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A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation ☆

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Abstract

We study a nonlocal mixed problem for a nonlinear pseudoparabolic equation, which can, for example, model the heat conduction involving a certain thermodynamic temperature and a conductive temperature. We prove the existence, uniqueness and continuous dependence of a strong solution of the posed problem. We first establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of deriving the a priori estimate is based on constructing a suitable multiplicator. From the resulted energy estimate, it is possible to establish the solvability of the linear problem. Then, by applying an iterative process based on the obtained results for the linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem.

Keywords: Integral condition; Nonlinear pseudoparabolic equation; Strong solution; Energy estimate

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1. Introduction

We deal with a nonlinear mixed problem having a nonlocal condition, the so-called energy specification. The problem of parameter identification from nonstandard boundary conditions in boundary value problems, originating from various engineering disciplines, is of growing interest. That is, a large number of physical phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems, such as problems in partial differential equations with integral conditions. These nonlocal boundary conditions such as the integral condition $\int_a^b u(x,t)\,dx = f(t)$, arise mainly when the data on the boundary cannot be measured directly, but their average values are known. More precisely, standard (Dirichlet, Neumann and Robin type) conditions which are prescribed pointwise are not always adequate as it depends on the physical context which data can be measured at the boundary of the physical domain. In some cases it is not possible to prescribe the solution u (pressure, temperature, ...) pointwise, because only the average value of the solution can be measured along the boundary or along some part of it. These kinds of problems are very important in the transport of reactive and passive contaminates in aquifer, an area of active interdisciplinary research of mathematicians, engineers and life scientists. For ample information, and for the derivation of mathematical models and for the precise hypothesis and analysis, the reader should refer to Cushmand and Ginn [8], Cushmand et al. [9]. The presence of an integral term in the boundary conditions can greatly complicate the application of standard functional or numerical methods, owing to the fact that the elliptic differential operator with integral condition is no longer positive definite in the usual function spaces, which poses the major source of difficulty. The physical significance of these conditions (total energy, total mass, mean, moments, etc.) has served as a fundamental reason for the increasing interest carried to this kind of problems. The first work, devoted to second order partial differential equations with nonlocal integral conditions goes back to Cannon [5]. Later, problems with integral conditions for parabolic equations were treated by Kamynin [15], Ionkin [14], Yurchuk [29], Bouziani [3], Mesloub and Bouziani [17], Mesloub [16]. Other parabolic problems also arise in plasma physics Samarskii [24], heat conduction Cannon [5], Ionkin [14], dynamics of ground waters, Nakhushev [21], Vodakhova [28], thermoelasticity Muravei [20], can be reduced to the nonlocal problem with integral conditions. An interesting collection of nonlocal parabolic problems in one-dimensional space is discussed in Fairweather [12]. Problems for elliptic equations with operator nonlocal conditions were considered by Scubachevski [25], Paneiah [22]. Then Gordeziani and Avalishvili [13], Mesloub and Bouziani [18], Mesloub and Lekrine [19], Pulkina [23], Beilin [1] devoted some papers to nonlocal problems for hyperbolic equations.

The pseudoparabolic equation and others

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} - \Delta u = 0, \tag{1.1}$$

have been extensively investigated, and many important results concerning existence, uniqueness and other properties of solutions have been published, see, for example, DiBenedetto [10,11], Coleman [7], Bouziani [4], and Showalter [26]. Equation (1.1) arises in various physical phenomena. It can, for example, model the diffusion of fluids in fractured porous media: Barenblatt [2], DiBenedetto [10], Coleman [7]. It can also model the heat conduction involving a thermodynamic temperature $T = u - k\ \Delta u$ and a conductive temperature u, Chen and Gurtin [6], Ting [27]. Motivated by this, we study a nonlocal nonlinear mixed problem for Eq. (1.1) in the case where the Laplacian operator is replaced by the Bessel operator $\frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial}{\partial x})$ and a term $f(x,t,u,u_x)$ is added to its right-hand side.

2. Problem setting

In the rectangular domain

$$D_T = \Omega \times (0,T) = \{(x,t) \in \mathbb{R}^2,\ 0 < x < l,\ 0 < t < T\},$$

we consider the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial t \partial x}\left(x\frac{\partial u}{\partial x}\right) = f\left(x,t,u,\frac{\partial u}{\partial x}\right), \tag{2.1}$$

with the initial data

$$u(x,0) = u_0(x), \tag{2.2}$$

Neumann boundary condition

$$u_x(l,t) = 0 \tag{2.3}$$

and the nonlocal weighted boundary condition

$$\int_0^l xu\,dx = 0, \tag{2.4}$$

with

$$\frac{\partial u_0(l,t)}{\partial x} = 0, \qquad \int_0^l xu_0\,dx = 0. \tag{2.5}$$

Here u_0 and f are given functions.

We shall assume: there exists a positive constant d such that

$$\left|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)\right| \leqslant d\left(|u_1 - u_2| + |v_1 - v_2|\right), \tag{A}$$

for all $(x,t) \in D_T$.

This paper is organized as follows: In Section 3, we state and pose the linear problem associated to (2.1)–(2.4) and introduce the function spaces used throughout the paper as well. Then in Section 4, we prove the uniqueness of the solution of the linear problem. And in Section 5, we show the existence of solutions. Finally, in Section 6, on the basis of the results obtained in Sections 4 and 5, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the nonlinear problem (2.1)–(2.4). The method used here is a further elaboration of that in [16].

3. Statement of the associated linear problem

Let us in this section give the position of the linear problem and introduce the different function spaces needed to investigate the mixed nonlocal problem given by the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial u}{\partial x}\right) = f(x,t), \tag{3.1}$$

and supplemented by conditions (2.2)–(2.4). The method used here is one of the most efficient functional analysis methods in solving partial differential equations with nonlocal boundary conditions, the so-called a priori estimate method or the energy–integral method. This method is essentially based on the construction of suitable multiplicators for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and on the density of the range of the operator generated by the abstract formulation of the stated problem.

To investigate the posed problem, we introduce the needed function spaces. We denote by $L^2_\rho(\Omega)$ the Hilbert space of weighted square integrable functions with inner product

$$(u,v)_{L^2_\rho(\Omega)} = (xu,v)_{L^2(\Omega)} = \int_\Omega xuv\,dx,$$

and with associated norm

$$\|u\|_{L^2_\rho(\Omega)} = \|\sqrt{x}u\|_{L^2(\Omega)} = \left(\int_\Omega xu^2\,dx\right)^{1/2}.$$

Let X be a Banach space with norm $\|u\|_X$, and let $u:(0,T)\to X$ be an abstract function. By $\|u(.,t)\|_X$ we denote the norm of $u(.,t)\in X$ for fixed t. Let $L^2(0,\acute{T};X)$ be the set of all measurable abstract functions $u(.,t):(0,T)\to X$ such that

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(.,t)\|^2_X\,dt < \infty.$$

If X is a Hilbert space, then $L^2(0,\acute{T};X)$ is also a Hilbert space. Let $C(0,T;X)$ be the set of all continuous functions $u:(0,T)\to X$ such that

$$\|u\|_{C(0,T;X)} = \max_{t\in[0,T]}\|u(.,t)\|_X < \infty.$$

And denote by $H^1_\rho(\Omega)$ the weighted Sobolev space with

$$\|u\|^2_{H^1_\rho(\Omega)} = \|u\|^2_{L^2_\rho(\Omega)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_\rho(\Omega)} < \infty.$$

The given problem (3.1), (2.2)–(2.4), can be viewed as the problem of solving the operator equation

$$Lu = (f,u_0), \quad \forall u\in D(L), \tag{3.2}$$

where L is the operator given by $L=(\mathcal{L},\ell)$, and $D(L)$ is the set of all functions

$$u\in L^2\big(0,\acute{T};H^1_\rho(\Omega)\big):\quad \frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t\partial x},\frac{\partial^3 u}{\partial t\partial x^2}\in L^2\big(0,\acute{T};H^1_\rho(\Omega)\big),$$

and u satisfies conditions (2.3) and (2.4). The operator L acts from B to F, where B is the Banach space obtained by enclosing the set $D(L)$ with respect to the finite norm

$$\|u\|_B^2=\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2_\rho(\Omega))}+\|u\|^2_{C(0,T;H^1_\rho(\Omega))}.$$

Functions $u\in B$ are continuous on $[0,T]$ with values in $H^1_\rho(\Omega)$. Hence the mapping

$$\ell:B\ni u\to \ell u=u(x,0)\in H^1_\rho(\Omega)$$

is defined and continuous on B. And F is the Hilbert space $L^2(0,T;L^2_\rho(\Omega))\times H^1_\rho(\Omega)$ consisting of vector valued functions $\mathcal{F}=(f,u_0)$ for which the norm

$$\|\mathcal{F}\|_F=\big(\|f\|^2_{L^2(0,T;L^2_\rho(\Omega))}+\|u_0\|^2_{H^1_\rho(\Omega)}\big)^{1/2}$$

is finite. Let $\overline{L}$ be the closure of the operator L with domain of definition $D(\overline{L})$.

Definition. We call a strong solution of the problem (3.1), (2.2)–(2.4), the solution of the operator equation

$$\overline{L}u=\mathcal{F}\quad \text{for all } u\in D(\overline{L}).$$

We establish an energy inequality for the operator L, and extend the obtained estimate to the closure $\overline{L}$, of the operator L. Finally, we prove the density of the range $R(L)$ of the operator L in the space F.

4. A priori estimate

In this section, we establish an a priori estimate for the operator L from which we conclude the uniqueness and continuous dependence of the solution upon the initial condition (2.2) and the right-hand side of (3.1). First observe that $\frac{\partial}{\partial x}\Im_x(f)=f$, and $\frac{\partial}{\partial x}\Im_0(f)=\Im_0^2(f)=0$, where $\Im_x(f)=\int_0^x f(\xi)\,d\xi$, and $\Im_x^2(\xi f(\xi))=\Im_x(\Im_\xi(\eta f(\eta)))$. By taking the inner product in $L^2_\rho(\Omega)$ of Eq. (3.1) and the integro-differential operator $Mu=x\frac{\partial u}{\partial t}-x\Im_x^2(\xi u)$ and then integrating over $(0,\tau)$, with $0\leqslant\tau\leqslant T$, with $\Im_x(f)$ coincides with $\Im^1_x(f)$, we obtain

$$\int_0^\tau\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)}dt-\int_0^\tau\int_0^l\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)dx\,dt$$
$$-\int_0^\tau\int_0^l x\frac{\partial u}{\partial t}\Im_x^2(\xi u)\,dx\,dt+\int_0^\tau\int_0^l\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\Im_x^2(\xi u)\,dx\,dt$$

$$+\int_0^\tau\int_0^l \frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial u}{\partial x}\right)\Im_x^2(\xi u)\,dx\,dt-\int_0^\tau\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial u}{\partial x}\right)dx\,dt$$

$$=\int_0^\tau\int_0^l xf(x,t)\frac{\partial u}{\partial t}\,dx\,dt-\int_0^\tau\int_0^l xf(x,t)\Im_x^2(\xi u)\,dx\,dt. \tag{4.1}$$

Standard integration by parts of each integral in (4.1) leads to

$$-\int_0^\tau\int_0^l \frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)dx\,dt=\frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial x}\right\|^2_{L^2_\rho(\Omega)}-\frac{1}{2}\left\|\frac{\partial u_0}{\partial x}\right\|^2_{L^2_\rho(\Omega)}, \tag{4.2}$$

$$-\int_0^\tau\int_0^l x\frac{\partial u}{\partial t}\Im_x^2(\xi u)\,dx\,dt=\frac{1}{2}\left\|\Im_x\big(\xi u(.,\tau)\big)\right\|^2_{L^2(\Omega)}-\frac{1}{2}\left\|\Im_x(\xi u_0)\right\|^2_{L^2(\Omega)}, \tag{4.3}$$

$$\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\Im_x^2(\xi u)\,dx\,dt=-\int_0^\tau\int_0^l x\frac{\partial u}{\partial x}\Im_x(\xi u)\,dx\,dt, \tag{4.4}$$

$$\int_0^\tau\int_0^l \frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial u}{\partial x}\right)\Im_x^2(\xi u)\,dx\,dt=-\int_0^\tau\int_0^l \frac{\partial}{\partial t}\left(x\frac{\partial u}{\partial x}\right)\Im_x(\xi u)\,dx\,dt, \tag{4.5}$$

$$-\int_0^\tau\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial u}{\partial x}\right)dx\,dt=\int_0^\tau\left\|\frac{\partial^2 u(.,t)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)}dt. \tag{4.6}$$

Substitution of (4.2)–(4.6) into (4.1) yields

$$\begin{aligned}&\int_0^\tau\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)}dt+\frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial x}\right\|^2_{L^2_\rho(\Omega)}\\&\quad+\int_0^\tau\left\|\frac{\partial^2 u(.,t)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)}dt+\frac{1}{2}\left\|\Im_x\big(\xi u(.,\tau)\big)\right\|^2_{L^2(\Omega)}\\&\quad=\frac{1}{2}\left\|\frac{\partial u_0}{\partial x}\right\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\left\|\Im_x(\xi u_0)\right\|^2_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}&\int_0^\tau\int_0^l x\frac{\partial u}{\partial x}\Im_x(\xi u)\,dx\,dt+\int_0^\tau\int_0^l x\frac{\partial^2 u}{\partial x\partial t}\Im_x(\xi u)\,dx\,dt\\&\quad+\int_0^\tau\int_0^l xf(x,t)\frac{\partial u}{\partial t}\,dx\,dt-\int_0^\tau\int_0^l xf(x,t)\Im_x^2(\xi u)\,dx\,dt.\end{aligned} \tag{4.7}$$

By virtue of the elementary inequalities

$$\int_0^l \left(\Im_x(\xi u)\right)^2 dx \leqslant \frac{l^3}{2}\|u(.,t)\|^2_{L^2_\rho(\Omega)},$$

$$\int_0^l \left(\Im_x^2(\xi u)\right)^2 dx \leqslant \frac{l^2}{2}\|\Im_x(\xi u)\|^2_{L^2(\Omega)},$$

$$\int_0^l x\left(\Im_x(\xi u)\right)^2 dx \leqslant l\|\Im_x(\xi u)\|^2_{L^2(\Omega)} \tag{4.8}$$

(see [3]) and the Cauchy's ε-inequality

$$\alpha\beta \leqslant \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \tag{4.9}$$

the last four terms of the right-hand side of (4.7) can be (respectively) estimated as follows:

$$\int_0^\tau\int_0^l x\frac{\partial u}{\partial x}\Im_x(\xi u)\,dx\,dt$$
$$\leqslant \frac{\varepsilon_1}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial x}\right\|^2_{L^2_\rho(\Omega)} dt + \frac{l}{2\varepsilon_1}\int_0^\tau \left\|\Im_x\left(\xi u(.,t)\right)\right\|^2_{L^2(\Omega)} dt, \tag{4.10}$$

$$\int_0^\tau\int_0^l x\frac{\partial^2 u}{\partial x\partial t}\Im_x(\xi u)\,dx\,dt$$
$$\leqslant \frac{\varepsilon_2}{2}\int_0^\tau \left\|\frac{\partial^2 u(.,t)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)} dt + \frac{l}{2\varepsilon_2}\int_0^\tau \left\|\Im_x\left(\xi u(.,t)\right)\right\|^2_{L^2(\Omega)} dt, \tag{4.11}$$

$$\int_0^\tau\int_0^l xf(x,t)\frac{\partial u}{\partial t}\,dx\,dt$$
$$\leqslant \frac{\varepsilon_3}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)} dt + \frac{1}{2\varepsilon_3}\int_0^\tau \|f(.,t)\|^2_{L^2_\rho(\Omega)} dt, \tag{4.12}$$

$$-\int_0^\tau\int_0^l xf(x,t)\Im_x^2(\xi u)\,dx\,dt$$
$$\leqslant \frac{l^3\varepsilon_4}{4}\int_0^\tau \left\|\Im_x\left(\xi u(.,t)\right)\right\|^2_{L^2(\Omega)} dt + \frac{1}{2\varepsilon_4}\int_0^\tau \|f(.,t)\|^2_{L^2_\rho(\Omega)} dt. \tag{4.13}$$

By taking $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, $\varepsilon_3 = 1$, $\varepsilon_4 = 1$, and by combining (4.7) and (4.10)–(4.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)} dt + \frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial x}\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\big\|\Im_x\big(\xi u(.,\tau)\big)\big\|^2_{L^2(\Omega)} \\
&\quad\leqslant \frac{1}{2}\left\|\frac{\partial u_0}{\partial x}\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\big\|\Im_x(\xi u_0)\big\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial x}\right\|^2_{L^2_\rho(\Omega)} dt \\
&\qquad + \int_0^\tau \big\|f(.,t)\big\|^2_{L^2_\rho(\Omega)} dt + \left(\frac{l^3}{4} + \frac{3l}{4}\right)\int_0^\tau \big\|\Im_x\big(\xi u(.,t)\big)\big\|^2_{L^2(\Omega)} dt.
\end{aligned}\tag{4.14}$$

Adding the following elementary inequality

$$\frac{1}{4}\big\|u(.,\tau)\big\|^2_{L^2_\rho(\Omega)} \leqslant \frac{1}{4}\|u_0\|^2_{L^2_\rho(\Omega)} + \frac{1}{4}\int_0^\tau \big\|u(.,t)\big\|^2_{L^2_\rho(\Omega)} dt + \frac{1}{4}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)} dt$$

to (4.14), and using the first inequality of (4.8), we obtain

$$\begin{aligned}
&\big\|\Im_x\big(\xi u(.,\tau)\big)\big\|^2_{L^2(\Omega)} + \big\|u(.,\tau)\big\|^2_{H^1_\rho(\Omega)} + \int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)} \\
&\quad\leqslant C\Bigg(\int_0^\tau \big\|\Im_x\big(\xi u(.,t)\big)\big\|^2_{L^2(\Omega)} dt + \int_0^\tau \big\|u(.,t)\big\|^2_{H^1_\rho(\Omega)} dt \\
&\qquad + \|u_0\|^2_{H^1_\rho(\Omega)} + \int_0^\tau \big\|f(.,t)\big\|^2_{L^2_\rho(\Omega)} dt\Bigg),
\end{aligned}\tag{4.15}$$

where

$$C = \max\big(3l + l^3, 4\big).$$

We now need to eliminate the sum $\int_0^\tau \|\Im_x(\xi u(.,t))\|^2_{L^2(\Omega)}\,dt + \int_0^\tau \|u(.,t)\|^2_{H^1_\rho(\Omega)}\,dt$ from the right-hand side of (4.15). To do this, we use the following version of Gronwall's lemma [19, Lemma 2.2].

Lemma 4.1. *If $g_1(t)$, $g_2(t)$ and $g_3(t)$ are nonnegative functions on the interval $[0,T]$, $g_1(t)$ and $g_2(t)$ are integrable on $[0,T]$, and $g_3(t)$ is bounded nondecreasing on $[0,T]$, and C is a positive constant, then*

$$\int_0^\tau g_1(t)\,dt + g_2(\tau) \leqslant e^{C\tau} g_3(\tau),$$

is a direct consequence of the inequality

$$\int_0^\tau g_1(t)\,dt + g_2(\tau) \leqslant g_3(\tau) + C\int_0^\tau g_2(t)\,dt.$$

By putting in (4.15)

$$g_1(t) = \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2_\rho(\Omega)},$$

$$g_2(\tau) = \left\| \Im_x\big(\xi u(.,\tau)\big)\right\|^2_{L^2(\Omega)} + \left\|u(.,\tau)\right\|^2_{H^1_\rho(\Omega)},$$

and

$$g_3(\tau) = C\left(\|u_0\|^2_{H^1_\rho(\Omega)} + \int_0^\tau \left\|f(.,t)\right\|^2_{L^2_\rho(\Omega)}\,dt \right),$$

we obtain

$$\begin{aligned} &\left\| \Im_x\big(\xi u(.,\tau)\big)\right\|^2_{L^2(\Omega)} + \left\|u(.,\tau)\right\|^2_{H^1_\rho(\Omega)} + \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2_\rho(\Omega)} \\ &\quad \leqslant Ce^{CT}\left(\|u_0\|^2_{H^1_\rho(\Omega)} + \int_0^\tau \left\|f(.,t)\right\|^2_{L^2_\rho(\Omega)}\,dt \right). \end{aligned} \tag{4.16}$$

If we discard the first term on the left-hand side of (4.16), and since its right-hand side does not depend on τ, we take the upper bound on the left-hand side with respect to τ from 0 to T, and we have the a priori estimate

$$\begin{aligned} &\left\|u(.,\tau)\right\|^2_{C(0,T;H^1_\rho(\Omega))} + \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2(0,T;L^2_\rho(\Omega))} \\ &\quad \leqslant Ce^{CT}\big(\|u_0\|^2_{H^1_\rho(\Omega)} + \left\|f(.,t)\right\|^2_{L^2(0,T;L^2_\rho(\Omega))}\big). \end{aligned}$$

Thus we have established the following theorem.

Theorem 4.2. *If $u \in D(L)$, then we have the a priori estimate*

$$\|u\|_B \leqslant c\|Lu\|_F, \tag{4.17}$$

where c is a positive constant independent of u given by

$$c = \sqrt{Ce^{CT}}, \quad \textit{with } C = \max\big(3l + l^3, 4\big).$$

Since we have no information concerning the range of the operator L, except that $R(L) \subset F$, we must extend L so that estimate (4.17) holds for the extension and its range is the whole space F. To this end, we establish the following proposition.

Proposition 4.3. *The operator $L : E \to F$ admits a closure $\overline{L}$.*

Proof. The proof is analogous to that in [18]. $\square$

Since points of the graph of the operator $\overline{L}$ are limits of sequences of points of the graph of L, then take the limit in (4.17) to obtain an a priori estimate for the operator $\overline{L}$, that is

$$\|u\|_B \leqslant c\|\overline{L}u\|_F \quad \forall u \in D(\overline{L}),$$

from which we conclude the results.

Corollary 4.4. *A strong solution of problem* (3.1), (2.2)–(2.4) *is unique and depends continuously on the data $(f, u_0) \in F$.*

Corollary 4.5. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and is equal to the closure $\overline{R(L)}$ of $R(L)$, that is $R(\overline{L}) = \overline{R(L)}$.*

5. Solvability of the linear problem

Now, we are in a position to state the main result for the linear problem.

Theorem 5.1. *Problem* (3.1), (2.2)–(2.4)*, has a unique strong solution $u = \overline{L}^{-1}(f, u_0) = \overline{L^{-1}}(f, u_0)$, that depends continuously on the data, for all $f \in L^2(0, T; L^2_\rho(\Omega))$ and $u_0 \in H^1_\rho(\Omega)$.*

Proof. According to Corollary 4.5, we deduce that to prove the existence of the strong solution, it is sufficient to show that $\overline{R(L)} = F$, that is $\overline{L}$ is one to one (injective). To this end, we need to establish the following proposition.

Proposition 5.2. *Let $D_0(L)$ be the set of all $u \in D(L)$ vanishing in a neighborhood of $t = 0$. If for $g \in L^2(0, T; L^2_\rho(\Omega))$ and for all $u \in D_0(L)$, we have*

$$(\mathcal{L}u, g)_{L^2(0,T;L^2_\rho(\Omega))} = 0, \tag{5.1}$$

then the function g vanishes almost everywhere in D_T.

Proof of Proposition 5.2. Assume that (5.1) holds for any $u \in D_0(L)$. Using this fact, we can express (5.1) in a special form. First define the function

$$\varphi(x, t) = \int_t^T g(x, \nu)\, d\nu. \tag{5.2}$$

Let $\frac{\partial u}{\partial t}$ be a solution of the equation

$$\frac{\partial u}{\partial t} + \Im_x^2(\xi u) = \varphi(x, t). \tag{5.3}$$

And let

$$u=\begin{cases}\int_\nu^t \frac{\partial u}{\partial \tau}\,d\tau, & \nu\leqslant t\leqslant T,\\ 0, & 0\leqslant t\leqslant \nu.\end{cases}\tag{5.4}$$

From (5.2) and (5.3), it follows that

$$g(x,t)=-\frac{\partial^2 u}{\partial t^2}-\Im_x^2(\xi u_t).\tag{5.5}$$

We have the following result:

Lemma 5.3. *The function $g(x,t)$ defined by* (5.5) *is in $L^2(0,T;L^2_\rho(\Omega))$.*

Proof. The proof can be derived as in [16]. □

To continue the proof of Proposition 5.2, we replace $g(x,t)$ in (5.1) by its representation (5.5); we have

$$\begin{aligned}&-\left(\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}+\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}\\&\quad+\left(\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right),\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}-\left(\frac{\partial u}{\partial t},\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}\\&\quad+\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}\\&\quad+\left(\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right),\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}=0.\end{aligned}\tag{5.6}$$

Invoking (5.3), (5.4) and the boundary conditions (2.3), (2.4), and then carrying out appropriate integrations by parts of each term of (5.6), we obtain

$$-\left(\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}=\frac{1}{2}\left\|\frac{\partial u(x,\nu)}{\partial t}\right\|^2_{L^2_\rho(\Omega)},\tag{5.7}$$

$$\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}=\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))},\tag{5.8}$$

$$\left(\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right),\frac{\partial^2 u}{\partial t^2}\right)_{L^2(0,T;L^2_\rho(\Omega))}=\frac{1}{2}\left\|\frac{\partial^2 u(x,\nu)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)},\tag{5.9}$$

$$-\left(\frac{\partial u}{\partial t},\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}=\left\|\Im_x(\xi u_t)\right\|^2_{L^2(\nu,T;L^2(\Omega))},\tag{5.10}$$

$$\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}=-\left(\frac{\partial u}{\partial x},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))},\tag{5.11}$$

$$\left(\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right),\Im_x^2(\xi u_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}=-\left(\frac{\partial^2 u}{\partial x\partial t},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))}.\tag{5.12}$$

Combination of (5.7)–(5.12) and (5.6) yields

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u(x,\nu)}{\partial t}\right\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\left\|\frac{\partial^2 u(x,\nu)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)}\\
&\qquad+\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))}+\left\|\Im_x(\xi u_t)\right\|^2_{L^2(\nu,T;L^2(\Omega))}\\
&\quad=\left(\frac{\partial u}{\partial x},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))}+\left(\frac{\partial^2 u}{\partial x\partial t},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))}.
\end{aligned}\tag{5.13}$$

By virtue of inequality (4.9), we can estimate the right-hand side of (5.13) as follows:

$$\begin{aligned}
&\left(\frac{\partial u}{\partial x},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))}\\
&\quad\leqslant\frac{l}{2}\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))}+\frac{1}{2}\left\|\Im_x(\xi u_t)\right\|^2_{L^2(\nu,T;L^2(\Omega))},
\end{aligned}\tag{5.14}$$

$$\begin{aligned}
&\left(\frac{\partial^2 u}{\partial x\partial t},\Im_x(\xi u_t)\right)_{L^2(\nu,T;L^2_\rho(\Omega))}\\
&\quad\leqslant\frac{l}{2}\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))}+\frac{1}{2}\left\|\Im_x(\xi u_t)\right\|^2_{L^2(\nu,T;L^2(\Omega))}.
\end{aligned}\tag{5.15}$$

Inserting (5.14), (5.15) and the Poincaré inequality

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))}\leqslant 24T^2\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))},$$

into (5.13) and omitting the third term on the left-hand side of the obtained inequality, we get

$$\left\|\frac{\partial u(x,\nu)}{\partial t}\right\|^2_{L^2_\rho(\Omega)}+\left\|\frac{\partial^2 u(x,\nu)}{\partial x\partial t}\right\|^2_{L^2_\rho(\Omega)}\leqslant l\big(1+24T^2\big)\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|^2_{L^2(\nu,T;L^2_\rho(\Omega))}.\tag{5.16}$$

If we denote the integral term on the right-hand side of (5.16) by $\theta(\nu)$, then we have

$$-\frac{d}{d\nu}\big(\theta(\nu)\exp\big(l\big(1+24T^2\big)\nu\big)\big)\leqslant 0.\tag{5.17}$$

Taking into account that $\theta(T)=0$, (5.17) gives

$$\theta(\nu)\exp\big(l\big(1+24T^2\big)\big)\nu\leqslant 0.\tag{5.18}$$

It follows from (5.18) that $g=0$ a.e. in $D_{T-\nu}=\Omega\times[T-\nu,T]$. Proceeding in this way step by step along the cylinders of height ν, we prove that $g=0$ a.e. in D_T. This completes the proof of Proposition 5.2. □

To complete the proof of Theorem 5.1, we suppose that for some element $G = (g, g_0) \in R(L)^{\perp}$,

$$(\mathcal{L}u, g)_{L^2(0,T;L^2_\rho(\Omega))} + (\ell u, g_0)_{H^1_\rho(\Omega)} = 0. \tag{5.19}$$

We must prove that $G = 0$. If we put $u \in D_0(L)$ into (5.19), we have

$$(\mathcal{L}u, g)_{L^2(0,T;L^2_\rho(\Omega))} = 0, \quad u \in D_0(L). \tag{5.20}$$

Applying Proposition 5.2 to (5.20), it follows that $g = 0$. Thus (5.19) takes the form

$$(\ell u, g_0)_{H^1_\rho(\Omega)} = 0. \tag{5.21}$$

But since the set $R(\ell)$ is everywhere dense in the space $H^1_\rho(\Omega)$, then relation (5.21) implies that $g_0 = 0$. Consequently $G = 0$, and Theorem 5.1 follows. □

6. The nonlinear problem

This section is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem (2.1)–(2.4). Let us consider the following auxiliary problem with homogeneous equation:

$$\mathcal{L}u = \frac{\partial U}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial U}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial t \partial x}\left(x\frac{\partial U}{\partial x}\right) = 0, \tag{6.1}$$

$$\ell U = U(x, 0) = u_0(x), \tag{6.2}$$

$$\frac{\partial U}{\partial x}(l, t) = 0, \tag{6.3}$$

$$\int_0^l xU\,dx = 0. \tag{6.4}$$

If u is a solution of problem (2.1)–(2.4) and U is a solution of problem (6.1)–(6.4), then $w = u - U$ satisfies

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial t \partial x}\left(x\frac{\partial w}{\partial x}\right) = F\left(x, t, w, \frac{\partial w}{\partial x}\right), \tag{6.5}$$

$$w(x, 0) = 0, \tag{6.6}$$

$$\frac{\partial w}{\partial x}(l, t) = 0, \tag{6.7}$$

$$\int_0^l xw\,dx = 0, \tag{6.8}$$

where $F(x, t, w, \frac{\partial w}{\partial x}) = f(x, t, w + U, \frac{\partial w}{\partial x} + \frac{\partial U}{\partial x})$. The function F satisfies the condition

$$\left|F(x, t, u_1, v_1) - F(x, t, u_2, v_2)\right| \leqslant d\big(|u_1 - u_2| + |v_1 - v_2|\big), \tag{B}$$

for all $(x, t) \in D_T$.

According to Theorem 5.1, problem (6.1)–(6.4) has a unique solution that depends continuously on $u_0 \in H^1_\rho(\Omega)$. It remains to solve the problem (6.5)–(6.8). We shall prove that problem (6.5)–(6.8) has a unique weak solution.

First let

$$\tilde{C}^1(D_T) = \left\{ \upsilon \in C^1(D_T),\ \text{such that } \frac{\partial^2 \upsilon}{\partial t \partial x} \in C(D_T) \right\}.$$

Assume that υ and $w \in \tilde{C}^1(D_T)$, $\upsilon(x,T)=0$, $w(x,0)=0$, $\int_0^l xw\,dx = \int_0^l x\upsilon\,dx = 0$. For $\upsilon \in \tilde{C}^1(D_T)$, we have

$$\begin{aligned} &-\left(\mathcal{L}w, \Im_x(\xi\upsilon)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\quad = -\left(\frac{\partial w}{\partial t}, \Im_x(\xi\upsilon)\right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right), \Im_x(\xi\upsilon)\right)_{L^2(D_T)} \\ &\qquad + \left(\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial w}{\partial x}\right), \Im_x(\xi\upsilon)\right)_{L^2(D_T)}. \end{aligned} \tag{6.9}$$

By using conditions on w and υ, a quick computation of each term on the right- and left-hand side of (6.9), gives

$$-\left(\frac{\partial w}{\partial t}, \Im_x(\xi\upsilon)\right)_{L^2(0,T;L^2_\rho(\Omega))} = -\left(\frac{\partial \upsilon}{\partial t}, \Im_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.10}$$

$$\left(\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right), \Im_x(\xi\upsilon)\right)_{L^2(D_T)} = -\left(x\frac{\partial w}{\partial x}, \upsilon\right)_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.11}$$

$$\left(\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial w}{\partial x}\right), \Im_x(\xi\upsilon)\right)_{L^2(D_T)} = \left(x\frac{\partial w}{\partial x}, \frac{\partial \upsilon}{\partial t}\right)_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.12}$$

$$-\left(\mathcal{L}w, \Im_x(\xi\upsilon)\right)_{L^2(0,T;L^2_\rho(\Omega))} = \left(\upsilon, \Im_x(\xi F)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.13}$$

Insertion of (6.10)–(6.13) into (6.9) yields

$$H(w,\upsilon) = \left(\upsilon, \Im_x(\xi F)\right)_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.14}$$

where

$$\begin{aligned} H(w,\upsilon) = {} & \left(x\frac{\partial w}{\partial x}, \frac{\partial \upsilon}{\partial t}\right)_{L^2(0,T;L^2_\rho(\Omega))} - \left(\frac{\partial \upsilon}{\partial t}, \Im_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & - \left(x\frac{\partial w}{\partial x}, \upsilon\right)_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.15}$$

Definition 6.1. A function $w \in L^2(0, \acute{T}; H^1_\rho(\Omega))$ is called a weak solution of problem (6.5)–(6.8) if (6.7) and (6.14) hold.

Let us construct an iteration sequence in the following way. Starting with $w^{(0)} = 0$, the sequence $(w^{(n)})_{n\in\acute{N}}$ is defined as follows: given the element $w^{(n-1)}$, then for $n = 1, 2, \ldots$ solve the problem:

$$\frac{\partial w^{(n)}}{\partial t}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w^{(n)}}{\partial x}\right)-\frac{1}{x}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial w^{(n)}}{\partial x}\right)$$
$$=F\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right), \tag{6.16}$$
$$w^{(n)}(x,0)=0, \tag{6.17}$$
$$\frac{\partial w^{(n)}}{\partial x}(l,t)=0, \tag{6.18}$$
$$\int_0^l xw^{(n)}(x,t)\,dx=0. \tag{6.19}$$

Theorem 5.1 asserts that for fixed n, each problem (6.16)–(6.19) has a unique solution $w^{(n)}(x,t)$. If we set $V^{(n)}(x,t)=w^{(n+1)}(x,t)-w^{(n)}(x,t)$, then we have the new problem

$$\frac{\partial V^{(n)}}{\partial t}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial V^{(n)}}{\partial x}\right)-\frac{1}{x}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial V^{(n)}}{\partial x}\right)=\sigma^{(n-1)}(x,t), \tag{6.20}$$
$$V^{(n)}(x,0)=0, \tag{6.21}$$
$$\frac{\partial V^{(n)}}{\partial x}(l,t)=0, \tag{6.22}$$
$$\int_0^l xV^{(n)}(x,t)\,dx=0, \tag{6.23}$$

where

$$\sigma^{(n-1)}(x,t)=F\left(x,t,w^{(n)},\frac{\partial w^{(n)}}{\partial x}\right)-F\left(x,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial x}\right).$$

Lemma 6.2. *Assume that condition* (B) *holds, then for the linearized problem* (6.20)–(6.23), *we have the a priori estimate*

$$\left\|V^{(n)}\right\|_{L^2(0,\acute{T};H^1_\rho(\Omega))}\leqslant K\left\|V^{(n-1)}\right\|_{L^2(0,\acute{T};H^1_\rho(\Omega))}, \tag{6.24}$$

where K is a positive constant given by

$$K=2\sqrt{T}de^{K_1T/2},\quad \textit{with } K_1=\max\left(1,\frac{3l+l^3}{2}\right).$$

Proof. Taking the inner product in $L^2(0,\tau;L^2_\rho(\Omega))$, with $0\leqslant\tau\leqslant T$, of Eq. (6.20) and the integro-differential operator

$$MV=x\frac{\partial V^{(n)}}{\partial t}-x\Im_x^2\left(\xi V^{(n)}\right),$$

we have

$$\int_0^\tau \left\| \frac{\partial V^{(n)}(.,t)}{\partial t} \right\|^2_{L^2_\rho(\Omega)} dt - \int_0^\tau \int_0^l \frac{\partial V^{(n)}}{\partial t} \frac{\partial}{\partial x}\left(x \frac{\partial V^{(n)}}{\partial x} \right) dx\,dt$$
$$- \int_0^\tau \int_0^l x \frac{\partial V^{(n)}}{\partial t} \Im_x^2\left(\xi V^{(n)}\right) dx\,dt + \int_0^\tau \int_0^l \frac{\partial}{\partial x}\left(x \frac{\partial V^{(n)}}{\partial x} \right) \Im_x^2\left(\xi V^{(n)}\right) dx\,dt$$
$$+ \int_0^\tau \int_0^l \frac{\partial^2}{\partial t \partial x}\left(x \frac{\partial V^{(n)}}{\partial x} \right) \Im_x^2\left(\xi V^{(n)}\right) dx\,dt - \int_0^\tau \int_0^l \frac{\partial V^{(n)}}{\partial t} \frac{\partial^2}{\partial t \partial x}\left(x \frac{\partial V^{(n)}}{\partial x} \right) dx\,dt$$
$$= \int_0^\tau \int_0^l x \frac{\partial V^{(n)}}{\partial t} \sigma^{(n-1)}(x,t)\,dx\,dt - \int_0^\tau \int_0^l x\sigma^{(n-1)}(x,t) \Im_x^2\left(\xi V^{(n)}\right) dx\,dt. \tag{6.25}$$

In light of conditions (6.22) and (6.23), successive integrations by parts of each term of (6.25) leads to

$$\int_0^\tau \left\| \frac{\partial V^{(n)}(.,t)}{\partial t} \right\|^2_{L^2_\rho(\Omega)} dt + \frac{1}{2} \left\| \frac{\partial V^{(n)}(.,\tau)}{\partial x} \right\|^2_{L^2_\rho(\Omega)}$$
$$+ \int_0^\tau \left\| \frac{\partial^2 V^{(n)}(.,t)}{\partial x \partial t} \right\|^2_{L^2_\rho(\Omega)} dt + \frac{1}{2} \left\| \Im_x\left(\xi V^{(n)}(.,\tau)\right) \right\|^2_{L^2(\Omega)}$$
$$= \left(\frac{\partial V^{(n)}}{\partial x}, \Im_x(\xi V^{(n)}) \right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(\frac{\partial^2 V^{(n)}}{\partial x \partial t}, \Im_x\left(\xi V^{(n)}\right) \right)_{L^2(0,T;L^2_\rho(\Omega))}$$
$$+ \left(\sigma^{(n-1)}, \frac{\partial V^{(n)}}{\partial t} \right)_{L^2(0,T;L^2_\rho(\Omega))} - \left(\sigma^{(n-1)}, \Im_x^2\left(\xi V^{(n)}\right) \right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.26}$$

By using inequality (4.9), each term on the right-hand side of (6.26), can be respectively controlled by

$$\int_0^\tau \left\| \frac{\partial V^{(n)}(.,t)}{\partial x} \right\|^2_{L^2_\rho(\Omega)} dt + \frac{l}{4} \int_0^\tau \left\| \Im_x\left(\xi V^{(n)}\right) \right\|^2_{L^2(\Omega)} dt, \tag{6.27}$$

$$\int_0^\tau \left\| \frac{\partial^2 V^{(n)}(.,t)}{\partial x \partial t} \right\|^2_{L^2_\rho(\Omega)} dt + \frac{l}{4} \int_0^\tau \left\| \Im_x\left(\xi V^{(n)}\right) \right\|^2_{L^2(\Omega)} dt, \tag{6.28}$$

$$d^2 \left(\int_0^T \left\| V^{(n-1)} \right\|^2_{L^2_\rho(\Omega)} + \int_0^T \left\| \frac{\partial V^{(n-1)}(.,t)}{\partial x} \right\|^2_{L^2_\rho(\Omega)} \right)$$
$$+ \frac{1}{2} \int_0^\tau \left\| \frac{\partial V^{(n)}(.,t)}{\partial t} \right\|^2_{L^2_\rho(\Omega)} dt, \tag{6.29}$$

$$d^2\left(\int_0^T \left\|V^{(n-1)}\right\|^2_{L^2_\rho(\Omega)}\,dt + \int_0^T \left\|\frac{\partial V^{(n-1)}(.,t)}{\partial x}\right\|^2_{L^2_\rho(\Omega)}\,dt\right) + \frac{l^3}{4}\int_0^\tau \left\|\Im_x\left(\xi V^{(n)}\right)\right\|^2_{L^2(\Omega)}\,dt. \tag{6.30}$$

It is obvious that

$$\frac{1}{2}\left\|V^{(n)}(.,\tau)\right\|^2_{L^2_\rho(\Omega)} \leqslant \frac{1}{2}\int_0^\tau \left\|V^{(n)}\right\|^2_{L^2_\rho(\Omega)}\,dt + \frac{1}{2}\int_0^\tau \left\|\frac{\partial V^{(n)}(.,t)}{\partial t}\right\|^2_{L^2_\rho(\Omega)}\,dt. \tag{6.31}$$

Combining (6.26)–(6.30) and adding side-to-side the resulted inequality and (6.31), it follows that

$$\left\|V^{(n)}(.,\tau)\right\|^2_{H^1_\rho(\Omega)} + \left\|\Im_x\left(\xi V^{(n)}(.,\tau)\right)\right\|^2_{L^2(\Omega)} \leqslant K_1\left(\int_0^\tau \left\|\Im_x\left(\xi V^{(n)}\right)\right\|^2_{L^2(\Omega)}\,dt + \int_0^\tau \left\|V^{(n)}\right\|^2_{H^1_\rho(\Omega)}\,dt\right) + 4d^2\int_0^\tau \left\|V^{(n-1)}\right\|^2_{H^1_\rho(\Omega)}\,dt, \tag{6.32}$$

where

$$K_1 = \max\left(1, \frac{3l + l^3}{2}\right).$$

We now apply Lemma 4.1 to (6.32) to get

$$\left\|V^{(n)}(.,\tau)\right\|^2_{H^1_\rho(\Omega)} + \left\|\Im_x\left(\xi V^{(n)}(.,\tau)\right)\right\|^2_{L^2(\Omega)} \leqslant 4d^2 e^{K_1 T}\int_0^T \left\|V^{(n-1)}\right\|^2_{H^1_\rho(\Omega)}\,dt. \tag{6.33}$$

After discarding the second term on the left-hand side of (6.33) and integrating the resulted inequality over the interval $(0, T)$, we obtain the desired a priori estimate (6.24), that is

$$\left\|V^{(n)}\right\|^2_{L^2(0,T;H^1_\rho(\Omega))} \leqslant 4Td^2e^{K_1T}\left\|V^{(n-1)}\right\|^2_{L^2(0,T;H^1_\rho(\Omega))}.$$

From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty} V^{(n)}$ converges if $4Td^2e^{K_1T} < 1$, that is if $d < \frac{1}{2\sqrt{T}}e^{-K_1T/2}$. Since $V^{(n)}(x,t) = w^{(n+1)}(x,t) - w^{(n)}(x,t)$, then it follows that the sequence $(w^{(n)})_{n\in N}$ defined by

$$\begin{aligned} w^{(n)}(x,t) &= \sum_{k=1}^{n-1} V^{(k)} + w^{(0)}(x,t) \\ &= \sum_{k=1}^{n-1} \big(w^{(k+1)}(x,t) - w^{(k)}(x,t)\big) + w^{(0)}(x,t), \quad k = 1, 2, \ldots \end{aligned}$$

converges to an element $w \in L^2(0,T; H_\rho^1(\Omega))$.

Now to prove that this limit function w is a solution of problem under consideration (6.20)–(6.23), we should show that w satisfies (6.7) and (6.14) as mentioned in Definition 6.1.

For problem (6.16)–(6.19), we have

$$H\big(w^{(n)}, \upsilon\big) = \left(\upsilon, \Im_x\left(\xi F\left(\xi, t, w^{(n-1)}, \frac{\partial w^{(n-1)}}{\partial \xi}\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.34}$$

From (6.34), we have

$$\begin{aligned} &H\big(w^{(n)} - w, \upsilon\big) + H(w, \upsilon) \\ &\quad = \left(\upsilon, \Im_x\left(\xi F\left(\xi, t, w^{(n-1)}, \frac{\partial w^{(n-1)}}{\partial \xi}\right)\right)\right. \\ &\qquad \left. - \Im_x\left(\xi F\left(\xi, t, w, \frac{\partial w}{\partial \xi}\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\qquad + \left(\upsilon, \Im_x\left(\xi F\left(\xi, t, w, \frac{\partial w}{\partial \xi}\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.35}$$

From the partial differential equation (6.16), we have

$$\begin{aligned} &\left(\upsilon, \frac{\partial}{\partial t}\Im_x\big(\xi\big(w^{(n)} - w\big)\big)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\qquad - \left(\upsilon, \Im_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\big(w^{(n)} - w\big)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\qquad - \left(\upsilon, \frac{\partial}{\partial t}\Im_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\big(w^{(n)} - w\big)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\quad = H\big(w^{(n)} - w, \upsilon\big). \end{aligned} \tag{6.36}$$

Integration by parts of each term on the left-hand side of (6.36), and use of conditions on υ and w transform (6.36) to

$$\begin{aligned} &-\left(\frac{\partial \upsilon}{\partial t}, \Im_x\big(\xi\big(w^{(n)} - w\big)\big)\right)_{L^2(0,T;L^2_\rho(\Omega))} - \left(x\upsilon, \frac{\partial}{\partial \xi}\big(w^{(n)} - w\big)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\quad + \left(x\frac{\partial \upsilon}{\partial t}, \frac{\partial}{\partial \xi}\big(w^{(n)} - w\big)\right)_{L^2(0,T;L^2_\rho(\Omega))} = H\big(w^{(n)} - w, \upsilon\big). \end{aligned} \tag{6.37}$$

We apply Cauchy–Schwarz inequality to terms on the left-hand side of (6.37) to get

$$H\big(w^{(n)}-w,\upsilon\big) \leqslant C\big\|w^{(n)}-w\big\|_{L^2(0,T;H^1_\rho(\Omega))}\left(\|\upsilon\|_{L^2(0,T;L^2_\rho(\Omega))}+\left\|\frac{\partial\upsilon}{\partial t}\right\|_{L^2(0,T;L^2_\rho(\Omega))}\right), \tag{6.38}$$

where

$$C=\frac{l^2}{\sqrt{2}}+l.$$

On the other side we have

$$\left(\upsilon,\Im_x\left(\xi F\left(\xi,t,w^{(n-1)},\frac{\partial w^{(n-1)}}{\partial\xi}\right)\right)-\Im_x\left(\xi F\left(\xi,t,w,\frac{\partial w}{\partial\xi}\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \leqslant \frac{ld}{\sqrt{2}}\big\|w^{(n)}-w\big\|_{L^2(0,T;H^1_\rho(\Omega))}\|\upsilon\|_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.39}$$

Taking into account (6.38) and (6.39), and passing to the limit in (6.37) as $n\to\infty$ to obtain

$$H(w,\upsilon)=\left(\upsilon,\Im_x\left(\xi F\left(\xi,t,w,\frac{\partial w}{\partial\xi}\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}.$$

Now to conclude that problem (6.20)–(6.23) has a weak solution, we show that (6.7) holds. Since $w\in L^2(0,T;H^1_\rho(\Omega))$, then $\int_0^t\frac{\partial w(x,s)}{\partial x}\,ds\in C(\overline{D_T})$, and we conclude that $\frac{\partial w}{\partial x}(l,t)=0$, a.e. □

Thus, we have proved the following:

Theorem 6.3. *Suppose that condition* (B) *holds, and that* $d<\frac{1}{2\sqrt{T}}e^{-K_1T/2}$, *then problem* (6.5)–(6.8), *has a weak solution belonging to* $L^2(0,T;H^1_\rho(\Omega))$.

It remains to prove that problem (6.5)–(6.8) admits a unique solution.

Theorem 6.4. *If condition* (B) *is satisfied, then the solution of problem* (6.5)–(6.8) *is unique.*

Proof. Suppose that w_1, $w_1\in L^2(0,T;H^1_\rho(\Omega))$ are two solution of (6.5)–(6.8), the $V=w_1-w_2$ is in $L^2(0,T;H^1_\rho(\Omega))$ and satisfies

$$\frac{\partial V}{\partial t}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right)-\frac{1}{x}\frac{\partial^2}{\partial t\partial x}\left(x\frac{\partial V}{\partial x}\right)=\sigma(x,t), \tag{6.40}$$

$$V(x,0)=0, \tag{6.41}$$

$$\frac{\partial V}{\partial x}(l,t)=0, \tag{6.42}$$

$$\int_0^l xV(x,t)\,dx=0, \tag{6.43}$$

where

$$\sigma(x,t) = F\left(x, t, w_1, \frac{\partial w_1}{\partial x}\right) - F\left(x, t, w_2, \frac{\partial w_2}{\partial x}\right).$$

Taking the inner product in $L^2(0, T; L^2_\rho(\Omega))$, of Eq. (6.40) and the integro-differential operator

$$MV = x\frac{\partial V}{\partial t} - x\Im_x^2(\xi V),$$

and following the same procedure done in establishing the proof of Lemma 6.2, we have

$$\|V\|_{L^2(0,\acute{T};H^1_\rho(\Omega))} \leqslant K\|V\|_{L^2(0,\acute{T};H^1_\rho(\Omega))}, \tag{6.44}$$

where

$$K = 2\sqrt{T}de^{K_1T/2}, \quad \text{with } K_1 = \max\left(1, \frac{3l + l^3}{2}\right).$$

Since $K < 1$, it follows from (6.44) that

$$(1 - K)\|V\|_{L^2(0,\acute{T};H^1_\rho(\Omega))} = 0,$$

which implies that $V = w_1 - w_2 = 0$, and hence $w_1 = w_2 \in L^2(0, T; H^1_\rho(\Omega))$. $\square$

Remark. It seems that our results still hold for the more general mixed nonlinear nonlocal problem

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial t\partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right) = f\left(x, t, u, \frac{\partial u}{\partial x}\right), \tag{6.45}$$

$$u(x,0) = u_0(x), \tag{6.46}$$

$$u_x(l,t) = \phi(x), \qquad \int_0^l u\,dx = E(t). \tag{6.47}$$

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