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Analysis of Rigid Frictionless Indentation on Half-Space with Surface Elasticity

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Abstract

The formulation of a boundary value problem of an axisymmetric, rigid, frictionless indentation acting on an elastic half-space is presented. The novel feature of this formulation is associated with the treatment of surface energy effects by employing a complete Gurtin-Murdoch continuum model for surface elasticity. With use of standard Love’s representation and Hankel integral transform, such boundary value problem can be reduced into a set of dual integral equations associated with the mixed boundary conditions on the surface of the half-space. Once these dual integral equations are transformed into a Fredholm integral equation of the second kind, selected numerical procedures based upon Galerkin approximation and a collocation technique are proposed to construct its solution numerically. The complete elastic fields are to be obtained for punches of different profiles and with smooth and non-smooth contact. The insight into the influence of surface free energy and size-dependency on material properties, gained in this particular study, should indicate and shed some lights on the implications related to soft elastic solids and nanotechnology.

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1. Introduction

Nanotechnology has remarkably become one of the most interesting fields in biology, chemistry, physics and engineering. Because of its focus on the very small size of particles, this kind of technology can significantly improve the quality of standard products to be superior ones. Nanostructures, for

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example, enable great applications in chemical, mechanical and electronic industries. The study of mechanical behavior at the nanoscale can be investigated by two basic approaches, namely, experimental methods and theoretical simulations. For the latter approach, the continuum-based model is considered attractive due to its lesser complexity when compared with atomistic simulations. In addition, several studies have been carried out and verified that modified continuum models accounted for surface energy effects and size-dependency are capable of studying nanoscale elements and correctly predicting responses of soft solids.

The concept of surface phenomena i.e. the thermodynamics of solid surfaces was noticed for more than one hundred years ago by Gibbs (Gibbs 1906). An explanation of the surface energy effects can be found in typical researches of surface and interface stresses (e.g. Cammarata 1994; Cammarata 1997). Gurtin and his co-workers (Gurtin and Murdoch 1975; Gurtin and Murdoch 1978; Gurtin et al. 1998) developed a mathematical framework for a continuum with surface stresses and proposed a linearized stress-strain constitutive relation for an isotropic elastic surface. This model has been applied successfully by several researchers to examine various issues involving nanoscale structures (e.g. Miller and Shenoy 2000; Lu et al. 2006; Intarit et al. 2010).

An indentation problem is one of fundamental problems in mechanics of solids and has been continuously investigated by several researchers due to its vast practical applications. The classical problem of axisymmetric rigid punch indenting on an elastic half-space is first considered by Boussinesq (1885). Sneddon (1965) later solved Boussinesq’s problem by applying Hankel integral transform techniques to overcome the limitations of the indented profile. The indentation problems associated with an elastic layer perfectly bonded to an elastic half-space have also been examined by using the Hankel transformation and then solved numerically by various techniques (Lebedev and Ufliand 1958; Dhaliwal and Rau 1970). Recently, Zhao (2009) has accounted the influence of surface energy effects for axisymmetric indentation problems. Although the out-of-plane terms in Gurtin-Murdoch model were not included in his formulation, numerical solutions still exhibits the size-dependent behavior due to surface energy effects.

The key objective of the present study is to generalize the work of Zhao (2009) to investigate responses of axisymmetric, rigid, frictionless indentation acting on a half-space by employing a complete Gurtin-Murdoch continuum model to account for the surface energy effects. The corresponding boundary value problem is formulated in terms of Love’s strain potential and then solved by the Hankel integral transform technique. The mixed boundary conditions on the surface of the half-space are transformed into a set of dual integral equations governing a single function sufficient for determining all field quantities. These dual integral equations are further reduced to a Fredholm integral equation of the second kind well-suited for constructing its solution numerically. The complete elastic fields within the half-space, especially very near the free surface, will be thoroughly investigated.

2. Formulation

Consider a homogeneous, isotropic, linearly elastic half-space indented by an axisymmetric rigid frictionless punch as shown schematically in Figure 1. The profile of the punch, denoted by a function \( \delta = \delta (r) \), is defined for convenience and without loss by choosing \( \delta = 0 \) at \( r = 0 \). The radius of a contact region and the indentation depth resulting from a resultant force \( P \) at the center of the punch are denoted by \( a \) and \( d \), respectively. In this study, the profile of the punch is assumed to be smooth (i.e. the unit normal vector to the surface of the punch or, equivalently, \( d\delta/dr \) is well-defined) at any point within the contact region except along the boundary \( r = a \) where the profile is allowed to be non-smooth. A punch with well-defined \( d\delta/dr \) for \( r \leq a \) is termed a smooth-contact punch (see Figure 1a) whereas a punch with well-defined \( d\delta/dr \) only for \( r < a \) is termed a non-smooth-contact punch (see Figure 1b).
Figure 1: Indentation of half-space by an axisymmetric rigid frictionless punch with (a) smooth contact and (b) non-smooth contact.

Behavior of the half-space (bulk) is governed by a classical theory of linear elasticity. In the absence of body forces, the governing field equations (i.e. equilibrium equations, constitutive relations and strain-displacement relations) can be expressed respectively as

\[ \sigma_{ij,} = 0, \quad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,} + u_{j,}) \]  

(1)

where \( u_i \), \( \sigma_{ij} \) and \( \varepsilon_{ij} \) denote the components of displacement, stress and strain tensors, respectively; \( \delta_{ij} \) is a Kronecker-delta symbol; and \( \mu \) and \( \lambda \) are Lamé constants of the bulk material.

A surface of the half-space is regarded as a negligibly thin membrane adhered perfectly to the bulk without slipping and its behavior (which is different from the bulk) is modeled by Gurtin-Murdoch continuum model for surface elasticity. The equilibrium conditions on the surface in terms of the generalized Young-Laplace equation (Povestenko 1993), the surface constitutive relations and the strain-displacement relationship, when specialized to this particular case, are given by (Gurtin and Murdoch 1975; Gurtin and Murdoch 1978; Gurtin et al. 1998)
\[ \sigma'_{(i_1,i_3,a)} + \sigma_{i_3} + t^0 = 0 \]

\[ \sigma'_{(i_1,a)} = \tau' \delta'_{(i_1,a)} + 2 \left( \mu' - \tau' \right) \varepsilon'_{(i_1,a)} + \left( \mu' + \tau' \right) \varepsilon'_{(i_1,a)} \delta'_{(i_1,a)} + \tau' u'_{(i_1,a)} \]

\[ \varepsilon'_{(i_1,i_1)} = \frac{1}{2} \left( u'_{(i_1,i_1)} + u'_{(i_1,i_1)} \right) \]

where the superscript ‘s’ is used to denote quantities corresponding to the surface; \( \mu' \) and \( \lambda' \) are surface Lamé constants; \( \tau' \) is the residual surface tension under unstrained conditions; and \( t^0 \) denotes the applied traction on the surface of the half-space. Throughout this paper, the usual summation convention is adopted for repeated indices, where Latin indices run from 1 to 3 while Greek ones take the value of 1 or 2. A comma denotes the differentiation with respect to the suffix coordinates.

When specialized to an axisymmetric case, the general solution for the displacements and stresses within the half-space can be expressed in terms of a function \( G(\xi, z) \) and its derivatives by using the Hankel integral transform technique along with the solution representation in terms of Love’s strain potential (Sneddon, 1951; Selvadurai, 2000):

\[ \sigma_{rr} = \int_{0}^{\infty} \left[ \lambda \frac{d G}{dz} + \left( \lambda + 2 \mu \right) \varepsilon^2 \frac{d G}{dz} \right] \frac{J_0(\xi r)}{d \xi} \left( \frac{2(\lambda + \mu)}{r} \right) \int_{0}^{\infty} \frac{d G}{dz} \frac{J_1(\xi r)}{d \xi} \]

\[ \sigma_{\theta \theta} = \lambda \int_{0}^{\infty} \left[ \lambda \frac{d G}{dz} + \left( \lambda + 2 \mu \right) \varepsilon^2 \frac{d G}{dz} \right] \frac{J_0(\xi r)}{d \xi} \left( \frac{2(\lambda + \mu)}{r} \right) \int_{0}^{\infty} \frac{d G}{dz} \frac{J_1(\xi r)}{d \xi} \]

\[ \sigma_{zz} = \int_{0}^{\infty} \left[ \lambda \frac{d G}{dz} + \left( \lambda + 2 \mu \right) \varepsilon^2 G \right] \frac{J_1(\xi r)}{d \xi} \]

\[ u_r = \frac{\lambda + \mu}{\mu} \int_{0}^{\infty} \frac{d G}{dz} \frac{J_1(\xi r)}{d \xi}, \quad u_z = \int_{0}^{\infty} \left[ \frac{d G}{dz} - \frac{\lambda + 2 \mu}{\mu} \varepsilon G \right] \frac{J_1(\xi r)}{d \xi} \]

where \( G(\xi, z) = (A + Bz)e^{\xi z} + (C + Dz)e^{\xi z}, J_n(\xi) \) denotes the first order Bessel functions of order \( n \); \( A, B, C \) and \( D \) are arbitrary functions of a transform parameter \( \xi \), which can be determined from boundary conditions. It should be noted that \( u_{\theta}, \sigma_{r \theta} \) and \( \sigma_{z \theta} \) vanish due to the axial symmetry and all non-zero field variables are independent of \( \theta \).

By invoking the remote conditions associated with the vanishing displacements and stresses as \( z \to \infty \), \( C \) and \( D \) must vanish and the function \( G(\xi, z) \) is therefore reduced to

\[ G(\xi, z) = (A + Bz)e^{\xi z} \]

For the indentation problem shown in Figure 1, the surface of the half-space can be decomposed into a surface outside the contact region (i.e. \( r > a \)) on which the traction identically vanishes and a surface inside the contact region (i.e. \( r \leq a \)) on which the normal displacement is prescribed while, resulting from
the frictionless surface of the punch, the shear traction vanishes. These mixed boundary conditions can be expressed as

$$ u_z \big|_{z=0} = d - \delta(r), \quad 0 \leq r \leq a $$  \hspace{1cm} (7)

$$ \sigma_{zz} \big|_{z=0} + \kappa \left( \frac{d^2 u_z}{dr^2} + \frac{1}{r} \frac{du_z}{dr} - \frac{u_z}{r^2} \right) = 0, \quad a < r < \infty $$  \hspace{1cm} (8)

$$ \sigma_{zz} \big|_{z=0} + \kappa \left( \frac{d^2 u_z}{dr^2} + \frac{1}{r} \frac{du_z}{dr} - \frac{u_z}{r^2} \right) = 0, \quad -\infty < r < \infty $$  \hspace{1cm} (9)

where \( \kappa = 2\mu \sigma + \lambda \sigma \) is a surface material constant. Upon substituting (6) into (5) and inserting the result into the boundary condition (9), it leads to

$$ A\xi \left( 1 + \Lambda_0 \xi \right) = \left( \frac{\lambda}{\lambda + \mu} + \Lambda_0 \xi \right) B $$  \hspace{1cm} (10)

where \( \Lambda_0 = \kappa / 2\mu \). By enforcing the mixed boundary conditions (7) and (8) along with the relation (10), it yields a system of dual integral equations:

$$ -\int_{\xi=0}^{\infty} \xi^2 \left[ \frac{\xi + 3\mu}{\xi (1 + \Lambda_0 \xi)} \right] B \xi (\xi) r d\xi = \Xi - \delta \left( r / a \right), \quad 0 \leq r \leq a $$  \hspace{1cm} (11)

$$ \int_{\xi=0}^{\infty} \xi^3 \left[ \frac{2\mu \xi + (\lambda + 3\mu) \Lambda_0 \xi}{\xi (1 + \Lambda_0 \xi)} \right] B \xi (\xi) r d\xi = 0, \quad a < r < \infty $$  \hspace{1cm} (12)

The dual integral equations (11) and (12) constitute a complete set of governing equations for determining the function \( B = B(\xi) \). By introducing two new functions \( \phi = \phi(\xi) \) and \( w = w(\xi) \) such that

$$ \phi(\xi) = \xi^3 \left[ \frac{2\mu \xi + (\lambda + 3\mu) \Lambda_0 \xi}{\xi (1 + \Lambda_0 \xi)} \right] B $$  \hspace{1cm} (13)

$$ \xi^{-1} \phi(\xi) \left[ 1 + w(\xi) \right] = \xi^2 \left[ \frac{(\lambda + 2\mu) \xi + (\lambda + 3\mu) \Lambda_0 \xi}{\mu (1 + \Lambda_0 \xi)} \right] B $$  \hspace{1cm} (14)

the dual integral equations (11) and (12) simply reduces to

$$ \int_{\xi=0}^{\infty} \xi^3 \phi(\xi) \left[ 1 + w(\xi) \right] B \xi (\xi) r d\xi = \Xi - \delta \left( r / a \right), \quad 0 \leq r \leq a $$  \hspace{1cm} (15)

$$ \int_{\xi=0}^{\infty} \phi(\xi) B \xi (\xi) r d\xi = 0, \quad a < r < \infty $$  \hspace{1cm} (16)

where \( \delta(r) = - \left[ (d - \delta(r)) \right] \) is a known function involving only the profile of the punch. It is important to remark that the function \( \phi = \phi(\xi) \) is now the primary unknown of the dual integrals (15) and (16) while
the function \( w = w(\xi) \) is known and can readily be obtained by solving equations (13) and (14). The explicit expression of \( w \) is given by

\[
 w(\xi) = \frac{\mu \left[ \left( \lambda + 2\mu \right) + \left( \lambda + 3\mu \right) \Lambda_0 \xi \right]}{2\mu \left[ \left( \lambda + \mu \right) + \left( \lambda + 2\mu \right) \Lambda_0 \xi \right] + \tau^* \xi \left[ \left( \lambda + 2\mu \right) + \left( \lambda + 3\mu \right) \Lambda_0 \xi \right]} - 1
\]  

(17)

It is evident from (17) that the function \( w \) has a limit value equal to \(-1\) as \( \xi \to \infty \).

Methods for determining the solution of the dual integral equations of the form (15) and (16) have been extensively studied by Mandal (1988) and Sneddon (1966). By following their procedures, such a set of dual integral equations can directly be reduced to a Fredholm integral equation of the second kind:

\[
 \phi(\xi) + \int_{0}^{\infty} K(\xi, u) \phi(u) du = F(\xi)
\]

(18)

where a kernel \( K(\xi, u) \) and a function \( F(\xi) \) are defined by

\[
 K(\xi, u) = \frac{\xi w(u)}{\pi u} \left( \frac{\sin(u + \xi)}{u + \xi} + \frac{\sin(u - \xi)}{u - \xi} \right)
\]

(19)

\[
 F(\xi) = \frac{2\xi}{\pi} \int_{0}^{\pi/2} \cos(\xi t) \left[ \frac{uf(u)}{\sqrt{t^2 - u^2}} dt \right]
\]

(20)

Note that the kernel \( K(\xi, u) \) is singular at \( u = 0 \) and is related to the surface stress effects via the function \( w \) while the function \( F(\xi) \), which is given in terms of prescribed data (i.e. the punch profile), contains a weakly singular inner integral. By applying a special variable transformation to regularize the inner integral of (20), it leads to

\[
 F(\xi) = \frac{2\xi^2}{\pi} \int_{0}^{\pi/2} \sin(\xi t) \bigg|_{t=\sin \theta} d\theta dt + \frac{2\xi \cos(\xi a)}{\pi} \int_{0}^{\pi/2} uf(u) \bigg|_{t=\sin \theta} d\theta
\]

(21)

By substituting the function \( f(r) \) (when the punch profile \( \delta(r) \) is known) into (21), the function \( F(\xi) \) can be obtained for the cases of a flat-ended cylindrical indenter, a conical indenter with an apex angle \( 2\alpha \) and a paraboloid indenter where \( \delta_0 \) is a constant, respectively as

\[
 \delta(r) = 0 \quad \Rightarrow \quad F(\xi) = \frac{-2d}{\pi} \sin(\xi a)
\]

(22)

\[
 \delta(r) = r \cot(\alpha) \quad \Rightarrow \quad F(\xi) = \frac{-2d}{\pi} \sin(\xi a) + \frac{\cot(\alpha)}{\xi} \left[ \xi a \sin(\xi a) + \cos(\xi a) - 1 \right]
\]

(23)

\[
 \delta(r) = \delta_0 \left( \frac{r}{a} \right)^2 \quad \Rightarrow F(\xi) = \frac{-2d}{\pi} \sin(\xi a) + \frac{4\delta_0}{\pi \xi^2 a^2} \left[ 2a \xi \cos(\xi a) + (-2 + a^2 \xi^2) \sin(\xi a) \right]
\]

(24)

In the absence of surface energy effects (i.e. \( \Lambda_0 = 0 \) and \( \tau^* = 0 \)), the above formulation simply reduces to that of a classical indentation problem. Specifically, the function \( w = w(\xi) \) defined by equation (17) become a constant:
and the dual integral equations (15) and (16) reduce to

\[ \int_0^a \frac{1}{r} \phi \left( \frac{\xi}{r} \right) J_0 \left( \frac{\xi}{r} \right) d\xi = f^*(r), \quad 0 \leq r \leq a \]

(26)

\[ \int_0^a \phi \left( \frac{\xi}{r} \right) J_0 \left( \frac{\xi}{r} \right) d\xi = 0, \quad a < r < \infty \]

(27)

where \( f'(r) = f(r)/(1+w^*) \). A set of dual integral equations (26) and (27) was solved analytically by Sneddon (1965).

3. Numerical implementation

Due to the complexity of the governing integral equation (18), it cannot be solved analytically and this therefore necessitates the use of numerical techniques to construct numerical solutions. In the present study, standard Galerkin approximation along with the collocation method is proposed to discretize the Fredholm integral equation (18) and establish a set of linear algebraic equations governing the approximate solutions for \( \phi = \phi(\xi) \). It should be noted that the function \( \phi \) vanishes at the origin of order \( O(r^u) \) and this condition can be employed to remove the singularity of the kernel \( K(r,u) \) at \( u = 0 \). Once the function \( \phi = \phi(\xi) \) is solved, the functions \( B \) and \( A \) can be subsequently determined from (13) and (10) respectively. By applying Hankel inversion, the complete elastic fields within the half-space can also be obtained from equations (5) along with the relation (6). In the case of smooth-contact punches, the radius of a contact region \( a \) is unknown a priori and a physically admissible condition must be properly invoked to determine such parameter. In the present investigation, the normal traction exerted by the punch must vanish along the boundary of the contact region \( r = a \) is utilized.

4. Conclusions

The classical problem of an axisymmetric rigid punch indenting on an elastic half-space has been re-examined by using a complete version of Gurtin-Murdoch continuum model to incorporate the influence of surface energy effects. By employing the solution representation in terms of Love’s strain potential and the Hankel integral transform technique, the associated boundary value problem has been reduced to a set of dual integral equations. To suit the construction of numerical solutions, such dual integral equations have further been transformed into a Fredholm integral equation of the second kind. Responses are to be obtained for a punch of different profiles and with smooth and non-smooth contact. When the complete elastic fields are obtained, various behaviors, for example, the pressure distribution under the punch, singularity at the end of the contact region, and the displacement and stress within the half-space especially very near the free surface can thoroughly be examined. In order to show the size-dependent behavior due to the influence of surface energy effects, the relationship between the contact pressure and the radius of the contact region under a punch of different profiles have to be compared with the classical case to see how values change when the radius of contact area becomes smaller. The applications of these findings on mechanics of nanoindentation and contact mechanics of soft elastic solids are currently in progress. Details of the numerical schemes and numerical results will be presented in a future publication.
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