-\Delta u \geq C\delta^{-1}(x) \log^{-1}\left(\frac{A}{\delta(x)}\right), & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

satisfies

$$
u(x) \geq c\delta(x) \log\left[\log\left(\frac{A}{\delta(x)}\right)\right] \text{ in } \Omega. \quad (17)$$

Proof. Let

$$w(x) = \varphi_1(x) \log\left[\log\left(\frac{A}{\varphi_1(x)}\right)\right], \quad x \in \Omega.$$

An easy computation yields

$$-\Delta w = \lambda_1 \varphi_1 \log\left[\log\left(\frac{A}{\varphi_1(x)}\right)\right] + \frac{1}{\varphi_1(x)} \frac{1}{\log\left(\frac{A}{\varphi_1(x)}\right)} \left|\nabla \varphi_1\right|^2 - \frac{1}{\varphi_1^2(x)} \frac{1}{\log\left(\frac{A}{\varphi_1(x)}\right)} \right|\nabla \varphi_1\right|^2 \leq \frac{c_0}{\varphi_1(x)} \log\left(\frac{A}{\varphi_1(x)}\right) \text{ in } \Omega,$$

for some $c_0 > 0$. Using (8) we can find $m > 0$ small enough such that

$$-\Delta mw \leq \frac{C}{\delta(x) \log\left(\frac{A}{\delta(x)}\right)} \text{ in } \Omega.$$

Now by maximum principle we deduce $u \geq mw$ in $\Omega$ and by (8) we obtain that $u$ satisfies the estimate (17). □

Theorem 2.10. Let $p \geq 0$, $A > \text{diam}(\Omega)$ and $a \in \mathbb{R}$. Then, problem

$$
\begin{cases}
-\Delta u = \delta(x)^{-2} \log^{-a}\left(\frac{A}{\delta(x)}\right) u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

has solutions if and only if $a > 1$. Furthermore, if $a > 1$ then (21) has a unique solution $u$ and there exist $c_1, c_2 > 0$ such that

$$c_1 \log^{1-p} \left(\frac{A}{\delta(x)}\right) \leq u(x) \leq c_2 \log^{1-p} \left(\frac{A}{\delta(x)}\right) \text{ in } \Omega. \quad (19)$$
Proof. Fix $B > A$ be such that the function $k : (0, B) \to \mathbb{R}$, $k(t) = t^{-2} \log^{-a}(\frac{B}{t})$ is decreasing on $(0, A)$. Then, any solution $u$ of (18) satisfies
\[
\begin{cases}
-\Delta u \geq c k(\delta(x)) u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{ on } \partial \Omega,
\end{cases}
\]
where $c > 0$. By virtue of Theorem 2.4 we deduce $\int_0^A k(t) \, dt < \infty$, that is, $a > 1$.

For $a > 1$, let
\[w(x) = \log^b \left( \frac{B}{\varphi_1(x)} \right), \quad x \in \Omega,\]
where $b = \frac{1-a}{1+p} < 0$. It is easy to see that
\[
-\Delta w = -b(1) \left| \nabla \varphi_1 \right|^2 + \lambda_1 \varphi_1^2 \log^{b-1} \left( \frac{B}{\varphi_1(x)} \right)
- b(b-1) \left| \nabla \varphi_1 \right|^2 \varphi_1^{-2} \log^{b-2} \left( \frac{B}{\varphi_1(x)} \right) \quad \text{in } \Omega.
\]
Choosing $B > 0$ large enough, we may assume
\[
\log \left( \frac{B}{\varphi_1(x)} \right) \geq 2(1-b) \quad \text{in } \Omega. \tag{20}
\]
Therefore, from (9) and (20) there exist $C_1, C_2 > 0$ such that
\[
C_1 \varphi_1^{-2} \log^{b-1} \left( \frac{B}{\varphi_1(x)} \right) \leq -\Delta w \leq C_2 \varphi_1^{-2} \log^{b-1} \left( \frac{B}{\varphi_1(x)} \right) \quad \text{in } \Omega,
\]
that is,
\[
C_1 \varphi_1^{-2} \log^{-a} \left( \frac{B}{\varphi_1(x)} \right) w^{-p} \leq -\Delta w \leq C_2 \varphi_1^{-2} \log^{-a} \left( \frac{B}{\varphi_1(x)} \right) w^{-p} \quad \text{in } \Omega.
\]
As before, from (8) it follows that $\underline{u} = mw$ and $\overline{u} = Mw$ are respectively subsolution and supersolution of (18) provided $m > 0$ is small and $M > 1$ is large enough. The rest of the proof is the same as for Theorem 2.7. $\square$

Corollary 2.11. Let $C > 0$, $p \geq 0$, $A > \text{diam}(\Omega)$ and $a > 1$. Then, there exists $c > 0$ such that any solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of
\[
\begin{cases}
-\Delta u \geq C \delta(x)^{-2} \log^{-a} \left( \frac{A}{\varphi_1(x)} \right) u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{21}
\]
satisfies
\[u(x) \geq c \log^{\frac{1-a}{1+p}} \left( \frac{A}{\delta(x)} \right) \quad \text{in } \Omega.
\]
3. Proof of Theorem 1.1

Since the system (1) is invariant under the transform \((u, v, p, q, r, s) \rightarrow (v, u, s, r, q, p)\), we only need to prove (i) and (iii).

(i) Assume that there exists \((u, v)\) a solution of system (1). Note that from (i) we have \(0 < q < 2\). Also, using Proposition 2.3, we can find \(c > 0\) such that (6) holds.

Case 1: \(p + q < 1\). From our hypothesis (i) we deduce \(r \geq 2\). Using the estimates (6) in the first equation of the system (1) we find

\[
\begin{align*}
-\Delta u & \leq c_1 \delta(x)^{-q} u^{-p}, & u > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

for some \(c_1 > 0\). From Proposition 2.6(i) we now deduce \(u(x) \leq c_2 \delta(x)\) in \(\Omega\), for some \(c_2 > 0\). Using this last estimate in the second equation of (1) we find

\[
\begin{align*}
-\Delta v & \geq c_3 \delta(x)^{-r} v^{-s}, & v > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

where \(c_3 > 0\). According to Corollary 2.5, this is impossible, since \(r \geq 2\).

Case 2: \(p + q > 1\). From hypothesis (i) we also have \(r \geq \frac{2-q}{1+p} \geq 2\). In the same manner as above, \(u\) satisfies (22). Thus, by Proposition 2.6(iii), there exists \(c_4 > 0\) such that

\[
\begin{align*}
-\Delta u & \leq c_4 \delta(x)^{-q} u^{-p}, & u > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

for some \(c_4 > 0\). Using Proposition 2.6(ii), there exists \(c_5 > 0\) such that

\[
\begin{align*}
-\Delta v & \geq c_5 \delta(x)^{-r} v^{-s}, & v > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

for some \(c_5 > 0\), which is impossible in view of Corollary 2.5, since \(r \geq \frac{2-q}{1+p} \geq 2\).

Case 3: \(p + q = 1\). From (i) it follows that \(r \geq 2\). As in the previous two cases, we easily find that \(u\) is a solution of (22), for some \(c_1 > 0\). Using Proposition 2.6(ii), there exists \(c_6 > 0\) such that

\[
\begin{align*}
-\Delta v & \geq c_6 \delta(x)^{-r} \log \left( \frac{A}{\delta(x)} \right) u^{-s}, & v > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

for some \(A > 3 \text{diam}(\Omega)\). Using this estimate in the second equation of (1) we obtain

\[
\begin{align*}
-\Delta v & \geq c_7 \delta(x)^{-r} \log \left( \frac{A}{\delta(x)} \right) v^{-s}, & v > 0 & \text{in } \Omega, \\
u & = 0 & & \text{on } \partial \Omega, \\
\end{align*}
\]

where \(c_7\) is a positive constant. From Theorem 2.4 it follows that
\[ \int_0^1 t^{1-r} \log^{-\frac{r}{1+p}} \left( \frac{A}{t} \right) dt < \infty. \]

Since \( r \geq 2 \), the above integral condition implies \( r = 2 \). Now, using (24) (with \( r = 2 \)) and Corollary 2.11, there exists \( c_8 > 0 \) such that

\[ v(x) \geq c_8 \log^\frac{p-1}{1+(p+1)} \left( \frac{A}{\delta(x)} \right) \quad \text{in } \Omega. \]  

(25)

Using the estimate (25) in the first equation of system (1) we deduce

\[ \begin{cases} -\Delta u \leq c_9 \log^\frac{a(1-p)}{(1+p)(1+s)} \left( \frac{A}{\delta(x)} \right) u^{-p}, & u > 0 \quad \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega, \end{cases} \]  

(26)

for some \( c_9 > 0 \). Fix \( 0 < a < 1 - p \). Then, from (26) we can find a constant \( c_{10} > 0 \) such that \( u \) satisfies

\[ \begin{cases} -\Delta u \leq c_{10} \delta(x)^{-a} u^{-p}, & u > 0 \quad \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega. \end{cases} \]

By Proposition 2.6(i) (since \( a + p < 1 \)) we derive \( u(x) \leq c_{11} \delta(x) \) in \( \Omega \), where \( c_{11} > 0 \). Using this last estimate in the second equation of (1) we finally obtain (note that \( r = 2 \)):

\[ \begin{cases} -\Delta v \geq c_{12} \delta(x)^{-2} v^{-s}, & v > 0 \quad \text{in } \Omega, \\
 v = 0 & \text{on } \partial \Omega, \end{cases} \]

which is impossible according to Corollary 2.5. Therefore, the system (1) has no solutions.

(iii) Suppose that the system (1) has a solution \((u, v)\) and let \( M = \max_{x \in \overline{\Omega}} v \). From the first equation of (1) we have

\[ \begin{cases} -\Delta u \geq c_1 u^{-p}, & u > 0 \quad \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega, \end{cases} \]

where \( c_1 = M^{-q} > 0 \). Using Proposition 2.6(iii) there exists \( c_2 > 0 \) such that \( u(x) \geq c_2 \delta(x)^{\frac{2}{1+p}} \) in \( \Omega \). Combining this estimate with the second equation of (1) we find

\[ \begin{cases} -\Delta v \leq c_3 \delta(x)^{- \frac{2r}{1+p}} v^{-s}, & v > 0 \quad \text{in } \Omega, \\
 v = 0 & \text{on } \partial \Omega. \end{cases} \]

Since \( \frac{2r}{1+p} + s > 1 \), again by Proposition 2.6(iii) we obtain that the function \( v \) satisfies

\[ v(x) \leq c_4 \delta(x)^{\frac{2(1+p-r)}{(1+p)(1+s)}} \quad \text{in } \Omega, \]

for some \( c_4 > 0 \). Using the above estimate in the first equation of (1) we find \( c_5 > 0 \) such that
\[
\begin{aligned}
\begin{cases}
-\Delta u \geq c_5 \delta(x) \frac{2q(1+p-r)}{(1+p)(1+s)} u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
which contradicts Corollary 2.5 since \( q(1 + p - r) > (1 + p)(1 + s) \). Thus, the system (1) has no solutions. This ends the proof of Theorem 1.1.

4. Proof of Theorem 1.2

(i) We divide the proof into six cases according to the boundary behavior of singular elliptic problems of type (3), as described in Proposition 2.6.

Case 1: \( r + s > 1 \) and \( \alpha = p + \frac{q(2-r)}{1+s} < 1 \). By Proposition 2.6(i) and (iii) there exist \( 0 < c_1 < c_2 < 1 \) such that:

- Any subsolution \( u \) and any supersolution \( \bar{u} \) of the problem

\[
\begin{cases}
-\Delta u = \delta(x) \frac{q(2-r)}{1+s} u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(27)

satisfy

\[
\bar{u}(x) \geq c_1 \delta(x) \quad \text{and} \quad \underline{u}(x) \leq c_2 \delta(x) \quad \text{in } \Omega.
\]

(28)

- Any subsolution \( v \) and any supersolution \( \bar{v} \) of the problem

\[
\begin{cases}
-\Delta v = \delta(x)^{-r} v^{-s}, & v > 0 \text{ in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(29)

satisfy

\[
\bar{v}(x) \geq c_1 \delta(x)^{\frac{2-r}{1+s}} \quad \text{and} \quad \underline{v}(x) \leq c_2 \delta(x)^{\frac{2-r}{1+s}} \quad \text{in } \Omega.
\]

(30)

We fix \( 0 < m_1 < 1 < M_1 \) and \( 0 < m_2 < 1 < M_2 \) such that

\[
M_1^{\frac{r}{1+s}} m_2 \leq c_1 < c_2 \leq M_1 m_2^{\frac{q}{1+p}}.
\]

(31)

and

\[
M_2^{\frac{q}{1+p}} m_1 \leq c_1 < c_2 \leq M_2 m_1^{\frac{r}{1+s}}.
\]

(32)

Note that the above choice of \( m_i, M_i \) \( (i = 1, 2) \) is possible in view of (5). Set

\[
\mathcal{A} = \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \begin{array}{l}
m_1 \delta(x) \leq u(x) \leq M_1 \delta(x) \quad \text{in } \Omega, \\
m_2 \delta(x)^{\frac{2-r}{1+s}} \leq v(x) \leq M_2 \delta(x)^{\frac{2-r}{1+s}} \quad \text{in } \Omega
\end{array} \right\}.
\]

For any \( (u, v) \in \mathcal{A} \), we consider \((Tu, Tv)\) the unique solution of the decoupled system
\[
\begin{aligned}
- \Delta (T u) &= v^{-q} (T u)^{-p}, \quad T u > 0 \quad \text{in } \Omega, \\
- \Delta (T v) &= u^{-r} (T v)^{-s}, \quad T v > 0 \quad \text{in } \Omega, \\
T u &= T v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(33)

and define

\[
\mathcal{F} : \mathcal{A} \rightarrow C(\overline{\Omega}) \times C(\overline{\Omega}) \quad \text{by } \mathcal{F}(u, v) = (T u, T v) \text{ for any } (u, v) \in \mathcal{A}.
\]  

(34)

Thus, the existence of a solution to system (1) follows once we prove that \( \mathcal{F} \) has a fixed point in \( \mathcal{A} \). To this aim, we shall prove that \( \mathcal{F} \) satisfies the conditions:

\( \mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}, \quad \mathcal{F} \) is compact and continuous.

Then, by Schauder’s fixed point theorem we deduce that \( \mathcal{F} \) has a fixed point in \( \mathcal{A} \), which, by standard elliptic estimates, is a classical solution to (1).

**Step 1:** \( \mathcal{F}(\mathcal{A}) \subseteq \mathcal{A} \).

Let \( (u, v) \in \mathcal{A} \). From

\[
v(x) \leq M_2 \delta(x)^{\frac{2-r}{1+p}} \quad \text{in } \Omega,
\]

it follows that \( T u \) satisfies

\[
\begin{aligned}
- \Delta (T u) &\geq M_2^{-q} \delta(x)^{-\frac{q(2-r)}{1+p}} (T u)^{-p}, \quad T u > 0 \quad \text{in } \Omega, \\
T u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Thus, \( \bar{u} := M_2^{\frac{q}{1+p}} T u \) is a supersolution of (27). By (28) and (32) we obtain

\[
T u = M_2^{-\frac{q}{1+p}} \bar{u} \geq c_1 M_2^{-\frac{q}{1+p}} \delta(x) \geq m_1 \delta(x) \quad \text{in } \Omega.
\]

From \( v(x) \geq m_2 \delta(x)^{\frac{2-r}{1+p}} \) in \( \Omega \) and the definition of \( T u \) we deduce that

\[
\begin{aligned}
- \Delta (T u) &\leq m_2^{-q} \delta(x)^{-\frac{q(2-r)}{1+p}} (T u)^{-p}, \quad T u > 0 \quad \text{in } \Omega, \\
T u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Thus, \( u := m_2^{\frac{q}{1+p}} T u \) is a subsolution of problem (27). Hence, from (28) and (31) we obtain

\[
T u = m_2^{-\frac{q}{1+p}} u \leq c_2 m_2^{-\frac{q}{1+p}} \delta(x) \leq M_1 \delta(x) \quad \text{in } \Omega.
\]

We have proved that \( T u \) satisfies

\[
m_1 \delta(x) \leq T u \leq M_1 \delta(x) \quad \text{in } \Omega.
\]

In a similar manner, using the definition of \( \mathcal{A} \) and the properties of the sub and supersolutions of problem (29) we show that \( T v \) satisfies

\[
m_2 \delta(x)^{\frac{2-r}{1+s}} \leq T v \leq M_2 \delta(x)^{\frac{2-r}{1+s}} \quad \text{in } \Omega.
\]

Thus, \((T u, T v) \in \mathcal{A}\) for all \((u, v) \in \mathcal{A}\), that is, \( \mathcal{F}(\mathcal{A}) \subseteq \mathcal{A} \).
Step 2: $\mathcal{F}$ is compact and continuous. Let $(u, v) \in \mathcal{A}$. Since $\mathcal{F}(u, v) \in \mathcal{A}$, one can find $0 < a < 2$ such that
\[
0 \leq -\Delta (Tu), -\Delta (Tv) \leq c \delta(x)^{-a} \quad \text{in } \Omega,
\]
for some positive constant $c > 0$. Using Proposition 2.2 we now deduce $Tu, Tv \in C^{0, \gamma}(\overline{\Omega})$ ($0 < \gamma < 1$). Since the embedding $C^{0, \gamma}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ is compact, it follows that $\mathcal{F}$ is also compact.

It remains to prove that $\mathcal{F}$ is continuous. To this aim, let $\{(u_n, v_n)\} \subset \mathcal{A}$ be such that $u_n \to u$ and $v_n \to v$ in $C(\overline{\Omega})$ as $n \to \infty$. Using the fact that $\mathcal{F}$ is compact, there exists $(U, V) \in \mathcal{A}$ such that up to a subsequence we have
\[
Tu_n \to U, \quad Tv_n \to V \quad \text{in } C(\overline{\Omega}) \text{ as } n \to \infty.
\]
On the other hand, by standard elliptic estimates, the sequences $\{Tu_n\}$ and $\{Tv_n\}$ are bounded in $C^{2, \beta}(\overline{\omega})$ ($0 < \beta < 1$) for any smooth open set $\omega \subset \Omega$. Therefore, up to a diagonally subsequence, we have
\[
Tu_n \to U, \quad Tv_n \to V \quad \text{in } C^{2}(\overline{\omega}) \text{ as } n \to \infty,
\]
for any smooth open set $\omega \subset \Omega$. Passing to the limit in the definition of $Tu_n$ and $Tv_n$ we find that $(U, V)$ satisfies
\[
\begin{aligned}
-\Delta U &= v^{-q}U^{-p}, \quad U > 0 \quad \text{in } \Omega, \\
-\Delta V &= u^{-r}V^{-s}, \quad V > 0 \quad \text{in } \Omega, \\
U &= V = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
By uniqueness of (33), it follows that $Tu = U$ and $Tv = V$. Hence
\[
Tu_n \to Tu, \quad Tv_n \to Tv \quad \text{in } C(\overline{\Omega}) \text{ as } n \to \infty.
\]
This proves that $\mathcal{F}$ is continuous.

We are now in a position to apply the Schauder’s fixed point theorem. Thus, there exists $(u, v) \in \mathcal{A}$ such that $\mathcal{F}(u, v) = (u, v)$, that is, $Tu = u$ and $Tv = v$. By standard elliptic estimates, it follows that $(u, v)$ is a solution of system (1).

The remaining five cases will be considered in a similar way. Due to the different boundary behavior of solutions described in Proposition 2.6, the set $\mathcal{A}$ and the constants $c_1, c_2$ have to be modified accordingly. We shall point out the way we choose these constants in order to apply the Schauder’s fixed point theorem.

Case 2: $r + s = 1$ and $\alpha = p + q < 1$. According to Proposition 2.6(i)–(ii) there exist $0 < a < 1$ and $0 < c_1 < 1 < c_2$ such that:

- Any subsolution $\underline{u}$ of the problem
\[
\begin{aligned}
-\Delta u &= \delta(x)^{-q}u^{-p}, \quad u > 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
satisfies
\[ \underline{u}(x) \leq c_2 \delta(x) \quad \text{in } \Omega. \]

- Any supersolution \( \bar{u} \) of the problem
\[
\begin{cases}
-\Delta u = \delta(x)^{-q(1-\alpha)}u^{-p}, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

satisfies
\[ \bar{u}(x) \geq c_1 \delta(x) \quad \text{in } \Omega. \]

- Any subsolution \( \underline{v} \) and any supersolution \( \bar{v} \) of problem (29) satisfy the estimates
\[ \underline{v}(x) \leq c_2 \delta(x)^{1-a} \quad \text{and} \quad \bar{v}(x) \geq c_1 \delta(x) \quad \text{in } \Omega. \]

We now define
\[ \mathcal{A} = \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : m_1 \delta(x) \leq u(x) \leq M_1 \delta(x) \quad \text{in } \Omega, \\
m_2 \delta(x) \leq v(x) \leq M_2 \delta(x)^{1-a} \quad \text{in } \Omega \right\}, \]

where \( 0 < m_i < 1 < M_i \) \((i = 1, 2)\) satisfy (31), (32) and
\[ m_2 \left[ \text{diam}(\Omega) \right]^a < M_2. \] (35)

We next define the operator \( \mathcal{F} \) in the same way as in Case 1 by (33) and (34). The fact that \( \mathcal{F}(\mathcal{A}) \subseteq \mathcal{A} \) and that \( \mathcal{F} \) is continuous and compact follows in the same manner.

Case 3: \( r + s < 1 \) and \( \alpha = p + q < 1 \). In the same manner we define
\[ \mathcal{A} = \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : m_1 \delta(x) \leq u(x) \leq M_1 \delta(x) \quad \text{in } \Omega, \\
m_2 \delta(x) \leq v(x) \leq M_2 \delta(x)^{1-a} \quad \text{in } \Omega \right\}, \]

where \( 0 < m_i < 1 < M_i \) \((i = 1, 2)\) satisfy (31)–(32) for suitable constants \( c_1 \) and \( c_2 \).

Case 4: \( r + s < 1 \) and \( \alpha = p + q = 1 \). The approach is the same as in Case 2 above if we interchange \( u \) with \( v \) in the initial system (1).

Case 5: \( r + s > 1 \) and \( \alpha = p + q = 1 \). Let \( 0 < a < 1 \) be fixed such that \( ar + s > 1 \). From Proposition 2.6(i), (iii), there exist \( 0 < c_1 < 1 < c_2 \) such that:

- Any subsolution \( \underline{u} \) of the problem
\[
\begin{cases}
-\Delta u = \delta(x)^{-q(1-\alpha)}u^{-p}, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

satisfies
\[ \underline{u}(x) \leq c_2 \delta(x)^a \quad \text{in } \Omega. \]
Any supersolution \( \bar{u} \) of the problem
\[
\begin{cases}
-\Delta u = \delta(x) \frac{q(2-r)}{1+r} u^{-p}, & u > 0 \text{ in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
satisfies
\[\bar{u}(x) \geq c_1 \delta(x) \text{ in } \Omega.\]

Any subsolution \( v \) of problem (29) satisfies
\[v(x) \leq c_2 \delta(x)^{\frac{2-r}{1+r}} \text{ in } \Omega.\]

Any supersolution \( \bar{v} \) of the problem
\[
\begin{cases}
-\Delta v = \delta(x)^{-ar} v^{-s}, & v > 0 \text{ in } \Omega, \\
 v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
satisfies
\[\bar{v}(x) \geq c_1 \delta(x)^{\frac{2-ar}{1+r}} \text{ in } \Omega.\]

We now define
\[
\mathcal{A} = \left\{ (u, v) \in C(\Omega) \times C(\Omega): \begin{align*}
 m_1 \delta(x) &\leq u(x) \leq M_1 \delta(x)^{a} & \text{ in } \Omega, \\
 m_2 \delta(x)^{\frac{2-ar}{1+r}} &\leq v(x) \leq M_2 \delta(x)^{\frac{2-r}{1+r}} & \text{ in } \Omega
\end{align*} \right\},
\]
where \( 0 < m_i < 1 < M_i \) (\( i = 1, 2 \)) satisfy (31)–(32) in which the constants \( c_1, c_2 \) are those given above and
\[m_1 \left[ \text{diam}(\Omega) \right]^{1-a} < M_1, \quad m_2 \left[ \text{diam}(\Omega) \right]^{\frac{r(1-a)}{1+r}} < M_2.\]

**Case 6:** \( r + s = 1 \) and \( \alpha = p + q = 1 \). We proceed in the same manner as above by considering
\[
\mathcal{A} = \left\{ (u, v) \in C(\Omega) \times C(\Omega): \begin{align*}
 m_1 \delta(x) &\leq u(x) \leq M_1 \delta(x)^{1-a} & \text{ in } \Omega, \\
 m_2 \delta(x) &\leq v(x) \leq M_2 \delta(x)^{1-a} & \text{ in } \Omega
\end{align*} \right\},
\]
where \( 0 < a < 1 \) is a fixed constant and \( m_i, M_i \) (\( i = 1, 2 \)) satisfy (31)–(32) for suitable \( c_1, c_2 > 0 \) and
\[m_i \left[ \text{diam}(\Omega) \right]^{a} < M_i, \quad i = 1, 2.\]

(iii) Let
\[a = \frac{2(1 + s - q)}{(1 + p)(1 + s) - qr}, \quad b = \frac{2(1 + p - r)}{(1 + p)(1 + s) - qr}.\]
Then
\[(1 + p)a + bq = 2, \quad ar + (1 + s)b = 2. \tag{36}\]

Since \(p + bq > 1\) and \(s + ar > 1\), from Proposition 2.6(iii) and (36) above we can find \(0 < c_1 < 1 < c_2\) such that:

- Any subsolution \(u\) and any supersolution \(\bar{u}\) of the problem

\[
\begin{cases}
-\Delta u = \delta(x)^{-bq}u^{-p}, & u > 0 \quad \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]

satisfy
\[
\bar{u}(x) \geq c_1 \delta(x)^a \quad \text{and} \quad \underline{u}(x) \leq c_2 \delta(x)^a \quad \text{in} \ \Omega.
\]

- Any subsolution \(v\) and any supersolution \(\bar{v}\) of the problem

\[
\begin{cases}
-\Delta v = \delta(x)^{-ar}v^{-s}, & v > 0 \quad \text{in} \ \Omega, \\
v = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]

satisfy
\[
\bar{v}(x) \geq c_1 \delta(x)^b \quad \text{and} \quad \underline{v}(x) \leq c_2 \delta(x)^b \quad \text{in} \ \Omega.
\]

As before, we now define
\[
\mathcal{A} = \left\{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : m_1 \delta(x)^a \leq u(x) \leq M_1 \delta(x)^a \quad \text{in} \ \Omega, \right.
\]
\[
\left. m_2 \delta(x)^b \leq v(x) \leq M_2 \delta(x)^b \quad \text{in} \ \Omega \right\},
\]

where \(0 < m_1 < 1 < M_1\) and \(0 < m_2 < 1 < M_2\) satisfy (31)–(32). This concludes the proof of Theorem 1.2.

5. Proof of Theorem 1.4

(i) Assume first that the system (1) has a solution \((u, v)\) with \(u \in C^1(\bar{\Omega})\). Then, there exists \(c > 0\) such that \(u(x) \leq c\delta(x)\) in \(\Omega\). Using this fact in the second equation of (1), we derive that \(v\) satisfies the elliptic inequality (23) for some \(c_3 > 0\). By Corollary 2.5 this entails \(r < 2\).

In order to prove that \(\alpha < 1\) we argue by contradiction. Suppose that \(\alpha \geq 1\) and we divide our argument into three cases.

Case 1: \(r + s > 1\). Then, \(\alpha = p + \frac{a(2-r)}{1+s} \geq 1\). From Proposition 2.3 we have \((x) \geq c\delta(x)\) in \(\Omega\), for some \(c > 0\). Then \(v\) satisfies
\[
\begin{cases}
-\Delta v \leq c_1 \delta(x)^{-r}v^{-s}, & v > 0 \quad \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]
where \( c_1 > 0 \). Since \( r < 2 \), from Proposition 2.6(iii) we find \( v(x) \leq c_2 \delta(x)^{\frac{2-r}{1+s}} \) in \( \Omega \), for some \( c_2 > 0 \). Using this estimate in the first equation of system (1) we deduce

\[
\begin{aligned}
-\Delta u &\geq c_3 \delta(x)^{\frac{q(2-r)}{1+s}} u^{-p}, \\
u &= 0
\end{aligned}
\]  
(38)

where \( c_3 > 0 \). Now, if \( \frac{q(2-r)}{1+s} \geq 2 \), from Corollary 2.5 the above inequality is impossible. Assume next that \( \frac{q(2-r)}{1+s} < 2 \).

If \( \alpha > 1 \), from (8), (38) and Proposition 2.6(iii) we find

\[
u(x) \leq c_4 \delta(x)^{\tau} \geq c_4 \varphi_1(x)^{\tau} \text{ in } \Omega,
\]  
(39)

where

\[
\tau = \frac{2 - \frac{q(2-r)}{1+s}}{1+p} \in (0, 1) \quad \text{and} \quad c_4 > 0.
\]

Fix \( x_0 \in \partial \Omega \) and let \( n \) be the outer unit normal vector on \( \partial \Omega \) at \( x_0 \). Using (39) and the fact that \( 0 < \tau < 1 \) we have

\[
\frac{\partial u}{\partial n}(x_0) = \lim_{t \to 0} \frac{u(x_0 + tn) - u(x_0)}{t} \leq c_4 \lim_{t \to 0} \frac{\varphi_1(x_0 + tn) - \varphi_1(x_0)}{t} \varphi_1^{\tau-1}(x_0 + tn)
\]

\[
= \frac{\partial \varphi_1}{\partial n}(x_0) \lim_{t \to 0} \varphi_1^{\tau-1}(x_0 + tn)
\]

\[
= -\infty.
\]

Hence, \( u \notin C^1(\Omega) \).

If \( \alpha = 1 \) we proceed in the same manner. From (38) and Proposition 2.6(ii) we deduce

\[
u(x) \geq c_5 \delta(x) \log^{\frac{1-p}{1+p}} \left( \frac{A}{\delta(x)} \right) \geq c_6 \varphi_1(x) \log^{\frac{1-p}{1+p}} \left( \frac{A}{\varphi_1(x)} \right) \text{ in } \Omega,
\]

where \( c_5, c_6 > 0 \). As before, we obtain \( \frac{\partial u}{\partial n}(x_0) = -\infty \), \( x_0 \in \partial \Omega \), which contradicts \( u \in C^1(\Omega) \).

**Case 2:** \( r + s < 1 \). Then, \( \alpha = p + q \geq 1 \). As in Case 1, \( v \) fulfills (37) and by Proposition 2.6(i) we find \( v(x) \leq c_7 \delta(x) \) in \( \Omega \), for some \( c_7 > 0 \). Thus, \( u \) satisfies

\[
\begin{aligned}
-\Delta u &\geq c_8 \delta(x)^{-q} u^{-p}, \\
u &= 0
\end{aligned}
\]  
\text{in } \Omega,
\]

where \( c_8 > 0 \). From Corollary 2.5 it follows \( q < 2 \). Since \( \alpha = p + q \geq 1 \), it follows that \( u \) satisfies either the estimate (ii) (if \( p + q = 1 \)) or the estimate (iii) (if \( p + q > 1 \)) in Proposition 2.6. Proceeding in the same way as before we derive that the outer unit normal derivative of \( u \) on \( \partial \Omega \) is \( -\infty \), which is impossible.
Case 3: $r + s = 1$. This also yields $\alpha = p + q \geq 1$. As before $v$ satisfies (37) and by Proposition 2.6(ii) we deduce

$$v(x) \leq c_9 \delta(x) \log^{\frac{1}{1+r}} \left( \frac{A}{\delta(x)} \right) \quad \text{in } \Omega,$$

where $c_9 > 0$. It follows that $u$ satisfies

$$
\begin{cases}
-\Delta u \geq c_{10} \delta(x)^{-q} \log^{\frac{q}{1+r}} \left( \frac{A}{\delta(x)} \right) u^{-p}, & u > 0 \quad \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $c_{10} > 0$. If $q - b \geq 2$ the above inequality is impossible in the light of Corollary 2.5. Assume next that $q - b < 2$. If $\alpha = p + q > 1$, we fix $0 < b < \min\{q, p + q - 1\}$ and from (40) we have that $u$ satisfies

$$
\begin{cases}
-\Delta u \geq c_{11} \delta(x)^{-(q-b)} u^{-p}, & u > 0 \quad \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

for some $c_{11} > 0$. Now, since $p + q - b > 1$, from Proposition 2.6(iii) we find

$$u(x) \geq c_{12} \delta(x) \frac{2-(q-b)\log \log (A \delta(x))}{(1+p)(1+s)} \quad \text{in } \Omega,$$

where $c_{12} > 0$. Since $0 < \frac{2-(q-b)\log \log (A \delta(x))}{(1+p)(1+s)} < 1$, we obtain as before that the normal derivative of $u$ on $\partial \Omega$ is infinite which is impossible.

It remains to consider the case $\alpha = p + q = 1$, that is, $p + q = r + s = 1$. First, if $q < 1 + s$, that is, $q \neq 1$ and $s \neq 0$, by (40) and Corollary 2.8 we deduce

$$u(x) \geq c_{13} \delta(x) \frac{1+s-q}{(1+p)(1+s)} \left( \frac{A}{\delta(x)} \right) \quad \text{in } \Omega,$$

for some $c_{13} > 0$. Proceeding as before we obtain $\frac{\partial u}{\partial n} = -\infty$ on $\partial \Omega$, which is impossible.

If $q = 1$ and $s = 0$ then we apply Proposition 2.9 to obtain

$$u(x) \geq c_{14} \delta(x) \log \left( \frac{A}{\delta(x)} \right) \quad \text{in } \Omega,$$

where $c_{14} > 0$. This also leads us to the same contradiction $\frac{\partial u}{\partial n} = -\infty$ on $\partial \Omega$. Thus, we have proved that if the system (1) has a solution $(u, v)$ with $u \in C^1(\Omega)$ then $\alpha < 1$ and $r < 2$.

Conversely, assume now that $\alpha < 1$ and $r < 2$. By Theorem 1.2(i) (Cases 1, 2 and 3) there exists a solution $(u, v)$ of (1) such that

$$u(x) \geq c \delta(x) \quad \text{in } \Omega,$$

and

$$v(x) \geq c \delta(x) \quad \text{in } \Omega, \text{ if } r + s \leq 1,$$
or

\[ v(x) \geq c\delta(x)^{\frac{2r}{r+s}} \quad \text{in } \Omega, \text{ if } r+s > 1, \]

for some \( c > 0 \). Using the above estimates we find

\[ -\Delta u = u^{-p}v^{-q} \leq C\delta(x)^{-\alpha} \quad \text{in } \Omega, \]

for some \( C > 0 \). By Proposition 2.2, we now deduce \( u \in C^{1,1-\alpha}(\overline{\Omega}) \). The proof of (ii) is similar.

(iii) Assume first that the system (1) has a solution \((u, v)\) with \( u, v \in C^1(\overline{\Omega}) \). Then, there exists \( c > 0 \) such that \( v(x) \leq c\delta(x) \) in \( \Omega \). Using this estimate in the first equation of (1) we find that

\[
\begin{aligned}
-\Delta u &\geq C\delta(x)^{-q}u^{-p}, \quad u > 0 \quad \text{in } \Omega, \\
u & = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where \( C \) is a positive constant. If \( p + q \geq 1 \), then we combine the result in Proposition 2.6(ii)–(iii) with the techniques used above to deduce \( \frac{\partial u}{\partial n} = -\infty \) on \( \partial\Omega \), so \( u \notin C^1(\overline{\Omega}) \). Thus, \( p + q < 1 \) and in a similar way we obtain \( r + s < 1 \).

Assume now that \( p + q < 1 \) and \( r + s < 1 \). By Theorem 1.2(i) (Case 3) we have that (1) has a solution \((u, v)\) such that \( u(x), v(x) \geq c\delta(x) \) in \( \Omega \), for some \( c > 0 \). This yields

\[
\begin{aligned}
-\Delta u &\leq C\delta(x)^{-(p+q)} \quad \text{in } \Omega, \\
-\Delta v &\leq C\delta(x)^{-(r+s)} \quad \text{in } \Omega,
\end{aligned}
\]

where \( C > 0 \). Now Proposition 2.2 implies \( u, v \in C^1(\overline{\Omega}) \). This concludes the proof.

6. Proof of Theorem 1.5

We shall prove only (i); the case (ii) follows in the same manner.

Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions of system (1). Using Proposition 2.3 there exists \( c_1 > 0 \) such that

\[ u_i(x), v_i(x) \geq c_1\delta(x) \quad \text{in } \Omega, \quad i = 1, 2. \]  

(41)

Hence, \( u_i \) satisfies

\[
\begin{aligned}
-\Delta u_i &\leq c_2\delta(x)^{-q}u_i^{-p}, \quad u_i > 0 \quad \text{in } \Omega, \\
u_i & = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

for some \( c_2 > 0 \). By Proposition 2.6(i) and (41) there exists \( 0 < c < 1 \) such that

\[ c\delta(x) \leq u_i(x) \leq \frac{1}{c}\delta(x) \quad \text{in } \Omega, \quad i = 1, 2. \]

(42)

This means that we can find a constant \( C > 1 \) such that \( Cu_1 \geq u_2 \) and \( Cu_2 \geq u_1 \) in \( \Omega \).
We claim that $u_1 \geq u_2$ in $\Omega$. Supposing the contrary, let

$$M = \inf \{ A > 1 : Au_1 \geq u_2 \text{ in } \Omega \}.$$ 

By our assumption, we have $M > 1$. From $Mu_1 \geq u_2$ in $\Omega$, it follows that

$$-\Delta v_2 = u_2^{r-1} v_2^{-s} \geq M^{-r} u_1^{r-1} v_2^{-s} \text{ in } \Omega.$$ 

Therefore $v_1$ is a solution and $M^\frac{r}{1+r} v_2$ is a supersolution of

$$\begin{cases} 
-\Delta w = u_1^{-r} w^{-s}, & w > 0 \text{ in } \Omega, \\
0 & \text{on } \partial \Omega.
\end{cases}$$

By Proposition 2.1 we obtain

$$v_1 \leq M^\frac{r}{1+r} v_2 \text{ in } \Omega.$$ 

The above estimate yields

$$-\Delta u_1 = u_1^{-p} v_1^{-q} \geq M^{-\frac{qr}{1+r}} u_1^{-p} v_2^{-q} \text{ in } \Omega.$$ 

It follows that $u_2$ is a solution and $M^\frac{qr}{(1+p)(1+q)} u_1$ is a supersolution of

$$\begin{cases} 
-\Delta w = v_2^{-q} w^{-p}, & w > 0 \text{ in } \Omega, \\
0 & \text{on } \partial \Omega.
\end{cases}$$

By Proposition 2.1 we now deduce

$$M^\frac{qr}{(1+p)(1+q)} u_1 \geq u_2 \text{ in } \Omega.$$ 

Since $M > 1$ and $\frac{qr}{(1+p)(1+q)} < 1$, the above inequality contradicts the minimality of $M$. Hence, $u_1 \geq u_2$ in $\Omega$. Similarly we deduce $u_1 \leq u_2$ in $\Omega$, so $u_1 \equiv u_2$ which also yields $v_1 \equiv v_2$. Therefore, the system has a unique solution. This completes the proof of Theorem 1.4.

References