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Analyticity for multi-Regge limits of the Bern–Dixon–Smirnov amplitudes

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Abstract

As a consequence of the AdS/CFT correspondence, planar $\mathcal{N}=4$ super-Yang–Mills $SU(N)$ theory is expected to exhibit stringy behavior and multi-Regge asymptotic. In this paper we extend our recent investigation to consider issues of analyticity, a central feature of Regge asymptotics. We contrast flat-space open string theory in the planar limit with the $\mathcal{N}=4$ super-Yang–Mills theory, as represented by the Bern, Dixon and Smirnov (2005) [1] (BDS) conjecture for n-gluon scattering, believed to be exact for $n=4,5$ and modified only by a function of cross-ratios for $n \geqslant 6$. It is emphasized that multi-Regge factorization should be applied to trajectories with definite signature. A variety of analyticity and factorization constraints realized in flat space string theory are not satisfied by the BDS conjecture, at least when the exponential factors are truncate in the infra-red regulator below $O(\epsilon)$.

1. Introduction

Regge asymptotics, combined with analyticity and crossing symmetry is a potentially powerful tool in understanding the planar limit of Yang–Mills theory. Indeed Regge constraints played a major role in the original S-matrix program that led to the discovery of string theory in flat

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space and in the context of $\mathcal{N}=4$ super-Yang–Mills and $\mathcal{N}=8$ SUGRA, the recent work of Arkani-Hamed, Cachazo and Kaplan [2] has shown again the utility of strong asymptotic constraints on the S-matrix. The full power of Regge asymptotics also includes factorization which imposes self-consistency conditions as one considers amplitudes with increasing number of external lines. The reason is familiar in the use of Feynman diagrams. The 4-point amplitude defines the Regge exchange "propagator" and the Reggeon two particle vertex. Then through the use of factorization and cutting rules (or unitarity), these same propagators and vertex functions form building blocks for a variety of multi-Regge limits of the n-point functions. In fact the process is iterative. In the 5-point function, one encounters a new double Regge vertex, which occurs in higher point functions and in the 6-point function a new Regge-particle scattering amplitude. This hierarchy places severe non-perturbative constraints on the theory. The properties of these are well established in flat space string theory, but as we will show have unexpected realization in the conjecture by Bern, Dixon and Smirnov (BDS) [1] for the maximal helicity violating (MHV) planar n-point $\mathcal{N}=4$ gluon amplitudes for all coupling λ, at least when the exponential factors are truncate below $O(\epsilon)$, the infra-red regulator.

In the modern context of gauge/string duality, the use of Regge properties is only beginning to be exploited, however there are some interesting results for the classic example of gauge/string duality which maps $\mathcal{N}=4$ super-Yang–Mills theory into gravity (or IIB super strings) in $AdS_5 \times S_5$. For example in the closed string sector, Brower, Polchinski, Strassler and Tan [3] have shown that the weak coupling BFKL Pomeron is mapped at strong coupling into a dual BFKL Pomeron with very similar properties. Direct extrapolation from the weak coupling perturbative sum to the strong coupling limit has also been made by Kotikov, Lipatov, Onishchenko, and Velizhanin [4]. In both limits conformal symmetry[1] requires that the leading Regge singularity is a fixed J-plane cut at intercept $j_0(\lambda)$. At weak coupling the intercept, $j_0 = 1 + O(\lambda)$, is near 1 corresponding to BFKL Reggized two gluon exchange while for strong coupling the intercept, $j_0 = 2 - O(1/\sqrt{\lambda})$, is near 2 for the AdS_5 graviton. Interesting interpolation between these weak and strong coupling limits has also been made by Stasto [8].

In the open string (or gluon scattering) sector, there is a new opportunity due to the BDS conjecture [1,9]. (For a recent review, see [10]; for older developments, see [11,12]). The BDS 4-point gluon amplitude exhibits a remarkably simple Regge asymptotic form [13,14], without even taking the high energy limit. Just as in the flat space super string theory, the J-plane is meromorphic with simple J-plane poles.

Recent work supports the view that the BDS amplitudes may also be formulated as a world sheet sigma model for strings propagating in $AdS_5 \times S_5$. Specifically, Alday and Maldacena [15, 16] (see also [17]) computed the wide angle scattering at strong coupling from a minimal surface, in close analogy with earlier calculations of flat space superstring amplitudes. Subsequently Berkovits and Maldacena [18] have demonstrated the equivalence of the gluon MHV planar amplitudes with Wilson loops at all values of the coupling using a fermionic T-duality, and noted that the MHV planar n-gluon scattering amplitude (world sheet tree amplitudes) are greatly simplified using a stringy generalization of the spinor helicity formalism (see Appendix A, Ref. [18]).

On the other hand, the $\mathcal{N}=4$ SYM gluon amplitudes are IR divergent, which require a cutoff, usually treated in dimensional regularization with $D = 4 - 2\epsilon$, both in SYM and in its gravity

[1] For the weak coupling BFKL equation this is referred as Möbius invariance which in strong coupling is realized [5–7] as the $SL(2,C)$ isometries of Euclidean AdS_3 subspace of AdS_5.

dual background. In particular, the Regge trajectory of gluons is both IR divergent as $1/\epsilon$ and divergent at $t = 0$ as $\log(-t/\mu^2)$, which as we will see complicates the details of the Regge identification.[2] The gluon amplitudes are then to be treated as ingredients in IR safe quantities, where we can take the cut-off to zero and obtain physical results.

All of this makes a comparison of the Regge limit for n-gluon BDS and the planar approximation to flat space superstring intriguing. (Throughout this paper, flat space string theory and gluon scattering are compared for the leading planar and large N_c approximation.) It is interesting to understand how the two theories realize Regge asymptotics and how these expressions differ, particularly as many generic features of Regge amplitudes reflect very general analyticity constraints on the planar amplitudes for any renormalizable field theory with leading Regge asymptotics. Ultimately these differences should be traceable to consequences of string scattering in the $AdS_5 \times S_5$ background dual to $\mathcal{N} = 4$ SYM versus the usual flat space background, although aside from a few comments, this will be postponed to future investigations.

Since it is widely believed that the BDS amplitude is exact up to $\mathcal{O}(\epsilon)$ terms for $n = 4$ and $n = 5$ and can only be corrected by a function of cross-ratios for $n > 5$, departures from conventional Regge expectation deserve careful scrutiny. Dual conformal invariance for the Wilson loop, was originally found in perturbation theory [20–23]. Subsequently it has also been observed at strong coupling [24,25] and extended to dual superconformal invariance in Ref. [18,26,27]. The need for a function of cross-ratios at $n > 5$ is seen at strong coupling in [16,28,29], and at low order perturbation theory for $n = 6$ [30–33]. Consequently this comparison with flat space string theory casts light on the special properties inherent in planar $\mathcal{N} = 4$ gluon scattering amplitudes. Differences should be traceable to the world sheet formulations of the flat space and gravity dual string.

In a recent paper [34] the authors investigated a limited number of constraints in the single Regge and linear multi-Regge behavior in the Euclidean region. Here we take up the issue of analytic continuation to regions describing physical processes, which has also been considered by Bartels et al. [35]. The combined constraints of Regge asymptotics, factorization and analytic continuation are subtle and very powerful. For an extensive analysis of this subject for the open string amplitude in flat space, one may see the review [36]. The reader is referred to this and Ref. [34] for crucial results which are not repeated here.

It is important to stress the focus of this paper. The BDS amplitudes are written as a product of two terms,

$$\mathcal{A}_{\mathrm{BDS}} = \mathcal{A}_{\mathrm{tree}} M(\epsilon), \tag{1.1}$$

the tree amplitudes with all polarization dependence and a scalar amplitude, $M(\epsilon)$, that is factorized into an IR divergent part and a finite part as $\epsilon \to 0$. Our analysis is done for the Regge limits *after* truncating $\log(\mathcal{M})$ below $O(\epsilon)$.

The organization and main conclusions of the paper are as follows. We begin in Section 3 by reviewing the Regge form for the 4-point function, remarking on the singular structure of the Regge trajectory at the gluon pole, the continuation to the physical region and the definition of Regge exchanges of definite signature. In Section 4, we consider the 5-point function and contrast the analytic properties of the double Regge vertex relative to flat space string theory. We note that the double Regge vertex does not obey the constraint needed for the absence of overlapping physical region discontinuities of the Steinman relation. In Section 5, we explain

[2] Gluon Regge trajectory in supersymmetric models was first calculated to two loops in [19].

how multi-Regge factorization of the 6-point function is realized in the signatured amplitudes for flat-space string theory by properly taking into account singularity in "cross-ratio" variables and the failure of the BDS amplitudes to satisfy this property. In Section 6, we incorporate color traces as well as gluon polarizations and show how factorization is realized for a general n-point amplitudes in the multi-Regge limit for flat space open string theory. Finally in Section 7, we discuss discontinuities in crossed-channel invariants, and find, paradoxically, the absence of Regge contribution in the "triple-Regge" limit for n-point amplitudes, $n \geqslant 6$. In Section 8 we conclude with some general discussions, including the possibility that some or most of these unconventional analyticity properties of multi-Regge amplitudes may be a result of the truncation of the log of BDS amplitude below $O(\epsilon)$.

Before proceeding to a detailed discussion of the BDS amplitudes, in Section 2, we present a general method for continuing individual planar multi-particle amplitudes. Following the kinematical approach of Alday and Maldacena [15,16], one can define for planar amplitudes the Regge limit away from all the physical singularities where amplitudes are real. *We refer to this limit as the Euclidean Regge limit*. Subsequently one can analytically continue each planar amplitude in the complex plane of invariants to a particular physical region above all unitarity thresholds to properly define the complex phases. Readers familiar with this subject may wish to proceed to Sections 4–6 where these methods are applied to higher point functions.

2. Analytic continuation of planar amplitudes

Both flat space open string theory amplitudes and the BDS n-gluon amplitudes are defined as on-shell scattering amplitudes with a restricted set of physical singularities due to the planar structure. This allows one to define the phases of multi-Regge limits by a systematic procedure. As we explain shortly, this procedure takes two steps. First, for each planar amplitude, the Regge limit is taken in the deep "Euclidean" region where it is real and analytic. Second, this amplitude can be defined in the physical scattering region by an analytic continuation in the "upper half plane". While in principle the analytic continuation can be performed on an independent set of $3n-10$ Mandelstam invariances that respect the constraints for on-shell scattering amplitudes, there is a subtlety involving cross ratios that approach unity in the extreme Regge limit, which we will explain briefly here for the 6-point function and more fully in Section 5.

The leading term in the large N limit for gauge theories, as emphasized first by 't Hooft, restricts the perturbative expansion to planar diagrams. This topological feature is shared by open superstring scattering amplitudes in flat space and was one of the first indications that Yang–Mills theory, and even QCD, might be equivalent to a string theory. The BDS conjecture also refers to the planar approximation for $\mathcal{N}=4$ super-Yang–Mills theory and therefore may well share some properties with open string theory. One consequence of the planar approximation is that the n-point gluon amplitude, $\mathcal{A}_n(k_i,\epsilon_i,a_i)$, is a sum over single color traces for each permutation $\pi(i)$ modulo pure cyclic ordering:

$$\mathcal{A}_n(k_i,\epsilon_i,a_i) = \sum_\pi \mathrm{Tr}\big[T^{a_{\pi(1)}}T^{a_{\pi(2)}}\cdots T^{a_{\pi(n)}}\big]A_n(k_{\pi(1)},\epsilon_{\pi(1)},k_{\pi(2)},\epsilon_{\pi(2)},\ldots,k_{\pi(n)},\epsilon_{\pi(n)}). \tag{2.1}$$

In (2.1), the T^a are generators of $SU(N)$ in the fundamental representation. For convenience, we will in what follows extend the analysis to $U(N)$, with normalization: $\sum_a T^a_{ij}T^a_{lm} = 2\delta_{im}\delta_{jl}$.

The MHV n-gluon scattering amplitudes for $\mathcal{N}=4$ SYM and for the open superstring theory [18,37] may be factored into a product of the Born term, the planar n-gluon tree amplitude and a "reduced amplitude", $M_n(1,2,\dots,n)$:

$$A_n(k_1,\epsilon_1,\dots,k_n,\epsilon_n) = A_{n,\,\mathrm{tree}}(k_1,\epsilon_1,\dots,k_n,\epsilon_n)M_n(k_1,\dots,k_n). \tag{2.2}$$

All the polarization dependence[3] is in the conformal invariant MHV tree amplitude. The reduced amplitude, $M_n = M_{n,\,\mathrm{BDS}}[t_i^{[r]}/\mu^2]$ or $M_n = M_{n,\,\mathrm{string}}[\alpha' t_i^{[r]}]$, is a Lorentz scalar function of invariants for adjacent momenta (i.e. $t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$), with an intrinsic scale (the string tension $1/\alpha'$ for flat space string theory and the IR cut-off μ^2 for BDS). Since it is well known that the zero slope limit ($\alpha' \to 0$) for flat space string scattering gives tree-level Yang–Mills theory, comparison with BDS can be restricted to the reduced amplitude.

The planar amplitude $A(1,2,\dots,n)$ is real and analytic (no cuts or poles) for Euclidean or space-like invariants, $t_i^{[r]} < 0$. Consequently it is convenient, when studying the Regge limits, to first take the limit in the Euclidean region followed by analytic continuation to the physical region to establish the complex phase. The subtlety is that one must do this continuation in an *independent set* of Mandelstam invariants staying on the mass and energy–momentum shell. For an n-particle planar amplitude, there are $n(n-3)/2$ BDS invariants but only $3n-10$ are independent.

The procedure we choose to use closely follows the approach introduced by Alday and Maldacena [15,16] for the deep Euclidean region (or wide angle scattering) for the planar n-point amplitude. We will extend this to allow us to approach the Regge limit, while avoiding all unitarity thresholds as needed in the subsequent discussion. Following Refs. [15,16,34] we first introduce light-cone variables on the external legs: $k_i^{\pm} = k_i^{(0)} \pm k^{(3)}$ and $\vec{k}_i^{\perp} = (k_i^{(1)}, k_i^{(2)})$. This represents $4n$ variables that must be constrained to give $3n-10$ invariants by enforcing (i) the mass shell $k_i^2 = -k_i^+k^- + \vec{k}_i^{\perp} \cdot \vec{k}_i^{\perp} = 0$, (ii) energy momentum conservation $\sum_i k_i = 0$ and (iii) Lorentz invariance. The last is guaranteed for the BDS amplitudes because they are explicit functions of Lorentz scalars. Consequently in light-cone coordinates, only the mass shell and energy momentum constraints need to be explicitly respected to satisfy all the non-linear constraints. The general solution for arbitrary n, as realized by both Alday–Maldacena [15,16] and Arkani-Hamed–Kaplan [38] requires analytically continuing to a $(2,2)$ metric by taking $k^{(1)} \to ik^{(1)}$ pure imaginary. Thus there is a space-like geometry in the 2–3 plane and a time-like geometry in the 0–1 plane. To satisfy energy–momentum conditions we construct closed "polygons" in each plane. Next to satisfy the on-shell condition, $(k_i^{(0)})^2 + (ik_i^{(1)})^2 = (k_i^{(2)})^2 + (k_i^{(3)})^2$ we must have the sides of equal length for the i-th gluon in the two planes. As an illustration consider the 5-point function in Fig. 1.

We choose a pentagon in the space-like 2–3 plane and a star in the time like 0–1 plane. This ensures that all BDS invariants, which in this case are two body invariants $(k_i + k_{i+1})^2$, are Euclidean! In fact this construction works for any odd n by choosing an n-gon in the space like plane and a "star" where cyclic order takes you to the "opposite" side. The case of even n is simpler because the "star" can now be taken to be a line back and forth in $k^{(0)}$ at $k^{(1)} = 0$ so one does not need the additional "energy-like" component (see Fig. 2).

In the multi-Regge limit, particles are either right and left movers, with large $k^{\pm}$ components respectively. It is possible to approach the limit, $k^{\pm} \to \infty$, while staying in the Euclidean region

[3] We shall adopt "all-incoming" momentum convention. However, occasionally, for convenience, we will switch to "all-outgoing" convention. For convenience we shall often use the shorthand notations such as $A_n(\pi(1),\pi(2),\dots,\pi(n))$ for $A_n(k_{\pi(1)},\epsilon_{\pi(1)},k_{\pi(2)},\epsilon_{\pi(2)},\dots,k_{\pi(n)},\epsilon_{\pi(n)})$ or similarly for $A_{n,\,\mathrm{tree}}$ and M_n when the full set of cyclically ordered arguments is obvious.

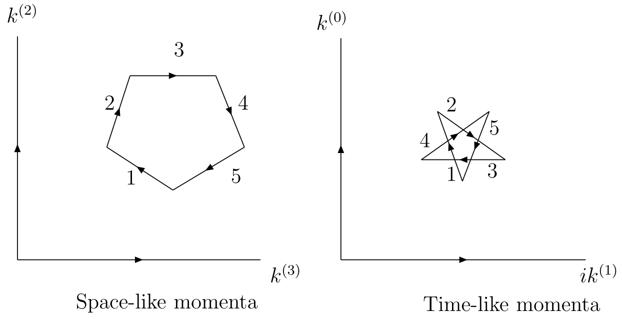

Fig. 1. The 5-point gluonic amplitude, $\mathcal{A}_5(k_1, k_2, k_3, k_4, k_5)$ evaluated on shell ($k_i^2 = 0$, $\sum_i k_i = 0$) with all BDS invariants ($t_i^{[r]} < 0$) space-like.

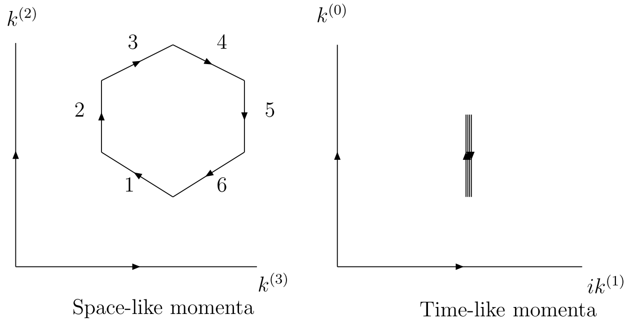

Fig. 2. The 6-point gluonic amplitude, $A_6(k_1, k_2, k_3, k_4, k_5, k_6)$, evaluated on shell ($k_i^2 = 0$, $\sum_i k_i = 0$) with all BDS invariants ($t_i^{[r]} < 0$) space-like. Note here it was possible to set $ik_i^{(1)}$ to a constant.

as depicted, for example for the 5-point function, in Fig. 3. This can clearly be generalized to any n. (For n even, the necessary deformation involves primarily the left-hand side of Fig. 2 with a corresponding elongation for the figure on the right.) Equivalently, for a general n-point amplitude in the multi-Regge limit, instead of using independent on-shell momenta, one can use a set of $3n - 10$ independent invariants. A natural set, $s_1, s_2, \ldots, t_1, t_2, \ldots, \kappa_{12}, \kappa_{23}, \ldots,$ appropriate for a given multi-Regge region, has been discussed in Ref. [34], and this is also illustrated in Fig. 4 (with $t_i = -q_i^2$). This set of independent variables was first introduced by Bali, Chew and Pignotti (BCP), [39], and we shall refer to these as the BCP variables. The BCP set is equivalent to the usual set of BDS variables for $n = 4$ and 5. For $n \geqslant 6$, a BDS variable is either already a BCP variable or can be expressed in terms of the BCP variables through a set of cross ratios, with an accompanying set of 4-dimensional Gram-determinant constraints.

Multi-Regge limit of planar amplitudes in the Euclidean region has been shown to factorize for both the flat-space string theory [36] and for the BDS n-gluon amplitudes [34] when the limit is taken in terms of an independent set of BCP invariants. In Ref. [34], we have focused on this Euclidean limit appropriate for a particular color ordering as for example depicted in Fig. 4. Moreover the factorization of the multi-Regge limit was achieved precisely because all “cross ratios” either vanish or approach 1 in the Euclidean multi-Regge region. For example, there are

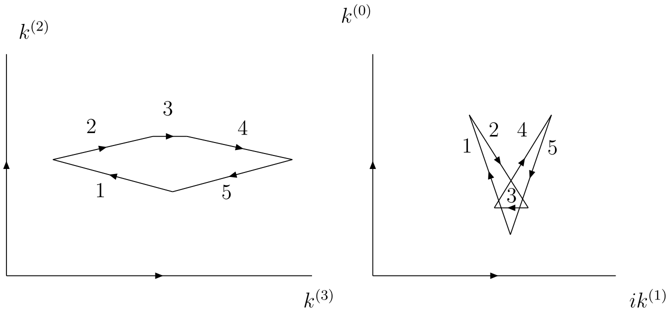

Fig. 3. The 5-point gluonic amplitude, $\mathcal{A}_5(k_1, k_2, k_3, k_4, k_5)$ in the double-Regge region, evaluated on shell ($k_i^2 = 0$, $\sum_i k_i = 0$) with all BDS invariants ($t_i^{[r]} < 0$) space-like.

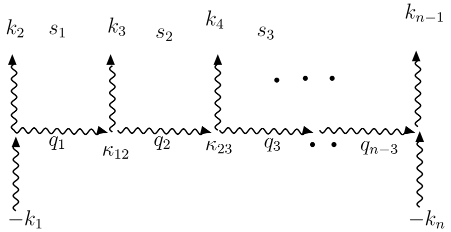

Fig. 4. Multiperipheral limit for the 2 to n-2 gluon scattering amplitude in the tree approximation.

three cross ratios for a 6-point BDS amplitude, u_1, u_2 and u_3. In the multi-Regge limit,

$$\Phi \equiv u_3 = \frac{ss_2}{\Sigma_1 \Sigma_2} \to 1, \tag{2.3}$$

with $u_1, u_2 \to 0$. (See Eq. (5.13) and Appendix B of Ref. [34].) The fact that $u_3 \to 1$ follows from a non-linear Gram-determinant constraint in the multi-Regge limit.

In this paper we focus on how to analytically continue these planar amplitudes back to the physical region with momenta depicted in Fig. 4. This will be done in some details for the 4-point and 5-point amplitudes in Section 3 and Section 4.1 respectively. The continuation for general n-point amplitudes can in principle be done by a suitable generalization of the procedure described in Section 4.1 for the 5-point amplitude. Although this procedure provides a general construction to handle 6-point and higher amplitudes where the non-linear constraints of the BDS invariants would otherwise be formidable, it has long been recognized that the simplicity of the multi-Regge amplitudes often masks the underlying analytic structure of the original planar amplitudes [40]. While the full amplitude can be correctly continued to the physical region obeying all non-linear constraints and the $+i\epsilon$ prescription for normal thresholds, this cannot be done for the leading Regge asymptotic term. In the Regge limit an alternate prescription requires relaxing the nonlinear Gram-determinant constraints during the course of the analytic continuation. For instance, for the 6-point amplitude, each planar amplitude should be considered a function of Φ, with a branch point singularity at $\Phi = 0$. Depending on the color-ordering involved, the variable Φ can be continued to different points,

$$\Phi \to 1, e^{-2\pi i}, e^{2\pi i} \tag{2.4}$$

by circling around the branch point at $\Phi = 0$. Of the 8 independent color orderings, the constraint $\Phi = 1$ can be maintained for 6, but not for 2, in the multi-Regge region [40]. In short the "on-shell" constraints and the "multi-Regge limit" do not commute and one cannot make use of the simplified nonlinear Gram-determinant constraints in the course of analytic continuation for some set of BDS amplitudes. One possible approach is to follow strictly using independent momentum components under $O(2,2)$ continuation. The conceptual advantage of this approach has recently been stressed by Nima Arkani-Hamed. In practice, this problem can be evade if we simply treat Φ's as independent variables until the physical region is reached. This procedure has been adopted consistently in past in establishing analyticity and factorization for the total amplitude, i.e., after summing over all planar orderings, for flat-space string theory [40]. We shall return to this point in Section 5.

3. Regge behavior of 4-point function

Let us review and contrast Regge properties for the 4-point amplitude in flat space string theory vs the BDS amplitude for gluons. Our motivation is to strike a cautionary note on comparing traditional Regge behavior with the BDS amplitude.

The 4-point amplitude has 3 Mandelstam invariants $s = -(k_1 + k_4)^2$, $t = -(k_1 + k_2)^2$ and $u = -(k_1 + k_3)^2$, with $s + t + u = 0$. In the planar limit for open strings in flat space, there are 3 planar amplitudes with singularities for positive Mandelstam invariants s–t, t–u and s–u corresponding to 3 independent permutations of the trace Tr[1234], Tr[1243] and Tr[1423]. To take the Regge limit while avoiding these singularities, in this case discrete poles, we consider $s \to -\infty$, ($u \simeq -s$), with $t < 0$ for the s–t permutation. The "gluonic" open string amplitude[4] gives [37]

$$\begin{aligned} A(s,t)/A_{\text{tree}}(s,t) &= \frac{\Gamma[2-\alpha(t)]\Gamma[2-\alpha(s)]}{\Gamma[3-\alpha(s)-\alpha(t)]} \\ &= [1-\alpha' t-\alpha' s]\int_0^1 dz\, z^{-\alpha(t)+1}(1-z)^{-\alpha(s)+1}, \end{aligned} \tag{3.1}$$

with trajectory $\alpha(t) = 1 + \alpha' t$. The tree-amplitude contains both s-channel and t-channel gluon poles,

$$A_{\text{tree}} \sim \frac{s}{t} + \frac{t}{s}. \tag{3.2}$$

In the Regge limit $s \to -\infty$, the integral is dominated by the region $z = O(-1/\alpha' s)$ and is easily computed

$$\begin{aligned} M_4(s,t) = A(s,t)/A_{\text{tree}} &\simeq (-\alpha' s)^{\alpha(t)-1}\int_0^\infty dy\, y^{-\alpha(t)+1}e^{-y} \\ &= \Gamma\big[2-\alpha(t)\big](-\alpha' s)^{\alpha(t)-1}. \end{aligned} \tag{3.3}$$

[4] We suppress here the color and helicity dependence of the external gluon lines. Also the 6 extra dimensions of the super string are assumed to be compactified on a 6-d torus.

No purely conformal theory can have a leading Regge pole, because the trajectory function requires a mass scale. In the limit $\alpha' \to 0$, $M_4(s,t) \to 1$ as it must. Comparing this with the Regge limit of the BDS amplitude, this scale must be provided by the IR cut-off. In what follows, we will often introduce a notational simplification,

$$\omega(t) = \alpha(t) - 1, \tag{3.4}$$

and this will be used for both flat-space string theory and for BDS.

The Reggeization of the gluon in non-supersymmetric Yang–Mills theories [13,41–44], as well as supersymmetric Yang–Mills [19,45–48], has a long history. Consider the $\mathcal{N}=4$ SYM together with the BDS conjecture for the corresponding planar on-shell 2-to-2 gluon scattering amplitude, $A_4(k_1+k_2 \to -k_3-k_4)$,

$$M_4 = A_4/A_{\text{tree}} = A^2_{\text{div}}(s)A^2_{\text{div}}(t)e^{\frac{f(\lambda)}{8}\log^2(s/t)+\tilde{c}(\lambda)} \tag{3.5}$$

where $\lambda = g^2N$, $f(\lambda)$ is proportional to the cusp anomalous dimension [1,15,49–54], and $\tilde{c}(\lambda)$ is a constant. The Sudakov form factor is

$$A_{\text{div}}(s) = \exp\left\{-\frac{1}{16}f(\lambda)\log^2\left(-s/\mu^2\right) + \left[\frac{1}{8\epsilon}f^{(-1)}(\lambda) + \frac{1}{4}g(\lambda)\right]\log\left(-s/\mu^2\right)\right\} \tag{3.6}$$

up to an s- and t-independent divergent factor. Remarkably the cancellation of the $\log^2(-s/\mu^2)$ and $\log^2(-t/\mu^2)$ in $A^2_{\text{div}}(s)$ and $A^2_{\text{div}}(t)$ respectively with the $\log^2(s/t)$ in Eq. (3.5) immediately gives the Regge amplitude [14,22],

$$M_4(s,t) = \beta(t)\left(\frac{-s}{\mu^2}\right)^{\alpha(t)-1} = \beta(t)\left(\frac{-s}{\mu^2}\right)^{\omega(t)}, \tag{3.7}$$

without taking the Regge limit: $-s$ large at fixed t. The gluon trajectory function

$$\alpha(t) = 1 + \omega(t) = 1 + \frac{1}{4\epsilon}f^{(-1)}(\lambda) - \frac{1}{4}f(\lambda)\log\left(-t/\mu^2\right) + \frac{1}{2}g(\lambda) + O(\epsilon) \tag{3.8}$$

depends on μ, the IR cut-off.[5] With Regge residue

$$\beta(t) \equiv \gamma^2(t) = \text{constant}\left(\frac{-t}{\mu^2}\right)^{\omega(-\mu^2)}, \tag{3.9}$$

$M_4(s,t)$ is manifestly symmetric in $s \leftrightarrow t$. (See Fig. 5.)

A moving Regge trajectory requires the scale breaking introduced by the IR cut-off playing the role of the Regge slope parameter α' in flat space string theory. In this sense the Regge behavior of $\mathcal{N}=4$ gluonic scattering is clearly a subtle affair. Note that the trajectory (3.8), at t large and positive goes as $-\log(-t/\mu^2)$, turning complex rather than rising linearly. Unlike the flat-space string theory, there are no Regge recurrences at positive real t and no scale for the slope α', consistent with a $\mathcal{N}=4$ conformal theory with no massive states. Conversely, in the deep Euclidean region where $t<0$, the trajectory is real and unbounded from below.

However due to the IR cut-off of a conformal theory there are some unusual features to this Regge amplitude. First the physical $J=1$ gluon pole at $t=0$ *does not* lie on the trajectory. In fact the trajectory is singular at $t=0$, presumably due to the multi-gluon channel. A more

[5] For the rest of the paper, we will mostly put $\mu^2 = 1$.

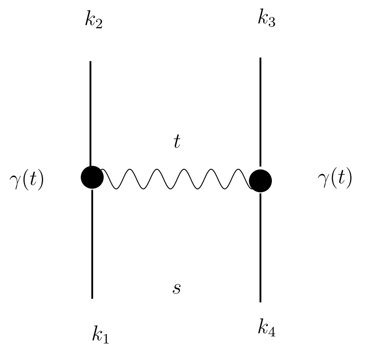

Fig. 5. The Regge limit for the elastic *planar* 4-point amplitude $A_4(s,t)$ with thresholds for $s \geqslant 0, t \geqslant 0$ with a "twisted' Regge limit" is $s \simeq -u \to -\infty$ and $t < 0$.

appropriate cut-off (as explained in [14,45,46]) would be to fix the trajectory at $J = 1$ for $t = 0$, or use a "Higgsing" regularization scheme [41,42] which consistently gives mass to the gluonic state, keeping it on the trajectory and separated from the multi-gluon channel. All this is just to remind ourselves that the conventional Reggeization of a field should be considered in the context of a proper renormalizable field theory. To avoid dependence on the IR cut-off it is better to restrict the Regge hypothesis here to the combined limit with $s \ll t \ll 0$. Alternatively we may take $d > 4 - 2\epsilon$ with negative ϵ which is IR finite, [55] and then the coupling is driven to zero in the IR, there is no Regge behavior, but the gluon at $t = 0$ does have $J = 1$ as it should.

Finally if we compare the flat space Regge limit with the Alday–Maldacena approach at strong coupling we see another important contrast. As demonstrated in Ref. [3], in flat space (and strong coupling $\mathcal{N} = 4$ closed strings) the Regge limit is a result of the world sheet operator product expansion as the particle vertices, $V(k_i, z_i) = e^{ikX(z_i)}$, approach one another, which allows one to introduce a new on-shell vertex operator

$$\mathcal{V}^{\pm}(k,z) = \left(\partial_z X^{\pm}(z)\right)^{1+\alpha' t} e^{\mp kX(z)} \tag{3.10}$$

for the emission and absorption of a Reggeon. This world sheet approach naturally leads to the standard properties of flat space string multi-Regge amplitudes. In the case of the 4-point BDS amplitude, at least at strong coupling following the Alday–Maldacena world sheet approach, there does not appear to be a similar result because the Regge limit does not require such a limiting procedure for the external operators. This already underlines differences between flat space string theory and MSYM.

3.1. Analytic continuation and 4-point signatured amplitude

As an illustration, let us consider the continuation of 4-point gluon Regge amplitude: $\gamma(t)(-s)^{\alpha(t)}\gamma(t)$. We know the answer to analytic continuation so this is a practice problem doing it the hard way! The answer is to continue $-s = e^{-i(\pi-\theta)}|s|$ on an arc in the upper-half s-plane (UHP), (θ from π to 0), to get,

$$\gamma(t)(-s)^{\alpha(t)}\gamma(t) \to \gamma(t)(s)^{\alpha(t)} e^{-i\pi\alpha(t)}\gamma(t). \tag{3.11}$$

Now let's do it in light-cone co-ordinates.

Start with $k_i = (k_i^+, k^-, k_\perp)$. The $A_4(s,t)$ BDS amplitude has an Euclidean Regge limit for $u = -s - t \to \infty, t < 0$ or $u = -(k_2 + k_3)^2 \to \infty, t = -(k_1 + k_2)^2 < 0$. This is satisfied by

$$-k_1^+ = k_2^+ \to \infty, \qquad k_3^- = -k_4^- \to \infty. \tag{3.12}$$

To have these strict equalities (which is nice but not absolutely necessary) we work in the brick-wall frame with $q^\perp = k_1^\perp = k_2^\perp$ (and therefore $q^\perp = -k_3^\perp = -k_4^\perp$). So the conservation on E–p is exactly,

$$k_1^\pm + k_2^\pm = -k_3^\pm - k_4^\pm = 0, \tag{3.13}$$

so we can stay on the E–p and mass-shell $k_i^2 = 0$ with the continuation,

$$k_1^\pm \to e^{\pm i\theta} k_1^\pm, \qquad k_2^\pm \to e^{\pm i\theta} k_2^\pm, \tag{3.14}$$

and $k_3^\pm, k_4^\pm$ unchanged, and we see that $t = -2k_1 k_2$ is not changed in the continuation, but $s = -2k_1 k_4 \simeq 2e^{i\theta} k_1^+ k_4^-$ is continued around the UHP as it should be!

Finally it is important to realize that more than one planar amplitude contributes to the same Regge limit. For example for the 4-point amplitude there are 3 distinct planar amplitude corresponding different cyclic orders of the external lines: $A(1,2,3,4)$, $A(2,1,3,4)$ and $A(1,3,2,4)$ with singularities for positive Mandelstam variables s,t; u,t and s,u respectively. The first two (with t-channel exchanges) contribute to the Regge limit $s \to \infty$ at fixed t. It is useful to introduce a variable $\eta = \pm 1$ to distinguish between $A_{\eta=1} = A(1,2,3,4)$ with singularities for positive s and $A_{\eta=-1} = A(2,1,3,4)$ with singularities for negative $s \sim -u$. In the physical region, the leading Regge term is

$$A_\eta(s,t) \sim (-\eta s)^{\alpha(t)}, \tag{3.15}$$

a complex phase, $(-)^{\alpha(t)} = e^{-i\pi\alpha(t)}$, for $\eta = 1$ and real for $\eta = -1$. In general a non-degenerate Regge singularity in the J-plane contributes only to even or odd linear combination of these two amplitudes,

$$\tilde{A}_\sigma(s,t) = A_{\eta=1}(s,t) + \sigma A_{\eta=-1}(s,t) \sim \big(e^{-i\pi\alpha(t)} + \sigma\big) s^{\alpha(t)} \tag{3.16}$$

distinguished by a quantum number referred to as "signature" ($\sigma = \pm$). Factorization only applies to Regge exchanges with definite signature. In open string theory, signature corresponds to state even and odd under world sheet parity — the eigenvalues of the twist operator Ω. In the present context this operator, Ω, defines the signature factor,

$$\xi_\sigma = e^{-i\pi\alpha(t)} + \sigma \equiv \sqrt{2}\,\Omega_{\sigma,\eta}(-\eta)^{\alpha(t)} \tag{3.17}$$

or applied to the amplitude, $\tilde{A}_\sigma = \sqrt{2}\,\Omega_{\sigma\eta} A_\eta$, where $\Omega_{\sigma\eta} = [(1+\eta) + (1-\eta)\sigma]/2\sqrt{2}$ or

$$\Omega_{\sigma\eta} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{3.18}$$

with $(+,-)$ ordering of vectors indices of both σ and η. Various $\sqrt{2}$ factors have been inserted so that $\Omega^2 = 1$.

For definiteness, we shall express the signatured amplitudes as

$$\tilde{A}_\sigma(s,t) = \tilde{\Pi}_\sigma(t) s^{\alpha(t)}, \tag{3.19}$$

where

$$\tilde{\Pi}_\sigma(t) = \xi_\sigma(t)\Gamma(t). \tag{3.20}$$

For flat-space string theory, we have

$$\Gamma(t) = \Gamma\bigl(1-\alpha(t)\bigr) \tag{3.21}$$

and, for BDS,

$$\Gamma(t) = \gamma^2(t)\left(\frac{\mu^2}{-t}\right). \tag{3.22}$$

Also, for future reference, note $\Omega^\dagger = \Omega$. It follows that one can easily invert the process so that, in the physical region,

$$A_\eta(s,t) = \frac{1}{\sqrt{2}}\Omega_{\eta\sigma}\tilde{A}_\sigma(s,t) = \Gamma(t)(-\eta)^{\alpha(t)}s^{\alpha(t)}. \tag{3.23}$$

4. Analyticity of the double Regge vertex

It has long been recognized that analyticity, when combined with other general principles such as unitarity, provides a powerful tool for constraining the allowed behavior for scattering amplitudes. For instance, one-loop MHV gluon amplitudes can be obtained using unitarity techniques starting from tree-graphs. Conversely, given the BDS conjecture, analyticity and unitarity can in principle be used to test its validity. In this section, we discuss some aspects of analyticity constraints on 5-point amplitudes in various Regge limits. We contrast the properties of the BDS amplitudes with the stringy expectations based on flat-space open-string amplitudes. We first focus on the BDS Reggeon-particle–Reggeon vertex, G_2, which, unlike the case of flat-space string theory, has a rather special analyticity structure in the κ variable. We demonstrate that the BDS vertex, $G_{2,\,\mathrm{BDS}}(t_1,t_2,\kappa_{12})$ (see Fig. 6b, with $\kappa_{12} = s_1 s_2/s$), obtained from 5-point amplitude computed to $O(\epsilon^0)$, does not satisfy the Steinmann rules [56–60]. This structure is shown to lead to an unusual behavior for the 5-point BDS amplitude in certain limiting regions,[6] e.g., $|s| \gg |s_1| \gg |s_2|$, or $|s| \gg |s_2| \gg |s_1|$. The implication of this discussion for $n \geqslant 6$ will be addressed in Section 7. In particular, we find the absence of a Regge contribution in the inclusive "triple Regge" limit for the 6-point function.

4.1. Single- and double-Regge limits for 5-gluon amplitude

In Ref. [34], it was demonstrated how BDS n-point amplitudes, $n \geqslant 5$, led to Regge behavior in various Regge limits taken in the Euclidean region. For example, a color-ordered 5-point amplitude, aside from lower-order terms coming from the asymptotic expansion of the tree-amplitude, again has the remarkable property observed in the 4-point function of an exact single- and double-Regge form

$$\begin{aligned} M_5(s_1,s_2,s,t_1,t_2) = A_5/A_{\mathrm{tree}} &= \gamma(t_1)\bigl(-s_1/\mu^2\bigr)^{\omega_1} G_1^{[3]}(s_2,t_1,t_2,\kappa_{12}) \\ &= \gamma(t_1)\bigl(-s_1/\mu^2\bigr)^{\omega_1} G_2(t_1,t_2,\kappa_{12})\bigl(-s_2/\mu^2\bigr)^{\omega_2}\gamma(t_2), \end{aligned} \tag{4.1}$$

[6] The limit where $|s| \gg |s_1| \to \infty$ and $|s_1| \gg |s_2|$, with s_2 either large or fixed, is referred to historically as the "helicity-pole" limit [36,58,61,62]. For 5-point function, this lies outside the physical region. However, this is no longer the case for $n \geqslant 6$.

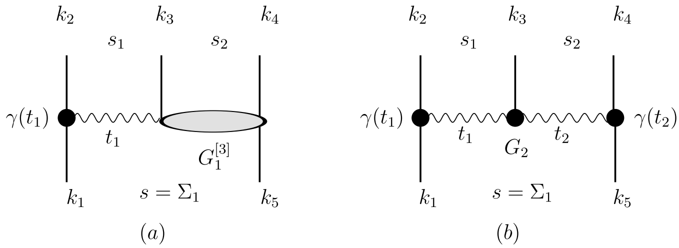

Fig. 6. Regge limits for 5-point amplitude. On the left, the single Regge limit factorizes defining a new single Regge 3-particle vertex, $G_1^{[3]}(s_2, t_1, t_2, \kappa_{12})$ and on the right, the double Regge limit defines a new two-Reggeon vertex, $G_2(t_1, t_2, \kappa_{12})$.

where $\alpha(t_1) = \alpha_1 = 1 + \omega_1$, $\alpha(t_2) = \alpha_2 = 1 + \omega_2$ and $\kappa = s/s_1 s_2$. The two expressions can be interpreted as either the single- or double-Regge limit as illustrated in Fig. 6. The expression is in fact symmetric under the interchange $s_1 \leftrightarrow s_2$. In addition to the Regge trajectory, $\alpha(t)$, two particle vertex, $\gamma(t)$, one encounters for the first time in the 5-point function the Reggeon-particle–Reggeon vertex, $G_2(t_1, t_2, \kappa_{12})$, and the three particle Reggeon vertex, $G_1^{[3]}(s_2, t_1, t_2, \kappa_{12})$), which must re-occur in higher point function if factorization is valid. We discuss below the analytic structure of vertices, in particularly the Reggeon-particle–Reggeon vertex, G_2, in the κ variable, and the continuation to the physical region for the color-ordered (12345), Fig. 6. The issue of physical region *factorization* in the linear multi-Regge limits for $n \geqslant 5$ in summing over all color orderings will be discussed in Section 6.

The particular color-ordered 5-point amplitude considered in Eq. (4.1) is real in the Euclidean region, $s_1, s_2, s, t_1, t_2 < 0$. The physical region where $s_1, s_2, s > 0$ and $t_1, t_2 < 0$ is reached by analytic continuation of the *on-shell* scattering amplitude. The procedure follows the strategy outlined in Section 2.

On shell, there are 5 independent BDS invariants, s, s_1, s_2, t_1, t_2. As we mentioned in Section 2, the analytic continuation from the deep Euclidean region to the physical region can best be understood in terms of light-cone momentum components a la Alday and Maldacena [16], and Arkani-Hamed and Kaplan [38]. Let us first provide an estimate. For $s_1, s_2, s \to -\infty$, consider the Euclidean region where

$$-k_1^+ \sim k_2^+ \to \infty, \qquad -k_5^- \sim k_4^- \to -\infty, \quad \text{with } k_3^- < 0,\ k_3^+ > 0. \tag{4.2}$$

However, due to the on-shell condition, $k_3^2 = 0$, we must have

$$k_3^+ k_3^- = \left(k_3^\perp\right)^2 < 0, \qquad \kappa = s_1 s_2 / s \simeq \left(k_3^\perp\right)^2 < 0. \tag{4.3}$$

We can follow what [16] and [38] did by continuing to a (2, 2) metric with $k_3^1 = i|k_3^1|$ to allow for this. We can be in a frame with all components of $k_3^2 = O(\mu)$. Now the trick is almost the same. Continue

$$k_1^\pm \to e^{\pm i\theta} k_1^\pm, \qquad k_2^\pm \to e^{\pm i\theta} k_2^\pm \tag{4.4}$$

to get to $s > 0$, $s_1 > 0$ and $\kappa < 0$. Next continue,

$$k_3^1 = e^{-i\theta/2} i \left|k_3^1\right| \tag{4.5}$$

so that κ goes to $\kappa > 0$ through the upper-half plane (UHP).[7]

From Ref. [34], we have found that the Reggeon–Reggeon–Gluon vertex, $G_2(t_1, t_2, \kappa)$ takes on the following dependence on κ,

$$G_2 = C(t_1, t_2) e^{A \log^2(-\kappa/\mu^2) - B(t_1,t_2)\log(-\kappa/\mu^2)} \tag{4.6}$$

where $A = -f(\lambda)/16$, $B(t_1, t_2) = (\omega(t_1) + \omega(t_2))/2$, and

$$\begin{aligned} C(t_1, t_2) = {} & \mathrm{const}\big(-t_1/\mu^2\big)^{-\frac{1}{2}\omega(-\mu^2)}\big(-t_2/\mu^2\big)^{-\frac{1}{2}\omega(-\mu^2)} \\ & \times \exp\big[f(\lambda)\log^2\big(t_1 t_2/\mu^4\big)/16\big]. \end{aligned} \tag{4.7}$$

As expected, with $t_1, t_2 < 0$ fixed, G_2 is real in the Euclidean region where $-\infty < \kappa < 0$. The analytic structure can be specified by a cut-plane, with a branch cut drawn along the positive axis, from $\kappa = 0$ to $\kappa = +\infty$. For color-ordering (12345), as indicated in Fig. 6, the physical region, where $s_1, s_2, s > 0$, can be reached from the Euclidean region via the procedure outlined earlier. This corresponds to continuing κ to the positive axis, $0 < \kappa < \infty$, via the upper-half plane, i.e., $\kappa \to |\kappa| + i\epsilon$ and $\log(-\kappa/\mu^2) \to \log(|\kappa|/\mu^2) - i\pi$. That is, in the physical region, the color-ordered amplitude is

$$\begin{aligned} M_5 & \simeq \gamma_1\gamma_2(-s_1)^{\omega_1}(-s_2)^{\omega_2} G_2\big(t_1, t_2, |\kappa| + i\epsilon\big) \\ & = \gamma_1\gamma_2 e^{-i\pi\omega_1 - i\pi\omega_2} s_1^{\omega_1} s_2^{\omega_2} G_2\big(t_1, t_2, |\kappa| + i\epsilon\big). \end{aligned} \tag{4.8}$$

Note that, in addition to the product of two Regge phase factors, G_2 is also complex in the physical region.

4.2. Double-Regge representation and analyticity

The singularity of $G_2(t_1, t_2, \kappa)$ in κ is a reflection of the singularities in s_1, s_2, s, for each color-ordered amplitude. For the ordering $(1, 2, 3, 4, 5)$, M_5 has right-hand branch cuts in s, s_1, and s_2. From (4.8), its discontinuity in s comes entirely from the vertex G_2. On the other hand, the discontinuity in s_1 is more involved, receiving contributions from both the Regge factor, $(-s_1)^{\omega_1}$ and the vertex G_2. The same applies for the discontinuity in s_2. Before exploring the unusual singularity structure of the BDS vertex, Eq. (4.6), it is instructive to contrast it with that for the flat-space string theory.

The 5-gluon super string MHV amplitude can be written

$$\begin{aligned} M_{5,\,\mathrm{string}} & = \frac{A_{5,\,\mathrm{string}}}{A_{\mathrm{tree}}} \\ & = \int_0^1 dx \int_0^1 dy\, x^{-\alpha' t_1} y^{-\alpha' t_2} (1-x)^{-\alpha' s_1} (1-y)^{-\alpha' s_2} (1-xy)^{-\alpha'(s - s_1 - s_2)} K(x, y), \end{aligned} \tag{4.9}$$

where $K = \alpha'^2\{t_1 t_2/xy + \frac{1}{2}[t_1 s_1 + s_2 t_2 - s(t_1 + t_2) - s_1 s_2 + 2i\epsilon_{\mu\nu\sigma\lambda} k_1^\mu k_2^\nu k_3^\sigma k_4^\lambda]/(1 - xy)\}$. By holding all invariants Euclidean, one again can verify that $M_{5,\,\mathrm{string}} \to 1$ when $\alpha' \to 0$. Using

[7] However, there are some details: we need to guarantee that $t_1 < 0$ and $t_2 < 0$. To do this properly, we should follow the procedure outlined in Section 2. We will not go into those details here. We also note that we have reversed our convention, from $\kappa = s/s_1 s_2$ used in Ref. [34], to $\kappa = s_1 s_2/s$.

the technique of Vertex Operator, $\mathcal{V}$, introduced by Brower, Polchinski, Strassler and Tan, [3] or more directly from (4.10), one finds for the double Regge limit [36,58,62]

$$G_{2,\,\mathrm{string}}(t_1,t_2,\kappa)\sim\kappa^{-1}(-t_1-t_2+\cdots)\int_0^\infty dx_1\int_0^\infty dx_2\,x_1^{-\omega_1}x_2^{-\omega_2}e^{-x_1-x_2+(1/\alpha'\kappa)x_1x_2}. \tag{4.10}$$

Here we have used the same notation $\omega_1=\alpha_1-1$ and $\omega_2=\alpha_2-1$ appropriate for a linear Regge trajectory with intercept 1. Note that in $G_{2,\,\mathrm{string}}$ both integrals converge for $\kappa<0$, thus defining an analytic function which is real for $\kappa<0$. The function for $\kappa>0$ must be defined by analytic continuation, and one finds a branch cut from $\kappa=0$ to $\kappa=+\infty$. This cut-plane analytic structure is shared by both the flat-space string theory and the BDS vertex, (4.6). However, they differ significantly in the nature of singularity in κ.

It can be shown that (4.10) can be expressed as

$$G_{2,\,\mathrm{string}}(t_1,t_2,\kappa)=(-\kappa)^{-\omega_2}V_1(t_1,t_2,\kappa)+(-\kappa)^{-\omega_1}V_2(t_1,t_2,\kappa), \tag{4.11}$$

where V_1 and V_2 each admits a power series expansion in κ. One finds that V_1 and V_2 are *single-valued* for $-\infty<\kappa<\infty$, thus remaining *real* for $\kappa>0$. In fact, V_1 and V_2 are *entire functions* of κ. It follows that a double-Regge expansion leads to a decomposition for the five-point amplitude as a sum of two terms,

$$\begin{aligned}M_{5,\,\mathrm{string}}\sim\gamma(t_1)\big[&(-s)^{\omega_2}(-s_1)^{\omega_1-\omega_2}V_1(t_1,t_2,\kappa)\\&+(-s)^{\omega_1}(-s_2)^{\omega_2-\omega_1}V_2(t_1,t_2,\kappa)\big]\gamma(t_2).\end{aligned} \tag{4.12}$$

The first term has right-hand cuts in invariants s and s_1 and has no singularity in s_2, since V_1 is analytic in κ. Similarly, the second term has singularities in s and s_2, but not in s_1.

As one moves from the Euclidean region into the physical region where $s_1,s_2,s>0$ and $t_1,t_2<0$, the amplitude becomes complex-valued. Since V_1 and V_2 are entire in κ, it follows that there is a separation in the singularity structure in invariants s_1 and s_2, with

$$\begin{aligned}&\mathrm{Disc}_{s_1}M_{5,\,\mathrm{string}}(s,s_1,s_2,t_1,t_2)\simeq 2i\gamma_1\gamma_2\sin\pi(\omega_2-\omega_1)(-s)^{\omega_2}s_1^{\omega_1-\omega_2}V_1(t_1,t_2,\kappa),\\&\mathrm{Disc}_{s_2}M_{5,\,\mathrm{string}}(s,s_1,s_2,t_1,t_2)\simeq 2i\gamma_1\gamma_2\sin\pi(\omega_1-\omega_2)(-s)^{\omega_1}s_2^{\omega_2-\omega_1}V_2(t_1,t_2,\kappa).\end{aligned} \tag{4.13}$$

In contrast, both terms in (4.12) contribute to the discontinuity in s,

$$\begin{aligned}&\mathrm{Disc}_s M_{5,\,\mathrm{string}}\\&\quad\simeq -2i\big(\sin\pi\omega_2 s^{\omega_2}(-s_1)^{\omega_1-\omega_2}V_1(t_1,t_2,\kappa)+\sin\pi\omega_1 s^{\omega_1}(-s_2)^{\omega_2-\omega_1}V_2(t_1,t_2,\kappa)\big).\end{aligned} \tag{4.14}$$

It is well known that, given its discontinuity, an analytic function can be re-constructed through a dispersion relation. Similarly, given $\mathrm{Disc}_{s_1}M_5$, $\mathrm{Disc}_{s_2}M_5$, and $\mathrm{Disc}_s M_5$, it is in principle possible to reconstruct the full amplitude through repeated dispersion integrals. It is therefore useful to refer to (4.12) as a “dispersive decomposition” for the 2-to-3 amplitude in the Regge limit.

Let us now return to the BDS vertex, Eq. (4.6). Given $G_2(t_1,t_2,\kappa)$, as a complex number, it is always possible to express this vertex as a sum of two terms

$$G_2(t_1,t_2,\kappa)=(-\kappa)^{-\omega_2}C_1(t_1,t_2,\kappa)+(-\kappa)^{-\omega_1}C_2(t_1,t_2,\kappa) \tag{4.15}$$

and

$$\begin{aligned}M_{5,\,\mathrm{BDS}}\sim\gamma(t_1)\big[&(-s)^{\omega_2}(-s_1)^{\omega_1-\omega_2}C_1(t_1,t_2,\kappa)\\&+(-s)^{\omega_1}(-s_2)^{\omega_2-\omega_1}C_2(t_1,t_2,\kappa)\big]\gamma(t_2).\end{aligned} \tag{4.16}$$

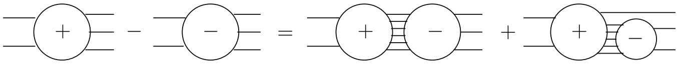

Fig. 7. 2-to-3 unitarity having two types of discontinuities.

By demanding that C_2 and C_1 be *real in the physical region* where $\kappa = |\kappa| + i\epsilon$, these two coefficients can be expressed in terms of the magnitude, $|G_2|$, and its phase, ϕ,

$$C_1(t_1, t_2, \kappa) = \kappa^{\omega_2} |G_2| \frac{\sin(\pi\omega_1 - \phi)}{\sin\pi(\omega_2 - \omega_1)},$$
$$C_2(t_1, t_2, \kappa) = \kappa^{\omega_1} |G_2| \frac{\sin(\pi\omega_2 - \phi)}{\sin\pi(\omega_1 - \omega_2)}. \quad (4.17)$$

However, it is important to examine the analytic structure of $C_1(t_1, t_2, \kappa)$ and $C_2(t_1, t_2, \kappa)$ in κ. The key difference between $G_{2,\,\mathrm{string}}(t_1, t_2, \kappa)$, the BDS vertex G_2, (4.6), is the presence of the $\log^2(-\kappa/\mu^2)$ factor in the exponent, leading to $M_5 \sim (-\kappa)^{A\log(-\kappa/\mu^2)}$ at $\kappa = 0$. It is easy to verify that $C_1(t_1, t_2, \kappa)$ and $C_2(t_1, t_2, \kappa)$ contain branch points both at $\kappa = 0$ and $\kappa = \infty$. It follows that, unlike the corresponding functions $V_1(t_1, t_2, \kappa)$ and $V_2(t_1, t_2, \kappa)$ for flat-space string theory, $C_1(t_1, t_2, \kappa)$ and $C_2(t_1, t_2, \kappa)$ are not single-valued over the positive κ-axis. Since both C_1 and C_2 contain a branch point at $\kappa = 0$, both terms will contribute to discontinuities in s_1, s_2 and s. That is, with C_1 and C_2 replacing V_1 and V_2, (4.13) no longer holds for $M_{5,\,\mathrm{BDS}}$. In particular, one cannot associate $\mathrm{Disc}_{s_1} M_{5,\,\mathrm{BDS}}$ with that from the C_1 and $\mathrm{Disc}_{s_1} M_{5,\,\mathrm{BDS}}$ with that from C_2, as is the case for the flat-space string theory.

4.3. Analyticity and unitarity

To appreciate the importance of the discussion above, it is useful to briefly review the constraints coming from enforcing analyticity and unitarity. It is well known that the total imaginary part of an amplitude in the physical region can be interpreted as the sum of discontinuities. On the other hand, through unitarity, each discontinuity can be expressed as a product of other amplitudes. These general relations, with appropriate qualifications, can also be applied to color-ordered amplitudes. Consider the color-ordered 4-point amplitude, $A(s, t)$, with color ordering (1234) in the physical region where $s > 0$ and $t < 0$. From 2-to-2 unitarity, the s-channel discontinuity can be expressed as a sum, with contribution from allowed planar multi-particle intermediate states in the s-channel. That is, each contribution can be associated with an allowed re-scattering process.

For 2-to-n amplitudes, $n > 2$, there are many different re-scattering processes allowed, leading to discontinuities in various different invariants. For instance, for a 2-to-3 process, $a + b \to c_1 + c_2 + c_3$, the unitarity condition in the physical region can be represented schematically by Fig. 7. Each term on the right can again be associated with a physically realizable re-scattering process. There are now two types of discontinuities. The first term represents the discontinuity in the total energy invariant, $s_{ab} = -(p_a + p_b)^2$. The second is a sum of three separate terms, each represents the discontinuity in one of three sub-energy invariants, $s_{ij} = -(p_{c_i} + p_{c_j})^2$. Although these discontinuities co-exist in the physical region, each can be extracted by taking the imaginary part of the amplitude by first holding all other invariants in the Euclidean region and then returning back to the physical region by analytic continuation. As illustrated by the flat-space string amplitude $A_{5,\,\mathrm{string}}$ for color-ordering (12345), in a double-Regge expansion, (4.12), we have discontinuities in invariants s_1, s_2 and s, given by (4.13) and (4.15) respectively.

Clearly, for $n > 4$, there can be simultaneous discontinuities in several invariants in the physical region. However, one must distinguish between compatible and overlapping invariants. Since each discontinuity can be identified with an allowed "re-scattering" process, a simultaneous discontinuity can exist in the physical region only for compatible invariants, e.g., simultaneous in s and one of the sub-energy variable. For our color-ordered amplitude, (4.10), the allowed pairs are (s, s_1) and (s, s_2). For flat-space string theory, from either (4.13) or (4.15), the associated double-discontinuities are given by

$$\begin{aligned} &\mathrm{Disc}_s \mathrm{Disc}_{s_1} M_{5,\,\mathrm{string}}(s, s_1, s_2, t_1, t_2) \sim \sin\pi\omega_2 \sin\pi(\omega_1 - \omega_2) s^{\omega_2} s_1^{\omega_1-\omega_2} V_1(t_1, t_2, \kappa), \\ &\mathrm{Disc}_s \mathrm{Disc}_{s_2} M_{5,\,\mathrm{string}}(s, s_1, s_2, t_1, t_2) \sim \sin\pi\omega_1 \sin\pi(\omega_2 - \omega_1) s^{\omega_1} s_2^{\omega_2-\omega_1} V_2(t_1, t_2, \kappa). \end{aligned} \tag{4.18}$$

In contrast, there cannot be simultaneous discontinuities for overlapping invariants, e.g., the pair (s_1, s_2). The double-discontinuities in this pair of overlapping invariants must vanish since it would not correspond to an allowed re-scattering process. For flat-space string theory, it follows from (4.13)

$$\mathrm{Disc}_{s_1} \mathrm{Disc}_{s_2} M_{5,\,\mathrm{string}}(s, s_1, s_2, t_1, t_2) = 0, \tag{4.19}$$

which is often referred to as the Steinmann relation [56–60]. The Steinmann relation has played an important role historically in establishing inclusive distributions as discontinuities and in understanding the Mueller–Regge hypothesis [56,58,59]. The traditional proof for the Steinmann relation relies on having a mass gap, e.g., [56]. For a theory with a mass gap, double discontinuities in overlapping singularities are associated with higher order Landau singularities, not normal thresholds, and they vanish in the physical regions [56,59].

However, for 5-point BDS amplitude, $M_{5,\,\mathrm{BDS}}$, the singularities of C_1 and C_2 at $\kappa = 0$ will contribute to discontinuities in s_1 and s_2. Expressing $M_{5,\,\mathrm{BDS}}$ as a function of independent variables (s, s_1, s_2, t_1, t_2), in place of κ, it follows from (4.6) that $M_{5,\,\mathrm{BDS}}$ will contain in the exponent a term of the form $\sim \log(-s_1)\log(-s_2)$. Such a term clearly leads to

$$\mathrm{Disc}_{s_1} \mathrm{Disc}_{s_2} e^{[2A \log(-s_1)\log(-s_2)+\cdots]} \neq 0, \tag{4.20}$$

where this double discontinuity is taken with s, t_1, t_2 fixed. The remainder in the exponent includes quadratic terms, $-2A[\log(-s_1)\log(-s) + \log(-s_2)\log(-s)]$, as well as terms linear in $\log(-s)$, $\log(-s_1)$ and $\log(-s_2)$. That is, the BDS 5-point amplitude does not share the simple separability property exhibited by (4.13), (4.18) and (4.19) for flat space string theory.

It is tempting to identify the difficulty noted above as a manifestation of massless theory where Landau singularities coalesce. This issue has also been commented on in a recently updated version of [35], where it has been argued that failure of the Steinmann relation would lead to "gluon instability".[8] Since one could treat a massless YM theory as the zero mass limit of a non-Abelian Higgs model where the gauge bosons are massive [41,42], and since the Steinmann rule is expected to hold before taking the massless limit, the difficulty of the singularity at $\kappa = 0$ noted by us should be traceable to how the limit is approached.[9] We will return to comment on this issue further in Section 8.

[8] More recently [63], it has been emphasized that n-point BDS amplitudes violate Steinmann rules for $n \geqslant 6$. However, it has been argued that this violation is due to a different mechanism, unrelated to the issue of the presence of massless particles. Here we are concerned with 5-point BDS amplitude.

[9] We would like to thank Prof. Lipatov for sharing his insight on this issue with us. In such an approach, the vertex G_2, will likely have further singularity in κ, e.g., at $\kappa_1 < 0$, in addition to the branch point at $\kappa_0 = 0$. Steinmann relation

It is worth noting that this analytic representation for 5-point gluon amplitudes, (4.15), has been studied previously. It was emphasized in [64] that, to one-loop, both C_1 and C_2 are real in the physical region, as expected. However, our emphasis here is on the analytic structure in κ. In particular, we stress that both C_1 and C_2 are singular at $\kappa = 0$, leading to (4.20). It has been suggested in [35], (v.4), that the vanishing of overlapping discontinuity in (s_1, s_2), $\mathrm{Disc}_{s_1}\mathrm{Disc}_{s_2} M_{5,\,\mathrm{BDS}}(s, s_1, s_2, t_1, t_2) = 0$, can be maintained by not identifying singularities of C_1 and C_2 at $\kappa = 0$ with singularities at $s_1 = 0$ and $s_2 = 0$. That is, $\mathrm{Disc}_{s_1}\mathrm{Disc}_{s_2} M_{5,\,\mathrm{BDS}}(s, s_1, s_2, t_1, t_2)$ should be taken at fixed $\kappa \neq 0$. This suggested procedure differs from that used here, which is more conventional, and is one normally used for analyzing singularity structure for flat-space string amplitudes. Clearly, this is important issue which deserves further investigation.

We stress here, although (4.20) follows from the singularity at $\kappa = 0$, our observation on the Steinmann rule violation can be attributed directly to the term $\sim \log(-s_1)\log(-s_2)$, coming from 1-loop contribution to $\log M_5$. When expressed in terms of the κ variable, this leads to the $\log^2(-\kappa/\mu^2)$ term in the exponent in (4.6). Therefore, at least to 1-loop, one cannot avoid identifying singularity at $\kappa = 0$ with singularities in s_1 and s_2 respectively.[10] A practical difficulty of this observation manifests itself for the 5- and higher-point amplitudes in the helicity-pole limit [58,61,62]. Unlike the case of flat-space string theory, a planar amplitude no longer takes on a simple form in this limit. Since this limit is structurally related to the more general triple-Regge limit, a topic we discuss in Section 7, we turn to this analysis for flat-space string theory next.

4.4. Helicity pole limit for 5-point amplitudes

We now discuss the helicity pole limit for five-point amplitude, which provides a simpler illustration of the Steinmann rules. This also serves as a prelude to the discussion for physical region discontinuities in "crossed-invaraints" for $n \geqslant 6$ carried out in Section 7, but can be omitted at a first reading.

This limit corresponds to $s \to \infty$ with s_1 and/or s_2 fixed [58,61,62]. To simplify the analysis further, let us first consider a combined helicity-pole and Regge limit where $s_1, s \to \infty$ but $s \gg s_1$, while holding s_2 fixed, i.e., $s \gg s_1 \gg s_2$. (Of course, one can also treat the opposite limit by interchanging indices 1 and 2.) In term of s_1, s_2, κ, this corresponds to taking the limit,

$$s_1 \to \infty, \qquad \kappa \to 0, \tag{4.21}$$

holds only when the discontinuity is taken across κ_0, but not across κ_1. In the massless limit, as we pointed out above, these singularities coincide, preventing one from verifying the Steinmann relation by analytic continuity in κ.

[10] It is well known that one-loop gluon amplitudes can be re-constructed via a cut-unitarity procedure for box-diagram. The general procedure of cut-unitarity should in principle work also for all orders. It has long been recognized that the analytic properties of multiparticle amplitudes in a massless theory are complicated, even in the so-called "multi-Regge kinematic" (MRK) region. However, it has been demonstrated that, in the MRK region, by confining to the "next-to-leading logarithmic approximation", (NLLA), certain analytic simplification can be achieved, which can be cast in the form of a set of bootstrap conditions and prescription for taking discontinuities has been given, consistent with the Steinmann relations [65,66]. It remains a challenge on demonstrating how these could be realized beyond NLLA and on how they apply to the BDS ansatz which is supposed to be exact to all order. Nevertheless, it has been stressed by some that "there is no doubt in the fulfillment of the Steinmann relation in QCD and in supersymmetric gauge models". Our concern here relates to how to properly interpret a simultaneous discontinuity in overlapping invariants bordering the physical region and the overall singularity structure for 5-point BDS amplitude, as exhibited by Eq. (4.20).

with s_2 held fixed. This limit can be approached in several ways. For instance, we can first take the single-Regge limit where $s_1 \to \infty$, with s_2 and κ fixed, Fig. 6a, leading to a 4-point vertex, $G_1^{[3]}(s_2, t_1, t_2, \kappa)$, with one external Reggeon. From this, one can take the small κ limit.

In flat-space string theory, this 4-point vertex $G_1^{[3]}(s_2, t_1, t_2, \kappa)$ can be easily found [58],

$$G_1^{[3]}(s_2, t_1, t_2, \kappa) \sim \kappa^{-1} \int_0^1 dx\, x^{-\omega_2} (1-x)^{-\omega(s_2)} \big[1 - (1 - s_2/\kappa)x\big]^{\omega_1 - 1}, \tag{4.22}$$

which is shown kinematically in Fig. 6. Note that, in this representation, the s-dependence for $M_{5,\,\text{string}}$ enters through κ. In the small κ limit, one finds that the amplitude can again be expressed as a sum of two terms

$$M_{5,\,\text{string}} \sim \gamma_1 (-s)^{\omega_1} \mathcal{A}_{\text{string}}(s_2, t_2; t_1) + (-s)^{\omega_2} (-s_1)^{\omega_1 - \omega_2} \Gamma(-\omega_2 + 1) \Gamma(\omega_2 - \omega_1), \tag{4.23}$$

where $\mathcal{A}_{\text{string}}(s_2, t_2; t_1)$ takes on the interpretation of Reggeon-particle-to-particle–particle amplitude. This decomposition is a generalization of the result obtained earlier, Eq. (4.12), but now valid for s_2 held at a finite value. Note that the second term is independent of s_2 and all the singularity in s_2 is reflected in the first term. We can also identify the product $\Gamma(-\omega_2 - 1)\Gamma(\omega_1 - \omega_2)$ with $\gamma_1 \gamma_2 V_1(t_1, t_2, 0)$.

The amplitude $\mathcal{A}_{\text{string}}$ has a series of poles in s_2 when $\alpha(s_2) = \omega(s_2) + 1$ takes on positive integers,

$$\begin{aligned} \mathcal{A}_{\text{string}}(s_2, t_2; t_1) &= \int_0^\infty dx\, x^{-\omega_2 + \omega_1 - 1} (1-x)^{-\alpha(s_2) + 1} \\ &= \frac{\Gamma(\omega_1 - \omega_2) \Gamma(-\alpha(s_2) + 2)}{\Gamma(2 - \alpha(s_2) - \omega_2 + \omega_1)}. \end{aligned} \tag{4.24}$$

Recall $\alpha_1 = \alpha(t_1)$ and $\alpha_2 = \alpha(t_2)$. When $\alpha_1 = 2$ ($\omega_1 = 1$), it reduces to an on-shell 4-point amplitude, without tachyon. The Steinmann relation again holds trivially

$$\begin{aligned} &\text{Disc}_s \text{Disc}_{s_1} M_{5,\text{string}} \sim s^{\omega_2} s_1^{\omega_1 - \omega_2} V_1(t_1, t_2, 0) \neq 0, \\ &\text{Disc}_s \text{Disc}_{s_2} M_{5,\text{string}} \sim s^{\omega_1} \text{Disc}_{s_2} \mathcal{A}_{\text{string}}(s_2, t_2; t_1) \neq 0, \\ &\text{Disc}_{s_1} \text{Disc}_{s_2} M_{5,\,\text{string}} = 0. \end{aligned} \tag{4.25}$$

In the Regge limit where s_2 is large, all the poles in s_2 collapse into a right-hand cut, with

$$\mathcal{A}_{\text{string}}(s_2, t_2; t_1) \sim (-s_2)^{\omega_2 - \omega_1}, \tag{4.26}$$

which again leads to Eq. (4.12). However, the helicity-pole limit is more general, and it holds for s_2 finite.

For completeness, we record here that, for the flat-space string theory, the helicity-pole limit for a 5-point function where $s \to -\infty$ with s_1 and s_2 both fixed is

$$M_{5,\,\text{string}} \sim \gamma_1 (-s)^{\omega_1} \mathcal{A}_{\text{string}}(s_2, t_2; t_1) + \mathcal{A}_{\text{string}}(s_1, t_1; t_2) (-s)^{\omega_2} \gamma_2. \tag{4.27}$$

In this representation, the Steinmann condition is manifest.

5. Analytic continuation and factorization constraints

Factorization for Reggeon (just as for particle) diagrams places strong recursive constraints on higher point functions. The first time Regge factorization plays a crucial role in gluon (or open string) scattering is at the level of the 6-point function. Here the identical double Regge particle vertex defined in the 5-point function enters again. How the multi-Regge limit factorizes is in fact a subtle interplay of the issues of analytical continuation of the individual planar amplitudes and the projection onto trajectories of definite signature.

Here we examine this by comparing the 5-point and 6-point multi-Regge amplitudes. Fortunately this analysis is independent of whether or not the Steinmann relations hold.

5.1. Signatured Regge–Regge particle vertex

To determine the correct phase for each the individual permutations of the planar amplitudes, $A_n(\pi(1), \pi(2), \ldots, \pi(5))$, contributing to the physics region for $-k_1, -k_5, \to k_2, k_3, k_4$ we begin in the Euclidean region as described in Section 2 and analytically continue in *independent* invariants to the physical region.[11]

For general n-point amplitude in the Euclidean region, a natural choice for the $3n-10$ independent invariants is the BCP set, $s_1, s_2, \ldots,\ t_1, t_2, \ldots,\ \kappa_{12}, \kappa_{23}, \ldots$ as illustrated in Fig. 4, where $\kappa_{i,i+1} = s_i s_{i+1}/\Sigma_{i,i+1}$ with $\Sigma_{i,i+1} = -(k_{i+1} + k_{i+2} + k_{i+3})^2$. As an example consider the amplitude the planar permutation, $A_n(1, 2, \ldots, n)$, whose multi-Regge limit is

$$M_n = A_n/A_{n,\,\text{tree}} \simeq \gamma_1(t_1)(-s_1)^{\omega_1} G_2(t_1, t_2, \kappa_{12}) \cdots (-s_{n-3})^{\omega_{n-3}} \gamma_{n-3}(t_{n-1}). \tag{5.1}$$

This color-ordered amplitude in the Euclidean Regge limit ($s_i \to -\infty$ at fixed $t_i < 0, \kappa_{i,i+1} < 0$) must be purely real, since it can be demonstrated that all BDS invariants remain negative, $t_i^{[r]} < 0$, away from all singularities. In (5.1), we have set the mass scale $1/\alpha' = 1$ and $\mu^2 = 1$ for flat space string and BDS amplitudes respectively. An illustration in the deep Euclidean region is given for $n = 5$ and $n = 6$ in Fig. 1 and Fig. 2 respectively. In the multi-Regge limit we must take $k^{\pm} \to \infty$ for right and left movers respectively while staying in the Euclidean region as depicted, for example for the 5-point function, in Fig. 3. As mentioned earlier, this can clearly be generalized to any n.

The continuation to the physical region can be done in light-cone variables by a suitable generalization of the procedure described in Section 4.1 for the 5-point amplitude. Focus first on the natural color-ordering $(1, 2, 3, 4, \ldots, n)$. Observing that $s_i \simeq k^+_{i+2} k^-_{i+3}$ and $\kappa_{i,i+1} \simeq (k^{\perp}_{i+2})^2$, we may begin by continuing the longitudinal components so that $s_i \to s_i > 0$ in a large semicircle in the UHP holding κ's essentially fixed. Then each κ is continued into its UHP. For $n = 5$, as seen earlier, one gets

$$M_5(1, 2, 3, 4, 5) \simeq \gamma_1 s_1^{\omega_1} e^{-i\pi\omega_1} G_2(t_1, t_2, \kappa_{12} + i\epsilon) s_2^{\omega_2} e^{-i\pi\omega_1} \gamma_2. \tag{5.2}$$

Next, we need to treat the continuation for other inequivalent orderings. We will illustrate the procedure first for the 5-point functions here, which in fact suffices to define the general rule. For 5-point amplitudes, there are 8 inequivalent color orderings, but anticyclic reversal reduces the independent set to four, which can be characterized by two indices (η_1, η_2), as depicted in Fig. 8, where $\eta_i = -1$ is indicated by a “cross”. The double-Regge limit for each planar diagram

[11] For convenience, we now switch in this section to an all-outgoing momentum convention.

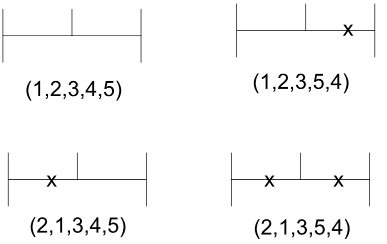

Fig. 8. Four cyclic orderings characterized by twistings.

in the physical region has powers $(-\eta_i s_i)^{\omega_i}$ with $\eta_i = \pm 1$, $i = 1, 2$. which contribute a complex phase $e^{-i\pi\omega_i}$ for $\eta_i = 1$ (right-hand cuts) and real factors for $\eta_i = -1$ (left-hand cuts). That is, for $\eta_i = -1$, there is no need for continuation.

One important new feature is the fact that, by *on-shell* analytic continuation, the two Reggeon vertices become complex in the physical region, but can take on two possible values

$$G_2(1, 2; \pm) = G_2\big(t_1, t_2; |\kappa_{12}| \pm i\epsilon\big). \tag{5.3}$$

For $n = 5$, except for $\eta_1 = \eta_2 = -1$, κ_{12} is continued to the positive real axis via the UHP so that the two-Reggeon-vertex takes on $G_2(+)$. However, when $\eta_1 = \eta_2 = -1$, $\kappa_{12} = s_1 s_2/\Sigma_{12}$ is continued via the LHP. This is because both s_1 and s_2 have left-hand cuts and no continuation is required in s_1 and s_2. However, Σ_{12} is continued into the UHP to its final positive value, leading to $G_2(-)$. The general factorization pattern in the linear multi-Regge limit will invariably involve the discontinuity $\Delta G = G_2(+) - G_2(-)$. When particles are on-shell, one finds ΔG vanishes and the factorization pattern simplifies.

To summarize, in the physical region,

$$M_{\eta_1\eta_2} \sim s_1^{\omega_1} s_2^{\omega_2} (-\eta_1)^{\omega_1} (-\eta_2)^{\omega_2} \gamma_1\gamma_2 G_2\big(1, 2; \epsilon(\eta_1, \eta_2)\big), \tag{5.4}$$

or, equivalently for the planar amplitudes,

$$A_{\eta_1\eta_2} \sim s_1^{\alpha_1} s_2^{\alpha_2} (-\eta_1)^{\alpha_1} (-\eta_2)^{\alpha_2} \Gamma(t_1)\Gamma(t_2)(\gamma_1\kappa_{12}\gamma_2)^{-1} G_2\big(1, 2; \epsilon(\eta_1, \eta_2)\big), \tag{5.5}$$

where $\epsilon(+,+) = \epsilon(-,+) = \epsilon(+,-) = +$ and $\epsilon(-,-) = -$, i.e.,

$$\epsilon(\eta, \eta') = \frac{1 + \eta + \eta' - \eta\eta'}{2}. \tag{5.6}$$

Note that, with our convention for the propagator $\Gamma(t)$, (3.22), a factor of $(\gamma_1\gamma_2)^{-1}$ has to be inserted.[12] Dependence on polarizations has been suppressed and will be made explicit in the next section.[13]

As emphasized earlier, multi-Regge factorization is generally expected only for "signatured" amplitudes [40]. For the 5-point amplitudes, we can introduce "signatured" amplitudes,

$$\tilde{A}_{\sigma_1\sigma_2} = 2\Omega_{\sigma_1\eta_1}\Omega_{\sigma_2\eta_2} A_{\eta_1,\eta_2}. \tag{5.7}$$

[12] For flat-space string theory, recall that $\Gamma(t) = \Gamma(1 - \alpha(t))$ and $\gamma(t) = 1$.

[13] In going from $M_{\eta_1\eta_2}$ to $A_{\eta_1\eta_2}$, a factor of $\eta_1\eta_2$ has been supplied, which will be accounted for in the next section when treating the color-trace.

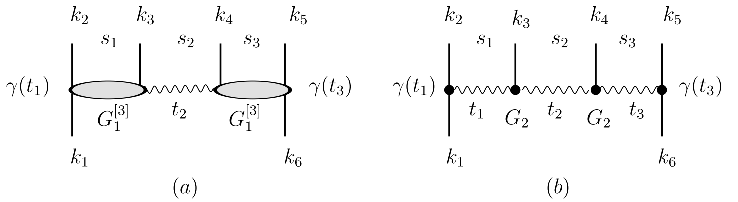

Fig. 9. Regge limits for 6-point amplitude are completely determined by factorization from the 5-point amplitude. On the left, (a), single-Regge limit with vertices $G_1^{[3]}(s_1, t_2, t_1, \kappa_{12})$ and $G_1^{[3]}(s_3, t_2, t_3, \kappa_{23})$. On the right, (b), linear triple-Regge limit with internal vertices $G_2(t_1, t_2, \kappa_{12})$ and $G_2(t_2, t_3, \kappa_{23})$.

Expressing these signatured amplitudes as

$$\tilde{A}_{\sigma_1\sigma_2} = s_1^{\alpha(1)} s_2^{\alpha(2)} \tilde{\Pi}_{\sigma_1}(1) \tilde{G}_{\sigma_1\sigma_2}(1,2) \tilde{\Pi}_{\sigma_2}(2) \tag{5.8}$$

this allow us to define "signatured" vertex $\tilde{G}_{\sigma_1\sigma_2}$,

$$\tilde{G}_{\sigma_1\sigma_2}(1,2) = (\gamma_1\kappa_{12}\gamma_2)^{-1}\big[G_2(1,2;+) - \Delta G \sigma_1\sigma_2 \xi_1(\sigma_1)^{-1} \xi_2(\sigma_2)^{-1}\big]. \tag{5.9}$$

Here, $\tilde{\Pi}_{\sigma_i}(i) = \xi_{\sigma_i}(t_i)\Gamma(t_i)$ is a signatured propagator, generalizing Eq. (3.20) for each exchange. Conversely, given signatured amplitudes $\tilde{A}_{\sigma_1\sigma_2}$, one can recover the planar amplitudes, A_{η_1,η_2}, by inversion,

$$A_{\eta_1,\eta_2} = (1/2)\Omega_{\eta_1\sigma_1}\Omega_{\eta_2\sigma_2}\tilde{A}_{\sigma_1\sigma_2}. \tag{5.10}$$

5.2. Analyticity constraint on the 6-point function

We turn next to 6-point functions and continue to focus on the linear multi-Regge limits, as illustrated by Fig. 9b. In spite of the troubling observation on the analytic structure of the vertex G_2 in κ, we can proceed to discuss the question of continuation into the physical region. For the multi-Regge limit, there are now 8 inequivalent color configurations, which can be characterized by three indices, η_1, η_2, η_3, $\eta_i = \pm 1$. We will focus here on the *on-shell* continuation from Euclidean to the physical region for each color configuration. In particular, we show how this allows multi-Regge factorization for signatured amplitudes, i.e., generalizing (5.8) to $n = 6$,

$$\begin{aligned}\tilde{A}_{\sigma_1\sigma_2\sigma_3} = {} & s_1^{\alpha(t_1)} s_2^{\alpha(t_2)} s_3^{\alpha(t_3)} \Pi_{\sigma_1}(t_1) \tilde{G}_{\sigma_1\sigma_2}(t_1, t_2; \kappa_{12}) \\ & \times \tilde{\Pi}_{\sigma_2}(t_2) \tilde{G}_{\sigma_2\sigma_3}(t_2, t_3; \kappa_{23}) \tilde{\Pi}_{\sigma_3}(t_3).\end{aligned} \tag{5.11}$$

Again, color-trace as well as polarization factors will be ignored for now.

One of the key differences between the case $n \leqslant 5$ vs. $n \geqslant 6$ is the fact that BDS invariants are no longer independent. In Ref. [34], we have stressed that, in the Regge limit, constraints among BDS invariants can be understood in terms of constraints on cross ratios. Indeed, one obtains multi-Regge behavior in the Euclidean regions only if these constraints are imposed. To be on-shell, one can work with independent invariants, in terms of which these constraints are automatically satisfied. To be precise, we have used the set of $3n - 10$ independent BCP invariants, $s_1, s_2, \ldots, t_1, t_2, \ldots, \kappa_{12}, \kappa_{23}, \ldots$, and have shown, in the *Euclidean multi-Regge region where* $s_i, t_j, \kappa_{i,i+1}$ *are all negative*, that

$$M_n \simeq \gamma_1(-s_1)^{\omega_1} G_2(t_1, t_2, \kappa_{12}) \cdots (-s_{n-3})^{\omega_{n-3}} \gamma_{n-3}. \tag{5.12}$$

Note that $\kappa_{i,i+1} = s_i s_{i+1}/\Sigma_i < 0$, $\Sigma_i = -(k_{i-2} + k_{i-3} + k_{i-4})^2$. To illustrate, let us focus on the case $n = 6$ and the natural color ordering $(1, 2, 3, 4, 5, 6)$. As is well known, there are now three independent cross ratios,

$$u_1 = \frac{t_1^{[2]} t_6^{[4]}}{t_1^{[3]} t_6^{[3]}} = \frac{t_1 s_3}{t_2 \Sigma_2}, \qquad u_2 = \frac{t_2^{[2]} t_1^{[4]}}{t_2^{[3]} t_1^{[3]}} = \frac{t_3 s_1}{t_2 \Sigma_1}, \qquad u_3 = \frac{t_3^{[2]} t_2^{[4]}}{t_3^{[3]} t_2^{[3]}} = \frac{s_2 s}{\Sigma_1 \Sigma_2} \tag{5.13}$$

and, in the linear multi-Regge limit, nonlinear constraints among BDS variables together with the *on-shell conditions* lead to

$$u_1 \to 0, \qquad u_2 \to 0, \qquad u_3 \to 1. \tag{5.14}$$

Only when these conditions are enforced, multi-Regge behavior, Eq. (5.12), follows. This analysis leading to factorization in the Euclidean region applies to all color orderings, and it has been generalized to $n > 6$ in Ref. [34].

Let us now turn to the continuation to the physical region. Staying on-shell, we need to specify the paths of continuation for κ_{12} and κ_{23} separately. For the multi-Regge limit, generalizing the analysis for $n = 5$, there are 16 color configurations involving t_1, t_2 and t_3 as BDS invariants. Of these, 8 are inequivalent (due to anti-cyclic symmetry), which can be characterized by three indices, η_1, η_2, η_3, $\eta_i = \pm 1$, e.g., the color ordering $(1, 2, 3, 4, 5, 6)$ corresponds to $\eta_1 = \eta_2 = \eta_3 = +1$. Since κ_{12} and κ_{23} depend on (s_1, s_2, Σ_1) and (s_2, s_3, Σ_2) respectively, the physical region again corresponds to $\kappa_{12} \to |\kappa_{12}| + i\epsilon$ and $\kappa_{23} \to |\kappa_{23}| + i\epsilon$. That is, in the physical region where $s_i > 0$, the amplitude with color-order (123456), Fig. 9b, is

$$\begin{aligned} A_{+++} \simeq{}& (-s_1)^{\alpha_1}(-s_2)^{\alpha_2}(-s_3)^{\alpha_3} \\ &\times \Gamma(1)\Gamma(2)\Gamma(3)(\gamma_1\kappa_{12}\gamma_2)^{-1}(\gamma_2\kappa_{23}\gamma_2)^{-1} G_2(1, 2; +) G_2(2, 3; +), \end{aligned} \tag{5.15}$$

where $G_2(i, i+1; \pm) = G_2(t_i, t_{i+i}; |\kappa_{i,i+1}| \pm i\epsilon)$.

It is straightforward to generalize this continuation procedure to other color configurations, leading to apparent factorization for all color orderings in the physical region. However, for the BDS amplitudes, a non-factorizable result was obtained in Ref. [35] for the color configurations corresponding to $(-, +, -)$. Prompted by this discrepancy, we have examined more closely the corresponding issues in flat-space string theory. Indeed, it has been demonstrated for flat-space string theory that a straightforward generalization of the above procedure would not lead to a "faithful" representation of the analytic structure for some of the planar amplitudes. To be precise, in the case of $n = 6$, for configurations $(-, +, -)$ and $(-, -, -)$, holding $u_3 = 1$ would not account correctly the analytic structure of the original dual amplitudes. For instance, for the $(-+-)$, a planar amplitude contains right-hand cuts in $s_{34} = -(k_3 + k_4)^2$ and $s_{25} = -(k_2 + k_5)^2$. Holding $u_3 \simeq 1$ in the course of continuation would not allow both s_{34} and s_{25} to reach the physical region simultaneously via UHP.

An effective procedure for keeping track of the analytic structure in the multi-Regge limit is to retain the u_3 dependence so that,

$$M_{\eta_1\eta_2\eta_3} \simeq s_1^{\omega_1} s_2^{\omega_2} s_3^{\omega_3} (-\eta_1)^{\omega_1} (-\eta_2)^{\omega_2} (-\eta_3)^{\omega_3} \gamma_1\gamma_3 B_{\eta_1\eta_2\eta_3}(1, 2, 3, \Phi_{\eta_1\eta_2\eta_3}), \tag{5.16}$$

where we have re-written Φ for u_3. For all planar configurations other than these two exceptions listed above, $\Phi = 1$ is consistent with the process of continuation. For configurations $(-, +, -)$ and $(-, -, -)$, Φ takes on $e^{-2\pi i}$ and $e^{2\pi i}$ respectively. Due to a branch-cut singularity at $\Phi = 0$, holding $\Phi = 1$ for these two configurations in the course of continuation to the physical region would lead to incorrect results.

Following the analysis of [40], one can show for flat-space string theory that $B_{\eta_1\eta_2\eta_3}$ can be expressed as a sum of two terms,

$$B_{\eta_1\eta_2\eta_3}(1,2,3,\Phi_{\eta_1\eta_2\eta_3}) = G\big(1,2;\epsilon(\eta_1,\eta_2)\big)G\big(2,3;\epsilon(\eta_2,\eta_3)\big) + \Delta B_{\eta_1\eta_2\eta_3}(1,2,3,\Phi_{\eta_1\eta_2\eta_3}), \tag{5.17}$$

where

$$\Delta B_{\eta_1\eta_2\eta_3} = \big[(\Phi_{\eta_1\eta_2\eta_3})^{\alpha_1} - 1\big]\frac{e^{i\epsilon(\eta_1,\eta_2)\pi\alpha_1}}{\sin\pi\alpha_1}\frac{e^{i\epsilon(\eta_2,\eta_3)\pi\alpha_2}}{\sin\pi\alpha_2}\frac{\Delta G(1,2)}{2i}\frac{\Delta G(2,3)}{2i}. \tag{5.18}$$

For color configurations $(-,\pm,-)$, we have $\Phi = e^{\mp 2\pi i}$ and ΔB non-zero, thus breaking naive factorization. It is also clear that this fact depends on $G(t,t';\kappa)$ having a branch-cut for $\kappa > 0$ and $\Delta G \neq 0$.

If one were to set $\Phi_{\eta_1\eta_2\eta_3} = 1$, independent of η_i, (5.17) would become a product of two vertices, thus leading to "naive factorization",

$$\begin{aligned}A_{\eta_1\eta_2\eta_3} &\simeq (-\eta_1)^{\alpha_1}(-\eta_2)^{\alpha_2}(-\eta_3)^{\alpha_3}s_1^{\alpha_1}s_2^{\alpha_2}s_3^{\alpha_3}\\ &\quad\times\Gamma(1)\Gamma(2)\Gamma(3)(\gamma_1\kappa_{12}\gamma_2)^{-1}(\gamma_2\kappa_{23}\gamma_3)^{-1}\\ &\quad\times G_2\big(1,2;\epsilon(\eta_1,\eta_2)\big)G_2\big(2,3;\epsilon(\eta_2,\eta_3)\big).\end{aligned} \tag{5.19}$$

Alas, analyticity, i.e., causality, dictates that we must retain the correct $\Phi_{\eta_1\eta_2\eta_3}$ dependence. This invalidates the naive factorization for $A_{\eta_1\eta_2\eta_3}$. Instead, it leads to factorization for signatured amplitudes, which we turn to next.

5.3. Signature factorization and the BDS amplitudes

Let us next examine the consequence of signature factorization, (5.11). It is easy to transform signature factorization back to $A_{\eta_1\eta_2\eta_3}$,

$$A_{\eta_1\eta_2\eta_3} = 2^{-3/2}\Omega_{\eta_1\sigma_1}\Omega_{\eta_2\sigma_2}\Omega_{\eta_3\sigma_3}\tilde{A}_{\sigma_1\sigma_2\sigma_3}. \tag{5.20}$$

This leads to a two-term recursion relation,

$$B_{\eta_1,\eta_2,\eta_3}(1,2,3) = B_{\eta_1,\eta_2}(1,2)J_1(2,3) + B_{\eta_1,(\eta_3\eta_2)}(1,2)e^{i\epsilon(\eta_2,\eta_3)\pi\alpha_2}J_2(2,3), \tag{5.21}$$

i.e., relating B_{η_1,η_2,η_3} to $B_{\eta_1,\pm\eta_2}$, where $B_{\eta_1,\eta_2}(1,2) = G(1,2;\epsilon(\eta_1,\eta_2))$. The coefficient functions $J_1(2,3)$ and $J_2(2,3)$ are real in the physical region, related to the Reggeon–Reggeon vertex by

$$G(2,3;+) = J_1(2,3) + e^{i\pi\alpha_2}J_2(2,3). \tag{5.22}$$

More directly, one can express J_1 and J_2 in terms of G and ΔG, e.g.,

$$J_2(i,i+1) = \frac{\Delta G(i,i+1)}{2i\sin\pi\alpha_{i+1}}. \tag{5.23}$$

It is easy to check that (5.21) directly reproduces the non-factorization (for color configurations $(-,\pm,-)$), emphasized earlier. Using the fact that, in the physical region, the phase $\Phi_{\eta_1\eta_2\eta_3}$ can be expressed as $e^{-i(1-\eta_1)\eta_2(1-\eta_3)\pi/2}$, one finds that (5.17) directly leads to the recursion relation, (5.21). Alternatively, one can check that (5.18) directly leads to signature factorization, (5.11), for the flat-space string theory.

This two-term recursion relation can also be generalized to higher point functions. Assuming factorization for higher point signatured amplitudes, it follows that $B_{\eta_1,\eta_2,\dots,\eta_{n-1},\eta_n}$ can be expressed as a linear combination of $B_{\eta_1,\eta_2,\dots,\eta_{n-1}}$ and $B_{\eta_1,\eta_2,\dots,-\eta_{n-1}}$,

$$\begin{aligned} B_{\eta_1,\eta_2,\dots,\eta_{n-1},\eta_n} &= B_{\eta_1,\eta_2,\dots,\eta_{n-1}} J_1(n-1,n) \\ &\quad + B_{\eta_1,\eta_2,\dots,-\eta_{n-1}} e^{i\epsilon(\eta_{n-1},\eta_n)\pi\alpha_{n-1}} J_2(n-1,n). \end{aligned} \tag{5.24}$$

It has been shown inductively [40] that indeed signature factorization holds for $n > 6$ for flat-space string theory.

Let us turn next to an examination of 6-point BDS amplitudes. As emphasized earlier, we adopt the procedure of dropping $O(\epsilon)$ for

$$\log M_6 = \log \frac{A_6}{A_{6,\,\mathrm{tree}}} = I_6^{(1)}(\epsilon) + F_6^{(1)}(0) \tag{5.25}$$

in taking the multi-Regge limit. In this case, BDS amplitudes for $n \geqslant 6$ reduce to simple combinations of products of logarithms and dilogarithmic functions. As pointed in Ref. [34], these dilog functions do not contribute in the Euclidean multi-Regge limit and Regge factorization can be achieved. This relies on the observation that all cross ratios either vanish or approaching 1 in this limit.

However, as pointed above, analyticity consideration forces one to relax the constraint on the cross ratios in the course of continuation back to the physical region. In the case of $n = 6$, there are only three such cross ratios, and the one which requires special attention is the variable Φ, or u_3 in (5.13). Since its nontrivial dependence enters explicitly, continuation into the physical region can be carried out unambiguously. In [35], one finds for A_{-+-} that, in the course of continuation where $\Phi: 1 \to e^{-2\pi i}$, $\log M_6$ picks up an extra piece

$$\begin{aligned} &\Delta \log M_6(-,+,-) \\ &\quad = \frac{f(\lambda)}{8}\left(\frac{1}{\epsilon} + \log\frac{\mu^2 s_2}{(-t_1)(-t_3)}\right)\log\Phi + \frac{f(\lambda)}{4}\pi i \log\left(\frac{\Phi}{1-\Phi}\right) \\ &\quad = \frac{f(\lambda)}{4}\pi i\left(-\frac{1}{\epsilon} + \log\left(\frac{(-t_1)(-t_3)}{\mu^2 s_2}\left[\frac{\Phi}{1-\Phi}\right]\right)\right). \end{aligned} \tag{5.26}$$

Here the term $\log(\Phi/1-\Phi)$ comes from analytic continuation of the dilog $\mathrm{Li}_2(z)$, $z = 1-\Phi$, onto its second sheet at $z \simeq 0$. With this addition to $M_6(-,+,-)$ due to analytic continuation, it breaks the naive factorization, (5.20). We note that the term $\log\Phi$ is equally important in arriving at a finite result, and this term cannot be expressed as a function of cross ratios. A similar analysis can also be carried out for the $M_6(-,-,-)$, and naive factorization again breaks down.

Let us next examine how BDS fares with respect to MR factorization expected for signatured amplitudes. Either from (5.21) or directly from (5.17), one can show that the condition for factorization for signatured amplitudes can be expressed as

$$e^{\Delta\log M_6(-,\pm,-)} = 1 \pm (2i)\frac{e^{\pm i(\phi_{1,2}+\phi_{2,3})}\sin\phi_{1,2}\sin\phi_{2,3}}{\sin\pi\alpha_2} \tag{5.27}$$

where angles $\phi_{i,i+1}$ are defined by

$$G_2(i,i+1;+) = |G_2| e^{i\phi_{i,i+1}}. \tag{5.28}$$

It can be checked the modification to BDS amplitude due to continuation in Φ through $\mathrm{Li}_2(1-\Phi)$, (5.26), does not satisfy the condition above. It follows that signature factorization, (5.11), also fails.

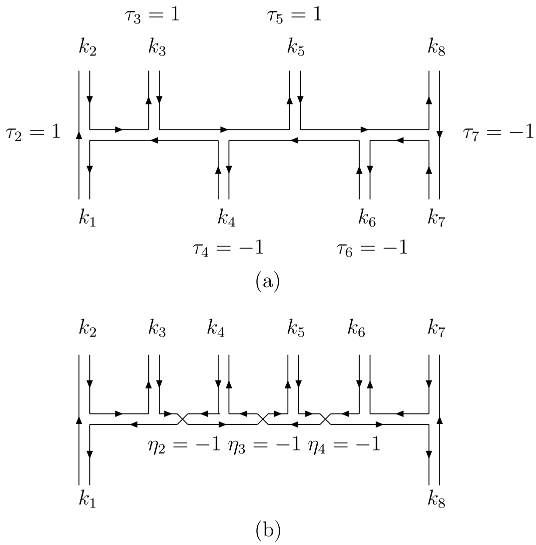

Fig. 10. Example of planar diagram $A(12358764)$ contributing to the multi-Regge limit for $-k_1 - k_8 \to k_2 + k_3 + k_4 + k_5 + k_6 + k_7$ with twisted vertices $\tau_4 = \tau_6 = \tau_7 = -1$ in the top diagram, (a), implying twisted links $\eta_2 = \tau_3\tau_4 = -1$, $\eta_3 = \tau_4\tau_5 = -1$ and $\eta_4 = \tau_5\tau_6 = -1$ in the bottom diagram, (b).

6. Factorization for multi-Regge limits

Factorization is an iterative property. Propagators and vertices encountered in lower point functions must be present in higher point functions, along with new higher vertices in non-polynomial expansion such as those present in the multi-Regge (or Gribov) effective field theory diagrams. Here we focus on factorization for the general linear multi-Regge for 2 to $n-2$ amplitudes. As we noted in section when we neglect the color trace the 4-point amplitude has two degenerate trajectories of opposite signature and in the 5-point function these two trajectories couple to a 2 by 2 Reggeon–Reggeon vertex. Here we include the color trace and demonstrate the form of multi-Regge factorization for the flat space string. For the BDS amplitudes factorization appears to fail at the 6-point level because due to unconventional analyticity properties. It seems likely that this is another aspect of the difficulties noted in section with the Steinmann relation for the BDS amplitudes.

The pattern which emerges can be most easily seen by considering the general case of the n-gluon amplitude in the linear multi-Regge limit. The full set of planar amplitudes contributing to the linear multi-Regge limit for $1 + n \to 3 + 4 + \cdots + n - 1$ scattering are 2^{n-2} permutation found by "flipping" any of the $n-2$ final particles to the opposite side of the trace, $\mathrm{Tr}[12\cdots n]$, as illustrated in Fig. 10. Here we introduce the notation: $\mathrm{Tr}[ijkl\cdots] = \mathrm{Tr}[T^{a_i}T^{a_j}T^{a_k}T^{a_l}\cdots]$. To count these configurations, let $\tau_i = \pm 1$ for $i = 2, \ldots, n-1$ for each outgoing line (see Fig. 10a), $\tau_i = -1$ indicates the ith gluon has been "flipped". There are 2^{n-3} choices with Regge powers $(\mp s_i)^{\alpha(t_i)} \equiv (-\eta_i s_i)^{\alpha(t_i)}$. The case of a real Regge power, $s_i^{\alpha}(t_i)$ ($\eta_i = -1$) requires a twist of one of the two adjacent lines so $\eta_i = \tau_{i+1}\tau_{i+2}$ (see Fig. 10b). In addition we note that the n-gluon planar amplitudes obey exact cyclic and anti-cyclic conditions,

$$A_n(2, \ldots, n, 1) = A_n(1, 2, \ldots, n),$$
$$A_n(1, 2, \ldots, n) = (-1)^n A_n(n, \ldots, 2, 1), \tag{6.1}$$

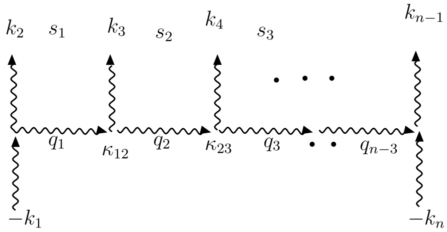

Fig. 11. Multiperipheral limit for the 2 to $n-2$ gluon scattering amplitude in the tree approximation.

respectively. As we will verify shortly, this implies an extra factor of $\tau_2\tau_3\cdots\tau_{n-1}$.

In the multi-Regge limit, the full amplitude (at least for MHV helicities) can be written as a sum over contributions from 2^{n-2} distinct color permutations, $\pi(\tau_2,\tau_3,\ldots)$. Each contribution is a product of three factors,

$$\mathcal{A}_n(1,2,\ldots,n) = \sum_{\pi([\tau])} \mathrm{Tr}\big[T^{a_{\pi(1)}}T^{a_{\pi(2)}}\cdots T^{a_{\pi(n)}}\big]A^{\mathrm{tree}}_{n,[\tau]}(k_1,\epsilon_1,\ldots,k_n,\epsilon_n)M_{n,[\eta]}(k_1,\ldots,k_n) \tag{6.2}$$

the color trace for each cyclic order, the tree diagram and the reduced amplitude M_n with no dependence on the polarization or color labels. Thus it is useful to start with the tree approximation.

6.1. Factorization of the MHV tree diagram

The factorization of $A_{n,\,\mathrm{tree}}$ can be understood both from perturbation theory and open string theory. The Regge limit of the planar Born term, as depicted in Fig. 11,

$$A^{\mathrm{tree}}_{n,[\tau]}(k_1,\epsilon_1,\ldots,k_n,\epsilon_n) = g^n s\tau_2\epsilon_1\cdot\epsilon_2\frac{1}{t_1}(\tau_3\epsilon_3\cdot\gamma_3)\frac{1}{t_2}\cdots\frac{1}{t_{n-3}}\tau_{n-1}\epsilon_{n-1}\cdot\epsilon_n, \tag{6.3}$$

where $\gamma_i(q_{i-2},q_{i-1}) \simeq -q^{\perp}_{i-2} - q^{\perp}_{i-1} + \cdots$ is a reduced vertex, related to the three gluon effective vertex for the peripheral multi-gluon high energy limit,

$$\Gamma^{\mu}_{\nu,\nu'}(q,q') = \frac{k_1^{\nu}k_n^{\nu'}}{s}\gamma^{\mu}(q,q') \tag{6.4}$$

which was first discussed in [67,68] for their treatment of perturbative Pomeron in QCD. (For an elementary treatment, see [69] and references therein.) The longitudinal components are fixed by the on shell gauge condition, $k_i^{\mu}\gamma_i^{\mu}(q_{i-2},q_{i-1})=0$, and a gauge choice, $\gamma_i^{\mu}\to\gamma_i^{\mu}+k_i^{\mu}$. We also note that, to leading order, using $s\simeq s_1\kappa_{12}^{-1}s_2\kappa_{23}^{-1}s_3\cdots\kappa_{n-4,n-3}^{-1}s_{n-3}$,

$$A^{\mathrm{tree}}_{n,[\tau]}(1,2,\ldots,n)\sim\tau_2\frac{s_1}{t_1}\tau_3\kappa_{12}^{-1}\frac{s_2}{t_2}\tau_4\kappa_{23}^{-1}\frac{s_3}{t_3}\cdots\tau_{n-3}\kappa_{n-4,n-3}^{-1}\frac{s_{n-3}}{t_{n-3}}\tau_{n-1}, \tag{6.5}$$

where we have dropped g^n and the polarizations factors to simplify the expression. (For a more explicit treatment, see [70].)

From the open string perspective (with co-ordinate parameter $w=\tau+i\sigma$, $\sigma\in[0,\pi]$) the signs ($\tau_i=\pm1$) are world sheet charge conjugations implemented by applying the twist operator, $\Omega=(-1)^N$, to permute the gluon vertex:

$$\epsilon_i(k)\cdot\partial\tau X(\sigma,\tau)e^{ik_iX(\sigma,\tau)}\to\epsilon_i(k)\cdot\partial\tau X(\pi-\sigma,\tau)e^{ik_iX(\pi-\sigma,\tau)}. \tag{6.6}$$

It is well known that in the zero slope limit $\alpha' \to 0$ that both the bosonic and super string reproduces planar trees for $N_c = \infty$ Yang–Mills theory. Indeed this is the original inspiration for the Parke–Taylor and MHV developments.

Next consider the factorization of the color trace for the 2^{n-2} permutations enumerated by τ_i. Using the procedure identical to the established analysis of Chan–Paton [71] factors in open string theory, the traces can be factored in the t-channel using completeness for $U(N_c)$, $\sum_a T^a_{ij} T^a_{lm} = 2\delta_{im}\delta_{jl}$, where we have adopted the normalization: $\mathrm{Tr}[T^a T^b] = 2\delta_{ab}$. Beginning with no color twists ($\tau_i = 1$) the trace is factorized as

$$\mathrm{Tr}[12\cdots n] = 2^{-(n-3)}\,\mathrm{Tr}[a_1 a_2 c_1]\,\mathrm{Tr}[c_1 a_3 c_2]\,\mathrm{Tr}[c_2 a_4 c_3]\cdots \mathrm{Tr}[c_{n-3} a_{n-1} a_n]. \tag{6.7}$$

The permutation corresponding to flipping the ith outgoing gluon ($\tau_i = -1$) in the color trace corresponds to complex conjugation: $\mathrm{Tr}[c_{i-1} a_i c_{i-2}] = \mathrm{Tr}^*[c_{i-2} a_i c_{i-1}]$. Thus the triple trace $\mathrm{Tr}[abc] = d^{abc} + i f^{abc}$ at each vertex is replace by a factor,

$$T^{abc}_\tau = d^{abc} + i\tau f^{abc}.$$

This combined with anti-cyclic symmetry of the n-gluon amplitudes $\mathcal{A}_n(1,2,\ldots,n) = (-1)^n \times \mathcal{A}_n(n,\ldots,2,1)$ implies that there are always an even number of D vertices in each monomial in accord with our explicit $n = 4$ and $n = 5$ forms.

Applying this to the Regge limit of the Born approximation alone, the sum over τ's remove all D-terms,

$$\begin{aligned}\mathcal{A}_{n,\,\mathrm{tree}} &\sim 2^{-(n-3)}(g)^{n-2}\sum_{\tau_i} \tau_2 T^{a_1 a_2 c_1}_{\tau_2}(s_1/t_1)\kappa_{12}^{-1}\tau_3 T^{c_1 a_3 c_2}_{\tau_3}(s_2/t_2)\kappa_{23}^{-1}\cdots(s_{n-3}/t_{n-3})\\ &= 2(ig)^{n-2} f^{a_1 a_2 c_1} f^{c_1 a_3 c_2}\cdots f^{c_{n-3} a_{n-1} a_n}(s_1/t_1)\kappa_{12}^{-1}(s_2/t_2)\kappa_{23}^{-1}\cdots(s_{n-3}/t_{n-3}),\end{aligned} \tag{6.8}$$

where for simplicity we have ignored the polarization factors. Without this extra factor, $\tau_2\cdots\tau_{n-1}$, the Born term would not agree with perturbation theory at the tree level.

6.2. *Full multi-Regge factorization*

To proceed to the full Regge limit we now must consider the factorization of the reduced amplitude. As discussed in section for both super string theory and the BDS amplitudes, each of the 2^{n-2} individual linear Regge amplitudes takes the form of multi-Regge dependence,

$$\begin{aligned}A_n(\tau_i)/A^{\mathrm{tree}}_{n,[\tau]} &\simeq (-\eta_1)^{\alpha_1-1}(-\eta_2)^{\alpha_2-1}\cdots(-\eta_{n-3})^{\alpha_{n-3}-1} s_1^{\alpha_1-1} s_2^{\alpha_2-1}\cdots s_{n-3}^{\alpha_{n-3}-1}\\ &\quad\times \gamma(t_1)\gamma(t_{n-3}) B_n(\eta_1,\eta_2,\ldots,t_1,\kappa_{12},\ldots).\end{aligned} \tag{6.9}$$

Putting the color traces (6.7) together with polarizations and the Regge amplitudes (6.9) leads to a separation between color-/helicity-factors and planar amplitudes

$$\mathcal{A}_{n,\,\mathrm{Regge}} \simeq \sum_\tau \tilde{t}^{a_1 a_2 c_1}_{\tau_2}\tilde{T}^{c_1 a_3 c_2}_{\tau_3}\tilde{T}^{c_2 a_4 c_3}_{\tau_4}\cdots\tilde{t}^{c_{n-3} a_{n-1} a_n}_{\tau_{n-1}} A_n(\eta_1,\eta_2,\ldots) \tag{6.10}$$

where $\eta_j = \tau_{j+1}\tau_{j+2}$, and $\{A_n(\eta_1,\ldots)\}$ are planar amplitudes discussed in Section 5,

$$\begin{aligned}A_n(\eta_1,\eta_2,\ldots) &= (-\eta_1)^{\alpha_1}(-\eta_2)^{\alpha_2}\cdots(-\eta_{n-3})^{\alpha_{n-3}} s_1^{\alpha_1} s_2^{\alpha_2}\cdots s_{n-3}^{\alpha_{n-3}}\\ &\quad\times(-t_1\kappa_{12})^{-1}(-t_2\kappa_{23})^{-1}\cdots\gamma(t_1)\gamma(t_{n-3})\\ &\quad\times B_n(\eta_1,\eta_2,\ldots,t_1,\kappa_{12},\ldots).\end{aligned} \tag{6.11}$$

In arriving at (6.10), we have supplied a factor of $\eta_1\eta_2\cdots\eta_{n-3}$, which was left out in going from M_n to A_n for $n=5,6$ in the previous section. We have also made use of the fact that $\eta_1\eta_2\cdots\eta_{n-3}=\tau_2\tau_{n-1}$, thus removing factors τ_2 and τ_{n-1} at the ends of the MR chain coming from the tree. The vertices now include color labels and polarization vectors,

$$\tilde{T}^{cac'}_{\tau_j}=(g\epsilon_j\cdot\gamma_j/2)\big(\tau_j d^{cac'}+if^{cac'}\big) \tag{6.12}$$

and at the ends reduce to

$$\begin{aligned}&\tilde{t}^{abc}_{\tau_2}(t_1)=(g\epsilon_1\cdot\epsilon_2/\sqrt{2})\big(d^{abc}+i\tau_2 f^{abc}\big),\\&\tilde{t}^{c'a'b'}_{\tau_{n-1}}(t_{n-3})=(g\epsilon_{n-1}\cdot\epsilon_n/\sqrt{2})\big(d^{a'b'c'}+i\tau_{n-1}f^{a'b'c'}\big)\end{aligned} \tag{6.13}$$

because one of the Reggeons is replaced by an on-shell gluon. Lastly, by inserting a factor $(\eta_j+\tau_{j+1}\tau_{j+2})/2$ for each η_j, we arrive at

$$\begin{aligned}\mathcal{A}_{n,\,\mathrm{Regge}}\simeq{}&\sum_\eta\sum_\tau\tilde{t}^{a_1a_2c_1}_{\tau_2}\tilde{T}^{c_1a_3c_2}_{\tau_3}\cdots\tilde{t}^{c_{n-3}a_{n-1}a_n}_{\tau_{n-1}}\\&\times\big[(\eta_1+\tau_2\tau_3)/2\big]\big[(\eta_2+\tau_3\tau_4)/2\big]\cdots A_n(\eta_1,\eta_2,\ldots),\end{aligned} \tag{6.14}$$

where τ and η are now independent sums.

So far, our discussion has been general, applicable to both BDS and flat-space string theory, (other that the replacement for the propagator $(-1/t)$ factor by $\Gamma(1-\alpha(t))$). Let us next turn to the assumption of factorization in signature space. Following the discussion in Section 5, the reduced amplitude factorizes in signature space. More directly, for the planar amplitudes A_n, we have

$$\tilde{A}_n(\sigma_1,\ldots)=\big[s_1^{\alpha_1}s_2^{\alpha_2}\cdots\big]\Pi_{\sigma_1}\tilde{G}_{\sigma_1\sigma_2}\tilde{\Pi}_{\sigma_2}\cdots\tilde{G}_{\sigma_{n-4}\sigma_{n-3}}\tilde{\Pi}_{\sigma_{n-3}}, \tag{6.15}$$

where $\tilde{\Pi}_{\sigma_j}$ stands for $\tilde{\Pi}_{\sigma_j}(j)$, $\tilde{G}_{\sigma_j\sigma_{j+1}}$ for $\tilde{G}_{\sigma_j\sigma_{j+1}}(j,j+1)$, and A_n is given by an inverse transform,

$$A_n(\eta_1,\ldots)=2^{-(n-3)/2}\sum_\sigma\Omega_{\eta_1,\sigma_1}\Omega_{\eta_2,\sigma_2}\cdots\tilde{A}_n(\sigma_1,\sigma_2,\ldots). \tag{6.16}$$

Substituting this into (6.14), the η sum can be carried out, leading to

$$\begin{aligned}\mathcal{A}_{n,\,\mathrm{Regge}}\simeq{}&\sum_\sigma\sum_\tau\Pi_j\bigg[\bigg(\frac{1+\sigma_j}{2}\bigg)\tau_{j+1}\tau_{j+2}+\bigg(\frac{1-\sigma_j}{2}\bigg)\bigg]\\&\times\tilde{t}^{a_1a_2c_1}_{\tau_2}\tilde{T}^{c_1a_3c_2}_{\tau_3}\cdots\tilde{T}^{c_{n-4}a_{n-2}c_{n-1}}_{\tau_{n-2}}\tilde{t}^{c_{n-3}a_{n-1}a_n}_{\tau_{n-1}}\tilde{A}_n(\sigma_1,\ldots).\end{aligned} \tag{6.17}$$

In more explicit form as a matrix product, first we define $V^{abc}_{\tau\sigma,\tau'\sigma'}\equiv\tilde{G}_{\sigma,\sigma'}\tilde{T}^{abc}_\tau\delta_{\tau,\tau'}$, $\gamma^{abc}_{\tau\sigma}\equiv\tilde{t}^{abc}_\tau$ and then we represent the Reggeon propagator in τ and σ

$$\Delta_{\tau\sigma,\tau'\sigma'}(s,t)=\frac{[(1-\sigma)+(1+\sigma)\tau\tau']}{2}\Gamma(t)\xi_\sigma(t)(s)^{\alpha(t)}. \tag{6.18}$$

Multi-Regge factorization,

$$\begin{aligned}\mathcal{A}_{n,\,\mathrm{Regge}}\simeq{}&s_1^{\alpha_1}s_2^{\alpha_2}\cdots s_{n-3}^{\alpha_{n-3}}\gamma^{a_1,a_2}(t_1)\Delta(s_1,t_1)V^{a_3}(t_1,\kappa_{12},t_2)\\&\times\Delta(s_2,t_2)V^{a_4}(t_1,\kappa_{23},t_3)\cdots\gamma^{a_{n-1}a_n}(t_{n-3}),\end{aligned} \tag{6.19}$$

now takes on the form of a product of 4 by 4 propagator matrices, with $\tau = \pm 1, \sigma = \pm 1$, and Reggeon–Reggeon particle vertices that are $4N_c^2$ by $4N_c^2$ matrices, if we include colors. (Helicity labels have been suppressed.)

A more convenient form is to diagonalize the Reggeon propagator. This is achieved by performing an $SU(2)$ rotation, $U = \exp[-i\pi\sigma_2/2]$ by 45° from the τ ("twist") basis to the χ ("color") basis

$$U_{\chi,\tau} = (1/\sqrt{2}) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}_{\chi,\tau}, \tag{6.20}$$

so the Reggeon propagator becomes (in the vector space $(\chi = +1, \chi = -1)$)

$$\Delta_{\chi\sigma,\chi'\sigma'} = \begin{pmatrix} (1+\sigma) & 0 \\ 0 & (1-\sigma) \end{pmatrix}_{\chi,\chi'} \xi_\sigma \Gamma(t) s^{\alpha(t)} \delta_{\sigma,\sigma'}. \tag{6.21}$$

More explicitly, we can express the propagator as a diagonal matrix in χ, σ, with diagonal elements:

$$\Delta_{\chi\sigma}(s,t) = \Delta_\sigma(s,t)\delta_{\sigma,\chi} = 2\xi_\sigma \Gamma(t) s^{\alpha(t)} \delta_{\sigma,\chi}. \tag{6.22}$$

Factorization in fact only involves a pair of degenerate trajectories ($\sigma = \pm 1$), or 2 trajectories in conventional parlance.

This rotation also separates the D- and F-color factors. Replacing χ by σ,

$$\gamma_\sigma^{abc}(t_1) = U_{\tau,\sigma} \gamma_\tau^{abc}(t) = (g\epsilon_1 \cdot \epsilon_2)\gamma_\sigma^{abc} = (g\epsilon_1 \cdot \epsilon_2) \begin{pmatrix} d^{abc} \\ -if^{abc} \end{pmatrix}_\sigma, \tag{6.23}$$

so that $\sigma = -1$ is the F-term, consistent with the vertex, (6.32), introduced earlier. We also must transform the 2-Reggeon vertex to color space, and, again replacing χ by σ,

$$V_{\sigma_1\sigma_2}^{abc}(t_1, \kappa_{12}, t_2) = (g\epsilon_3 \cdot \gamma_3/2) C_{\sigma_1\sigma_2}^{abc} \tilde{G}_{\sigma_1\sigma_2}(t_1, t_2; \kappa_{12}) \tag{6.24}$$

where

$$C_{\sigma_1\sigma_2}^{abc} = \begin{pmatrix} if^{abc} & d^{abc} \\ d^{abc} & if^{abc} \end{pmatrix}_{\sigma_1,\sigma_2}. \tag{6.25}$$

Finally, we have

$$\begin{aligned} \mathcal{A}_{n,\,\mathrm{Regge}} \simeq{}& s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{n-3}^{\alpha_{n-3}} \gamma_{\sigma_1}^{a_1a_2c_1}(1) \Delta_{\sigma_1}(1) V_{\sigma_1\sigma_2}^{c_1a_3c_2}(1,2) \\ & \times \Delta_{\sigma_2}(2) V_{\sigma_2\sigma_3}^{c_2a_4c_3}(2,3) \cdots \Delta_{\sigma_{n-3}}(n-3) \gamma_{\sigma_{n-3}}^{c_{n-3}a_{n-1}a_n}(n-3). \end{aligned} \tag{6.26}$$

Now the restriction to even number of D-vertices is explicit. Starting with an F-vertex (for example), the D-vertices are "kink" operators flipping the sign of σ so that kink/anti-kink pairs guarantees this condition. We also note that the 5-point function is special with only one vertex. In general the kink/anti-kink pairs can be separated.

6.3. Illustration: Signature representation for 4-gluon and 5-gluon amplitudes

To understand this somewhat formal construct, let us consider the special cases for $n = 4$ and 5 which are especially simple.

For the 4-point function, one immediately obtains

$$A_{4,\,\text{Regge}} = \sum_{c_1,\sigma_1} \gamma_{\sigma_1}^{a_1a_2c_1}(1)\Delta_{\sigma_1}(1)\gamma_{\sigma_1}^{c_1a_3a_4}(1)s^{\alpha(t)}$$
$$= g^2(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4)\big[\big(-f^{a_1a_2c}f^{ca_3a_4}\big)\xi_-(t)$$
$$+ \big(d^{a_1a_2c}d^{ca_3a_4}\big)\xi_+(t)\big]\Gamma(t)s^{\alpha(t)}. \tag{6.27}$$

Let us now see how this agrees with a more direct analysis. For $n=4$, there are $4! = 24$ color permutations for planar amplitudes in Eq. (2.1). Taking into account of cyclic symmetry reduces this to 6 independent contributions, and they can be enumerated as three pairs of planar amplitudes with singularities in the s–t, u–t and s–u Mandelstam invariants. Only the s–t and u–t amplitudes contribute to the Regge exchange in the t-channel,

$$\mathcal{A}_4(k_i,a_i) = \mathrm{Tr}[1234]A_4(1234) + \mathrm{Tr}[2134]A_4(2134)$$
$$+ \mathrm{Tr}[1243]A_4(1243) + \mathrm{Tr}[2143]A_4(2143) + (s\text{–}u)\ \text{terms}. \tag{6.28}$$

Note that, with $[T^a,T^b] = if_{abc}T^c$ and $\{T^a,T^b\} = d_{abc}T^c$, we obtain

$$\mathrm{Tr}\big[T^{a_i}T^{a_j}T^{a_k}T^{a_l}\big] = (1/2)\big(if^{ija}+d^{ija}\big)\big(if^{akl}+d^{akl}\big). \tag{6.29}$$

Combining this with the condition of invariance of the planar 4-gluon amplitude under anti-cyclic permutations: $A_{st} = A_{++} = A_4(1234) = A_4(2143) = A_{--}$ and $A_{ut} = A_{-+} = A_4(2134) = A_4(1243) = A_{+-}$, we have

$$\mathcal{A}_4(k_i,a_i) = -f^{ija}f^{akl}(A_{++}-A_{-+}) + d^{ija}d^{akl}(A_{++}+A_{-+}) + (s\text{–}u)\ \text{terms}. \tag{6.30}$$

In the Regge limit $s\to+\infty$, $t<0$ fixed, the amplitude (6.28) factorizes with two degenerate trajectories of opposite signature (or opposite charge conjugation)

$$\mathcal{A}_4(k_i,a_i)/A_{4,\,\text{tree}} \simeq \sum_{\sigma=\pm1}\gamma_\sigma^{a_1a_2c}(t)\big(e^{-i\pi\alpha(t)}+\sigma\big)s^{\alpha(t)-1}\gamma_\sigma^{ca_3a_4}(t) \tag{6.31}$$

where

$$\gamma_\sigma^{abc}(t) = \gamma(t)\begin{cases} d^{abc}, & \sigma=+1,\\ -if^{abc}, & \sigma=-1.\end{cases} \tag{6.32}$$

When compared with Eq. (6.28), the only modifications are factors from the tree specifying the polarizations.

Note that the gluon exchange corresponds to odd-signature exchange, with F-coupling, as expected. The F-coupling trajectory, which contains the gluon pole at $t=0$, has odd signature. We shall occasionally refer to this as the color "octet-trajectory". The D-coupling trajectory, containing both a color "singlet" and an octet component, has even signature and does not have a pole at $t=0$, and its contribution vanishes at the tree level. Nevertheless, at one-loop and beyond, the even-signature persists in the BDS and super string amplitudes. Indeed having contributions from leading trajectory with both signatures is also characteristic of type-II oriented open strings with the ends attached to D-branes.

Turning next to $n=5$. From (6.26), the 5-point function is

$$A_{5,\,\text{Regge}} = \gamma_{\sigma_1}^{a_1a_2c_1}(1)\Delta_{\sigma_1}(1)V_{\sigma_1,\sigma_2}^{c_1a_3c_2}(1,2)\Delta_{\sigma_2}(2)\gamma_{\sigma_2}^{c_2a_4a_5}(2)s_1^{\alpha_1}s_2^{\alpha_2}. \tag{6.33}$$

Substituting in various expressions, we obtain

$$A_{5,\,\mathrm{Regge}} = g^3(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\gamma_3/2)(\epsilon_4\cdot\epsilon_5)\big[\gamma_{\sigma_1}^{a_1a_2c_1}C_{\sigma_1\sigma_2}^{c_1a_3c_2}\gamma_{\sigma_2}^{c_2a_4a_5}\big] \times \tilde G_{\sigma_1\sigma_2}(t_1,t_2;\kappa_{12})\big(\xi_{\sigma_1}(t_1)\Gamma(t_1)s_1^{\alpha(t_1)}\big)\big(\xi_{\sigma_2}(t_2)\Gamma(t_2)s_2^{\alpha(t_2)}\big). \tag{6.34}$$

To clarify this result, let us again return to a more direct analysis for factorization in the double Regge limit of the 5-point function illustrated in Fig. 6. There are now 5! color configurations. To contribute to the double-Regge limit, the planar amplitudes must have Regge singularities in $t_1 = -(k_1+k_2)^2$ and $t_2 = -(k_4+k_5)^2$, so that $(1,2)$ and $(4,5)$ are adjacent. The sum over permutations of $\mathrm{Tr}[12345]A(12345)$, with $(1,2)$ and $(4,5)$ lines adjacent, yields 8 terms. The color traces can be factored on the Regge exchanges as $\mathrm{Tr}[12345] = (1/4)\,\mathrm{Tr}[12c_1]\,\mathrm{Tr}[c_13c_2]\,\mathrm{Tr}[c_245]$. Altogether, this leads to 8 combinations of F- and D-terms.

The 8 permutations of $A(12345)$ are analytically continued from the Euclidean to the physical scattering region independently to give the factorized form. For $n=5$, because amplitudes are odd under anti-cyclic permutations, $A(12345) = -A(54321)$, this further reduces to 4 independent contributions in the Regge limit as illustrated in Fig. 8. The total contribution in the physical region is

$$\mathcal{A}_5 \simeq \big[\mathrm{Tr}(12345) - Tr(54321)\big]A_{++} + \big[\mathrm{Tr}(12354) - \mathrm{Tr}(45321)\big]A_{+-} + \big[\mathrm{Tr}(21345) - \mathrm{Tr}(54312)\big]A_{-+} + \big[\mathrm{Tr}(21354) - \mathrm{Tr}(45312)\big]A_{--}, \tag{6.35}$$

where, in the double-Regge limit, $A_{\eta_1\eta_2}$ is given by Eq. (5.5). Again, dependence on polarizations has been suppressed.

Eq. (6.35) can directly be expressed in "signatured" representation, in terms of the 2-gluon Regge vertex (6.32) and a new double Regge vertex,

$$A_{5,\,\mathrm{Regge}} = \gamma_{\sigma_1}^{a_1a_2c_1}(t_1)\big(e^{-i\pi\alpha(t_1)}+\sigma_1\big)V_{\sigma_1,\sigma_2}^{c_1a_3c_2}\big(e^{-i\pi\alpha(t_2)}+\sigma_2\big)\gamma_{\sigma_2}^{c_2a_4a_5}(t_2), \tag{6.36}$$

where

$$V_{\sigma_1,\sigma_2}^{abc} = \begin{cases} if^{abc}\tilde G_{\sigma_1\sigma_2}(1,2), & \sigma_1=\sigma_2, \\ d^{abc}\tilde G_{\sigma_1\sigma_2}(1,2), & \sigma_1=-\sigma_2, \end{cases} \tag{6.37}$$

with $\tilde G_{\sigma_1\sigma_2}(1,2)$ given by Eq. (5.9). Again, when compared with Eq. (6.34), the only modifications are factors from the tree specifying the helicity and color configurations.

7. Discontinuity in "missing mass" M^2 for $n \geqslant 6$

In Section 4, we have discussed some aspects of analyticity constraints on the 5-point function in various Regge limits. In particular, we point out that, in the double-Regge limit, the BDS amplitude does not satisfies the Steinmann rules. Nevertheless, we have demonstrated in Sections 5 and 6 that this deficiency does not affect the discussion of signature factorization property in the physical region for n-point amplitudes in the linear multi-Regge limit. In this section, we return to a closer examination of analyticity and unitarity constraints for higher point amplitudes.

For $n \geqslant 6$, there now exists threshold singularities in the physical region in "crossed invariants" involving both initial and final momenta. Consider the amplitude for a 3-to-3 process,

$$a+b+x' \to a'+b'+x. \tag{7.1}$$

The 3-to-3 amplitude has discontinuity in $M^2 = -(p_a+p_b-p_x)^2$, the invariant in the so-called "missing mass" channel. Just as the 2-to-2 unitarity in the forward limit of $t=0$ leads to a

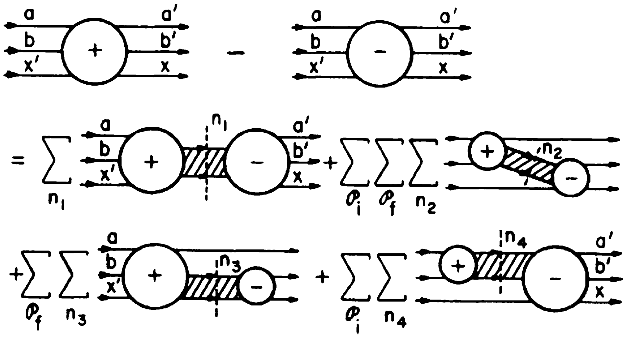

Fig. 12. 3-to-3 unitarity in the physical region. The sum over n_1 for the first term on the right represents all allowed intermediate states in the abx' channel. $\mathcal{P}_i$ and $\mathcal{P}_f$ represent sums over different initial- and final-state combinations for various intermediate states, labelled by sums n_2, n_3 and n_4.

total cross section, the discontinuity in M^2 in the forward limit where $p_a = p_{a'}$, $p_b = p_{b'}$ and $p_x = p_{x'}$ leads to the inclusive cross section for the process

$$a + b \to x + \text{anything}, \tag{7.2}$$

$$\frac{d\sigma}{dp_x} \sim (1/s)\mathrm{Disc}_{M^2} T_6(a, b, x' \to a', b', x), \tag{7.3}$$

where $M^2 > 0$. This is a generalized Optical Theorem. If multi-Regge applies to the 6-point function, taking the discontinuity in M^2 leads to a non-trivial prediction for the inclusive cross section.

We focus in this section on such discontinuities in the physical regions. Surprisingly, we find the absence of Regge contribution in the "triple-Regge" limit of the 6-point function. More generally, BDS amplitudes do not lead to a well-defined "Reggeon-particle" 4-point amplitude with the expected M^2-discontinuity, based on flat-space string expectation. The absence of such flat-space behavior also holds for $n > 6$.

We end this introduction by illustrating the relevant 3-to-3 unitarity condition which can be represented schematically by Fig. 12. There are now four types of terms on the right-hand side of this unitarity relation [56,59,60,62]. Each term can be associated with a physically realizable re-scattering process, and each is a discontinuity in an appropriate invariant. For the third and the fourth terms, they represent discontinuities in initial or final sub-energy invariants, a generalization of that discussed earlier for the 2-to-3 unitarity. The second term is new; it represents discontinuities in "crossed invariants" involving both initial and final momenta, e.g., the missing-mass invariant, $M^2 = -(p_a + p_b - p_x)^2$ introduced above. We examine in this section properties of the discontinuity in M^2 in various Regge limits.

Another important reason for studying the helicity-pole/triple-Regge limit is the fact that functions $\mathrm{Li}_2(1 - u_i)$, which enter in the n-point amplitudes for $n \geqslant 6$, now become even more important. In all the Regge limits that we have studied in [34], the cross ratios, u_1, u_2, u_3, (5.13), remain finite, moreover taking on values of either 0 or 1, in the linear multi-Regge limit, and as a consequence the terms $\mathrm{Li}_2(1 - u_i)$ in the BDS ansatz for $n \geqslant 6$ gave only constants, and thus did not influence the limit. Any other well-behaved function $f(u_1, u_2, u_3)$ of the cross ratios u_i at 0 and 1 that one could in principle add to the BDS ansatz while still respecting dual conformal invariance [20,22,30] would have become also irrelevant. It is therefore crucial for the consistency of the (corrected) BDS ansatz to examine limits where this does not happen, and the u_i's are

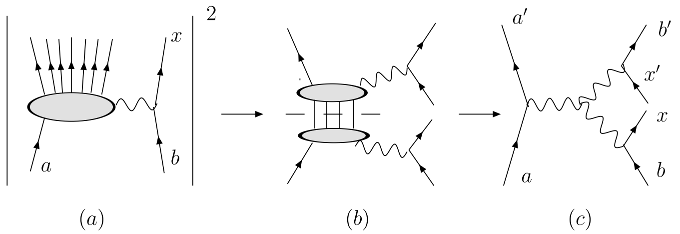

Fig. 13. Triple-Regge behavior of inclusive cross section as M^2-discontinuity.

infinite. Notice that from the point of view of BDS invariants, $t_i^{[r]}$'s, the simplest limit possible would involve only two variables (one independent and one dependent) going to infinity. This will turn out to be the helicity-pole/triple-Regge limit, and in this limit indeed we find that two of the u_i's will become large.

7.1. Triple-Regge limit and expectations from flat space string theory

Let us first consider the "triple-Regge" limit. Kinematically, an inclusive cross section, Eq. (7.3), can be treated as a 2-to-2 cross section, with M^2 the mass-squared for one of the two final particles, as illustrated schematically by the Fig. 13a. As such, it is a function of three independent invariants, two being the energy and momentum-transfer invariants, s and t, and the third being M^2. The triple-Regge limit corresponds to having

$$s/M^2 \to \infty, \qquad M^2 \to \infty \tag{7.4}$$

with t fixed. The standard Regge behavior first leads to a factor $|s/M^2|^{2\alpha(t)}$, with M^2 serving as a scale. There will be a second Regge factor, $(M^2)^{\alpha(0)}$, which accounts for the increasing multiplicity of final states as M^2 grows. (Fig. 13b.) As the M^2-discontinuity of a 6-point amplitude, the inclusive cross section thus takes on a triple-Regge form [61,62]

$$d\sigma \sim (1/s)\mathrm{Disc}_{M^2} A_6 \sim (1/s)G(t)\big(M^2\big)^{\alpha_0}\big|s/M^2\big|^{2\alpha_2}, \tag{7.5}$$

which can be represented schematically by Fig. 13c.

We next generalize this triple-Regge behavior to the non-forward limit for the 3-to-3 process, $a+b+x' \to a'+b'+x$. We can begin with any color ordering so long as (a,b,x) are adjacent, i.e., an amplitude with singularities in $M^2 = -(p_a+p_b-p_x)^2$. For definiteness, let us consider first the color-ordering (123456) identified with (a,a',b',x',x,b), as indicated by Fig. 14a. Here, a,b,x' are incoming and a',b',x are outgoing, as indicated by arrows in Figs. 13b, 13c and 14b. With an all-incoming momentum convention, one has $k_1 = -p_{b'}$, $k_2 = p_{x'}$, $k_3 = -p_x$, $k_4 = p_b$, $k_5 = p_a$, and $k_6 = -p_{a'}$. Amplitudes for other orderings can then be obtained by appropriate substitutions and analytic continuations.

Let us denote adjacent BDS invariants by

$$\begin{aligned} &t_4^{[2]} = s, \qquad t_6^{[2]} = s', \qquad t_5^{[2]} = t_1, \qquad t_3^{[2]} = t_2, \qquad t_1^{[2]} = t_2', \qquad t_2^{[2]} = s_{12}, \\ &t_3^{[3]} = t_6^{[3]} = M^2, \qquad t_1^{[3]} = t_4^{[3]} = \Sigma, \qquad t_2^{[3]} = t_5^{[3]} = \Sigma'. \end{aligned} \tag{7.6}$$

The triple-Regge limit corresponds to

$$s/M^2 \to \infty, \qquad s'/M^2 \to \infty, \qquad M^2 \to \infty, \tag{7.7}$$

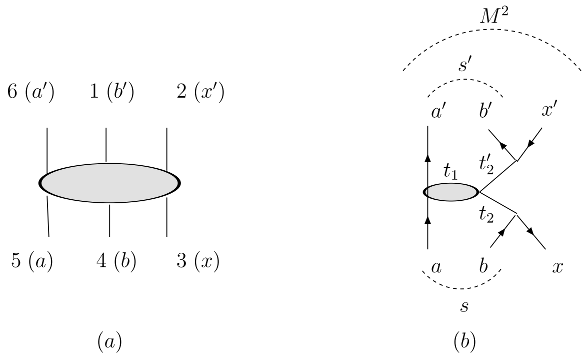

Fig. 14. 6-point amplitude with color-ordering specified by the left figure, (a). Some of the invariants appropriate for the triple-Regge limit are shown in (b).

with all other invariants fixed. Since there are only eight independent variables, there will be a constraint among these nine invariants. As we have shown in Ref. [34], the constraint simplifies in various Regge limits. To isolate the singularity in M^2, we shall first go to the Euclidean region and then analytically continue M^2 to the physical region where $M^2 > 0$.[14] The forward limit has $s = s'$, $t_2 = t_2'$, $t_1 = 0$, etc. Away from the forward limit, (7.5) generalizes to

$$\mathrm{Disc}_{M^2} A_6 \sim G(t_2, t_2'; t_1)\big(M^2\big)^{\alpha(t_1)-\alpha(t_2)-\alpha(t_2')}(-s)^{\alpha(t_2)}(-s')^{\alpha(t_2')}. \tag{7.8}$$

More precisely, for $M_6 = A_6/A_{6,\,\mathrm{tree}}$, one finds, for flat-space string theory,

$$\mathrm{Disc}_{M^2} M_6 \sim G(t_2, t_2'; t_1)\big(M^2\big)^{\omega(t_1)-\omega(t_2)-\omega(t_2')}(-s)^{\omega(t_2)}(-s')^{\omega(t_2')}, \tag{7.9}$$

where we recall that $\omega(t) = \alpha(t) - 1$.

It is also useful to first examine a more general limit:

$$s \to -\infty, \qquad s' \to -\infty \tag{7.10}$$

with $M^2 < 0$ fixed before taking the discontinuity in M^2. This is historically referred to as the helicity-pole limit [61]. For flat-space string theory, one finds [58]

$$M_6 \sim \mathcal{A}\big(M^2, t_2, t_2'; t_1\big)(-s)^{\omega(t_2)}(-s')^{\omega(t_2')} + \mathcal{B}, \tag{7.11}$$

where $\mathcal{B}$ has no discontinuity in M^2. $\mathcal{A}(M^2, t_2, t_2'; t_1)$ is known as the Reggeon-particle-to-Reggeon-particle amplitude.[15] In flat-space string theory, it takes on the form

$$\mathcal{A}_{\mathrm{string}}\big(M^2, t_2, t_2'; t_1\big) \sim \int_0^\infty dx\, x^{-\omega(t_1)+\omega(t_2)+\omega(t_2')-1}(1-x)^{-\omega(M^2)}, \tag{7.12}$$

Note that this is analogous to the limit taken for the 5-point function, (4.23), where the amplitude is also expressed as a sum of two pieces, each with unique singularity structure.

[14] For the physical region, some of the BDS invariants will have to be continued to positive values. For appropriate continuation procedure, see Refs. [59,60,62].

[15] The corresponding amplitude for the closed string sector plays an important role in an eikonal sum for multiple "graviton" exchanges. See, e.g., Refs. [5,6] and work by Amati, Ciafaloni and Veneziano [72,73].

$\mathcal{A}_{\text{string}}(M^2, t_2, t_2'; t_1)$ is structurally analogous to $\mathcal{A}_{\text{string}}(s_2, t_2; t_1)$, (4.24), the Reggeon-particle–particle–particle amplitude discussed earlier for the 5-point function.

We emphasize that the discontinuity of the Reggeon-particle-to-Reggeon-particle amplitude, $\mathcal{A}_{\text{string}}(M^2, t_2, t_2', t_1)$, in M^2 directly leads to the inclusive cross section in the forward helicity-pole limit, as illustrated in Fig. 13. This is a generic feature which should hold in general. For M^2 large, one can easily see that

$$\mathcal{A}_{\text{string}}\big(M^2, t_2, t_2', t_1\big) \sim \big(-M^2\big)^{\omega(t_1)-\omega(t_2)-\omega(t_2')}, \tag{7.13}$$

consistent with Eq. (7.9). From the perspective of a dispersion representation in M^2, the piece $\mathcal{B}$ in (7.11) represents a subtraction. For completeness, we record here for $\mathcal{B}$ for scalar tachyon amplitude, which is kinematically simpler. It consists of three terms, (Eq. (4.24) of Ref. [58]),

$$\mathcal{B} \sim (-s')^{\omega(t_1)}U_1 + (-s)^{\omega(t_1)}U_2 + (-s)^{(\omega(t_1)+\omega(t_2)-\omega(t_2'))/2}(-s')^{(\omega(t)+\omega(t_2')-\omega(t_2))/2}U_{12} \tag{7.14}$$

with U_1, U_2 and U_{12} independent of M^2.

7.2. BDS

We now turn to BDS and see if our flat-space based expectations are satisfied. We first consider the helicity-pole/triple-Regge limits. As pointed earlier, here we will deal with a situation where the di-logarithm functions begin to play an even more important role. We will demonstrate that, under BDS ansatz, one finds a surprising result where the Reggeon-particle-to-Reggeon-particle amplitude, $\mathcal{A}_{bds}(M^2, t_2, t_2', t_1)$, vanishes.

Before carrying out this analysis, it is useful to recall that, under dimensional regularization, the physical gluon pole does not lie on the Regge trajectory (see Section 3). In order to avoid dealing with such issues, we shall avoid approaching singular points, e.g., $t_1 = 0$ in (7.11). In general, in addressing various Regge/helicity pole limits, e.g., leading to (7.13), we shall keep all fixed variables Euclidean, e.g., $t_2, t_2', t_1, \Sigma, \Sigma' < 0$ in (7.6). In these regions, no unusual behavior is expected from the tree-amplitudes.

As mentioned earlier, in the notation used by BDS, for $n = 6$, with color order (123456), there are 9 invariants ($t_i^{[2]}, i = 1,\dots,6$ and $t_i^{[3]}, i = 1,2,3$) and one constraint among them. In our earlier study for the multi-Regge limits [34], we have seen that it is convenient to introduce $t_1, t_2, t_3, s_1, s_2, s_3$ and Σ_1, Σ_2 (or equivalently κ_1, κ_2), as independent variables, with s as dependent variable. As we have also shown in [34], the single-Regge limit, involves taking three $t_i^{[r]}$'s to infinity, two independent and one dependent, e.g. s_1, Σ_1 and s (or equivalently s_1, s to infinity and κ_1 fixed). For the helicity pole limit, it is even simpler since this limit involves only taking two invariants large. For this limit, we have found convenient to start with color-ordering indicated by Fig. 14a, and have also introduced a set of invariants, suggested by the inclusive cross-section, which are related to the BDS invariants by (7.6).

An equally useful set of notations for the BDS invariants has been introduced in [34] for the poly-Regge limit (which makes full use of the cyclical symmetry of the problem),

$$\begin{aligned} &t_6^{[2]} = s_{23}, \quad t_1^{[2]} = t_2, \quad t_2^{[2]} = s_{12}, \quad t_3^{[2]} = t_1, \quad t_4^{[2]} = s_{31}, \quad t_5^{[2]} = t_3, \\ &t_1^{[3]} = s_1, \quad t_2^{[3]} = s_2, \quad t_3^{[3]} = s_3. \end{aligned} \tag{7.15}$$

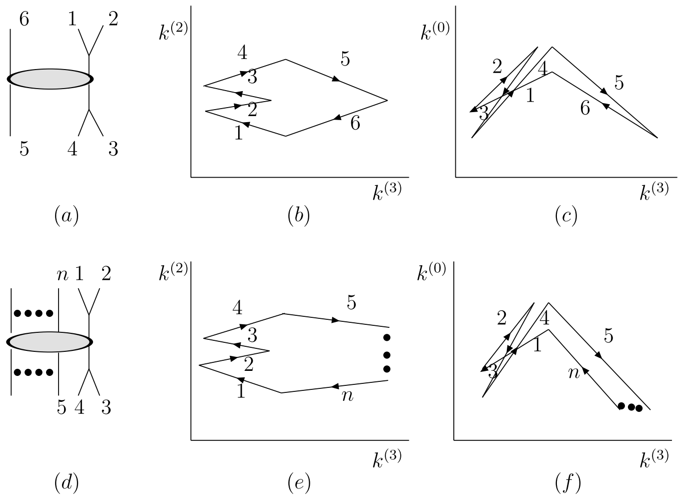

Fig. 15. Configuration of momenta in the helicity pole limit. (a) 6-point amplitude; (b) and (c) momenta for 6-point amplitude in the $(k^{(3)}, k^{(2)})$ and $(k^{(3)}, k^{(0)})$ planes. (d) n-point amplitude; (e) and (f) momenta for n-point amplitude in the $(k^{(3)}, k^{(2)})$ and $(k^{(3)}, k^{(0)})$ planes.

These notations have also been used in [36,58] (see Fig. 16), and they are related to (7.6), the M^2-discontinuity notation, by the substitutions $s_3 \leftrightarrow M^2$, $s_2 \leftrightarrow \Sigma'$, $s_1 \leftrightarrow \Sigma$, $t_3 \leftrightarrow t_1$, $t_2 \leftrightarrow t_2'$, $t_1 \leftrightarrow t_2$, $s_{23} \leftrightarrow s'$, $s_{31} \leftrightarrow s$ and $s_{12} \leftrightarrow s_{12}$.

For the helicity pole limit, we only have two $t_i^{[r]}$ invariants becoming large, $t_4^{[2]} = s_{31} = s \to -\infty$ and $t_6^{[2]} = s_{23} = s' \to -\infty$. The constraint among the BDS invariants implies in this limit that we have $s_{31} \simeq s_{23}$ ($s \simeq s'$), this being therefore the simplest limit one can take on the 6-point BDS ansatz. As mentioned earlier, all fixed variables will be held Euclidean, e.g., $t_1 < 0$, away from the singular point $t_1 = 0$. We can also characterize the limit by saying that s/M^2 is large, with M^2 and s/s' fixed, as discussed earlier. (Recall the analogous limit for $n = 5$.) In this limit, two of the three u_i cross ratios go to infinity, so it is in principle a good way to test for the presence of an additional function $f(u_1, u_2, u_3)$, as it could become important in the limit where its arguments are large (the same way as $\mathrm{Li}_2(1 - u_i)$ in the BDS 6-point amplitude does). For the determination of the $f(u_1, u_2, u_3)$ function via dual Wilson loops, it is of interest to exhibit the configuration of momenta (or dual Wilson loop) in the helicity pole limit. We therefore present it, together with the generalization to higher n-points, in Fig. 15.

We obtain in this limit for the BDS 6-point amplitude (expressing in both the notation appropriate for inclusive distribution and in the poly-Regge notation for comparison with the poly-Regge limit),

$$\begin{aligned} M_6^{\mathrm{BDS}} &\simeq (-s)^{(\omega(t_1)+\omega(t_2)-\omega(t_2'))/2}(-s')^{(\omega(t_2)+\omega(t_2')-\omega(t_2))/2} U(t_1, t_2, t_2', s_{12}, \Sigma, \Sigma') + \cdots \\ &\leftrightarrow (-s_{23})^{(\omega(t_2)+\omega(t_3)-\omega(t_1))/2}(-s_{31})^{(\omega(t_3)+\omega(t_1)-\omega(t_2))/2} U(t_1, t_2, t_3, s_{12}, s_1, s_2) + \cdots \end{aligned} \tag{7.16}$$

or, if we substitute $s = s'$,

$$\begin{aligned} M_6^{\mathrm{BDS}} &= (-s)^{\omega(t_1)} U(t_1, t_2, t_2', s_{12}, \Sigma, \Sigma') + \cdots \\ &\leftrightarrow (-s_{23})^{\omega(t_3)} U(t_1, t_2, t_3, s_{12}, s_1, s_2) + \cdots. \end{aligned} \tag{7.17}$$

Notice that (7.16) only corresponds to the last term in (7.14), and that there is no M^2 dependence at all. That is, the Reggeon-particle-to-Reggeon-particle amplitude, $\mathcal{A}_{bds}(M^2, t_2, t_2', t)$, vanishes.

We emphasize that in order to obtain $\mathcal{A}(M^2, t_2, t_2', t) \neq 0$, the first term in (7.11), we would need to add in $\log M_6^{\text{BDS}}$ a term

$$\Delta \log M_6^{\text{BDS}} \simeq -\frac{f}{8} \ln \frac{t_6^{[2]} t_2^{[2]}}{t_4^{[2]}} \ln \frac{t_3^{[2]} t_5^{[2]}}{t_1^{[2]}} + O(1) = -\frac{f}{8} \ln u_{6,4;2} \ln u_{1,3;5} + O(1), \tag{7.18}$$

where

$$u_{i,j;k} \equiv \frac{x_{i,k}^2 x_{j,k}^2}{x_{i,j}^2} \tag{7.19}$$

are not cross ratios, and cannot be written in terms of them. Thus such a term would be prohibited by dual conformal invariance [20,22,30].

The absence of proper M^2 discontinuity raises further concerns on the reliability of BDS ansatz for multi-gluon amplitudes. There are several possibilities. It is important to point out, from Fig. 13, at 1-loop, the effective diagrams giving M^2 dependence would come from the Passarino–Veltman reduction to “two mass hard” ($2mh$) scalar boxes, i.e., scalar boxes with 2 adjacent external massive (virtual) lines and the other two external lines massless (on-shell). But it is known that for MHV amplitudes, there are no contributions to any 1-loop n-point functions from $2mh$ scalar boxes. In fact, the Passarino–Veltman reduction obtains only “two mass easy” ($2me$) scalar boxes, with non-adjacent external massive lines, for $\mathcal{N} = 4$ SYM at leading order in N. At higher loops it becomes more involved to show how the M^2 dependence vanishes, but at 2-loops, the explicit 6-point calculation of [31] finds the BDS ansatz is correct (up to a function of cross ratios, that cannot generate the M^2 dependence, as we argued), and at 3-loops, the explicit IR divergence formula agrees with the BDS ansatz [54]. Therefore, it is possible that the absence of proper M^2 discontinuities is the property of MHV amplitudes only.

However, our analogous findings in the helicity pole limit on the vanishing of “Reggeon-particle” amplitudes for both $n = 5$ and $n = 6$ BDS amplitudes suggest a more serious deficiency. An interesting possibility relates to the fact that, as pointed out earlier, to reconstruct the full amplitudes, it is technically insufficient to keep only $O(1)$ terms for $\log M_n$ as $\epsilon \to 0$. That is, from $M_n = e^{\log M_n}$, the $O(1)$ for M_n will receive contributions from terms in $\log M_n$ to all orders in ϵ, due to the presence of ϵ^{-2} and ϵ^{-1} terms in $\log M_n$. In fact, for $n > 4$, $O(\epsilon)$ terms at 1-loop will involve more than box-diagrams.[16] Therefore, it is conceivable that proper M^2 discontinuities can be restored in this more general setting. However, as we have also pointed out in the Introduction, we do not consider this more general treatment in our analysis here.

7.3. Poly-Regge limit

We now re-visit the poly-Regge limit, studied in [34], and we re-produce here the schematic depiction for this limit in Fig. 16. In that limit, in accordance with [58], we have found that

$$A_6 \simeq \prod_{j=1}^{3} (-s_j)^{\alpha(t_j)} V(t_i; s_{ij}), \tag{7.20}$$

[16] We would like to thank Marcus Spradlin and Anastasia Volovich for emphasizing this fact to us.

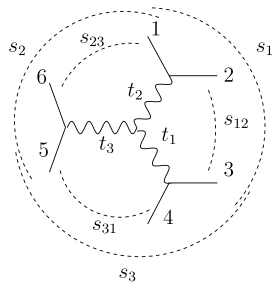

Fig. 16. Color-ordered amplitude with invariants appropriate for the poly-Regge limit.

where $V(t_i; s_{ij})$ is also referred to as the triple Regge vertex. The poly-Regge limit is defined as $|s_{ij}| \gg |s_i| \to \infty$, with t_i and $\eta_{ij} = s_{ij}/s_i s_j$ kept fixed. In Section 3 of [58], the triple Regge vertex V of flat space string theory was analyzed in the limit that η_{ij} are also large. This is a very interesting case to consider since $u_1 = \eta_{23} t_1, u_2 = \eta_{31} t_2, u_3 = \eta_{12} t_3$. This means we encounter a situation where all $u_i \gg 1$. This is therefore the best case to study for determining the relevance of the $\mathrm{Li}_2(1-u_i)$ terms in the BDS ansatz.

In [58], the amplitude of flat space string theory was found to become in the above limit, when re-expressed in terms of s_i and s_{ij},[17]

$$\begin{aligned} A_6 \sim\ & (-s_3)^{\alpha_3-\alpha_1-\alpha_2}(-s_{13})^{\alpha_1}(-s_{23})^{\alpha_2}\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(\alpha_1+\alpha_2-\alpha_3) \\ & + (-s_1)^{\alpha_1-\alpha_2-\alpha_3}(-s_{12})^{\alpha_2}(-s_{13})^{\alpha_3}\Gamma(-\alpha_2)\Gamma(-\alpha_3)\Gamma(\alpha_2+\alpha_3-\alpha_1) \\ & + (-s_2)^{\alpha_2-\alpha_1-\alpha_3}(-s_{23})^{\alpha_3}(-s_{21})^{\alpha_1}\Gamma(-\alpha_3)\Gamma(-\alpha_1)\Gamma(\alpha_3+\alpha_1-\alpha_2) \\ & + \frac{1}{2}(-s_{23})^{(\alpha_3+\alpha_2-\alpha_1)/2}(-s_{13})^{(\alpha_3+\alpha_1-\alpha_2)/2}(-s_{12})^{(\alpha_1+\alpha_2-\alpha_3)/2} \\ & \times \Gamma\big(-(\alpha_3+\alpha_2-\alpha_1)/2\big)\Gamma\big(-(\alpha_3+\alpha_1-\alpha_2)/2\big)\Gamma\big(-(\alpha_1+\alpha_2-\alpha_3)/2\big), \end{aligned} \tag{7.21}$$

where $\alpha_i = \alpha(t_i)$, $i = 1, 2, 3$. However, from the BDS result, we obtain again just the fourth term, namely

$$(A_6)_{\mathrm{BDS}} \sim (-s_{23})^{(\alpha_3+\alpha_2-\alpha_1)/2}(-s_{13})^{(\alpha_3+\alpha_1-\alpha_2)/2}(-s_{12})^{(\alpha_1+\alpha_2-\alpha_3)/2}\Gamma(t_1, t_2, t_3) + \ldots \tag{7.22}$$

and the product of Gamma functions in the last term in (7.21) is replaced by a new vertex

$$\Gamma(t_1, t_2, t_3) = \exp\left\{\left(\frac{f^{-1}(\lambda)}{8\epsilon} + \frac{g(\lambda)}{4}\right)(\ln t_1 + \ln t_2 + \ln t_3)\right\}. \tag{7.23}$$

Note that, in order to express (7.23) in terms of α_i, or equivalently ω_i, one would have to modify factors associated with IR divergent terms.

[17] This limit allows an interpolation between the poly-Regge limit and the helicity pole limit. The expression quoted here is for bosonic string with tachyons. The corresponding amplitude for superstring is similar and we will not report it here.

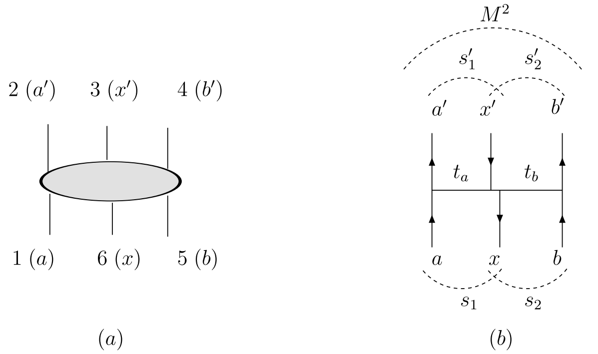

Fig. 17. 6-point amplitude with momentum-color-ordering appropriate for Mueller–Regge limit.

To restore all terms in (7.21), one needs

$$\Delta \log M_6^{\mathrm{BDS}} \simeq -\frac{f(\lambda)}{8} \log \frac{t_5^{[2]}}{t_3^{[2]} t_1^{[2]}} \log \frac{t_6^{[2]} t_4^{[2]}}{t_2^{[2]} t_3^{[3]}} + O(1)$$

$$= -\frac{f(\lambda)}{8} \log u_{1,5;3} \log \frac{u_{2,4;6}}{x_{3,6}^2} + O(1) \tag{7.24}$$

which cannot be expressed as functions of cross ratios. This result is analogous to the findings of Section 7.2, and makes it clearer that the full flat space string theory result does not appear in $\mathcal{N} = 4$ SYM, but rather only a subleading term in its expansion.

7.4. Mueller Regge limit

For completeness, we end by a discussion on the so-called "Mueller–Regge" limit where kinematically the momenta can be arranged in the following suggestive multi-Regge order, Fig. 17. To allow discontinuity in M^2, we need to consider color ordering with (axb) and $(a'x'b')$ adjacent. It is convenient to consider first the color-ordering, $(123456) = (aa'x'b'bx)$ (Fig. 17a); other color orderings can be obtained by appropriate substitutions and continuations.

Let us now denote adjacent BDS invariants as, (Fig. 17b),

$$t_5^{[2]} \equiv s_2, \qquad t_3^{[2]} \equiv s'_2, \qquad t_6^{[2]} \equiv s_1, \qquad t_2^{[2]} \equiv s'_1, \qquad t_1^{[2]} \equiv t_a, \qquad t_4^{[2]} \equiv t_b,$$
$$t_2^{[3]} \equiv M^2, \qquad t_4^{[3]} \equiv \Sigma, \qquad t_3^{[3]} = \Sigma'. \tag{7.25}$$

Note that, for this color-ordering,

$$s = -(p_a + p_b)^2 = -(k_1 + k_5)^2, \qquad s' = -(p_{a'} + p_{b'})^2 = -(k_2 + k_4)^2, \tag{7.26}$$

are non-adjacent invariants.

We consider $s_1 \simeq s'_1 \to -\infty$, and $s_2 \sim s'_2 \to -\infty$, with t_a, t_b, ... fixed. This is effectively a linear double-Regge limit. One finds that, [62]

$$M_{6,\,\mathrm{string}} \sim (-s_1)^{\omega(t_a)} (-s_2)^{\omega(t_b)} G_{2,\,\mathrm{string}}^{[4]}(\kappa, \kappa', \ldots), \tag{7.27}$$

where

$$\kappa = \frac{s_1 s_2}{M^2}, \qquad \kappa' = \frac{s'_1 s'_2}{M^2}, \quad \text{and} \quad \kappa \simeq \kappa' \tag{7.28}$$

are also kept fixed. By continuing M^2 back to the physical region, it is clear that taking the discontinuity in M^2 is the same as taking the discontinuity in κ and κ'. For the inclusive cross section where $\kappa = \kappa'$, one finds that $G^{[4]}_{2,\,\mathrm{string}}(\kappa, \kappa, \ldots)$ in flat-space string theory can be expressed as an integral over the Reggeon-particle–Reggeon vertex introduced earlier, $G_2(t_a, t_b, \kappa)$, [62]

$$G^{[4]}_{2,\,\mathrm{string}}(\kappa, \kappa, \ldots) \sim \int_0^1 dz\, z^{-\omega(\Sigma)}(1-z)^{-\omega(\Sigma')} G_{2,\,\mathrm{string}}\big(t_a, t_b, z(1-z)\kappa\big). \tag{7.29}$$

A similar but more involved expression for the non-forward limit can also be obtained. Note that $G^{[4]}_{2,\,\mathrm{string}}$ is real for $\kappa < 0$ in the Euclidean region, and has a right-hand cut for $\kappa > 0$. With $s_1 = s_1'$, $s_2 = s_2' < 0$, one has

$$\mathrm{Disc}_{M^2} M_{6,\,\mathrm{string}} \simeq (-s_1)^{\omega(t_a)}(-s_2)^{\omega(t_b)} \mathrm{Disc}_{M^2} G^{[4]}_{2,\,\mathrm{string}}(\kappa, \kappa, \ldots). \tag{7.30}$$

The discontinuity can then be taken, similar to that for a 5-point function. Unlike the case of 5-point function, however, this discontinuity now directly relates to an observable.

Let us next turn to the BDS amplitude. We can now do the Mueller–Regge limit in exactly the same way, using the variables in (7.25). In the limit $s_1 \simeq s_1' \to -\infty$; $s_2 \simeq s_2' \to -\infty$, $M^2 \to -\infty$, with $\kappa = s_1 s_2/M^2$ and $\kappa' = s_1' s_2'/M^2$ fixed, we obtain from the BDS amplitude

$$M_6 \simeq \big(\sqrt{s_1 s_1'}\,\big)^{\omega(t_a)}\big(\sqrt{s_2 s_2'}\,\big)^{\omega(t_b)} G^{[4]}_2(\kappa, \kappa', \ldots), \tag{7.31}$$

where the vertex $G^{[4]}_2(\kappa, \kappa', \ldots)$ is given by

$$\begin{aligned} \log G^{[4]}_2(\kappa, \kappa', \ldots) = {} & \frac{f(\lambda)}{8} \log(-t_a)\log(-t_b) + \frac{1}{2}\left(\frac{f^{(-1)}(\lambda)}{4\epsilon} + \frac{g(\lambda)}{2}\right)\log(t_a t_b) \\ & - \frac{f(\lambda)}{8}\left\{\mathrm{Li}_2\left(1 - \frac{\kappa' s_1}{\Sigma' s_1'}\right) + \mathrm{Li}_2\left(1 - \frac{t_a t_b}{\Sigma'\Sigma}\right) + \mathrm{Li}_2\left(1 - \frac{\kappa s_1'}{\Sigma s_1}\right)\right. \\ & \left. + \frac{1}{2}\left[\log^2\left(\frac{\kappa' s_1}{\Sigma' s_1'}\right) + \log^2\left(\frac{t_a t_b}{\Sigma'\Sigma}\right) + \log^2\left(\frac{\kappa s_1'}{\Sigma s_1}\right)\right]\right\}. \end{aligned} \tag{7.32}$$

Let us first note that, in the physical region, $\Sigma < 0$ and $\Sigma' < 0$. With $\kappa > 0$ and $\kappa' > 0$, those four terms in (7.32) involving κ or κ' will lead to discontinuities in M^2. Note that the last two lines of (7.32) go to zero if the arguments of the log's go to infinity. Thus if we have $s_1 \simeq s_1'$; $s_2 \simeq s_2'$ and $\kappa \simeq \kappa'$ is taken to be much larger than Σ, Σ' on top of the Mueller–Regge limit, the $\kappa \simeq \kappa'$ dependence completely drops out, and like in the previous cases, there is no M^2 dependence in the vertex G. Indeed, in flat-space string theory, one moves smoothly from the Mueller double-Regge limit to the triple-Regge/helicity-pole limits, with $\kappa \to \infty$, $\kappa' \to \infty$. Then, if $\kappa = \kappa'$ (forward limit), and much larger than Σ, Σ',

$$M_6 \simeq (-s_1)^{\omega(t_a)}(-s_2)^{\omega(t_b)} G^{[4]}_2(\kappa, \kappa', \ldots), \tag{7.33}$$

where

$$\begin{aligned}\log G_2^{[4]} &= \frac{f(\lambda)}{8}\log t_a \log t_b + \frac{1}{2}\left(\frac{f^{(-1)}(\lambda)}{4\epsilon} + \frac{g(\lambda)}{2}\right)\log(t_a t_b) \\ &\quad - \frac{f(\lambda)}{8}\left\{\mathrm{Li}_2\left(1 - \frac{t_a t_b}{\Sigma'\Sigma}\right) + \frac{1}{2}\log^2\left(\frac{t_a t_b}{\Sigma'\Sigma}\right)\right\} \\ &= \log\bigl[\gamma(t_a)\gamma(t_b)\bigr] + \frac{f(\lambda)}{8}\log t_a \log t_b \\ &\quad - \frac{f(\lambda)}{8}\left\{\mathrm{Li}_2\left(1 - \frac{t_a t_b}{\Sigma'\Sigma}\right) + \frac{1}{2}\log^2\left(\frac{t_a t_b}{\Sigma'\Sigma}\right)\right\}.\end{aligned} \tag{7.34}$$

Since there is no M^2 dependence in (7.34), there is no discontinuity, $\mathrm{Disc}_{M^2} A_6 = 0$, same as for the helicity pole limit or the poly-Regge limit.

For the Mueller double-Regge limit, however, κ and κ' should be kept fixed at finite values. In that case, the exact cancellation between discontinuities in M^2 from Li_2 and the terms in the last line in (7.32) no longer holds. The expression can be further simplified, but will not be provided here.

8. Discussion

A central issue for this paper, as well as others [35,74], is the continuation of multi-Regge amplitudes from the Euclidean region to various physical regions which are distinguished by different color orderings. The problem that must be dealt with is that the order in going on-shell in the continuation and in taking the multi-Regge limit do not commute. This is discussed in Section 2 and more explicitly in Section 5.2. In the body of this paper it is emphasized that multi-Regge factorization is expected only for signatured amplitude. Thus for the 6-point amplitude, 6 of the 8 inequivalent color configurations allow for the straightforward results that $\Phi = 1$ (in (5.16)) is consistent with the naive continuation. There are two other configurations, where Φ takes on $e^{2\pi i}$ and $e^{-2\pi i}$ respectively by circling the branch point at $\Phi = 0$. It is these latter two continuations which have been the subject of controversy [35,74]. We proceed to discuss the consequences of the two opposing positions.

8.1. *Comments on analytical continuation in the literature*

In the Euclidean multi-Regge limit, the finite parts of $\log M_6^{\mathrm{BDS}}$ contain $\log s$ and dilogs, and all cross ratios either vanish or approaching 1 in this limit. Thus in [34] we have found that dilogs don't contribute in the Euclidean region and factorization can be achieved.

However, as mentioned, in [35] it was found that analytical continuation of the $u_3 \equiv \Phi$ cross ratio in a physical region for $n = 6$ gluons leads to an extra term, that apparently breaks factorization.

The conventional procedure is to drop $O(\epsilon)$ for

$$\log M_6 = \log \frac{A_6}{A_{6,\,\mathrm{tree}}} = I_6^{(1)}(\epsilon) + F_6^{(1)}(0) \tag{8.1}$$

in taking the multi-Regge limit. In this case, BDS amplitudes for $n \geqslant 6$ reduce to simple combinations of products of logarithms and dilogarithmic functions. As pointed in Ref. [34], these dilog functions do not contribute in the Euclidean multi-Regge limit and naive Regge factorization can be achieved. This relies on the observation that all cross ratios either vanish or approaching 1 in this limit.

However, as pointed out above, analyticity consideration forces one to relax the constraint on the cross ratios in the course of continuation back to the physical region. In the case of $n = 6$, there only three such cross ratios, and the one which requires special attention is the variable Φ, or u_3 in (5.13). Since its nontrivial dependence enters in a single dilog term, continuation into the physical region can be carried out explicitly. In [35], one finds for A_{-+-} that, in the course of continuation where $\Phi : 1 \to e^{-2\pi i}$, $\log M_6$ picks up an extra piece

$$\Delta \log M_6(-,+,-) = \frac{f(\lambda)}{4}\pi i\left(-\frac{1}{\epsilon} + \log\left(\frac{(-t_1)(-t_3)}{\mu^2(s_2)}\left[\frac{\Phi}{1-\Phi}\right]\right)\right). \tag{8.2}$$

With this additional term in $M_6(-,+,-)$, it breaks both the naive factorization, (5.20), and that required for factorization in signature space, (5.27). A similar analysis can also be carried out for the $M_6(-,-,-)$. It follows that $M(-,\pm,-)$ violates both naive factorization and that required for signatured factorization, and factorization cannot be regained simply by adding terms which are functions of cross-ratios.

In the published version of the paper by Del Duca et al. [74], which appeared after the first version of our paper, a different proposal for the analytical continuation of the multi-Regge (asymptotic) form was presented (Appendix C). The BDS ansatz is defined in the ϵ-expansion, so it is a priori hard to check what would happen to the full amplitude if we keep ϵ finite, but in [74], v.5, this procedure was tested on the one-loop amplitude. It was claimed that the $\epsilon \to 0$ and multi-Regge limits don't commute, and the extra term found by [3] is not present if we keep ϵ finite and take it to zero after the multi-Regge limit. It should be emphasized that this is not the conventional way of understanding the BDS ansatz. We also note that in the latest update to [35], it has been argued that the above claim in [74], v.5, was invalid due to an arithmetic error. As of this writing, no retraction by the authors of [74], v.5, has appeared.

Independent of the discussion of [74], one might wonder why it might be reasonable to take $\epsilon \to 0$ after the Regge limit. Although conventional wisdom favors the continuation of [35], there is one example where that choice can be reconsidered. From general principles (Mandelstam counting [75], etc.) in an IR finite renormalizable Yang–Mills theory, the gluon Regge trajectory $\alpha(t)$ must satisfy $\alpha(0) = 1$. If one first makes a Laurent expansion in $\epsilon \to 0$, then for $\mathcal{N} = 4$ SYM we have $\alpha(t) \to 1 + \frac{1}{4\epsilon} f^{(-1)}(\lambda) - \frac{1}{4} f(\lambda)\log(-t/\mu^2)$. On the other hand, if one does not expand in $\epsilon \to 0$, but keeps $\epsilon \neq 0$ and finite, and takes $t \to 0$ instead, from general expectations, we should have $\alpha(t = 0) = 1$, contradicting the above relation, for which $\alpha(t \to 0) \to \infty$. A way to solve the contradiction is to introduce a gauge invariant IR cut-off at fixed ϵ, leading to a gluon mass m (for instance via Higgs mechanism, to guarantee consistency), with the result at small $\lambda = g^2 N \ll \epsilon$ [41,42], $\alpha(t) \simeq 1 - [(t - m^2)/t](g^2 N/(8\pi^2))\log(-t)$, consistent with the above $\alpha(t)$ in the Laurent ϵ expansion at $m^2 = 0$. Then $\alpha(t = m^2) = 1$ and only then we can take $m^2 \to 0$. Although this does not bear directly on [35] vs. [74], it emphasizes that the order of limits is often essential in drawing physical conclusions. In this connection, it is also worth noting that the conventional proof for the Steinmann relation relies on having a mass gap, e.g., [56]. It is therefore also possible difficulties of this and other related issues could in principle be resolved by calculations of the $\mathcal{O}(\epsilon)$ terms in the exponent of the IR divergent BDS amplitudes. Further analyses along these lines could help in clarifying the role of Steinmann relation in a conformal theory.

8.2. *Brief summary*

In this paper we have investigated the issue of analyticity of the $\mathcal{N}=4$ SYM amplitudes, using the BDS ansatz, in regard to the multi-Regge limits. In particular, we have looked at issues of analytical continuation that were not addressed in our previous paper [34], analyzed the behavior of universal Regge vertices appearing in all n-point amplitudes, and some Regge limits directly related to unitarity conditions, the helicity pole and triple-Regge limits. By way of comparison, flat space string theory amplitudes are generally used as a primer for Regge behavior, so we have compared them with the $\mathcal{N}=4$ SYM amplitudes. The results that we found are unusual.

We have found that the IR cut-off $\mathcal{N}=4$ SYM planar amplitudes, as characterized by BDS in the Regge limits differ from those of flat-space super string theory in several important aspects. Thus, intuition and specific properties of flat-space string theory can be applied to planar MSYM with at best a great deal of caution. For $n>5$, it has been suggested that BDS amplitudes should be modified by adding a function of the cross-ratios. However this cannot eliminate the various mismatches in properties of flat-space string theory and the various Regge limits of $\mathcal{N}=4$ SYM. This can be seen more directly by examining (5.26) and (5.27), i.e., terms which must be added do not appear to be expressible as functions of cross ratios. Thus, one must conclude that the Regge behavior of conformal $\mathcal{N}=4$ SYM theory is rather different from that of flat-space string theory. Perhaps this is not unexpected, as flat space string theory has a mass-scale, a mass-gap, and linearly rising Regge trajectories with recurrences: all absent from $\mathcal{N}=4$ SYM. With this in mind, the different properties of the Regge limits of the BDS amplitudes and flat-space string theory should not be surprising. However, since flat space string properties considered here follow both from generic rules of planar unitarity and the existence of on-shell Reggeon vertex operator [3] on the worldsheet, these difference are worth more careful study particularly with respect to implementation of the IR cut-off in the BDS construction.

Note added

The authors would like to thank Zvi Bern, Lance Dixon, and Lev Lipatov for conversations and correspondence and for pointing out the need to revise the discussion on analytic continuations in the original version of this paper. We also note that a paper by Del Duca, Duhr, and Glover [74], appeared after our paper was posted. Another recent paper by Bartels et al. [76] has further extended their work in [35]. We would also like to thanks Dr. Del Duca for bringing to our attention Refs. [70,77] which are relevant to our discussion in Section 6.1.

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