High-order approximation of Pearson diffusion processes

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\textbf{A B S T R A C T}

This paper focuses on Pearson diffusions and the spectral high-order approximation of their related Fokker–Planck equations. The Pearson diffusions is a class of diffusions defined by linear drift and quadratic squared diffusion coefficient. They are widely used in the physical and chemical sciences, engineering, rheology, environmental sciences and financial mathematics. In recent years diffusion models have been studied analytically and numerically primarily through the solution of stochastic differential equations. Analytical solutions have been derived for some of the Pearson diffusions, including the Ornstein–Uhlenbeck, Cox–Ingersoll–Ross and Jacobi processes. However, analytical investigations and computations for diffusions with so-called heavy-tailed ergodic distributions are more difficult to perform. The novelty of this research is the development of an accurate and efficient numerical method to solve the Fokker–Planck equations associated with Pearson diffusions with different boundary conditions. Comparisons between the numerical predictions and available time-dependent and equilibrium analytical solutions are made. The solution of the Fokker–Planck equation is approximated using a reduced basis spectral method. The advantage of this approach is that many models for pricing options in financial mathematics cannot be expressed in terms of a stochastic partial differential equation and therefore one has to resort to solving Fokker–Planck type equations.

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\textbf{1. Introduction}

The Fokker–Planck (or forward Kolmogorov) equation arises in a wide variety of problems of physical and biological interest including non-Newtonian flows, plasma physics, biophysics and financial mathematics. Despite all the work that has been undertaken by many scientists in this area there remain many unresolved problems associated with this equation. For example, there exists only a few exact solutions to transient problems. Beyond these, one has to resort to numerical investigations. The situation is exacerbated when the Fokker–Planck equation is nonlinear. In this paper we investigate numerical solutions of Fokker–Planck equations for so-called Pearson diffusions.

The Pearson class of diffusions is very attractive since their ergodic (stationary, equilibrium) solutions satisfy the famous Pearson equation [1]. The classification of the stationary solutions of Pearson diffusions gives six classes of probability distributions, three of them with an infinite system of moments (Gaussian, Gamma, Beta) and three others with only a finite number of moments (inverted Gamma, Student and Fisher–Snedecor). The latter are known as heavy-tailed distributions. Among the class of Pearson diffusions, the Ornstein–Uhlenbeck (OU), Cox–Ingersoll–Ross (CIR) [2], and Jacobi processes have been well studied and applied in many practical situations. However, the statistical analysis of heavy-tailed Pearson diffusions such as reciprocal gamma, Student and Fisher–Snedecor diffusions are relatively new (see [3–11], for example).
It is now generally accepted that heavy-tailed distributions are useful for the analysis of a range of problems in application areas such as communication networks, risky assets and insurance modelling. Since for most diffusion models a closed form representation of the transition density is unavailable and its statistical analysis is often very complicated, a numerical approach becomes unavoidable.

Most effort in recent years has been expended in studying different types of Pearson diffusions using stochastic differential equations (SDEs) applying both analytical and numerical methods (see [5,12] and references therein, for example). Although stochastic methods have a number of advantages, there are still a number of issues which need further consideration and work. One of them, for example, is the issue of the computation of the non-negative diffusion term in the stochastic representation of the CIR process, which has been discussed in [13]. To elaborate on this issue further, consider the well-known CIR diffusion as SDE driven by standard Brownian motion, \( B_t \), \( t \geq 0 \), that is

\[
dX_t = (a - bX_t)dt + \sigma \sqrt{X_t}dB_t, \quad t \geq 0,
\]

with real parameters \( a \geq 0 \), \( b \in \mathbb{R} \), and \( \sigma > 0 \). This model is widely used for modelling interest rates in financial mathematics. It is known that the strong solution is unique and preserves the non-negativity of the initial data. For the real-valued numerical implementation of the \( \sqrt{X_t} \) term in stochastic simulations, the non-negativity of the numerical approximation is of paramount importance. Standard numerical schemes do not conserve the non-negativity of the solutions and overcome the problem in an ad-hoc fashion by evaluating \( \sqrt{X_t} \) instead of \( \sqrt{X_t} \) (see [14,15], for example). Indeed using usual schemes, such as Euler or Milstein, when discretising the CIR process located at 0 can lead to negative values for which the square root is not defined. There are a number of possible schemes that avoid this problem and these have been discussed in [16]. This problem associated with non-negative diffusion terms is typical for all Pearson diffusions with the exception of the OU process.

Analytical solutions have been derived for some of the Pearson diffusions, including the OU, CIR and Jacobi processes, but there has been less success for time-dependent investigations and for diffusions with heavy-tails. In this paper we present a numerical scheme for solving the associated Fokker–Planck equations which have ergodic Fisher–Snedecor, reciprocal gamma or Student distributions. In spite of the simple and tractable analytical forms of stationary (ergodic) distributions, the analytical form of time-dependent solutions are either not tractable and very complicated or even unknown. In many practical situations one has to resort to numerical techniques to determine the solution of such equations.

In this paper we utilise techniques that have been developed to perform direct numerical simulations of the Fokker–Planck equation arising from atomistic and mesoscopic models in theoretical polymer rheology [17–20]. Typically, these models involve a large number of configurational degrees of freedom. This means that all the standard methods of discretisation require a substantial number of unknowns to obtain an accurate representation of the distribution function. For this reason, initial numerical work in this area concentrated on the solution of equivalent stochastic differential equations.

Recently, however, Ammar et al. [21] devised a method for significantly reducing the number of degrees of freedom involved in the solution of the Fokker–Planck equation in high dimensions using a basis reduction method. In their paper, Ammar et al. [21] approximated the solution of the Fokker–Planck equation using a finite element basis. This technique was extended to nonlinear rheological models [22] and to transient problems [23]. High-order spectral approximations were developed in [24]. Efficiency was achieved through a dynamic construction of the basis which ensured that only basis functions containing the most representative information of the solution are retained. This technique is used in the present paper to generate high-order approximations of Pearson diffusion processes.

2. Pearson’s diffusions and their classification

The study of time-dependent homogeneous diffusion processes with transition density, \( p(x, t) \), and invariant distributions from the Pearson family dates from the 1930’s, when Kolmogorov [25] investigated the Fokker–Planck or forward Kolmogorov equation

\[
\frac{\partial}{\partial t}p(x, t) = -\frac{\partial}{\partial x} [a(x)p(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x)p(x, t)], \quad x \in \mathbb{R}, \quad t \geq 0,
\]

(1)

with a linear drift, \( a(x) = a_1 x + a_0 \), and a quadratic squared diffusion, \( b(x) = 2d(x) \), and \( d(x) = b_2 x^2 + b_1 x + b_0 \) and observed that the invariant density \( p(\cdot) \) satisfies the differential equation

\[
\frac{p'(x)}{p(x)} = \frac{a(x) - d'(x)}{d(x)} = \frac{(a_1 - 2b_2)x + (a_0 - b_1)}{b_2 x^2 + b_1 x + b_0}, \quad x \in \mathbb{R},
\]

(2)

introduced in [1], in order to unify some of the most important statistical distributions.

It seems appropriate to call this important class of processes, Pearson diffusions. The class of Pearson diffusions is closed under transitions and scale transformations. Forman and Sørensen [4], for example, provide a classification of the stationary solutions of these processes (see also [5,26]).
A classification of Pearson diffusions in terms of six basic subfamilies may be achieved using criteria based on the degree, \(\text{deg}(d)\), of the polynomial \(d(x)\) appearing in the denominator of the Pearson equation (2) and the sign of the leading coefficient \(b_2\) and the discriminant \(\Delta(d)\) in the quadratic case

\[
\Delta(d) = b_1^2 - 4b_2b_0. \tag{3}
\]

The six cases can be described in the following way:

1. Ornstein–Uhlenbeck: \(\text{deg}(d) = 0\); invariant density normal.
2. CIR diffusion: \(\text{deg}(d) = 1\); invariant density gamma.
3. Jacobi diffusion: \(b_2 < 0\); \(\text{deg}(d) = 2\); \(\Delta(d) > 0\); invariant density beta.
4. Fisher–Snedecor diffusion: \(b_2 > 0\); \(\text{deg}(d) = 2\); \(\Delta(d) > 0\); invariant density Fisher–Snedecor.
5. Reciprocal gamma diffusion: \(b_2 > 0\); \(\text{deg}(d) = 2\); \(\Delta(d) = 0\) invariant density reciprocal gamma.
6. Student diffusion: \(b_2 > 0\); \(\text{deg}(d) = 2\); \(\Delta(d) < 0\); invariant density Student.

For the terminology and notation associated with Markov processes see [27], for example. Also note that to identify the diffusion processes boundary conditions should be specified in each case.

Boundary points \((D^1, D^2)\) for OU, CIR and Jacobi processes are described using classical Feller classification [28,27]. However, for the heavy-tailed processes the oscillatory/non-oscillatory (O/NO [29,30]) and Weyl’s limit-point/limit-circle (see [30,31]) classification schemes should be applied. These latter two classifications are not widely used and are not described in any of the classical texts in the field. The behaviour of the transition density, \(p(x, t)\), depends on the particular classification of the boundaries that is implemented within the high-order approximation. A numerical example is provided in the case of the Fisher–Snedecor diffusion to illustrate this point.

3. Spectral representation

A function \(f\) is an eigenfunction of \(\mathcal{G}\) (infinitesimal generator of the corresponding diffusion) if there exists a positive real number \(\lambda\) the eigenvalue, such that

\[
\mathcal{G}f + \lambda f = 0.
\]

Note that \(-\mathcal{G}\) is the Sturm–Liouville operator. In the Pearson case, this differential equation is of hypergeometric type

\[
d(x)f'' + a(x)f' + \lambda f = 0, \tag{4}
\]

where \(d(x)\) and \(a(x)\) are polynomials of at most second and first degrees, respectively, and \(\lambda\) is a real or complex spectral parameter.

It turns out that for \(\lambda = \lambda_n\), the equation has a particular solution of the form \(f(x) = Q_n(x)\) which is a polynomial of degree at most \(n\). These polynomials are known as the classical orthogonal polynomials of a continuous variable.

The analysis of the spectrum of the corresponding infinitesimal generators of the OU, CIR and Jacobi processes is simple and purely discrete with classical orthogonal polynomials as corresponding eigenfunctions: Hermite, Laguerre and Jacobi polynomials, respectively (see [32], for example). In the case of Pearson diffusions with heavy-tailed invariant distributions, the spectrum of the infinitesimal generator consists of two disjoint parts: the discrete spectrum (consisting of finitely many simple eigenvalues) and the essential spectrum. Furthermore, in all these cases the corresponding eigenfunctions are less known finite systems of orthogonal polynomials: Bessel polynomials for reciprocal gamma diffusion, Romanovski polynomials for Student diffusion [33] and Fisher–Snedecor polynomials for Fisher–Snedecor diffusion (see [32,34–37,10]).

Although a knowledge of the eigenfunctions of the Pearson diffusions is not required for the development of high-order approximations, they play a fundamental role in the analytical solutions. In the case of heavy-tailed Pearson diffusions, this information is not widely available and so it is provided here so that the problem is fully defined in terms of the infinitesimal generator and the associated eigenvalues and eigenfunctions.

4. Implicit scheme for the Fokker–Planck equation

In this section we present the numerical scheme that generates the fundamental solution of the Fokker–Planck Eq. (1) (see [38]).

4.1. Weak formulation of the problem

We consider the Fokker–Planck equation (FPE) (1) and this equation is valid for the conditional density, \(p(x, t)\), of the variable \(X_t = x|X_0 = x_0\) of the corresponding homogeneous Markov process with the state space, \(\Omega\), that is, for any initial \(x_0, t_0\), with the initial condition

\[
p(x, t)\big|_{t=t_0} = \delta(x - x_0). \tag{5}
\]
However, using the definition of the conditional probability one can see that it is also valid for $p(x, t)$ with the initial condition

$$p(x, t)|_{t=t_0} = p(x, t_0)$$

(6)

which is less singular than (5). In other words, the numerical solution is not sensitive to the choice of initial condition and therefore we can choose the initial condition to be any integrable function in $\Omega$.

We also use the following notation, $p(x_1; x_0, t) = \frac{\partial}{\partial x} P(X_t \leq x_1 | X_0 = x_0)$.

In this paper, a modification of the scheme that was used in [24] to predict the evolution of the configuration pdf associated with kinetic theory models in polymer dynamics is described. The method is based on an adaptive reduced basis approximation to the Fokker–Planck equation. The method was first proposed in [21] in the context of the finite element method. The idea underlying basis reduction comes from finding an approximate singular value decomposition (SVD) and it provides us with an efficient choice of basis functions that contains the most representative information about the solution. Leonenko and Phillips [24] have extended this procedure to develop spectral approximations to the solution of the high-dimensional Fokker–Planck equation derived from the equation of motion for the beads in an elastic dumbbell model for dilute polymer solutions. In this paper we use an implicit scheme to solve the one-dimensional problems arising in the study of Pearson diffusion processes.

The backward Euler discretisation of Eq. (1) is

$$\frac{p(x, t^{n+1}) - p(x, t^n)}{\Delta t} = -\frac{\partial}{\partial x}[a(x)p(x, t^{n+1})] + \frac{1}{2} \frac{\partial^2}{\partial x^2}[b(x)p(x, t^{n+1})],$$

(7)

where $\Delta t$ is the time step and $t^n = n\Delta t$. We have assumed that the drift and diffusion coefficients are independent of time. If $p^*$ is a suitable test function that satisfies homogeneous initial and boundary conditions, then we can formulate a weak formulation of this semi-discrete problem by multiplying Eq. (7) by $p^*$ and integrating over the interval $I = [c, d]$:

$$\int_I p^* \left( \frac{p(x, t^{n+1}) - p(x, t^n)}{\Delta t} \right) \ dx + \int_I p^* \frac{\partial}{\partial x}[a(x)p(x, t^{n+1})] \ dx - \frac{1}{2} \int_I p^* \frac{\partial^2}{\partial x^2}[b(x)p(x, t^{n+1})] \ dx = 0.$$  

(8)

Theoretically, $I = (-\infty, \infty)$ or $I = [0, \infty)$ for Pearson diffusions. However, for computational reasons, the interval is truncated to one of finite length. The truncation length is chosen so that there is only a small relative difference in the approximation when the problem is solved on larger intervals.

4.2. Representation of the solution and spectral discretisation

The solution of Eq. (8) is assumed to have the following form:

$$p(x, t^{n+1}) = \sum_{j=1}^{\infty} \alpha_j^{n+1} g_j^{n+1}(x),$$

(9)

where the coefficients $\alpha_j^{n+1}$ and basis functions $g_j^{n+1}(x)$ are time-dependent. To simplify the notation we drop the superscript $n + 1$ on terms on the right-hand side of (9). Note that the test function can be represented as follows:

$$p^*(x, t^{n+1}) = \sum_{j=1}^{\infty} \alpha_j^* g_j(x).$$

(10)

The basis functions $g_j(x)$ are represented using Lagrangian interpolating polynomials associated with the Gauss–Lobatto Legendre (GLL) nodes, i.e. the roots of $(1 - x^2)L_N'(x) = 0$, where $L_N(x)$ is the Legendre polynomial of degree $N$. More specifically, we have

$$g_j(x) = \sum_{k=0}^{N} g_{j,k} h_k(x)$$

(11)

and $g_{j,k} = g_j(x_k)$, $k = 0, \ldots, N$, are unknown coefficients and $x_k$, $k = 0, \ldots, N$, are the GLL nodes. In the case of an approximation of degree $N$, the Lagrangian interpolants, $h_k$, are defined by

$$h_k(x) = \frac{(1 - x^2)L_N'(x)}{N(N + 1)L_N(x_k)(x - x_k)}.$$  

(12)

The entries of the Legendre collocation differentiation matrix, $\hat{D}_N$, are given by

$$\hat{D}_{N,j} = h_k'(x_j) = \begin{cases} \frac{L_N(x_j)}{L_N(x_k)} \frac{1}{(x_j - x_k)} , & j \neq k, \\ \frac{(N + 1)N}{4} , & j = k = 0, \\ 0 , & \text{otherwise}. \end{cases}$$  

(13)
Note that the integrals in Eq. (8) are approximated using the Gauss–Lobatto Legendre quadrature rule
\[ \int_{-1}^{1} f(x)dx \approx \sum_{j=0}^{N} f(x_j)w_j, \] (14)
where the weights are given by
\[ w_j = \frac{2}{N(N + 1)(L_N(x_j))^2}. \] (15)
This quadrature rule is exact whenever \( f \) is a polynomial of degree \( 2N - 1 \) or less.

The procedure for determining the numerical approximation at each time step comprises two stages. The basis functions are determined dynamically using an enrichment procedure to augment the basis followed by the determination of the coefficients in the expansion using projection. The algorithm is initiated with the determination of the first basis function \( g_1(x) \) using enrichment.

At the \( J \)th stage of the algorithm, given the basis functions \( g_j(x), j = 1, \ldots, J \), the coefficients \( \alpha_j, j = 1, \ldots, J \), are computed using projection. This is followed by an enrichment of the basis in which \( g_{J+1}(x) \) is determined. The reduced basis method proceeds in an iterative manner until convergence is obtained. More precisely, the algorithm at a given time step is terminated when \( |\alpha_j| < 10^{-7} \).

4.3. Projection stage

The purpose of the projection stage is to compute the coefficients \( \alpha_j, j = 1, \ldots, J \), in the representation (9). In this stage, the basis functions \( g_j(x), j = 1, \ldots, J \), are known. Thus, inserting the approximation (9) and (10) into the discrete weak formulation of the problem (8) and simplifying the results yields the following linear system:
\[ M\alpha = \mathbf{v}, \] (16)
where \( \alpha = (\alpha_1, \ldots, \alpha_J)^T \) and \( M \) is the \( J \times J \) matrix with entries defined by
\[ M_{ij} = g_i^T (B + \Delta tA)g_j \] (17)
and the components of the right-hand side vector \( \mathbf{v} \) are given by
\[ v_i = g_i^T B\mathbf{p}^n. \] (18)
In Eqs. (17) and (18) the vectors \( \mathbf{g}_i \) and \( \mathbf{p}^n \) are given by
\[ \mathbf{g}_j = (g_j(x_0), \ldots, g_j(x_N))^T \]
and
\[ \mathbf{p}^n = (p(x_0, t^n), \ldots, p(x_N, t^n))^T. \]
The entries of the \( N \times N \) matrices \( A \) and \( B \) are defined by
\[ A_{m,n} = -w_n a(x_n) (\hat{D}_N)_{n,m} + \frac{1}{2} \sum_{l=0}^{N} b(x_l) w_l (\hat{D}_N)_{l,m}(\hat{D}_N)_{l,n}, \] (19)
\[ B_{m,n} = w_m \delta_{m,n}. \] (20)
Note that the basis functions \( \{\mathbf{g}_j\} \) are constructed to be orthonormal with respect to the discrete inner product, i.e. \( \mathbf{g}_i^T B\mathbf{g}_j = \delta_{ij} \).

Once \( \alpha \) has been determined by solving (16), we have the best rank \( J \) approximation to the solution and we proceed to the enrichment stage. In practice, we have found that \( J \leq 3 \) and so the system (16) is very small and computationally inexpensive to solve.

4.4. Enrichment stage

In this stage, the basis is enriched by adding an additional function \( r(x) \) to the basis. Assuming knowledge of \( g_j(x) \), and \( \alpha_j, j = 1, \ldots, J \), we write the new approximation in the form:
\[ p(x, t^{n+1}) = \sum_{j=1}^{J} \alpha_j g_j(x) + r(x), \] (21)
where \( r(x) \) is expressed in the form
\[ r(x) = \sum_{k=0}^{N} r_k h_k(x). \] (22)
The test functions used in the weak formulation are given by
\[ p^j(x) = h_j(x), \quad j = 0, \ldots, N. \]
This results in the following linear system for the unknown coefficients \( r_k, k = 0, \ldots, N \), in the representation of the new basis function:
\[
(B + \Delta tA)r = c,
\]
where
\[
c = Bp^n - \sum_{j=1}^{J} \alpha_j (B + \Delta tA)g_j.
\]
The solution of this linear system furnishes the coefficients \( r_k \) in the expansion of the new basis function \( r(x) \). To normalise the solution we evaluate
\[
r_{\text{new}} = \frac{r}{\sqrt{r^TBr}}.
\]
Finally, we set
\[
g_{J+1}(x) = r_{\text{new}}(x).
\]
Concerning the computational cost, this step consumes the main part of the global computing time. Note, however, that the solver normally converges very quickly and in the one-dimensional case only one enrichment step is usually required.

In the case of a \( d \)-dimensional Fokker–Planck equation the system (23) becomes non-linear and iterative techniques are required for its solution (see [24]).

5. Pearson diffusions

5.1. The Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process is obtained by setting \( d(x) = 1 \) in Eq. (2). This diffusion is a stationary or transient OU process (also known in the financial literature as the Vasiček model) and defined on \((-\infty, \infty)\), where \( a \in \mathbb{R}, b > 0, \sigma \in \mathbb{R}_+ \), with eigenvalues \( \lambda_n = nb \).

The Fokker–Planck representation has the form:
\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} (a - bx)p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2 p(x, t)
\]
(25)
The ergodic distribution is Gaussian
\[
\text{gauss}(x) = \frac{1}{\sqrt{\frac{\pi \sigma^2}{b}}} \exp \left( -\frac{b}{\sigma^2} \left( x - \frac{a}{b} \right)^2 \right).
\]
The boundary conditions \( D^1 = -\infty \) and \( D^2 = +\infty \) are natural boundaries for all choices of parameters.

It is well known that the Hermite polynomials, \( H_n(x) \) (eigenfunctions) (see [32], for example) are orthogonal with respect to standard Gaussian density.

The transition probability density has the expansion
\[
p(x_1; x_0, t) = \frac{1}{\sqrt{\frac{\pi \sigma^2}{b}}} \exp \left( -\frac{b}{\sigma^2} \left( x - \frac{a}{b} \right)^2 \right) \left( \sum_{n=0}^{\infty} \frac{e^{-bnt}}{2^n n!} H_n(y_0) H_n(y_1) \right),
\]
where
\[
y_i = \sqrt{\frac{b}{\sigma^2}} \left( x_i - \frac{a}{b} \right), \quad i = 0, 1.
\]
Also we can write the transient solution in terms of the Gaussian distribution as follows:
\[
p(x; x_0, t) = \frac{1}{\sqrt{\frac{\pi \sigma^2}{b} (1 - e^{-2bt})}} \exp \left( -\frac{(x - \frac{a}{b} - (x_0 - \frac{a}{b}) e^{-b t})^2}{\frac{a^2 b}{1 - e^{-2bt}}} \right).
\]
(26)
In Fig. 1(a) we plot the numerical solution of the time-dependent Fokker–Planck Eq. (25) obtained using a spectral approximation with \( N = 60 \). The model parameters were chosen to be \( a = 2, b = 1 \) and \( \sigma^2 = 1 \). At steady state a single basis function is sufficient to represent the solution. The exact solution to the problem is shown in Fig. 1(b). A comparison of
the numerical approximation with the exact solution at equilibrium is shown in Fig. 1(c). Excellent agreement is obtained confirming that the spectral approximation is able to provide an accurate representation of solutions to this equation.

Convergence of the numerical solution as the order of polynomial approximation, \( N \), is increased is shown in Fig. 2. Convergence in \( L_2 \)-norm is presented in Table 1. Here we present the convergence of the approximation at equilibrium. The approximation converges rapidly with \( N \) and very little difference can be seen between the profiles for \( N = 10 \) and \( N = 40 \). However, to ensure the transient solutions are approximated with good precision we choose \( N = 60 \) for all subsequent calculations.

### 5.2. The square root/CIR process

In financial mathematics this process is known as the CIR process and it is defined on domain \( D = [0, \infty) \).

The Fokker–Planck representation has the form:

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} (a - bx)p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2 x p(x, t).
\] (27)
The boundary conditions are:

\[
D^1 = \begin{cases} 
\text{exit}, & \alpha \leq -1 \\
\text{regular}, & -1 < \alpha < 0 \\
\text{entrance}, & 0 \leq \alpha.
\end{cases}
\]

The invariant distribution is the Gamma density

\[
\gamma(x) = \frac{\theta^\alpha}{\Gamma(\alpha + 1)} x^\alpha e^{-\theta x}, \quad x > 0,
\]

where

\[
\alpha = \frac{2a}{\sigma^2} - 1, \quad \theta = \frac{2b}{\sigma^2}.
\]

The orthogonal polynomials (eigenfunctions) are Laguerre polynomials [32] and \(\lambda_n = n\).

The eigenvalue expansion for the transition probability density is given by

\[
p(x; x_0, t) = \gamma(x) \sum_{n=0}^{\infty} e^{-bt} \frac{n!}{(\alpha + 1)n} L_n^{(\alpha)}(\theta x_0)L_n^{(\alpha)}(\theta x)
\]

or

\[
p(x; x_0, t) = c_t e^{-(u+v)} \left( \frac{v}{u} \right)^{q/2} I_q(2\sqrt{uv}),
\]

where

\[
c_t = \frac{2b}{\sigma^2(1-e^{-bt})} u = c_t x_0 e^{-bt}, \quad v = c_t x, \quad q = 2a/\sigma^2 - 1,
\]

and \(I_q\) is the modified Bessel function of the first kind

\[
I_q(x) = \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^{2k+q} \frac{1}{k!\Gamma(k+q+1)}.
\]

In Fig. 3(a) we plot the numerical solution of the time-dependent Fokker–Planck Eq. (27) obtained using a spectral approximation with \(N = 60\). The boundary behaviour is specified using the following classification of the endpoints: \(D^1\) is entrance and \(D^2\) is natural. The model parameters were chosen to be \(a = 1, b = 1\) and \(\sigma^2 = 1\). Again at steadystate a single basis function is sufficient to represent the solution. The exact solution to the problem is shown in Fig. 3(b). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 3(c). Excellent agreement is obtained confirming that the spectral approximation is able to provide an accurate representation of solutions to this equation and captures the correct behaviour as \(x \to \infty\).

### 5.3. The Jacobi diffusion

In financial mathematics, the Jacobi diffusion is used for modelling exchange rates in target zones [39] and it is defined on domain \(D = [0, A]\).

The Fokker–Planck representation has the form:

\[
\frac{\partial}{\partial t} p(x, t) = - \frac{\partial}{\partial x} (a - bx)p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2 x(A-x)p(x, t).
\]

The invariant distribution is

\[
\beta(x) = \frac{x^\beta (A-x)^\alpha}{A^{\alpha+1}B(\alpha+1,\beta+1)}, \quad s(x) = x^{-\beta-1}(A-x)^{-\alpha-1}, \quad x \in [0, A],
\]

where \(B^{(\alpha, \beta)}\) is the beta distribution

\[
B^{(\alpha, \beta)} = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}.
\]
and
\[
\alpha = \frac{2b}{\sigma^2} - \frac{2a}{\sigma^2 A} - 1, \quad \beta = \frac{2a}{\sigma^2} - 1, \quad \alpha > -1, \quad \beta > -1.
\]

The boundary behaviour for the Jacobi process is the same as for CIR process at the left boundary (28) with \( \beta \) replacing \( \alpha \). On the right boundary we impose (28). Note that in the case when \( a > 0 \) and \( b > 0 \) and \( \frac{a}{b} < A \) (which means that the mean-reverting level lies in the interval \((0, A)\)) both boundaries are not exit boundaries.

The discrete spectrum of the generator is
\[
\lambda_n = \frac{\sigma^2}{2} \left( n - 1 + \frac{2b}{\sigma^2} \right).
\]

The orthogonal polynomials (eigenfunctions) in this case are Jacobi \([32]\).

The eigenvalue expansion of the transition probability density is
\[
p(x_1; x_0, t) = \beta(x_1) \sum_{n=0}^{\infty} \frac{e^{-\lambda_n t}}{p_n^2} p^{(n, \beta)}_n(y_0) p^{n, \beta}_n(y_1),
\]
where
\[
p_n^2 = \frac{(\alpha + 1)_n (\beta + 1)_n}{(\alpha + \beta + 2)_n (2n + \alpha + \beta + 1)!},
\]
\[
y_i = \left( \frac{2x_i}{A} - 1 \right), \quad i = 0, 1 \quad \text{and} \quad (m)_n = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)}.
\]

Numerical solutions for the Fokker–Planck equation related to Jacobi diffusion can be obtained in the same manner as the OU and CIR processes.

5.4. The Fisher–Snedecor diffusion

The corresponding Fokker–Planck equation for the Fisher–Snedecor diffusion is:
\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \left[ \theta \left( x - \frac{\beta}{\beta - 2} \right) \right] p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ \frac{4\theta}{\alpha(\beta - 2)} x(\alpha x + \beta) \right] p(x, t),
\]
where \( \alpha > 0 \) and \( \beta > 2 \). The invariant distribution of this diffusion is the Fisher–Snedecor distribution \( FS(\alpha, \beta) \) with the probability density function

\[
fs(x) = \frac{\alpha^\frac{\alpha}{2}\beta^\frac{\beta}{2}}{B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} \frac{x^{\frac{\alpha}{2}-1}}{(\alpha x + \beta)^{\frac{\alpha}{2}+\frac{\beta}{2}}} \quad x > 0,
\]

and \( B(\alpha, \beta) \) is the beta function, while \( \alpha \) and \( \beta \) are some shape parameters.

The restriction \( \beta > 2 \) on the value of the parameter \( \beta \) ensures the existence of the mean of the invariant distribution. Therefore, the quadratic polynomial \( d(x) = x(\alpha x + \beta) \) with the positive first coefficient \( (b_2 > 0) \) and positive discriminant \( \Delta(d) > 0 \) characterises the Fisher–Snedecor diffusion in the class of Pearson diffusions.

In this case, the negative infinitesimal generator \( -\mathfrak{g} \) has a finite set of simple eigenvalues

\[
\lambda_n = \frac{\theta}{\beta - 2} n(\beta - 2n), \quad 0 \leq n \leq \left\lfloor \frac{\beta}{4} \right\rfloor, \quad \beta > 2,
\]

in \([0, \Lambda]\), and the corresponding finite number of eigenfunctions are given by the Rodrigues formula

\[
Q_n(x) = K_n x^{1-\frac{n}{2}} (\alpha x + \beta)^{\frac{\alpha}{2}+\frac{\beta}{2}} \frac{d^n}{dx^n} \left( 2^n x^{\frac{n}{2}+n-1} (\alpha x + \beta)^{n-\frac{\alpha}{2}} \right),
\]

where

\[
K_n = (-1)^n \sqrt{\frac{(n+1)}{n} n!}, \quad d_n = 2^n \frac{\alpha}{2} I_n \left( 2n - \frac{\beta}{2} \right) / \Gamma \left( n + \frac{\beta}{2} \right), \quad I_n = \left( \frac{2\beta^2}{\alpha} \right) \frac{n!}{B \left( \frac{\alpha}{2}, \frac{\beta}{2} \right)},
\]

The polynomials are orthogonal with respect to the density (36) [35,36,34,10,31] and purely absolutely continuous of multiplicity one in \((\Lambda, \infty)\) where

\[
\Lambda = \frac{\theta \beta^2}{8(\beta - 2)}.
\]

Therefore,

\[
p(x; x_0, t) = fs(x) \left[ \sum_{n=0}^{\left\lfloor \frac{\beta}{4} \right\rfloor} e^{-\lambda_n t} Q_n(x_0)Q_n(x) + \int_{\Lambda}^{\infty} e^{-\lambda t} \psi(x_0, -\lambda) \psi(x, -\lambda) d\lambda \right],
\]

where \( \psi \) are solutions of the Sturm–Liouville equation normalised with respect to the speed measure, \( m(x) \). These solutions can be defined in the form of Gauss hypergeometric functions \( _2F_1 \) and have a complicated form (see also [10]) which make them difficult to evaluate. Thus, numerical methods seem inescapable.

The boundary classification below permits us to compute a cutoff \( \Lambda \). Note the Feller classification is not enough in this case and we need to apply oscillatory/non-oscillatory and limit point/limit circle or Weyl's classifications (see [30]).

The boundaries of the state space of Fisher–Snedecor diffusion with parameters \( \alpha > 0 \) and \( \beta > 2 \) are classified as follows:

1. Boundary \( D^1 = 0 \) is regular for \( \alpha \leq 2 \) and entrance otherwise, while \( D^2 = \infty \) is natural boundary for all \( \alpha > 0 \).
2. For all \( \alpha > 0 \) boundary \( D^1 = 0 \) is non-oscillatory, while \( D^2 = \infty \) is oscillatory/non-oscillatory with the unique positive cutoff \( \Lambda \).
3. Boundary \( D^1 = 0 \) is of limit-circle type for \( \alpha < 4 \) and of limit-point type otherwise while boundary \( D^2 = \infty \) is of limit-point type for every positive value of \( \alpha \).

In Fig. 4(a) we plot the numerical solution of the time-dependent Fokker–Planck equation (35) obtained using a spectral approximation with \( N = 60 \). The boundary behaviour is specified using the classification (2) above. The model parameters were chosen to be \( \alpha = 5, \beta = 20 \) and \( \theta = 1 \). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 4(b). Note that this plot uses log scales. The numerical approximation is able to capture the behaviour of the analytical solution as \( x \to \infty \).

In Fig. 5(a) we plot the numerical solution of the time-dependent Fokker–Planck equation (35) obtained using a spectral approximation with \( N = 60 \). The boundary conditions are natural for \( D^2 = \infty \) and regular for \( D^1 \). The model parameters were chosen to be \( \alpha = 1.5, \beta = 10 \) and \( \theta = 1 \). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 5(b). Again the numerical approximation is able to capture the behaviour of the analytical solution as \( x \to \infty \).
5.5. The reciprocal gamma diffusion

The Fokker–Planck equation for the reciprocal gamma diffusion is

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \left[ \frac{\theta}{\beta - 1} \left( x - \frac{\alpha}{\beta - 1} \right) \right] p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ \frac{2\theta}{(\beta - 1) x^2} \right] p(x, t),
\]

(41)

where \( \alpha > 0 \) and \( \beta > 1 \). The restriction \( \beta > 1 \) ensures the existence of the mean of the invariant distribution. The equation in this form was first introduced in [40]. Peškir [41] observed that the Dothan model [42], which is one of the most popular short term interest rate models, corresponds to the FPE of this type.

The invariant distribution of this diffusion is the reciprocal gamma distribution with the probability density function

\[
r_g(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{-\beta-1} e^{-\frac{x}{\alpha}}, \quad x > 0,
\]

(42)

where \( \alpha > 0 \) and \( \beta > 0 \) are shape parameters. Moreover, the tail of the reciprocal gamma distribution with density (42) decreases like \( x^{-(1+\beta)} \) and this distribution is heavy-tailed.

Therefore, the quadratic polynomial \( d(x) = x^2 \) with positive coefficient \( (b_2 > 0) \) and zero discriminant \( \Delta(d) = 0 \) characterizes the reciprocal gamma diffusion in the class of Pearson diffusion.

In this case, the negative infinitesimal generator \( (-\mathcal{G}) \) has a finite set of simple eigenvalues

\[
\lambda_n = \theta n \left( \frac{\beta - n}{\beta - 1} \right), \quad 0 \leq n \leq \left\lceil \frac{\beta}{2} \right\rceil, \quad \beta > 1,
\]

(43)

in \( [0, \Lambda] \), and a corresponding finite number of eigenfunctions which are known as Bessel polynomials and given by the Rodrigues formula:

\[
Q_n(x) = K_n x^{\beta+1} e^{\alpha/x} \frac{d^n}{dx^n} \left[ x^{2n-(\beta+1)} e^{-\alpha/x} \right],
\]

(44)
where

\[ K_n = (-1)^n / \sqrt{(1-1)^n} n! d_n I_n, \]
\[ d_n = (-1)^n (\beta - n) \Gamma(\beta - n) / ((\beta - 2n) \Gamma(\beta - 2n)), \quad I_n = \alpha^{2n} \Gamma(\beta - 2n) / \Gamma(\beta), \]

These polynomials are orthogonal with respect to the density (42), see [31,34–36,10] and are known as Bessel polynomials. They are purely absolutely continuous of multiplicity one in \( \langle \Lambda, \infty \rangle \), where

\[ \Lambda = \frac{\theta \beta^2}{4(\beta - 1)}. \] (45)

That is

\[ p(x; x_0, t) = r g(x) \left[ \sum_{n=0}^{\infty} e^{-\lambda t} Q_n(x_0) Q_n(x) + \int_{\Lambda}^{\infty} e^{-\lambda t} \psi(x_0, -\lambda) \psi(x, -\lambda) d\lambda \right], \] (46)

where \( \psi \) are solutions of the Sturm–Liouville equation normalised with respect to the speed measure, \( m(x) \). These solutions can be defined in the form of Gauss hypergeometric functions \( _2F_1 \) and have a complicated form, see [10], which is hard to compute. Again, numerical methods are the only option to compute the solution of the Fokker–Planck equation.

The boundary classification below permits us to compute a cutoff \( \Lambda \). Note the Feller classification is not enough in this case and we need to apply oscillatory/non-oscillatory and limit point/limit circle or Weyl’s classifications [30]. The boundaries of the state space of the reciprocal gamma diffusion with parameters \( \alpha > 0 \) and \( \beta > 1 \) are classified as:

1. Boundary \( D^1 = 0 \) is entrance and \( D^2 = \infty \) is natural boundary.
2. Boundary \( D^1 = 0 \) is non-oscillatory and \( D^2 = \infty \) is oscillatory/non-oscillatory with unique positive cutoff \( \Lambda \).
3. Boundary \( D^1 = 0 \) is of limit-circle type and \( D^2 = \infty \) is of limit-point type.

In Fig. 6(a) we plot the numerical solution of the time-dependent Fokker–Planck equation (41) obtained using a spectral approximation with \( N = 100 \). The model parameters were chosen to be \( \alpha = 2, \beta = 20 \) and \( \theta = 0.5 \). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 6(b). The numerical approximation is able to capture the behaviour of the analytical solution as \( x \to \infty \) and also the very large peak around \( x = 0.1 \).

5.6. The Student diffusion

5.6.1. Symmetric version

The Student diffusion has corresponding Fokker–Planck equation:

\[ \frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \left[ -\theta (x - \mu) \right] p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ \frac{2\theta \delta^2}{(\nu - 1)} \left( 1 + \left( \frac{x - \mu}{\delta} \right)^2 \right) \right] p(x, t), \] (47)

where \( \nu > 1, \delta > 0, \) and \( \mu \in \mathbb{R} \). The Student diffusion was first studied in [43] in 1964 and recently in [3,7,8,10]. This diffusion can also be observed as special case of the so-called hypergeometric diffusion introduced in [29,31].
The invariant distribution of this diffusion is Student distribution with parameter \( \nu > 0 \) degrees of freedom with the probability density function

\[
st_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\delta \sqrt{\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \left(\frac{x - \mu}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R},
\]

(48)

where \( \delta > 0 \) is a scale parameter and \( \mu \in \mathbb{R} \) is a location parameter.

Moreover, the left and right tails of the Student distribution with density (48) decrease like \(|x|^{-(1+\nu)}\), and this distribution is heavy-tailed. The quadratic polynomial \( d(x) = 1 + ((x - \mu)/\delta)^2 \) with the positive first coefficient \( b_2 > 0 \) and negative discriminant \( \Delta(d) < 0 \) characterises the Student diffusion in the class of Pearson diffusions.

In this case, the negative infinitesimal generator \((-\mathcal{g})\) has a finite set of simple eigenvalues

\[
\lambda_n = \frac{\theta}{\nu - 1} n (\nu - n), \quad 0 \leq n \leq \left\lfloor \frac{\nu}{2} \right\rfloor, \quad \nu > 1
\]

(49)
in \([0, \Lambda]\), and a corresponding finite number of eigenfunctions

\[
Q_n(x) = K_n \delta^{2n} \left(1 + \left(\frac{x - \mu}{\delta}\right)^2\right)^{-\frac{n+1}{2}} \frac{d^n}{dx^n} \left\{ \left(1 + \left(\frac{x - \mu}{\delta}\right)^2\right)^{-\frac{n+1}{2}} \right\}.
\]

(50)

where

\[
K_n = (-1)^n \sqrt{n!} d_n, \quad d_n = (-1)^n \Gamma(n + 1)/\Gamma(n - 1), \quad I_n = \delta^{2n} \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu}{2} - n\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2} - n\right)}.
\]

These polynomials are orthogonal with respect to the density (48) [31,34–36,10]. These polynomials are known as Romanovski polynomials (see [37,33]). Also the corresponding Sturm–Liouville equation has a purely continuous part of the spectrum with cutoff \( \Lambda \), located on the interval \((\Lambda, \infty)\), where

\[
\Lambda = \frac{\theta \nu^2}{4(\nu - 1)}, \quad \nu > 1.
\]

That is

\[
p(x; x_0, t) = st_\nu(x) \sum_{n=0}^{[\nu/2]} e^{-\lambda_n t} Q_n(x_0) Q_n(x) + \int_\Lambda^\infty e^{-\lambda t} \psi(x_0, -\lambda) \psi(x, -\lambda) d\lambda,
\]

(52)

where \( \psi \) are solutions of the Sturm–Liouville equation normalised with respect to the speed measure, \( m(x) \). These solutions can be defined in the form of Gauss hypergeometric functions \(_2F_1\) and have a complicated form, which is hard to compute. Again, numerical methods are the only tools to compute the solution of the Fokker–Planck equation.

The boundary classification below permits us to compute a cutoff \( \Lambda \). Note the Feller classification is not enough in this case and we need to apply oscillatory/non-oscillatory and limit point/limit circle or Weyl’s classifications (see [30]).

The boundaries for Student diffusions are:

1. The boundaries \( D^1 = -\infty \) and \( D^2 = \infty \) are natural, oscillatory/non-oscillatory, limit-point type singular boundaries with the unique positive cutoff \( \Lambda \).
2. Both \( D^1 = -\infty \) and \( D^2 = \infty \) are non-oscillatory for \( \nu < \Lambda \) and oscillatory for \( \nu > \Lambda \).

In Fig. 7(a) we plot the numerical solution of the time-dependent Fokker–Planck equation (47) obtained using a spectral approximation with \( N = 100 \). The boundary behaviour is specified using the classification (1) above. The model parameters were chosen to be \( \alpha = 4, \beta = 1, \theta = 1 \) and \( \nu = 9 \). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 7(b). The numerical approximation is able to capture all the important properties of the analytical solution.

### 5.6.2. Skew-Student diffusion

Let us consider a more general version of the Student diffusion from our classification table, see [10,26].

The corresponding Fokker–Planck equation in this case is

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} [\theta(x - \mu)] p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[2\theta a(d^2(x - \mu)^2)\right] p(x, t).
\]

(53)
In this case the eigenvalues are
\[ \lambda_n = \theta n [1 - a(n - 1)], \quad 0 \leq n \leq \left[ \frac{1 + a}{2a} \right]. \]

The boundaries can be described in a similar manner to the previous section. The only difference is that the Romanovski polynomials will depend on skewness which can be seen from the Rodrigues formula:
\[ Q_n(x) = K_n \frac{1}{st_{v, \rho}(x)} \frac{d^n}{dx^n} [2a(\delta^2 + (x - \mu)^2)^n st_{v, \rho}(x)], \quad 0 \leq n \leq \left[ \frac{1 + a}{2a} \right], \]
where
\[ K_n = (-1)^n / \sqrt{(-1)^n n! d_n I_n}, \]
\[ d_n = \prod_{i=0}^{n-1} \left( -\frac{1}{a} + \frac{n + i - 1}{2} \right), \quad I_n = \int_{-\infty}^{\infty} [2a(\delta^2 + (x - \mu)^2)^n st_{v, \rho}(x)] dx. \]

The continuous part of the spectrum or essential spectrum seems unknown to the best of our knowledge and numerical methods are applied here.
Another skew-Student or Pearson type IV diffusion parametrisation was considered in [26]. In Fig. 8(a) we plot the numerical solution of the time-dependent Fokker–Planck equation (53) obtained using a spectral approximation with \( N = 100 \). The model parameters were chosen to be \( \mu = 8, \mu' = 2, \delta = 0.5, a = 10, \theta = 2 \). A comparison of the numerical approximation with the exact solution at equilibrium is shown in Fig. 8(b). The numerical approximation is able to capture all the important properties of the analytical solution.

6. Multivariate Pearson diffusion

The full potential of the reduced basis spectral method will be realised when solving high-dimensional Fokker–Planck equations (see [24]). In this case a significant reduction in the number of degrees of freedom involved required to obtain a numerical solution is obtained. Pearson diffusions in high-dimensions have received little attention to date but the demand for accurate simulations of multi-dimensional equations has increased in the application of stochastic processes to many areas including finance. We plan to continue to investigate Pearson-like diffusions in high-dimensions using the reduced basis method, that is to consider the Fokker–Planck equation of the form:

\[
\frac{\partial p(\vec{x}, t)}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} c_i(\vec{x}) p(\vec{x}, t) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \sigma_{ij}^2(\vec{x}) p(\vec{x}, t),
\]

where \( p(\vec{x}, t) = p(\vec{x}; \vec{x}_0, t) \) is the transition probability density of the corresponding \( d \)-dimensional Markov process with the linear drift:

\[
\vec{c}(\vec{x}) = (c_1(\vec{x}), \ldots, c_d(\vec{x})) = (\vec{c}, \vec{x}),
\]

and the diffusion term is quadratic:

\[ D = (\sigma_{ij}^2(\vec{x}))_{1 \leq i,j \leq d} = (\Sigma \vec{x}, \vec{x}), \]

where \( \Sigma \) is a \( d \times d \)-matrix.

In general, this is a challenging problem. However, for a diagonal \( D \) (that is the Brownian motions of the multidimensional diffusions are independent) one can proceed further. For simplicity, consider the situation where the eigenvalue problem has a pure discrete spectrum. The corresponding eigenvalue problem for a generator of multidimensional diffusions \( G = (G_1, \ldots, G_d) \) has a hypergeometric form

\[ G_{f_n^{(i)}} + \lambda_n^{(i)} f_n^{(i)} = 0, \quad i = 1, \ldots, d, \]

for a multi-index \( n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \). The eigenfunctions have a product form: \( f_n = f_{n_1} \times \cdots \times f_{n_d} \) and eigenvalues \( \lambda_n = \lambda_{n_1}^{(1)} + \cdots + \lambda_{n_d}^{(d)} \), where for \( i = 1, \ldots, d, \) \( \lambda_1^{(i)} < \lambda_2^{(i)} < \cdots < \lambda_{n_i}^{(i)} \), \( \lim_{n_i \to \infty} \lambda_{n_i}^{(i)} = \infty \).

Note that in this case the ergodic measure of components is of product form: \( \mu = \mu_1 \times \cdots \times \mu_d \), and we denote \( m(\vec{x}) \) its Radon–Nikodym derivatives. Then, the spectral representation of the transition density of the multidimensional diffusion with linear drift and quadratic square diffusion takes the form:

\[
p(\vec{x}; \vec{x}_0, t) = m(\vec{x}) \sum_{n \in \mathbb{Z}_+^d} e^{-\lambda_n t} f_n(\vec{x}) f_n(\vec{x}_0) .
\]

For the Hermite case \( f_n(\vec{x}) = H_{n_1}(x_1) \times \cdots \times H_{n_d}(x_d) \) is a product of corresponding one-dimensional orthogonal polynomials. There are similar representations for Laguerre and Jacobi polynomials. The situation for heavy-tailed multidimensional diffusions, however, seems to be very complicated due to difficulties related to the continuous part of the spectrum.
7. Conclusions

A reduced basis spectral method has been developed to investigate the solutions to the Fokker–Planck equation corresponding to six types of Pearson diffusion. These diffusion processes are popular throughout the applied sciences including physics, biology, biophysics and financial mathematics. The first three classes of Pearson diffusion (Ornstein–Uhlenbeck, Cox–Ingersoll–Ross and Jacobi processes) are well studied and have been applied widely. However, the other three classes which are ergodic distributions have heavy-tails and have been less studied — their transient solutions are either unknown or possess complicated analytical forms. Thus, the use of numerical methods to determine approximate solutions seems to be unavoidable. The reduced basis spectral method provides an accurate approximation to the known analytical solutions and rapid convergence. Also this approach circumvents the problem associated with non-negative diffusion terms that arise in the corresponding stochastic approach (see the introduction for an explanation).

References