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Minimal non-deletable sets and minimal non-codeletable sets in binary images

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Abstract

The concepts of *strongly* 8-*deletable* and *strongly* 4-*deletable* sets of 1s in binary images on the 2D Cartesian grid were introduced by Ronse in the mid-1980s to formalize the connectivity preservation conditions that parallel thinning algorithms are required to satisfy. In this paper we call these sets *deletable* and *codeletable*, respectively. To establish that a proposed parallel thinning algorithm for binary images on the 2D Cartesian grid preserves 8-(4-)connected foreground components and 4-(8-)connected background components, it is enough to prove that the set of 1s which are changed to 0s at each pass of the algorithm is always a deletable (codeletable) set. Ronse established results that are very useful in this context for proving that a finite set $\mathcal{D}$ of 1s is deletable or codeletable. In particular, he showed that $\mathcal{D}$ and its proper subsets are all codeletable in a binary image if each singleton and each pair of 8-adjacent pixels in $\mathcal{D}$ is codeletable. He further showed that $\mathcal{D}$ and its proper subsets are all deletable in a binary image if (1) each singleton and each pair of 4-adjacent pixels in $\mathcal{D}$ is deletable, and (2) no set of 2, 3, or 4 pairwise 8-adjacent pixels that is an 8-connected foreground component of the image is entirely contained in $\mathcal{D}$. In the 1990s and early 2000s analogous results were obtained by Hall, Ma, Gau, and the author for binary images on the 2D hexagonal grid, the 3D Cartesian and face-centered cubic grids, and the 4D Cartesian grid. This paper extends the above-mentioned work to binary images on almost any polytopal complex whose union is n-dimensional Euclidean space, for $n \leq 4$. Our main results generalize and unify the corresponding results of the earlier work.

Keywords: Digital topology; Binary image; Thinning; 4D; Deletable; Codeletable; Simple; Cosimple; Minimal non-deletable; Minimal non-simple; Polytopal complex; Xel complex

1. Introduction

In the 1980s Ronse [27,28] established fundamental results regarding the possible forms of (what this paper calls) minimal non-deletable and minimal non-codeletable sets in binary images on the 2D Cartesian grid complex. For such images, these results provided the basis for systematic and fairly general methods (which we discuss in Section 3) of conclusively verifying that a proposed parallel thinning algorithm "preserves topology". In the 1990s and early 2000s similar results were established by Hall, Ma, Gau, and the author [7–9,15,18] for the 3D and 4D Cartesian

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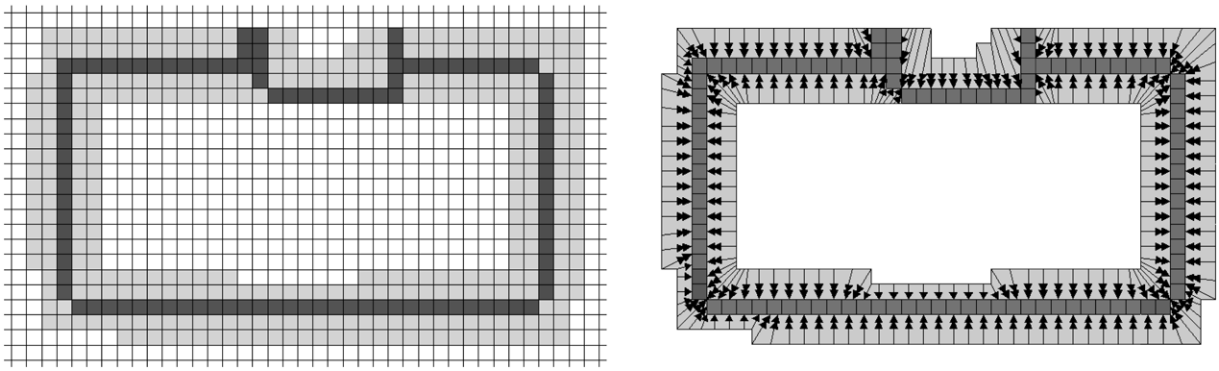

Fig. 1. The diagram on the left shows a binary image $\mathbb{I}$ on the 2D Cartesian grid complex, and a deletable set $\mathcal{D}$ of 1s of $\mathbb{I}$: the white pixels are 0s of $\mathbb{I}$, the light gray pixels are the 1s of $\mathbb{I}$ that belong to the deletable set $\mathcal{D}$, and the dark gray pixels are the 1s of $\mathbb{I}$ that do not belong to $\mathcal{D}$. The diagram on the right shows a deformation retraction of the foreground polyhedron of $\mathbb{I}$ onto the foreground polyhedron of $\mathbb{I} - \mathcal{D}$.

grid complexes, the 2D hexagonal grid complex, and the 3D face-centered cubic grid complex. The main goal of this paper is to establish generalizations of the above-mentioned results to almost any polytopal complex[1] whose union is Euclidean n-space $\mathbb{R}^n$, for $n \leq 4$. These generalizations are stated in our Main Theorem (Theorem 2.1).

1.1. Deletable, codeletable, simple, and cosimple sets

A *binary image* on a polytopal complex $\mathbf{K}$ is a function $\mathbb{I}$ that assigns a value of 1 or 0 to each grid cell of $\mathbf{K}$. Here the term *grid cell* of $\mathbf{K}$ means an element of $\mathbf{K}$ whose dimensionality is equal to the dimensionality of $\mathbf{K}$. (Note that the domain of a binary image on a polytopal complex $\mathbf{K}$ does not include the elements of $\mathbf{K}$ whose dimensionality is less than the dimensionality of $\mathbf{K}$.) We write $\mathcal{G}(\mathbf{K})$ for the set of all grid cells of $\mathbf{K}$. Thus a binary image on $\mathbf{K}$ is a function $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0, 1\}$. A grid cell P of $\mathbf{K}$ is called a 1 or a 0 of the image $\mathbb{I}$ according to whether $\mathbb{I}(P) = 1$ or $\mathbb{I}(P) = 0$. The set $\bigcup \mathbb{I}^{-1}[\{1\}]$ (i.e., the union of all the 1s of $\mathbb{I}$) will be called the *foreground polyhedron* of $\mathbb{I}$.

In this context, the most familiar n-dimensional polytopal complex is the nD *Cartesian grid complex*, the grid cells of which are the closed unit n-dimensional cubes whose vertices have integer coordinates. A grid cell of the 2D Cartesian grid complex is called a *pixel*, and a grid cell of the 3D Cartesian grid complex is called a *voxel*.

The concepts of deletable and codeletable sets are defined in terms of the foreground polyhedron. For this purpose, let $\mathbb{I}$ be a binary image on a polytopal complex $\mathbf{K}$, and let $\mathcal{D} \subseteq \mathbb{I}^{-1}[\{1\}]$ (i.e., let $\mathcal{D}$ be any set of 1s of $\mathbb{I}$). We write $\mathbb{I} - \mathcal{D}$ to denote the binary image on $\mathbf{K}$ that is defined by:

$$(\mathbb{I} - \mathcal{D})(P) = \begin{cases} 1 & \text{if } \mathbb{I}(P) = 1 \text{ and } P \notin \mathcal{D} \\ 0 & \text{if } \mathbb{I}(P) = 0 \text{ or } P \in \mathcal{D}. \end{cases}$$

The set $\mathcal{D}$ will be called a *deletable* set of $\mathbb{I}$ if $\mathcal{D}$ is finite and the foreground polyhedron of $\mathbb{I}$ can be continuously deformed over itself onto the foreground polyhedron of $\mathbb{I} - \mathcal{D}$, in such a way that all points of the latter remain fixed throughout the deformation process. More formally, we say $\mathcal{D}$ is a *deletable* set of $\mathbb{I}$ if $\mathcal{D}$ is finite and the foreground polyhedron of $\mathbb{I} - \mathcal{D}$ is a deformation retract[2] of the foreground polyhedron of $\mathbb{I}$. Fig. 1 shows a deletable set of 1s in a binary image, and a corresponding deformation retraction. Note that the empty set is a deletable set of any binary image.

We write $\mathbb{I}^{\mathbf{c}}$ to denote the binary image on $\mathbf{K}$ whose 1s and 0s are respectively the 0s and 1s of $\mathbb{I}$. The set $\mathcal{D}$ will be called a *codeletable* set of $\mathbb{I}$ if $\mathcal{D}$ is a deletable set of the binary image $(\mathbb{I} - \mathcal{D})^{\mathbf{c}}$; note that $(\mathbb{I} - \mathcal{D})^{\mathbf{c}}$ is just the binary image on $\mathbf{K}$ whose 1s are the 0s of $\mathbb{I}$ and the elements of $\mathcal{D}$.

[1] In this paper, a *polytopal complex* is a collection of polytopes in $\mathbb{R}^n$ that satisfies conditions **P1**–**P3** in Section 2.1. This is just the usual concept of a polytopal complex (see, e.g., [33]), except that we allow a polytopal complex to have infinitely many elements.

[2] In topology (see, e.g., [1]) a set Y is said to be a *deformation retract* of a set $X \supseteq Y$ if there exists a continuous map $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ for all $x \in X$, $h(x, 1) \in Y$ for all $x \in X$, and $h(y, t) = y$ for all $y \in Y$ and all $t \in [0, 1]$. Such a map h is called a *deformation retraction* of X onto Y.

On the 2D Cartesian grid complex, it is not hard to show that a finite set of 1s in a binary image $\mathbb{I}$ is deletable if and only if *none* of the following happens when those 1s are all changed to 0s:

- An 8-connected foreground component[3] of $\mathbb{I}$ is split.
- An 8-connected foreground component of $\mathbb{I}$ is completely eliminated.
- Distinct 4-connected background components of $\mathbb{I}$ are merged.
- A new 4-connected background component of $\mathbb{I}$ is created.

Similarly, a finite set of 1s in a binary image $\mathbb{I}$ on the 2D Cartesian grid complex is codeletable if and only if *none* of the following happens when those 1s are all changed to 0s:

- A 4-connected foreground component of $\mathbb{I}$ is split.
- A 4-connected foreground component of $\mathbb{I}$ is completely eliminated.
- Distinct 8-connected background components of $\mathbb{I}$ are merged.
- A new 8-connected background component of $\mathbb{I}$ is created.

The following fundamental fact is a straightforward consequence of the definitions of deletable and codeletable sets:

Fact 1.1. *If $\mathbb{I}$ is a binary image and, for $1 \leq i \leq m$, $\mathcal{D}_i$ is a deletable (codeletable) set of 1s of the binary image $\mathbb{I} - \bigcup\{\mathcal{D}_j \mid 1 \leq j < i\}$, then $\bigcup\{\mathcal{D}_j \mid 1 \leq j \leq m\}$ is a deletable (codeletable) set of 1s of $\mathbb{I}$.*

The concepts of deletable and codeletable sets are essentially generalizations (to binary images on arbitrary polytopal complexes) of Ronse's concepts of *strongly* 8-*deletable* and *strongly* 4-*deletable* sets in binary images on the 2D Cartesian grid complex [27,28]. Specifically, $\mathcal{D}$ is a deletable (codeletable) set of a binary image $\mathbb{I}$ on the 2D Cartesian grid complex if and only if, in Ronse's terminology, $\mathcal{D}$ is a strongly 8-deletable (strongly 4-deletable) subset of the set of 1s of $\mathbb{I}$.

If P is a 1 of a binary image $\mathbb{I}$ on the 2D Cartesian grid complex, then P is an 8-simple 1 in the usual sense (see, e.g., [30, pp. 232–3]) if and only if the singleton set $\{P\}$ is a deletable set of $\mathbb{I}$, and P is a 4-simple 1 in the usual sense if and only if $\{P\}$ is a codeletable set of $\mathbb{I}$. If P is a 1 in a binary image $\mathbb{I}$ on the 3D Cartesian grid complex, then P is a 26-simple 1 in the usual sense (see, e.g., [2,4,31]) if and only if $\{P\}$ is a deletable set of $\mathbb{I}$, and P is a 6-simple 1 in the usual sense if and only if $\{P\}$ is a codeletable set of $\mathbb{I}$. (These facts can be deduced from [13, Theorems 2.10 and 5.6] and [10, Proposition 0.16 and Corollary 0.20].)

Because of these equivalences, we define a *simple* grid cell of a binary image $\mathbb{I}$ on any polytopal complex to be a grid cell P such that $\mathbb{I}(P) = 1$ and $\{P\}$ is a deletable set of $\mathbb{I}$, and we similarly define a *cosimple* grid cell of $\mathbb{I}$ to be a grid cell P such that $\mathbb{I}(P) = 1$ and $\{P\}$ is a codeletable set of $\mathbb{I}$. Note that P is a cosimple 1 of $\mathbb{I}$ if and only if P is a simple 1 of $(\mathbb{I} - \{P\})^{\mathbf{c}}$.

We will see from Fact 4.4 and Theorem 4.5 below that, on any polytopal complex of dimensionality ≤ 4, one can give discrete (and computationally tractable) necessary and sufficient conditions for a 1 of a binary image to be simple or cosimple.

We say that a set $\mathcal{D}$ of grid cells is a *simple set* (*cosimple set*) of a binary image $\mathbb{I}$ if $\mathcal{D}$ is a finite set of 1s of $\mathbb{I}$ and it is possible to arrange the elements of $\mathcal{D}$ into a sequence $(d_i \mid 0 \leq i < |\mathcal{D}|)$ in which each element d_i is a simple (cosimple) 1 of the image $\mathbb{I} - \{d_j \mid 0 \leq j < i\}$. Note that the empty set is both simple and cosimple in any binary image, and that a singleton set $\{P\}$ is simple (cosimple) in $\mathbb{I}$ if and only if its only element P is a simple (cosimple) 1 of $\mathbb{I}$. Readily, $\mathcal{D}$ is a cosimple set of $\mathbb{I}$ if and only if $\mathcal{D}$ is a simple set of $(\mathbb{I} - \mathcal{D})^{\mathbf{c}}$. It follows from Fact 1.1 that, in any binary image on any polytopal complex, all simple (cosimple) sets are deletable (codeletable). The converse is false. In [26], Passat, Couprie, and Bertrand give examples of pairs of 1s in binary images on the 3D Cartesian grid complex that are deletable but not simple in our sense,[4] and their main result can be regarded as a characterization of such pairs.

[3] Recall that a set $\mathcal{D}$ of pixels is said to be a *k-connected foreground component* of $\mathbb{I}$ if $\mathcal{D}$ is a k-connected set of 1s of $\mathbb{I}$, and no pixel in $\mathcal{D}$ is k-adjacent to a 1 of $\mathbb{I}$ that is not in $\mathcal{D}$. Similarly, $\mathcal{D}$ is said to be a *k-connected background component* of $\mathbb{I}$ if $\mathcal{D}$ is a k-connected set of 0s of $\mathbb{I}$, and no pixel in $\mathcal{D}$ is k-adjacent to a 0 of $\mathbb{I}$ that is not in $\mathcal{D}$.

[4] Our usage of the term *simple* is consistent with (and generalizes) the usage of this term in [7,8,13,14,18], but differs from the usage of this term in [26]. In a binary image on the 3D Cartesian grid complex, a set of voxels that is simple in the sense of [26] is deletable in our sense, but need not be simple in our sense.

1.2. The main results

A *minimal non-deletable* (*minimal non-codeletable*) set of a binary image $\mathbb{I}$ is a set $\mathcal{D}$ of 1s of $\mathbb{I}$ such that $\mathcal{D}$ is a non-deletable (non-codeletable) set of $\mathbb{I}$, but every proper subset of $\mathcal{D}$ is a deletable (codeletable) set of $\mathbb{I}$. Note that a set consisting of just one 1 is minimal non-deletable (minimal non-codeletable) if and only if that 1 is non-simple (non-cosimple).

Ronse [28] identified all possible forms of minimal non-deletable set and minimal non-codeletable set in binary images on the 2D Cartesian grid complex. Specifically, it follows from the results of [28] that, on the 2D Cartesian grid complex:

R1 There exists a binary image in which $\mathcal{D}$ is a minimal non-deletable set of 1s if and only if $\mathcal{D}$ consists of just one pixel, or of two 8-adjacent pixels, or of three or four pairwise 8-adjacent pixels.

R2 There exists a binary image in which $\mathcal{D}$ is a minimal non-deletable set of 1s and $\mathcal{D}$ is not an 8-connected foreground component if and only if $\mathcal{D}$ consists of just one pixel or of two 4-adjacent pixels.

R3 There exists a binary image in which $\mathcal{D}$ is a minimal non-codeletable set of 1s if and only if $\mathcal{D}$ consists of just one pixel or of two 8-adjacent pixels.

R4 There exists a binary image in which $\mathcal{D}$ is a minimal non-codeletable set of 1s and $\mathcal{D}$ is not a 4-connected foreground component if and only if $\mathcal{D}$ consists of just one pixel or of two 8-adjacent pixels. (Note that this condition is the same as that of **R3**.)

The main goal of this paper is to establish generalizations of the results **R1**–**R4** that will apply to almost any polytopal complex whose union is $\mathbb{R}^n$, for $n \leq 4$. Specifically, in Section 2.3 we will define a class of polytopal complexes that satisfy three rather mild conditions, and our Main Theorem will answer the following questions (which **R1**–**R4** answered for the 2D Cartesian grid complex) for all such complexes **K** of dimensionality ≤ 4:

Q1 For exactly which sets $\mathcal{D}$ of grid cells of **K** does there exist a binary image on **K** in which $\mathcal{D}$ is a minimal non-deletable set of 1s?

Q2 For exactly which sets $\mathcal{D}$ of grid cells of **K** does there exist a binary image on **K** in which $\mathcal{D}$ is a minimal non-deletable set of 1s and $\mathcal{D}$ is not a weakly connected foreground component?

Q3 For exactly which sets $\mathcal{D}$ of grid cells of **K** does there exist a binary image on **K** in which $\mathcal{D}$ is a minimal non-codeletable set of 1s?

Q4 For exactly which sets $\mathcal{D}$ of grid cells of **K** does there exist a binary image on **K** in which $\mathcal{D}$ is a minimal non-codeletable set of 1s and $\mathcal{D}$ is not a strongly connected foreground component?

In **Q2** and **Q4**, the concepts of *weakly connected foreground component* and *strongly connected foreground component* are natural generalizations (to binary images on polytopal complexes) of the standard concepts of 8- and 4-connected foreground components in binary images on the 2D Cartesian grid complex. Precise definitions will be given in Section 2.2.

Polytopal complexes that belong to the class we define in Section 2.3 will be called *convex xel complexes*. In addition to the 2D Cartesian grid complex, there are a number of other simple convex xel complexes **K** for which questions **Q1**–**Q4** have already been answered in the literature. Hall [9] essentially answered these questions for the 2D hexagonal grid complex (whose grid cells are regular hexagons that tessellate the plane), and Ma [18] essentially answered them for the 3D Cartesian grid complex. Gau and Kong [7] answered the questions for the 3D face-centered cubic grid complex (whose grid cells are rhombic dodecahedra that tessellate 3-space), and more recently [8,15] for the 4D Cartesian grid complex. The main results of the present paper generalize and unify the corresponding results of this earlier work.

This paper covers much the same ground as the author's paper in the Proceedings of the DGCI 2006 conference (Szeged, October 2006) [14]. However, the proofs of the main results have been simplified. Another significant difference between this paper and [14] is that whereas [14] presented this theory in terms of the concept of a simple set, this paper presents the theory primarily in terms of the more fundamental concept of a deletable set. But we will see from Theorem 5.3 that the two approaches are in fact equivalent. To avoid some technicalities that might have been a distraction from the main thrust of our arguments, this paper states and proves the Main Theorem just for convex xel complexes, even though the cell complexes considered in [14] were allowed to contain non-convex cells.

One part of this paper that goes beyond [14] is Section 9, which gives a generalization of the Main Theorem to convex xel complexes of arbitrary dimensionality. This generalization is based on the use of weaker definitions

of deletable and codeletable sets; we call the sets which satisfy these weaker definitions *homology-deletable* and *homology-codeletable* sets. Deletable sets are homology-deletable, but a homology-deletable set need not be deletable, even in the 3D Cartesian grid complex. Nevertheless, on convex xel complexes of dimensionality ≤ 4, a set is minimal non-deletable if and only if it is minimal non-homology-deletable, and is minimal non-codeletable if and only if it is minimal non-homology-codeletable. (The definition of a homology-deletable set is an obvious generalization of a definition that was given by Niethammer et al. [23, Definition 3] for binary images on nD Cartesian grid complexes, and which is mathematically equivalent to a definition that was suggested by the author some years ago [12, Definition 13].) A "homology version" of the Main Theorem can be obtained by substituting "homology-deletable" and "homology-codeletable" for "deletable" and "codeletable". This version of the theorem holds in convex xel complexes of any dimensionality.

2. Statement of the Main Theorem

2.1. Polytopal complexes

A *polytope* is a set that is the convex hull of a finite set of points in a Euclidean space, or, equivalently [33], a bounded set that is the intersection of finitely many closed halfspaces of a Euclidean space.

Let P be any polytope. We write $\dim(P)$ to denote the dimensionality of P; this is the dimensionality of the affine hull of P if $P \neq \emptyset$, and is defined to be -1 if $P = \emptyset$. We say that a polytope F is a *proper face* of P if $P \neq \emptyset$ and $F = \emptyset$, or if $\dim(P) \geq 1$ and there is a $(\dim(P)-1)$-dimensional hyperplane H in the affine hull of P such that $P \setminus H$ is connected (i.e., H does not separate P) and $P \cap H = F$. Every proper face of P is a polytope of dimensionality $< \dim(P)$. If $\dim(P) \geq 1$ then it can be shown [33] that, in addition to the empty proper face, P has proper faces of dimensionality m for each integer m in the range $0 \leq m < \dim(P)$. If $Q = P$ or Q is a proper face of P, then we say Q is a *face* of P. If $\{v\}$ is a 0D face of P, then we say v is a *vertex* of P. If e is a 1D face of P, then we say e is an *edge* of P.

We write $\mathbf{faces}(P)$ to denote the set of all faces of the polytope P. For example, if P is a 3D cube then $\mathbf{faces}(P)$ has 28 elements—P itself, six 2D faces, twelve 1D faces (the edges of P), eight 0D faces (each consisting of a single vertex of P), and the empty set.

A *polytopal complex* is a collection $\mathbf{K}$ of polytopes, all of which lie in the same finite dimensional Euclidean space, that satisfies the following conditions:

P1: If $P \in \mathbf{K}$, then $\mathbf{faces}(P) \subseteq \mathbf{K}$.
P2: If $P_1, P_2 \in \mathbf{K}$, then $P_1 \cap P_2 \in \mathbf{faces}(P_1) \cap \mathbf{faces}(P_2)$.
P3: $\mathbf{K}$ is locally finite—i.e., each point $x \in \bigcup \mathbf{K}$ has a neighborhood that intersects only finitely many elements of $\mathbf{K}$.

As a consequence of **P2**, we have that if P and P' are any two elements of a polytopal complex, then $P \subseteq P'$ if and only if $P \in \mathbf{faces}(P')$.

The *dimensionality* of a polytopal complex $\mathbf{K}$, denoted by $\dim(\mathbf{K})$, is the integer $\max_{P \in \mathbf{K}} \dim(P)$. As mentioned in Section 1.1, we say that P is a *grid cell* of $\mathbf{K}$ if $P \in \mathbf{K}$ and $\dim(P) = \dim(\mathbf{K})$, and we write $\mathcal{G}(\mathbf{K})$ to denote the set of all grid cells of $\mathbf{K}$.

If P is any polytope, then it follows from standard results [33, Propositions 2.2 and 2.3] that P has just finitely many faces, that every intersection of faces of P is a face of P, and that the faces of each face F of P are exactly the faces of P that are contained in F. Thus if P is any polytope then $\mathbf{faces}(P)$ is a polytopal complex.

2.2. Foreground and background components

Let $\mathbf{K}$ be a polytopal complex such that $\dim(\mathbf{K}) \geq 1$, and let $P_1, P_2 \in \mathcal{G}(\mathbf{K})$. We say P_1 is *weakly adjacent* to P_2 if $P_1 \neq P_2$ and $P_1 \cap P_2 \neq \emptyset$, and we say P_1 is *strongly adjacent* to P_2 if $\dim(P_1 \cap P_2) = \dim(\mathbf{K}) - 1$. For example, if $\mathbf{K}$ is the 3D Cartesian grid complex, then P_1 is weakly adjacent to P_2 if and only if they are 26-adjacent voxels, and P_1 is strongly adjacent to P_2 if and only if they are 6-adjacent voxels. If $\mathbf{K}$ is the nD Cartesian grid complex then, in the notation of Herman [11], P_1 is weakly adjacent to P_2 if and only if they are α_n-adjacent grid cells, and P_1 is strongly adjacent to P_2 if and only if they are ω_n-adjacent grid cells.

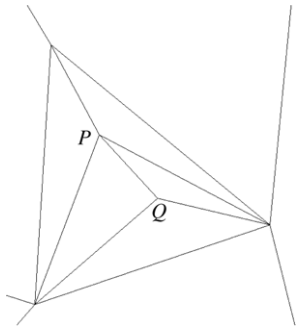

Fig. 2. This is an example of a configuration that cannot occur in a 2D convex xel complex. Here P and Q are 0D polytopes (i.e., each consists of a single vertex), and the four grid cells that contain P all intersect each of the three grid cells that contain Q. Thus condition **X3** is violated—there do *not* exist disjoint grid cells P' and Q' such that $P \in \mathbf{faces}(P')$ and $Q \in \mathbf{faces}(Q')$.

We say that a set $\mathcal{T} \subseteq \mathcal{G}(\mathbf{K})$ is *weakly connected* (*strongly connected*) if, for all $P, P' \in \mathcal{T}$, there exists a sequence of $m \geq 1$ grid cells $T_1, T_2, \ldots, T_m \in \mathcal{T}$ such that $T_1 = P$, $T_m = P'$, and T_i is weakly (strongly) adjacent to T_{i+1} for $1 \leq i < m$.

Now let $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0, 1\}$ be any binary image on $\mathbf{K}$. Then a *weakly connected foreground component* (*strongly connected foreground component*) of $\mathbb{I}$ is a nonempty set $\mathcal{T}$ of 1s of $\mathbb{I}$ such that $\mathcal{T}$ is weakly (strongly) connected and no element of $\mathcal{T}$ is weakly (strongly) adjacent to a 1 of $\mathbb{I}$ that is not in $\mathcal{T}$. Similarly, a *weakly connected background component* (*strongly connected background component*) of $\mathbb{I}$ is a nonempty set $\mathcal{T}$ of 0s of $\mathbb{I}$ such that $\mathcal{T}$ is weakly (strongly) connected and no element of $\mathcal{T}$ is weakly (strongly) adjacent to a 0 of $\mathbb{I}$ that is not in $\mathcal{T}$.

2.3. Convex xel complexes, and the Main Theorem

A *convex xel complex* is a polytopal complex $\mathbf{K}$ for which $\dim(\mathbf{K}) \geq 1$ and which satisfies the following additional conditions:

X1: Each $(\dim(\mathbf{K}) - 1)$-dimensional element of $\mathbf{K}$ is a face of exactly two grid cells of $\mathbf{K}$.
X2: For every two weakly adjacent grid cells P and P' of $\mathbf{K}$, there is a sequence $P = P_0, P_1, \ldots, P_k = P'$ of grid cells of $\mathbf{K}$ such that, for $0 \leq i < k$, $P \cap P' \subsetneq P_i$ and P_i is strongly adjacent to P_{i+1}.
X3: If $P, Q \in \mathbf{K}$ and $P \cap Q = \emptyset$, then there exist $P', Q' \in \mathcal{G}(\mathbf{K})$ such that $P \in \mathbf{faces}(P')$, $Q \in \mathbf{faces}(Q')$, and $P' \cap Q' = \emptyset$.

We mention that conditions **X1** and **X2** are satisfied by any polytopal complex $\mathbf{K}$ whose union is $\mathbb{R}^n$ for some $n \geq 1$ (or any other manifold of dimensionality ≥ 1). Condition **X3** excludes polytopal complexes that contain configurations such as the one shown in Fig. 2. Also, **X3** implies that if $P \in \mathbf{K}$ and P is not a grid cell, then P is a proper face of a grid cell—this follows from **X3** when we take Q to be any 0D element $\{v\}$ of $\mathbf{K}$ such that $v \notin P$.

We are now in a position to state our Main Theorem, which answers the questions **Q1–Q4** of Section 1.2 for any convex xel complex of dimensionality ≤ 4.

Theorem 2.1 (*The Main Theorem*). *Let* $\mathbf{K}$ *be a convex xel complex such that* $\dim(\mathbf{K}) \leq 4$, *and let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then:*

1. $\bigcap \mathcal{D} \neq \emptyset$ *is a necessary and sufficient condition for there to exist a binary image on* $\mathbf{K}$ *in which* $\mathcal{D}$ *is a minimal non-deletable set of* 1*s.*
2. $\dim(\bigcap \mathcal{D}) \geq 1$ *is a necessary and sufficient condition for there to exist a binary image on* $\mathbf{K}$ *in which* $\mathcal{D}$ *is a minimal non-deletable set of* 1*s and* $\mathcal{D}$ *is not a weakly connected foreground component.*
3. $[\bigcap \mathcal{D} \neq \emptyset] \wedge \neg(\exists P \in \mathcal{D})[\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}]$ *is a necessary and sufficient condition for there to exist a binary image on* $\mathbf{K}$ *in which* $\mathcal{D}$ *is a minimal non-codeletable set of* 1*s.*
4. $[|\mathcal{D}| \leq \dim(\mathbf{K})] \wedge [\bigcap \mathcal{D} \neq \emptyset] \wedge \neg(\exists P \in \mathcal{D})[\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}]$ *is a necessary and sufficient condition for there to exist a binary image on* $\mathbf{K}$ *in which* $\mathcal{D}$ *is a minimal non-codeletable set of* 1*s and* $\mathcal{D}$ *is not a strongly connected foreground component.*

We mention that condition **X3** in the definition of a convex xel complex will be used only in proving that the conditions in assertions 2, 3, and 4 of this theorem are *sufficient* for $\mathcal{D}$ to have the stated properties. So assertion 1 is valid for sets $\mathcal{D}$ of grid cells of any polytopal complex $\mathbf{K}$ of dimensionality ≤ 4 that satisfies $\bigcup \mathbf{K} = \mathbb{R}^{\dim(\mathbf{K})}$; and, if we remove the words "and sufficient", then so is each of the assertions 2–4.

In the important special cases where $\mathbf{K}$ is a Cartesian grid complex of dimensionality ≥ 2, the conditions of assertions 3 and 4 of the Main Theorem are actually equivalent. This is because in such complexes it is readily confirmed that any nonempty set $\mathcal{D}$ of grid cells of $\mathbf{K}$ which satisfies the condition of assertion 3 must also satisfy $|\mathcal{D}| \leq \dim(\mathbf{K})$, and therefore satisfy the condition of assertion 4.

3. Application of the Main Theorem to the problem of verifying the topological soundness of parallel thinning algorithms

Parallel thinning algorithms are used to simplify binary images by reducing the set of 1s to a "thin" set. Many such algorithms have been developed; most, but not all (see, e.g., [6]), of these algorithms have been for binary images on the 2D and 3D Cartesian grids. The algorithms of Palágyi and Kuba [24,25] are two quite frequently cited 3D examples.

Parallel thinning algorithms are iterative algorithms that make a series of passes over a binary image. Each pass finds all 1s that satisfy some deletion condition, and then changes all those 1s to 0s. The term *deletion condition* (for binary images on a polytopal complex $\mathbf{K}$) means a Boolean function $\delta : \mathcal{G}(\mathbf{K}) \times \{0, 1\}^{\mathcal{G}(\mathbf{K})} \to \{\mathbf{false}, \mathbf{true}\}$ such that $\delta(P, \mathbb{I}) \Rightarrow \mathbb{I}(P) = 1$; here $\{0, 1\}^{\mathcal{G}(\mathbf{K})}$ denotes the set of all binary images on $\mathbf{K}$. As an example, in the case where $\mathbf{K}$ is the 2D Cartesian grid complex,[5]

$$\delta(P, \mathbb{I}) = (P \text{ is a simple north border 1 of } \mathbb{I} \text{ that is 8-adjacent to at least two other 1s of } \mathbb{I})$$

is a deletion condition that has been studied in the context of parallel thinning [29].

Different passes may use different deletion conditions. Most parallel thinning algorithms cycle through a fixed set of k deletion conditions $\delta_0, \ldots, \delta_{k-1}$, for some small integer $k \geq 1$; the algorithm terminates when it sees that none of those k deletion conditions is satisfied by any of the remaining 1s of the image. A parallel thinning algorithm of this kind for binary images on a polytopal complex $\mathbf{K}$ can be specified by the following pseudocode, in which $\mathbb{I}_{\mathbf{in}}$ and $\mathbb{I}_{\mathbf{out}}$ denote the input and output binary images, and $\mathbb{I}_{\mathbf{in}}$ is assumed to have only finitely many 1s:

Algorithm PT (*Parallel Thinning with Deletion Conditions* $\delta_0, \ldots, \delta_{k-1}$).

```
𝕀 ← 𝕀_in
InactiveSubiterationCount ← 0
i ← −1
repeat
   i ← (i + 1) mod k
   𝒟 ← {P ∈ 𝒢(K) | δ_i(P, 𝕀)}
   if 𝒟 ≠ ∅ then
      𝕀 ← 𝕀 − 𝒟
      InactiveSubiterationCount ← 0
   else InactiveSubiterationCount ← InactiveSubiterationCount + 1
until InactiveSubiterationCount = k
𝕀_out ← 𝕀
```

These algorithms are expected to "preserve the topology" of the image. In this regard there are two main types of algorithms, which satisfy different "topology preservation" conditions. An algorithm of the first type for binary images on the 2D Cartesian grid complex preserves 8-connected foreground components and 4-connected background components. More generally, for binary images on a polytopal complex $\mathbf{K}$, an algorithm of this first type satisfies the following topology preservation condition:

[5] If P and Q are 4-adjacent pixels that are respectively centered at the points (x, y) and $(x, y + 1)$ in $\mathbb{R}^2$, and $\mathbb{I}(P) = 1$ but $\mathbb{I}(Q) = 0$, then we say that P is a *north border* 1 of $\mathbb{I}$.

T1: $\{P \in \mathcal{G}(\mathbf{K}) \mid \mathbb{I}_{\mathbf{in}}(P) = 1 \text{ and } \mathbb{I}_{\mathbf{out}}(P) = 0\}$ is a deletable set of $\mathbb{I}_{\mathbf{in}}$.

An algorithm of the second type for binary images on the 2D Cartesian grid complex preserves 4-connected foreground components and 8-connected background components. More generally, for binary images on a polytopal complex $\mathbf{K}$, an algorithm of this second type satisfies the following topology preservation condition:

T2: $\{P \in \mathcal{G}(\mathbf{K}) \mid \mathbb{I}_{\mathbf{in}}(P) = 1 \text{ and } \mathbb{I}_{\mathbf{out}}(P) = 0\}$ is a codeletable set of $\mathbb{I}_{\mathbf{in}}$.

In general, it is not a trivial matter to conclusively verify that a proposed parallel thinning algorithm does indeed satisfy **T1** or **T2**. The Main Theorem provides the basis for fairly general methods of doing this, in the case of thinning algorithms for binary images on a convex xel complex of dimensionality ≤ 4. Our statement of these verification methods will use some additional notation, which we now introduce.

Let $\mathbf{K}$ be any polytopal complex. For $i \in \{1, 2, 3, 4\}$ we will say that a set $\mathcal{D}$ of grid cells of $\mathbf{K}$ is a $\mathbf{Q}\boldsymbol{i}_{\mathbf{K}}$*-set* if $\mathcal{D}$ has the property stated in question $\mathbf{Q}\boldsymbol{i}$ of Section 1.2. So, if $\mathbf{K}$ is a convex xel complex of dimensionality ≤ 4, then $\mathcal{D}$ is a $\mathbf{Q}\boldsymbol{i}_{\mathbf{K}}$-set if and only if $\mathcal{D}$ has the property stated in assertion i of the Main Theorem. For $i, j \in \{1, 2, 3, 4\}$ we will say that $\mathcal{D}$ is a $(\mathbf{Q}\boldsymbol{i}_{\mathbf{K}} \setminus \mathbf{Q}\boldsymbol{j}_{\mathbf{K}})$*-set* if $\mathcal{D}$ is a $\mathbf{Q}\boldsymbol{i}_{\mathbf{K}}$-set but is not a $\mathbf{Q}\boldsymbol{j}_{\mathbf{K}}$-set. For example, when $\mathbf{K}$ is the 2D Cartesian grid complex, it follows from assertion 3 of the Main Theorem (or from the result stated as **R3** in Section 1.2) that $\mathcal{D}$ is a $\mathbf{Q3}_{\mathbf{K}}$-set if and only if $\mathcal{D}$ consists of just one pixel or of two 8-adjacent pixels, and it follows from assertions 1 and 2 of the Main Theorem (or from the results **R1** and **R2**) that $\mathcal{D}$ is a $(\mathbf{Q1}_{\mathbf{K}} \setminus \mathbf{Q2}_{\mathbf{K}})$-set if and only if $\mathcal{D}$ consists of two pixels that are 8-adjacent but not 4-adjacent, or of three or four pairwise 8-adjacent pixels.

Before stating the verification methods, we state and prove an easy lemma on which the methods depend.

Lemma 3.1. *Let $\mathcal{D}$ be a finite set of* 1*s in some binary image $\mathbb{I}$ on a polytopal complex $\mathbf{K}$. Then*

1. *$\mathcal{D}$ and its proper subsets are all deletable sets of $\mathbb{I}$ if the following conditions both hold:*
   (a) *Every $\mathbf{Q2}_{\mathbf{K}}$-set of grid cells of $\mathcal{D}$ is a simple set of $\mathbb{I}$.*
   (b) *$\mathcal{D}$ contains no $(\mathbf{Q1}_{\mathbf{K}} \setminus \mathbf{Q2}_{\mathbf{K}})$-set that is a weakly connected foreground component of $\mathbb{I}$.*
2. *$\mathcal{D}$ and its proper subsets are all codeletable sets of $\mathbb{I}$ if the following conditions both hold:*
   (a) *Every $\mathbf{Q4}_{\mathbf{K}}$-set of grid cells of $\mathcal{D}$ is a cosimple set of $\mathbb{I}$.*
   (b) *$\mathcal{D}$ contains no $(\mathbf{Q3}_{\mathbf{K}} \setminus \mathbf{Q4}_{\mathbf{K}})$-set that is a strongly connected foreground component of $\mathbb{I}$.*

Proof. To establish assertion 1, let $\mathcal{D}$ be a finite set of 1s of $\mathbb{I}$ that satisfies conditions 1a and 1b, and suppose some subset of $\mathcal{D}$ is a *non*-deletable set of $\mathbb{I}$. Let $\mathcal{S}$ be any subset of $\mathcal{D}$ that is a minimal non-deletable set of $\mathbb{I}$. Then it follows from the definition of a $\mathbf{Q1}_{\mathbf{K}}$-set that $\mathcal{S}$ must be a $\mathbf{Q1}_{\mathbf{K}}$-set. Moreover, since $\mathcal{S}$ is non-deletable whereas any simple set is deletable, it follows from condition 1a that $\mathcal{S}$ cannot be a $\mathbf{Q2}_{\mathbf{K}}$-set, and so $\mathcal{S}$ is a $(\mathbf{Q1}_{\mathbf{K}} \setminus \mathbf{Q2}_{\mathbf{K}})$-set. Hence, by condition 1b, $\mathcal{S}$ is not a weakly connected foreground component of $\mathbb{I}$, and now the fact that $\mathcal{S}$ is a minimal non-deletable set of $\mathbb{I}$ implies that $\mathcal{S}$ is a $\mathbf{Q2}_{\mathbf{K}}$-set, which is a contradiction. This establishes assertion 1. Assertion 2 follows from a symmetrical argument. □

We now state the above-mentioned verification methods for **T1** and **T2**. The essential idea is to confine our attention to $\mathbf{Q1}_{\mathbf{K}}$-sets when verifying **T1**, and to confine our attention to $\mathbf{Q3}_{\mathbf{K}}$-sets when verifying **T2**. Note that every $\mathbf{Q2}_{\mathbf{K}}$-set is a $\mathbf{Q1}_{\mathbf{K}}$-set, and that every $\mathbf{Q4}_{\mathbf{K}}$-set is a $\mathbf{Q3}_{\mathbf{K}}$-set.

To conclusively verify that an instance of Algorithm PT satisfies **T1**, show that each deletion condition $\delta \in \{\delta_i \mid 0 \leq i < k\}$ has the following two properties (where $\bigwedge$ denotes Boolean conjunction):

1A If $\mathbb{I}$ is any binary image on $\mathbf{K}$, and $\mathcal{S}$ is any $\mathbf{Q2}_{\mathbf{K}}$-set of 1s of $\mathbb{I}$ such that $\bigwedge_{P \in \mathcal{S}} \delta(P, \mathbb{I})$, then $\mathcal{S}$ is a simple set of $\mathbb{I}$.

1B If $\mathbb{I}$ is any binary image on $\mathbf{K}$, and $\mathcal{S}$ is any $(\mathbf{Q1}_{\mathbf{K}} \setminus \mathbf{Q2}_{\mathbf{K}})$-set that is a weakly connected foreground component of $\mathbb{I}$, then there is some $P \in \mathcal{S}$ for which $\neg\delta(P, \mathbb{I})$.

To conclusively verify that an instance of Algorithm PT satisfies **T2**, show that each deletion condition $\delta \in \{\delta_i \mid 0 \leq i < k\}$ has the following two properties[6]:

[6] When $\mathbf{K}$ is the 2D, 3D, or 4D Cartesian grid complex, we observed earlier that the conditions of assertions 3 and 4 of the Main Theorem are actually equivalent, and so there are no $(\mathbf{Q3}_{\mathbf{K}} \setminus \mathbf{Q4}_{\mathbf{K}})$-sets. Thus property **2B** is vacuous in these cases.

2A If $\mathbb{I}$ is any binary image on $\mathbf{K}$, and $\mathcal{S}$ is any $\mathbf{Q4_K}$-set of 1s of $\mathbb{I}$ such that $\bigwedge_{P\in\mathcal{S}} \delta(P,\mathbb{I})$, then $\mathcal{S}$ is a cosimple set of $\mathbb{I}$.

2B If $\mathbb{I}$ is any binary image on $\mathbf{K}$, and $\mathcal{S}$ is any $(\mathbf{Q3_K} \setminus \mathbf{Q4_K})$-set that is a strongly connected foreground component of $\mathbb{I}$, then there is some $P \in \mathcal{S}$ for which $\neg\delta(P,\mathbb{I})$.

To justify these methods of verifying that an instance of Algorithm PT satisfies **T1** or **T2**, we observe that if conditions **1A** and **1B** hold for a deletion condition δ, then the following is true whenever $\mathcal{S}$ is a finite set of 1s of a binary image $\mathbb{I}$ on $\mathbf{K}$:

$$\text{If } \bigwedge_{P\in\mathcal{S}} \delta(P,\mathbb{I}), \text{ then } \mathcal{S} \text{ is a deletable set of } \mathbb{I}. \tag{1}$$

This is because if conditions **1A** and **1B** hold then, for every binary image $\mathbb{I}$ on $\mathbf{K}$, when we take $\mathcal{D} = \{P \in \mathcal{G}(\mathbf{K}) \mid \delta(P,\mathbb{I})\}$ the conditions 1a and 1b of Lemma 3.1 also hold. We see from a symmetrical argument (based on assertion 2 of the lemma) that if conditions **2A** and **2B** hold for a deletion condition δ, then the following is true whenever $\mathcal{S}$ is a finite set of 1s of a binary image $\mathbb{I}$ on $\mathbf{K}$:

$$\text{If } \bigwedge_{P\in\mathcal{S}} \delta(P,\mathbb{I}), \text{ then } \mathcal{S} \text{ is a codeletable set of } \mathbb{I}. \tag{2}$$

In consequence of (1), if the deletion conditions $\delta_0, \ldots, \delta_{k-1}$ used in some instance of Algorithm PT are such that conditions **1A** and **1B** hold when δ is any δ_i, then at each iteration of the **repeat** loop the set of 1s of the binary image $\mathbb{I}$ which are changed to 0s must be a deletable set of the image $\mathbb{I}$ (at the start of that iteration), and so (by Fact 1.1) that instance of Algorithm PT satisfies **T1**. Analogously, in consequence of (2), if the deletion conditions $\delta_0, \ldots, \delta_{k-1}$ used in an instance of Algorithm PT are such that conditions **2A** and **2B** hold when δ is any δ_i, then that instance of Algorithm PT satisfies **T2**. This establishes the correctness of the above methods of conclusively verifying that a parallel thinning algorithm satisfies **T1** or **T2**.

These verification methods are not completely general, as it is possible for an instance of Algorithm PT to satisfy **T1** even if it is not the case that each deletion condition $\delta \in \{\delta_i \mid 0 \le i < k\}$ satisfies **1A** and **1B**,[7] and it is similarly possible for **T2** to hold even if it is not the case that **2A** and **2B** hold for every $\delta \in \{\delta_i \mid 0 \le i < k\}$. But, in both cases, the parallel thinning algorithm would then be a rather unusual one in the sense that few such algorithms have been considered in the literature.

4. Characterizations of simple and cosimple 1s

4.1. The attachment and coattachment sets of a grid cell

In this paper we use the term *polyhedron* to mean a set that is expressible as the union of a locally finite collection of polytopes (which may include polytopes of different dimensionalities).[8] We use the term *finite polyhedron* to mean a set that is expressible as the union of a finite collection of polytopes. For example, if P is any polytope, then any union of faces of P is a finite polyhedron.

The *boundary* of a polytope P, which we denote by ∂P, is the finite polyhedron that is given by the union of all the proper faces of P. If P is an $(n+1)$-dimensional polytope, then ∂P is homeomorphic to the space $\mathbb{R}^n \cup \{\infty\}$, where ∞ denotes an added point whose neighborhoods are the complements in $\mathbb{R}^n \cup \{\infty\}$ of the bounded subsets of $\mathbb{R}^n$.

Let $\mathbf{K}$ be a polytopal complex, let $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0,1\}$ be a binary image on $\mathbf{K}$, and let $P \in \mathcal{G}(\mathbf{K})$. Then we define:

$$\mathbf{Attach}(P,\mathbb{I}) = \bigcup \left\{\mathbf{faces}(P) \cap \mathbf{faces}(X) \mid X \in \mathbb{I}^{-1}[\{1\}] \setminus \{P\}\right\} \setminus \{\emptyset\}$$

$$\mathbf{Coattach}(P,\mathbb{I}) = \bigcup \left\{\mathbf{faces}(P) \cap \mathbf{faces}(X) \mid X \in \mathbb{I}^{-1}[\{0\}] \setminus \{P\}\right\} \setminus \{\emptyset\}.$$

Each of $\mathbf{Attach}(P,\mathbb{I})$ and $\mathbf{Coattach}(P,\mathbb{I})$ is a collection of nonempty proper faces of the polytope P. We call the finite polyhedron $\bigcup \mathbf{Attach}(P,\mathbb{I})$ the *attachment set* of P in $\mathbb{I}$. Similarly, we call the finite polyhedron $\bigcup \mathbf{Coattach}(P,\mathbb{I})$

[7] For an example, see [8, p. 118, footnote 1].

[8] Our usage of the term *polyhedron* is common in topology; see, e.g., [22]. However, this term is used in a far more restrictive sense in much of the literature on polytopes (e.g., [33]).

the *coattachment set* of P in $\mathbb{I}$. Both the attachment set and the coattachment set of P in $\mathbb{I}$ are subsets of the boundary of the polytope P. If P is a 1 of $\mathbb{I}$, and the foreground polyhedron of $\mathbb{I}$ is to be obtained by "gluing" P to the foreground polyhedron of $\mathbb{I} - \{P\}$, then the attachment set of P in $\mathbb{I}$ is the set of points on the boundary of P at which glue should be applied!

Evidently, if P is a 1 in a binary image $\mathbb{I}$ on a polytopal complex, then the attachment set of P in $\mathbb{I}$ depends only on the weakly connected foreground component of $\mathbb{I}$ that contains P. We now establish an analogous property of coattachment sets, namely that if P is a 1 in a binary image $\mathbb{I}$ on a convex xel complex, then the coattachment set of P in $\mathbb{I}$ depends only on the strongly connected foreground component of $\mathbb{I}$ that contains P.

Proposition 4.1. *Let* $\mathbf{K}$ *be a convex xel complex, let* $\mathcal{T}$ *be a strongly connected subset of* $\mathcal{G}(\mathbf{K})$*, let* $\mathbb{I}_1$ *and* $\mathbb{I}_2$ *be binary images on* $\mathbf{K}$ *such that* $\mathcal{T}$ *is a strongly connected foreground component both of* $\mathbb{I}_1$ *and of* $\mathbb{I}_2$*, and let* $T \in \mathcal{T}$*. Then* $\mathbf{Coattach}(T, \mathbb{I}_1) = \mathbf{Coattach}(T, \mathbb{I}_2)$.

Proof. Let $Y \in \mathbf{Coattach}(T, \mathbb{I}_1)$. Then $Y \in \mathbf{faces}(T)$ and there is a grid cell $Q \in \mathbb{I}_1^{-1}[\{0\}]$ such that $Y \in \mathbf{faces}(Q)$. Thus $Y \subseteq T \cap Q$. By condition **X2** in the definition of a convex xel complex, there exists a sequence $T = Q_0, Q_1, \ldots, Q_k = Q$ of grid cells of $\mathbf{K}$ such that, for $0 \leq i < k$, $T \cap Q \subsetneq Q_i$ and Q_i is strongly adjacent to Q_{i+1}. Now $Q_0 = T \in \mathcal{T}$ but $Q_k = Q \notin \mathcal{T}$ (since $Q \in \mathbb{I}_1^{-1}[\{0\}]$). Let Q_j be the first element of the sequence $Q_0, Q_1, \ldots, Q_k$ that does not belong to $\mathcal{T}$. Then $Q_{j-1} \in \mathcal{T}$ and Q_{j-1} is strongly adjacent to Q_j, but $Q_j \notin \mathcal{T}$. As $\mathcal{T}$ is a strongly connected foreground component of $\mathbb{I}_2$, it follows that $\mathbb{I}_2(Q_j) = 0$ and so (since $Y \subseteq T \cap Q \subsetneq Q_j$, which implies that $Y \in \mathbf{faces}(Q_j)$, and since $Y \in \mathbf{faces}(T)$) we have that $Y \in \mathbf{Coattach}(T, \mathbb{I}_2)$. As Y is an arbitrary element of $\mathbf{Coattach}(T, \mathbb{I}_1)$, this shows that $\mathbf{Coattach}(T, \mathbb{I}_1) \subseteq \mathbf{Coattach}(T, \mathbb{I}_2)$. By a symmetrical argument, $\mathbf{Coattach}(T, \mathbb{I}_2) \subseteq \mathbf{Coattach}(T, \mathbb{I}_1)$. $\square$

4.2. Contractible polyhedra

A set S is said to be *contractible* if $S \neq \emptyset$ and S is continuously deformable over itself to some point p in S. More precisely, S is contractible if and only if $S \neq \emptyset$ and there is a continuous mapping $h : S \times [0, 1] \to S$ such that, for every point $s \in S$ and some point p in S, $h(s, 0) = s$ and $h(s, 1) = p$. Nonempty convex sets are good (if rather trivial) examples of contractible sets. From results of topology (e.g., [10, Proposition 0.16, Corollary 0.20]) we can deduce:

Fact 4.2. *Let* Q *be a polyhedron. Then the following are equivalent:*

- Q *is contractible.*
- *There is some point* $q \in Q$ *for which* $\{q\}$ *is a deformation retract of* Q.
- $Q \neq \emptyset$ *and, for every point* $q \in Q$, $\{q\}$ *is a deformation retract of* Q.

An essentially discrete, and computationally convenient, characterization of contractible finite polyhedra in Euclidean 3-space is given by:

Fact 4.3. *Every contractible finite polyhedron* $Q \subset \mathbb{R}^n$ *has the following properties:*

1. Q *is connected.*
2. $\mathbb{R}^n \setminus Q$ *has no bounded components* (*i.e.,* Q *has no "internal cavities"*).
3. $\chi(Q) = 1$.

When $n \leq 3$*, a finite polyhedron* $Q \subset \mathbb{R}^n$ *is contractible if and only if* Q *has these three properties.*

In property 3, $\chi(Q)$ denotes the Euler characteristic of Q. If Q is any finite polyhedron and $\mathbf{L}$ is any finite polytopal complex such that $\bigcup \mathbf{L} = Q$, then $\chi(Q) = \sum_{k=0}^{\dim(\mathbf{L})} (-1)^k \, |\{P \in \mathbf{L} \mid \dim(P) = k\}|$—it follows from a standard result of topology [10, Theorem 2.44] that the value of this alternating sum is the same for every finite polytopal complex $\mathbf{L}$ such that $\bigcup \mathbf{L} = Q$. For a nonempty finite polyhedron Q in $\mathbb{R}^3$ that is connected and has no internal cavities, the condition $\chi(Q) = 1$ is equivalent [16] to the condition that Q is simply connected, and excludes polyhedra which have "holes" (of the kind that doughnuts have) or "tunnels".

The second assertion of Fact 4.3 has an analog for polyhedra in the boundary of any polytope of dimensionality ≤ 4:

Fact 4.4. *Let P be a polytope such that* $\dim(P) \leq 4$. *Then a finite polyhedron $Q \subseteq \partial P$ is contractible if and only if Q has the following three properties:*

1. *Q is connected.*
2. *$(\partial P) \setminus Q$ is connected.*
3. $\chi(Q) = 1$.

Facts 4.3 and 4.4 follow from results of algebraic topology.[9] If $n = 1$ in Fact 4.3 or $\dim(P) = 2$ in Fact 4.4, then properties 1 and 2 are both implied by property 3 (which in this case just says that Q has exactly one more vertex than it has edges). If $n = 2$ in Fact 4.3 or $\dim(P) = 3$ in Fact 4.4, then each of the properties 1 and 2 is implied by the conjunction of the other two properties.

4.3. Necessary and sufficient conditions for a 1 *to be simple or cosimple*

The following theorem characterizes simple and cosimple 1s in any binary image on a polytopal complex of dimensionality ≤ 4. Note that these characterizations have discrete (and computationally tractable) formulations, based on Fact 4.4.

Theorem 4.5. *Let* $\mathbf{K}$ *be a polytopal complex such that* $\dim(\mathbf{K}) \leq 4$, *and let* $P \in \mathcal{G}(\mathbf{K})$ *be a* 1 *of a binary image* $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0, 1\}$. *Then:*

1. *P is a simple* 1 *of* $\mathbb{I}$ *if and only if* $\bigcup \mathbf{Attach}(P, \mathbb{I})$ *is contractible.*
2. *P is a cosimple* 1 *of* $\mathbb{I}$ *if and only if* $\bigcup \mathbf{Coattach}(P, \mathbb{I})$ *is contractible.*

Since P is a cosimple 1 of $\mathbb{I}$ if and only if P is a simple 1 of $(\mathbb{I} - \{P\})^{\mathbf{c}}$, and since $\mathbf{Coattach}(P, \mathbb{I}) = \mathbf{Attach}(P, (\mathbb{I} - \{P\})^{\mathbf{c}})$, the two assertions of this theorem are really equivalent. The "if" parts of the theorem are special cases of Corollary 4.7 in [17] (when we take X and C in that result to be P and the foreground polyhedron of $\mathbb{I} - \{P\}$ or of $\mathbb{I}^{\mathbf{c}}$). The "only if" parts follow from well-known results of algebraic topology.[10]

5. Characterizations of minimal non-deletable and minimal non-codeletable sets

We say that a set $\mathcal{D}$ of 1s of a binary image $\mathbb{I}$ is *minimal non-simple* (*minimal non-cosimple*) if $\mathcal{D}$ is a non-simple (non-cosimple) set of $\mathbb{I}$ but every proper subset of $\mathcal{D}$ is a simple (cosimple) set of $\mathbb{I}$.

Ronse showed in [27] that in any binary image on the 2D Cartesian grid complex, a set of 1s is simple if and only if it is deletable, and is cosimple if and only if it is codeletable. In fact this is true in any binary image on a 2D convex xel complex. But these results do not generalize to convex xel complexes of dimensionality $\geq$3: For example, in a binary image on the 3D Cartesian grid complex, it is possible for a pair of strongly adjacent grid cells (i.e., 6-adjacent voxels) to be deletable without being simple [26] or to be codeletable without being cosimple. However, we will see from Theorem 5.3 that, in any binary image on a polytopal complex, a set of 1s is minimal non-simple if and only if it is minimal non-deletable, and is minimal non-cosimple if and only if it is minimal non-codeletable.

For X = "simple", "cosimple", "deletable", or "codeletable", we say that a set $\mathcal{S}$ of 1s of a binary image $\mathbb{I}$ is a *hereditarily* X set of $\mathbb{I}$ if every subset of $\mathcal{D}$ (including $\mathcal{D}$ itself) is an X set of $\mathbb{I}$. The following theorem characterizes such sets:

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[9] It follows from the Alexander Duality Theorem [20, Ch. 5] and other standard results that, both in Fact 4.3 and in Fact 4.4, the conditions 1–3 must hold if Q's reduced homology groups are all trivial, and must therefore hold if Q is contractible. Moreover, in Fact 4.4 and when $n \leq 3$ in Fact 4.3, the conditions 1–3 hold only if Q's reduced homology groups are all trivial. For a finite polyhedron Q in $\mathbb{R}^3$ or in the boundary of a 4D polytope, the latter property implies that Q is also simply connected (see, e.g., [16]), and hence (by the theorems of Whitehead and Hurewicz [20, Chs. 7 and 8]) that Q is contractible.

[10] As was mentioned in footnote 9, a finite polyhedron (such as $\bigcup \mathbf{Attach}(P, \mathbb{I})$) in $\mathbb{R}^3$ or in the boundary of a 4D polytope is contractible if all its reduced homology groups are trivial. The reduced homology groups of $\bigcup \mathbf{Attach}(P, \mathbb{I})$ are all trivial if and only if the relative homology groups $H_*(\bigcup \mathbb{I}^{-1}[\{1\}], \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\}))$ are all trivial—this follows from the reduced homology sequence of the pair $(P, \bigcup \mathbf{Attach}(P, \mathbb{I}))$ and the Excision Theorem [10, Sec. 2.1], because the reduced homology groups of the nonempty polytope P are all trivial. It follows from the reduced homology sequence of the pair $(\bigcup \mathbb{I}^{-1}[\{1\}], \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\}))$ that the groups $H_*(\bigcup \mathbb{I}^{-1}[\{1\}], \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\}))$ are all trivial if and only if the inclusion of $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\})$ in $\bigcup \mathbb{I}^{-1}[\{1\}]$ induces isomorphisms of all the reduced homology groups. If P is a simple 1 of $\mathbb{I}$, then the latter condition must hold.

Theorem 5.1. *Let $\mathcal{S}$ be a finite set of* 1*s of a binary image* $\mathbb{I}$ *on a polytopal complex* **K**. *Then:*

1. *The following are equivalent:*
   (a) *$\mathcal{S}$ is a hereditarily simple set of* $\mathbb{I}$.
   (b) *$\mathcal{S}$ is a hereditarily deletable set of* $\mathbb{I}$.
   (c) *Each $P \in \mathcal{S}$ is a simple* 1 *of $\mathbb{I} - \mathcal{S}'$ for all $\mathcal{S}' \subseteq \mathcal{S} \setminus \{P\}$.*
2. *The following are equivalent:*
   (a) *$\mathcal{S}$ is a hereditarily cosimple set of* $\mathbb{I}$.
   (b) *$\mathcal{S}$ is a hereditarily codeletable set of* $\mathbb{I}$.
   (c) *Each $P \in \mathcal{S}$ is a cosimple* 1 *of $\mathbb{I} - \mathcal{S}'$ for all $\mathcal{S}' \subseteq \mathcal{S} \setminus \{P\}$.*

We will deduce this theorem from the following property of polyhedra:

Fact 5.2. *For polyhedra $Z \subseteq Y \subseteq X$, any two of the following imply the third*[11]*:*

1. *Z is a deformation retract of Y.*
2. *Y is a deformation retract of X.*
3. *Z is a deformation retract of X.*

Proof of Theorem 5.1. Since simple sets are deletable sets, 1a implies 1b. Next, suppose 1b holds, and $\mathcal{S}' \subseteq \mathcal{S} \setminus \{P\}$ for some $P \in \mathcal{S}$. Then each of $\mathcal{S}'$ and $\mathcal{S}' \cup \{P\}$ is a deletable set of $\mathbb{I}$, so (by the definition of a deletable set) both the foreground polyhedron of $\mathbb{I} - \mathcal{S}'$ and the foreground polyhedron of $\mathbb{I} - (\mathcal{S}' \cup \{P\}) = (\mathbb{I} - \mathcal{S}') - \{P\}$ are deformation retracts of the foreground polyhedron of $\mathbb{I}$. Hence, by Fact 5.2, the foreground polyhedron of $(\mathbb{I} - \mathcal{S}') - \{P\}$ is a deformation retract of the foreground polyhedron of $\mathbb{I} - \mathcal{S}'$. The grid cell P is therefore a simple 1 of $\mathbb{I} - \mathcal{S}'$. This shows that 1b implies 1c.

Now suppose 1c holds. Then we can use induction on $|\mathcal{T}|$ to prove that, if $\mathcal{T}$ is any subset of $\mathcal{S}$, and $(t_i \mid 1 \leq i \leq |\mathcal{T}|)$ is any enumeration of the elements of $\mathcal{T}$, then each element t_i is a simple 1 of the image $\mathbb{I} - \{t_j \mid 1 \leq j < i\}$, so that $\mathcal{T}$ is a simple set of $\mathbb{I}$. This shows that 1c implies 1a.

By an analogous argument, the conditions 2a, 2b, and 2c are equivalent. □

In binary images on the 3D Cartesian grid complex, hereditarily deletable and hereditarily codeletable sets are special cases of Bertrand's $\mathcal{P}$-*simple* sets [3]. Specifically, if $\mathcal{P}$ is a finite set of 1s of a binary image $\mathbb{I}$ on the 3D Cartesian grid, then $\mathcal{P}$ is a hereditarily deletable (hereditarily codeletable) set of $\mathbb{I}$ if and only if $\mathcal{P}$ is $\mathcal{P}_{26}$-simple ($\mathcal{P}_6$-simple) in the set of all 1s of $\mathbb{I}$.

Our next theorem asserts that the concepts of minimal non-simple and minimal non-cosimple set are respectively equivalent to the concepts of minimal non-deletable and minimal non-codeletable set, and gives useful characterizations of these concepts:

Theorem 5.3. *Let $\mathcal{D}$ be a finite set of* 1*s of a binary image* $\mathbb{I}$ *on a polytopal complex* **K**. *Then:*

1. *The following are equivalent:*
   (a) *$\mathcal{D}$ is a minimal non-simple set of* $\mathbb{I}$.
   (b) *$\mathcal{D}$ is a minimal non-deletable set of* $\mathbb{I}$.
   (c) *$\mathcal{D} \neq \emptyset$ and both of the following subconditions hold for each $P \in \mathcal{D}$:*
      i. *P is a non-simple* 1 *of $\mathbb{I} - (\mathcal{D} \setminus \{P\})$.*
      ii. *P is a simple* 1 *of $\mathbb{I} - \mathcal{D}'$ for all $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{P\}$.*
2. *The following are equivalent:*
   (a) *$\mathcal{D}$ is a minimal non-cosimple set of* $\mathbb{I}$.
   (b) *$\mathcal{D}$ is a minimal non-codeletable set of* $\mathbb{I}$.

[11] This follows from standard results of topology (see, e.g., [10, Prop. 0.16, Cor. 0.20]) which imply that a polyhedron Q is a deformation retract of a polyhedron $Q' \supseteq Q$ if and only if the inclusion map of Q into Q' is a homotopy equivalence. Under the hypotheses of Fact 5.2, consider the inclusion maps $i_1 : Z \to Y$, $i_2 : Y \to X$, and $i_3 : Z \to X$. Here $i_3 = i_2 \circ i_1$. So if i_2 is a homotopy equivalence then i_1 is homotopic to the composition of a homotopy inverse of i_2 with i_3, and if i_1 is a homotopy equivalence then i_2 is homotopic to the composition of i_3 with a homotopy inverse of i_1. Consequently, if any two of i_1, i_2, and i_3 are homotopy equivalences, then so is the third (as the composition of two homotopy equivalences and the homotopy inverse of a homotopy equivalence are homotopy equivalences).

(c) *$\mathcal{D} \neq \emptyset$ and both of the following subconditions hold for each $P \in \mathcal{D}$:*
   i. *P is a non-cosimple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{P\})$.
   ii. *P is a cosimple* 1 *of* $\mathbb{I} - \mathcal{D}'$ *for all* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{P\}$.

Proof. $\mathcal{D}$ is minimal non-X if and only if both of the following are true:

A. $\mathcal{D}$ is not hereditarily X.
B. Every proper subset of $\mathcal{D}$ is hereditarily X.

By Theorem 5.1, a set of 1s of $\mathbb{I}$ is hereditarily deletable if and only if it is hereditarily simple. Therefore $\mathcal{D}$ is minimal non-deletable if and only if it is minimal non-simple. Hence conditions 1a and 1b are equivalent.

As a finite set of 1s of $\mathbb{I}$ is hereditarily deletable if and only if it satisfies condition 1c of Theorem 5.1, we see that if subcondition 1(c)i holds for some $P \in \mathcal{D}$ then $\mathcal{D}$ is not hereditarily deletable, and we also see that subcondition 1(c)ii holds for all $P \in \mathcal{D}$ if and only if every proper subset of $\mathcal{D}$ is hereditarily deletable. Hence condition 1c implies that $\mathcal{D}$ is minimal non-deletable. To prove the converse, suppose $\mathcal{D}$ is minimal non-deletable (and hence nonempty). Then every proper subset of $\mathcal{D}$ is hereditarily deletable, so subcondition 1(c)ii holds for all $P \in \mathcal{D}$. We claim that subcondition 1(c)i also holds for all $P \in \mathcal{D}$. For if some $P \in \mathcal{D}$ is a simple 1 of $\mathbb{I} - (\mathcal{D} \setminus \{P\})$ then, since the set $\mathcal{D} \setminus \{P\}$ is deletable (as it is a proper subset of $\mathcal{D}$), it follows from Fact 1.1 that $\mathcal{D}$ is deletable, which is impossible (since $\mathcal{D}$ is minimal non-deletable). This justifies our claim, and so we have shown that if $\mathcal{D}$ is minimal non-deletable then condition 1c holds.

We have now established the equivalence of conditions 1a, 1b, and 1c. By a symmetrical argument, the conditions 2a, 2b, and 2c are equivalent. □

From the preceding theorem and Theorem 4.5, we deduce the following characterizations of minimal non-deletable and minimal non-codeletable sets, which we will use to prove the Main Theorem:

Theorem 5.4. *Let $\mathcal{D}$ be a finite set of* 1*s of a binary image $\mathbb{I}$ on a polytopal complex* $\mathbf{K}$ *such that* $\dim(\mathbf{K}) \leq 4$. *Then:*

1. *$\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$ if and only if $\mathcal{D} \neq \emptyset$ and both of the following conditions hold for each $P \in \mathcal{D}$:*
   (a) $\bigcup \mathbf{Attach}(P, \mathbb{I} - (\mathcal{D} \setminus \{P\}))$ *is not contractible.*
   (b) $\bigcup \mathbf{Attach}(P, \mathbb{I} - \mathcal{D}')$ *is contractible for all* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{P\}$.
2. *$\mathcal{D}$ is a minimal non-codeletable set of $\mathbb{I}$ if and only if $\mathcal{D} \neq \emptyset$ and both of the following conditions hold for each $P \in \mathcal{D}$:*
   (a) $\bigcup \mathbf{Coattach}(P, \mathbb{I} - (\mathcal{D} \setminus \{P\}))$ *is not contractible.*
   (b) $\bigcup \mathbf{Coattach}(P, \mathbb{I} - \mathcal{D}')$ *is contractible for all* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{P\}$.

From this theorem and Proposition 4.1 we deduce:

Proposition 5.5. *Let $\mathbb{I}$ be a binary image on a convex xel complex of dimensionality* ≤ 4. *Then:*

1. *Each minimal non-deletable set of $\mathbb{I}$ is a subset of a weakly connected foreground component of $\mathbb{I}$.*
2. *Each minimal non-codeletable set of $\mathbb{I}$ is a subset of a strongly connected foreground component of $\mathbb{I}$.*

Proof. Let $\mathcal{D}$ be a minimal non-deletable set of 1s of $\mathbb{I}$, and let P be any element of $\mathcal{D}$. To prove assertion 1, let $\mathcal{W}$ be the weakly connected foreground component of $\mathbb{I}$ that contains P, and let $\mathcal{D}' = (\mathcal{D} \setminus \{P\}) \cap \mathcal{W}$. Then $\mathbf{Attach}(P, \mathbb{I} - \mathcal{D}') = \mathbf{Attach}(P, \mathbb{I} - (\mathcal{D} \setminus \{P\}))$. This implies $\mathcal{D}' = \mathcal{D} \setminus \{P\}$, for otherwise it would follow from assertion 1 of Theorem 5.4 that $\bigcup \mathbf{Attach}(P, \mathbb{I} - \mathcal{D}')$ is contractible but $\bigcup \mathbf{Attach}(P, \mathbb{I} - (\mathcal{D} \setminus \{P\}))$ is not. Hence $\mathcal{D} \subseteq \mathcal{W}$ and assertion 1 is proved.

Similarly, to prove assertion 2 we consider the strongly connected foreground component $\mathcal{S}$ of $\mathbb{I}$ that contains P, and let $\mathcal{D}'' = (\mathcal{D} \setminus \{P\}) \cap \mathcal{S}$. It follows from Proposition 4.1 that $\mathbf{Coattach}(P, \mathbb{I} - \mathcal{D}'') = \mathbf{Coattach}(P, \mathbb{I} - (\mathcal{D} \setminus \{P\}))$. Hence assertion 2 of Theorem 5.4 implies that $\mathcal{D}'' = \mathcal{D} \setminus \{P\}$, and so $\mathcal{D} \subseteq \mathcal{S}$. □

6. Properties of contractible polyhedra in $\mathbb{R}^3$ or in the boundary of a 4D polytope

Our proof of the Main Theorem will depend on a property of intersections and unions of polyhedra in $\mathbb{R}^3$ or in the boundary of a 4D polytope, which we now state. This property is a consequence of the reduced Mayer–Vietoris sequence (see, e.g., [20, pp. 128–129]) and the fact, mentioned in footnote 9, that a finite polyhedron in $\mathbb{R}^3$ or in the boundary of a 4D polytope is contractible if and only if its reduced homology groups are all trivial.

Fact 6.1. *For finite polyhedra A and B in $\mathbb{R}^3$ or the boundary of a 4D polytope, any two of the following imply the third:*

1. *Each of A and B is contractible.*
2. *$A \cup B$ is contractible.*
3. *$A \cap B$ is contractible.*

The next lemma states consequences of Fact 6.1 that will be used in the proof of the Main Theorem. Note that each of the five conditions in assertion 1 of this lemma trivially implies that every member of $\mathcal{S}$ is contractible.

Lemma 6.2. *Let $\mathcal{S}$ be a finite collection of finite polyhedra in $\mathbb{R}^3$ or the boundary of a 4D polytope. Then:*

1. *The following are equivalent:*
   (a) *$\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.*
   (b) *$\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.*
   (c) *$\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, and $\bigcup \mathcal{S}$ is contractible.*
   (d) *$\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, and $\bigcap \mathcal{S}$ is contractible.*
   (e) *Every member of $\mathcal{S}$ is contractible, and every set that is obtainable from members of $\mathcal{S}$ by applying one or more union and/or intersection operations is contractible.*
2. *The following are equivalent:*
   (a) *$\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$.*
   (b) *$\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$.*

Proof. We can deduce from Fact 6.1, by induction on $|\mathcal{S}|$, that if 1(a) holds then $\bigcup \mathcal{S}$ is contractible. (Such an inductive argument is given in detail in the proof of Lemma 1 of [14, Sect. 5].) But if 1(a) holds then 1(a) still holds when we replace $\mathcal{S}$ with any nonempty subset $\mathcal{T}$ of $\mathcal{S}$, so 1(a) implies 1(b). Symmetrically, 1(b) implies 1(a), and so 1(a) is equivalent to 1(b).

If 2(a) holds then 1(a) (and hence 1(b)) will hold when we replace $\mathcal{S}$ with any nonempty proper subset of $\mathcal{S}$. Hence 2(a) implies 2(b). Symmetrically, 2(b) implies 2(a), and so 2(a) and 2(b) are equivalent. Now 1(c) is the conjunction of 2(a) with the statement that $\bigcup \mathcal{S}$ is contractible, and 1(b) is the conjunction of 2(b) with the statement that $\bigcup \mathcal{S}$ is contractible. As 2(a) is equivalent to 2(b), we have that 1(c) is equivalent to 1(b). Symmetrically, 1(d) is equivalent to 1(a). Thus 1(a), 1(b), 1(c), and 1(d) are equivalent.

Evidently, 1(e) implies 1(a). To prove the converse, let D be any set that is obtainable from members of $\mathcal{S}$ by applying union and/or intersection operations. As every Boolean expression can be expressed in disjunctive normal form, there exists some collection $\mathcal{S}'$ of sets such that $\bigcup \mathcal{S}' = D$, and such that each member of $\mathcal{S}'$ is the intersection of a subcollection of $\mathcal{S}$. Now suppose 1(a) holds. Then 1(a) still holds when we replace $\mathcal{S}$ with $\mathcal{S}'$. So (since 1(a) is equivalent to 1(b)) we have that $\bigcup \mathcal{S}' = D$ is contractible. □

Another fact about contractible polyhedra that we will need is the following, which is a consequence of the Nerve Theorem [5, Thm. 10.6(i)][12]:

Fact 6.3. *Let P be an n-dimensional polytope for some $n \geq 1$, and let $\mathcal{S}$ be a finite collection of two or more contractible finite polyhedra in ∂P such that:*

[12] The hypotheses of Fact 6.3 imply that the polyhedron of the nerve complex of $\mathcal{S}$ is the boundary of an $(|\mathcal{S}| - 1)$-dimensional simplex, and so it follows from the Nerve Theorem [5, Thm. 10.6(i)] that the $(|\mathcal{S}| - 2)$nd reduced homology group of $\bigcup \mathcal{S}$ is nontrivial. But, since $\bigcup \mathcal{S}$ is a polyhedron in the polyhedral $(n-1)$-dimensional sphere ∂P, the latter implies $|\mathcal{S}| - 2 \leq n - 1$, or $|\mathcal{S}| - 1 \leq n$, which is the first assertion of Fact 6.3. Since ∂P is a polyhedral $(n-1)$-dimensional sphere, the $(n-1)$st reduced homology group of $\bigcup \mathcal{S} \subseteq \partial P$ is nontrivial only if $\bigcup \mathcal{S} = \partial P$, and the kth reduced homology group of ∂P is nontrivial only for $k = n - 1$. Combining these two observations with the fact that the $(|\mathcal{S}| - 2)$nd reduced homology group of $\bigcup \mathcal{S}$ is nontrivial, we deduce the second assertion of Fact 6.3.

1. $\bigcap \mathcal{S} = \emptyset$.
2. $\bigcap \mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$.

Then $|\mathcal{S}| - 1 \leq n$, *and* $\bigcup \mathcal{S} = \partial P$ *if and only if* $|\mathcal{S}| - 1 = n$.

7. Necessity of the conditions of the Main Theorem

In this section we prove five lemmas which together imply the "necessary" parts of the four assertions of the Main Theorem. Specifically, the "necessary" part of assertion 1 of the Main Theorem will follow from Lemma 7.2, the "necessary" part of assertion 2 from Lemmas 7.2 and 7.3, the "necessary" part of assertion 3 from Lemmas 7.4 and 7.5, and the "necessary" part of assertion 4 from Lemmas 7.4–7.6.

Throughout this section and the next, $\mathbf{K}$ will denote an arbitrary convex xel complex of dimensionality ≤ 4. We now introduce some notation that will be used in both sections. For every nonempty finite set $\mathcal{D}$ of grid cells of $\mathbf{K}$, every $Q \in \mathcal{D}$, and every integer i in the range $1 \leq i \leq |\mathcal{D}| - 1$, we define $T_Q^{\mathcal{D},i}$ so that $(T_Q^{\mathcal{D},i} \mid 1 \leq i \leq |\mathcal{D}| - 1)$ is an enumeration of the set $\mathcal{D} \setminus \{Q\}$. For $1 \leq i \leq |\mathcal{D}| - 1$, we shall write $X_Q^{\mathcal{D},i}$ for $Q \cap T_Q^{\mathcal{D},i}$. Note that each set $X_Q^{\mathcal{D},i}$ is a proper face of the grid cell Q, and is therefore a contractible polyhedron if it is nonempty. For any binary image $\mathbb{I}$ on $\mathbf{K}$, we shall write $C_{Q,\mathbb{I}}$ for $\bigcup \mathbf{Coattach}(Q, \mathbb{I})$, and write $A_{Q,\mathbb{I}}^{\mathcal{D}}$ for $\bigcup \mathbf{Attach}(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$. Thus, if $\mathcal{D}$ is any nonempty finite set of 1s of $\mathbb{I}$ and $\{i_1, \ldots, i_r\}$ is any subset of $\{1, \ldots, |\mathcal{D}| - 1\}$, then we have:

- $A_{Q,\mathbb{I}}^{\mathcal{D}} \cup X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r} = \bigcup \mathbf{Attach}(Q, \mathbb{I} - ((\mathcal{D} \setminus \{Q\}) \setminus \{T_Q^{\mathcal{D},i_1}, \ldots, T_Q^{\mathcal{D},i_r}\}))$
- $C_{Q,\mathbb{I}} \cup X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r} = \bigcup \mathbf{Coattach}(Q, \mathbb{I} - \{T_Q^{\mathcal{D},i_1}, \ldots, T_Q^{\mathcal{D},i_r}\})$
- $C_{Q,\mathbb{I}} \cup X_Q^{\mathcal{D},1} \cup \cdots \cup X_Q^{\mathcal{D},|\mathcal{D}|-1} = \bigcup \mathbf{Coattach}(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$.

Hence assertions 1 and 2 of Theorem 5.4 can be restated as follows:

Remark 7.1. Let $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0, 1\}$ be any binary image on $\mathbf{K}$, and let $\mathcal{D}$ be any nonempty finite set of 1s of $\mathbb{I}$. Then $\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$ if and only if, for all $Q \in \mathcal{D}$:

A. $A_{Q,\mathbb{I}}^{\mathcal{D}}$ is not contractible.
B. $A_{Q,\mathbb{I}}^{\mathcal{D}} \cup X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r}$ is contractible for all *nonempty* subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, |\mathcal{D}| - 1\}$.

Similarly, $\mathcal{D}$ is a minimal non-codeletable set of $\mathbb{I}$ if and only if, for all $Q \in \mathcal{D}$:

C. $C_{Q,\mathbb{I}} \cup X_Q^{\mathcal{D},1} \cup \cdots \cup X_Q^{\mathcal{D},|\mathcal{D}|-1}$ is not contractible.
D. $C_{Q,\mathbb{I}} \cup X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r}$ is contractible for all *proper* subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, |\mathcal{D}| - 1\}$.

We note that condition D can only be satisfied if $C_{Q,\mathbb{I}}$ is contractible (as we may take $\{i_1, \ldots, i_r\}$ to be the empty set), unless $|\mathcal{D}| = 1$ (in which case condition D holds vacuously).

Using the above notation, we now state and prove the above-mentioned lemmas. In the rest of this section, $\mathbb{I}$ will denote an arbitrary binary image on $\mathbf{K}$, and $\mathcal{D}$ will denote an arbitrary nonempty finite set of 1s of $\mathbb{I}$. We will use the hypothesis that $\dim(\mathbf{K}) \leq 4$ only when appealing to Proposition 5.5, Lemma 6.2, or Remark 7.1.

Lemma 7.2. *If* $\bigcap \mathcal{D} = \emptyset$, *then* $\mathcal{D}$ *is not a minimal non-deletable set of* $\mathbb{I}$.

Proof. Suppose $\bigcap \mathcal{D} = \emptyset$, and $\mathcal{D}$ is a minimal non-deletable set of 1s of $\mathbb{I}$. Then we must have that $|\mathcal{D}| \geq 2$. Let Q be any grid cell in $\mathcal{D}$. Then $X_Q^{\mathcal{D},1} \cap \cdots \cap X_Q^{\mathcal{D},|\mathcal{D}|-1} = Q \cap T_Q^{\mathcal{D},1} \cap \cdots \cap T_Q^{\mathcal{D},|\mathcal{D}|-1} = \bigcap \mathcal{D} = \emptyset$. Let $\mathcal{S} = \{A_{Q,\mathbb{I}}^{\mathcal{D}} \cup X_Q^{\mathcal{D},1}, \ldots, A_{Q,\mathbb{I}}^{\mathcal{D}} \cup X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$. Then $\bigcap \mathcal{S} = A_{Q,\mathbb{I}}^{\mathcal{D}} \cup (X_Q^{\mathcal{D},1} \cap \cdots \cap X_Q^{\mathcal{D},|\mathcal{D}|-1}) = A_{Q,\mathbb{I}}^{\mathcal{D}}$. By condition B, $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$. But, by condition A, $\bigcap \mathcal{S} = A_{Q,\mathbb{I}}^{\mathcal{D}}$ is not contractible. This contradiction of Lemma 6.2 proves Lemma 7.2. $\square$

Lemma 7.3. *If* $\dim(\bigcap \mathcal{D}) = 0$, *and* $\mathcal{D}$ *is a minimal non-deletable set of* $\mathbb{I}$, *then* $\mathcal{D}$ *is a weakly connected foreground component.*

Proof. Suppose $\dim(\bigcap \mathcal{D}) = 0$ and $\mathcal{D}$ is a minimal non-deletable set of 1s of $\mathbb{I}$, but $\mathcal{D}$ is not a weakly connected foreground component of $\mathbb{I}$. Then $|\mathcal{D}| \geq 2$, and $\bigcap \mathcal{D}$ consists of a single vertex v of a grid cell of $\mathbf{K}$. As $\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$, $\mathcal{D}$ is a subset of a weakly connected foreground component of $\mathbb{I}$, by Proposition 5.5. So, since $\mathcal{D}$ is not a weakly connected foreground component of $\mathbb{I}$, there is a grid cell $Q \in \mathcal{D}$ for which $A^{\mathcal{D}}_{Q,\mathbb{I}} = \bigcup \mathbf{Attach}(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\})) \neq \emptyset$. Now $X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q = Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q = \bigcap \mathcal{D} = \{v\}$. Let $\mathcal{S} = \{A^{\mathcal{D}}_{Q,\mathbb{I}} \cup X^{\mathcal{D},1}_Q, \ldots, A^{\mathcal{D}}_{Q,\mathbb{I}} \cup X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$. Then $\bigcap \mathcal{S} = A^{\mathcal{D}}_{Q,\mathbb{I}} \cup (X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q) = A^{\mathcal{D}}_{Q,\mathbb{I}} \cup \{v\}$. Hence $\bigcap \mathcal{S}$ is not contractible, for if $v \notin A^{\mathcal{D}}_{Q,\mathbb{I}}$ then $A^{\mathcal{D}}_{Q,\mathbb{I}} \cup \{v\}$ is disconnected (as $A^{\mathcal{D}}_{Q,\mathbb{I}} \neq \emptyset$), and if $v \in A^{\mathcal{D}}_{Q,\mathbb{I}}$ then $A^{\mathcal{D}}_{Q,\mathbb{I}} \cup \{v\} = A^{\mathcal{D}}_{Q,\mathbb{I}}$ is not contractible by condition A. But, by condition B, $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$. This contradiction of Lemma 6.2 proves Lemma 7.3. □

Lemma 7.4. *If $\bigcap \mathcal{D} = \emptyset$, then $\mathcal{D}$ is not a minimal non-codeletable set of $\mathbb{I}$.*

Proof. Suppose $\bigcap \mathcal{D} = \emptyset$, and $\mathcal{D}$ is a minimal non-codeletable set of 1s of $\mathbb{I}$. Then $|\mathcal{D}| \geq 2$. Let Q be any grid cell in $\mathcal{D}$. Then $X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q = Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q = \bigcap \mathcal{D} = \emptyset$. Let $\mathcal{S} = \{C_{Q,\mathbb{I}} \cup X^{\mathcal{D},1}_Q, \ldots, C_{Q,\mathbb{I}} \cup X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$. Then $\bigcap \mathcal{S} = C_{Q,\mathbb{I}} \cup (X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q) = C_{Q,\mathbb{I}}$. By condition D, $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, and $\bigcap \mathcal{S} = C_{Q,\mathbb{I}}$ is contractible. But condition C implies $\bigcup \mathcal{S} = C_{Q,\mathbb{I}} \cup X^{\mathcal{D},1}_Q \cup \cdots \cup X^{\mathcal{D},|\mathcal{D}|-1}_Q$ is not contractible. This contradiction of Lemma 6.2 proves Lemma 7.4. □

Lemma 7.5. *If there is some $P \in \mathcal{D}$ for which $\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}$, then $\mathcal{D}$ is not a minimal non-codeletable set of $\mathbb{I}$.*

Proof. Suppose $\mathcal{D}$ is a minimal non-codeletable set of 1s of $\mathbb{I}$, and there is a grid cell $P \in \mathcal{D}$ such that $\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}$. Then $|\mathcal{D}| \geq 2$. Let Q be any grid cell in $\mathcal{D} \setminus \{P\}$. Assume, as we may, that $P = T^{\mathcal{D},1}_Q$. Then $X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q = Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q = \bigcap \mathcal{D} = \bigcap(\mathcal{D} \setminus \{P\}) = Q \cap T^{\mathcal{D},2}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q = X^{\mathcal{D},2}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q$. Let $\mathcal{S} = \{C_{Q,\mathbb{I}} \cup X^{\mathcal{D},1}_Q, \ldots, C_{Q,\mathbb{I}} \cup X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$, and let $\mathcal{S}' = \{C_{Q,\mathbb{I}} \cup X^{\mathcal{D},2}_Q, \ldots, C_{Q,\mathbb{I}} \cup X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$. Then $\bigcup \mathcal{S}$ is not contractible, by condition C.

But $\bigcup \mathcal{S}'$ is contractible, by condition D, so $\mathcal{S}' \subsetneq \mathcal{S}$. Condition D implies, more generally, that $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$. So, by assertion 2 of Lemma 6.2, $\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$. Hence $\bigcap \mathcal{S}'$ is contractible, and therefore $\bigcap \mathcal{S} = C_{Q,\mathbb{I}} \cup (X^{\mathcal{D},1}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q) = C_{Q,\mathbb{I}} \cup (X^{\mathcal{D},2}_Q \cap \cdots \cap X^{\mathcal{D},|\mathcal{D}|-1}_Q) = \bigcap \mathcal{S}'$ is contractible. As $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, and $\bigcap \mathcal{S}$ is contractible, Lemma 6.2 implies that $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$. But we noted above that $\bigcup \mathcal{S}$ is not contractible. This contradiction proves the lemma. □

Lemma 7.6. *If $|\mathcal{D}| > \dim(\mathbf{K})$ and $\mathcal{D}$ is a minimal non-codeletable set of $\mathbb{I}$, then $|\mathcal{D}| = \dim(\mathbf{K}) + 1$ and $\mathcal{D}$ is a strongly connected foreground component.*

Proof. Let $\mathcal{D}$ be a minimal non-codeletable set of 1s of $\mathbb{I}$ such that $|\mathcal{D}| > \dim(\mathbf{K})$. Then $\bigcap \mathcal{D} \neq \emptyset$, by Lemma 7.4. Let Q be any grid cell in $\mathcal{D}$. There is no grid cell $P \in \mathcal{D}$ such that $\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}$, by Lemma 7.5. So no two consecutive members of the chain $Q \supseteq Q \cap T^{\mathcal{D},1}_Q \supseteq \cdots \supseteq Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q$ of nonempty faces of Q are equal. Thus $\dim(Q) > \dim(Q \cap T^{\mathcal{D},1}_Q) > \cdots > \dim(Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q) \geq 0$. As $>$ occurs $|\mathcal{D}| - 1$ times here, we must have that $\dim(\mathbf{K}) = \dim(Q) \geq |\mathcal{D}| - 1$, and therefore (since $|\mathcal{D}| > \dim(\mathbf{K})$) that $|\mathcal{D}| = \dim(\mathbf{K}) + 1$, as asserted by the lemma. As $|\mathcal{D}| - 1 = \dim(Q)$, the above chain of inequalities implies $\dim(Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q) = 0$, whence $\bigcap \mathcal{D} = Q \cap T^{\mathcal{D},1}_Q \cap \cdots \cap T^{\mathcal{D},|\mathcal{D}|-1}_Q = \{v\}$ for some vertex v of Q.

Let $\mathcal{S} = \{C_{Q,\mathbb{I}}, X^{\mathcal{D},1}_Q, \ldots, X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$, and let $\mathcal{S}' = \{X^{\mathcal{D},1}_Q, \ldots, X^{\mathcal{D},|\mathcal{D}|-1}_Q\}$. Whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}'$, the set $\bigcap \mathcal{T}$ is a polytope (as it is an intersection of finitely many polytopes) and is nonempty because $\bigcap \mathcal{S}' = \bigcap \mathcal{D} \neq \emptyset$. Therefore $\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}'$. Hence, by Lemma 6.2, $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}'$. This and condition D imply that:

$$\bigcup \mathcal{T} \quad \text{is contractible whenever} \quad \emptyset \neq \mathcal{T} \subsetneq \mathcal{S}. \tag{3}$$

However, condition C implies that $\bigcup \mathcal{S}$ is not contractible. Hence, by Lemma 6.2, $\bigcap \mathcal{S} = C_{Q,\mathbb{I}} \cap \bigcap \mathcal{D}$ is not contractible, and therefore, since the set $\bigcap \mathcal{D} = \{v\}$ has no nonempty non-contractible subset, we must have that $\bigcap \mathcal{S} = \emptyset$.

As conditions C and D imply that no two of the polyhedra $C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}$ can be equal, we have that $|\mathcal{S}| = |\mathcal{D}| = \dim(\mathbf{K}) + 1 = \dim(Q) + 1$. By Lemma 6.2 and (3), $\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$. So, since $\bigcap \mathcal{S} = \emptyset$, we deduce from Fact 6.3 that $\bigcup \mathbf{Coattach}(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\})) = \bigcup \mathcal{S} = \partial Q$. Thus every point of ∂Q lies on the boundary of at least one 0 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$. From this, and condition **X1** in the definition of a convex xel complex, we see that Q is not strongly adjacent to any 1 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$—for if F is the common $(\dim(\mathbf{K}) - 1)$-dimensional face of Q and any such 1, then no point of $F \setminus \partial F$ could lie on the boundary of a 0 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$. As Q is an arbitrary element of $\mathcal{D}$, it follows that $\mathcal{D}$ is not a proper subset of a strongly connected foreground component of $\mathbb{I}$. Hence, by Proposition 5.5, $\mathcal{D}$ is a strongly connected foreground component of $\mathbb{I}$. □

8. Sufficiency of the conditions of the Main Theorem

The three lemmas in this section imply the "sufficient" parts of the four assertions of the Main Theorem. Throughout this section $\mathbf{K}$ will continue to denote an arbitrary convex xel complex of dimensionality ≤ 4. The hypothesis that $\dim(\mathbf{K}) \leq 4$ will be used only when appealing to Fact 6.1, Lemma 6.2, or Remark 7.1.

Our first two lemmas imply the "sufficient" parts of assertions 1 and 2, respectively.

Lemma 8.1. *Let $\mathcal{D}$ be a nonempty set of grid cells of $\mathbf{K}$ that satisfies the condition $\bigcap \mathcal{D} \neq \emptyset$. Let $\mathbb{I}$ be the binary image on $\mathbf{K}$ such that $\mathbb{I}^{-1}[\{1\}] = \mathcal{D}$. Then $\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$.*

Proof. We use the notation defined in the second paragraph of Section 7. Let Q be any element of $\mathcal{D}$. Then $A_{Q,\mathbb{I}}^{\mathcal{D}} = \emptyset$ and is therefore not contractible. Now the intersection of any nonempty subcollection of $\{X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$ is a nonempty polytope (as $\bigcap \mathcal{D} \neq \emptyset$) and is therefore a contractible set. Hence, by Lemma 6.2, $A_{Q,\mathbb{I}}^{\mathcal{D}} \cup X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r} = X_Q^{\mathcal{D},i_1} \cup \cdots \cup X_Q^{\mathcal{D},i_r}$ is contractible for all nonempty subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, |\mathcal{D}| - 1\}$. Thus Q satisfies conditions A and B of Remark 7.1. As Q is an arbitrary element of $\mathcal{D}$, we have that $\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$. □

Lemma 8.2. *Let $\mathcal{D}$ be a nonempty set of grid cells of $\mathbf{K}$ that satisfies the condition $\dim(\bigcap \mathcal{D}) \geq 1$. Then there is a binary image $\mathbb{I}$ on $\mathbf{K}$ such that:*

1. *$\mathbb{I}^{-1}[\{1\}]$ is weakly connected.*
2. *$\mathcal{D} \subsetneq \mathbb{I}^{-1}[\{1\}]$.*
3. *$\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$.*

Proof. Let v_1 and v_2 be distinct vertices of the polytope $\bigcap \mathcal{D}$. By property **X3** of the definition of a convex xel complex there exist grid cells Q_1 and Q_2 of $\mathbf{K}$ such that $v_1 \in Q_1$, $v_2 \in Q_2$, and $Q_1 \cap Q_2 = \emptyset$. Let $\mathbb{I}$ be the binary image on $\mathbf{K}$ whose set of 1s is $\mathcal{D} \cup \{Q_1, Q_2\}$. Evidently, this set is weakly connected. Since $Q_1 \cap Q_2 = \emptyset$, but every element of $\mathcal{D}$ intersects each of Q_1 and Q_2 (at v_1 or v_2), neither Q_1 nor Q_2 lies in $\mathcal{D}$. Thus assertions 1 and 2 hold.

We now establish assertion 3. We again use the notation defined in the second paragraph of Section 7. Let Q be any element of $\mathcal{D}$. Then we have that $A_{Q,\mathbb{I}}^{\mathcal{D}} = Q \cap (Q_1 \cup Q_2)$, which is disconnected (since $Q_1 \cap Q_2 = \emptyset$) and is therefore not contractible. Hence Q satisfies condition A of Remark 7.1.

Now we establish condition B. In the special case $|\mathcal{D}| = 1$, condition B is vacuous, so let us assume $|\mathcal{D}| > 1$. Let $\mathcal{W}$ be any nonempty subset of $\{X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$. Let $\mathcal{W}_1 = \mathcal{W} \cup \{Q \cap Q_1\}$, and let $\mathcal{W}_2 = \mathcal{W} \cup \{Q \cap Q_2\}$. Then $(\bigcup \mathcal{W}_1) \cup (\bigcup \mathcal{W}_2) = \bigcup \mathcal{W} \cup A_{Q,\mathbb{I}}^{\mathcal{D}}$. Note that if $\mathcal{X}$ is $\mathcal{W}$, $\mathcal{W}_1$, or $\mathcal{W}_2$, then the intersection of any nonempty subcollection of $\mathcal{X}$ is a nonempty polytope, which implies (by Lemma 6.2) that $\bigcup \mathcal{X}$ is contractible. Thus each of $\bigcup \mathcal{W}$, $\bigcup \mathcal{W}_1$, and $\bigcup \mathcal{W}_2$ is contractible. Since $(\bigcup \mathcal{W}_1) \cap (\bigcup \mathcal{W}_2) = (\bigcup \mathcal{W}) \cup (Q \cap Q_1 \cap Q_2) = \bigcup \mathcal{W}$ is also contractible, it follows from Fact 6.1 that $A_{Q,\mathbb{I}}^{\mathcal{D}} \cup \bigcup \mathcal{W} = (\bigcup \mathcal{W}_1) \cup (\bigcup \mathcal{W}_2)$ is contractible. As $\mathcal{W}$ is an arbitrary nonempty subset of $\{X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, we have that Q satisfies condition B of Remark 7.1.

Since Q is an arbitrary element of $\mathcal{D}$, we conclude from Remark 7.1 that $\mathcal{D}$ is a minimal non-deletable set of $\mathbb{I}$, so assertion 3 holds. □

The next lemma implies the "sufficient" parts of assertions 3 and 4 of the Main Theorem. Note that assertion 2 of this lemma implies $|\mathcal{D}| \leq \dim(\mathbf{K}) + 1$, by Lemma 7.6.

Lemma 8.3. *Let $\mathcal{D}$ be a nonempty set of grid cells of* $\mathbf{K}$ *such that*

$$\left[\bigcap \mathcal{D} \neq \emptyset\right] \wedge \neg\,(\exists P \in \mathcal{D})\left[\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}\right].$$

Let $\mathbb{I}$ be the binary image on $\mathbf{K}$ *such that* $\mathbb{I}^{-1}[\{1\}] = \{P \in \mathcal{G}(\mathbf{K}) \mid P \cap \bigcap \mathcal{D} \neq \emptyset\}$. *Then:*

1. $\mathbb{I}^{-1}[\{1\}]$ *is strongly connected.*
2. $\mathcal{D}$ *is a minimal non-codeletable set of* 1*s of* $\mathbb{I}$.
3. $\mathcal{D}$ *is a proper subset of* $\mathbb{I}^{-1}[\{1\}]$ *if* $|\mathcal{D}| \leq \dim(\mathbf{K})$.

Proof. Assertion 1 follows easily from condition **X2** in the definition of a convex xel complex because, if P_0 is some element of $\mathcal{D}$, then for each $P' \in \mathbb{I}^{-1}[\{1\}]\setminus\{P_0\}$ we have that P_0 is weakly adjacent to P' and that $(P_0 \cap P') \cap \bigcap \mathcal{D} \neq \emptyset$.

To establish assertion 2, let Q be an arbitrary element of $\mathcal{D}$. As before, we will use the notation defined in the second paragraph of Section 7. We see from the definition of $\mathbb{I}$ that $C_{Q,\mathbb{I}} \cap \bigcap \mathcal{D} = \emptyset$. Moreover, condition **X3** in the definition of a convex xel complex implies that any proper face of Q that does not intersect $\bigcap \mathcal{D}$ must be a face of a grid cell of $\mathbf{K}$ that does not intersect $\bigcap \mathcal{D}$. It follows that $C_{Q,\mathbb{I}} = \bigcup\{F \in \mathbf{faces}(Q) \mid F \cap \bigcap \mathcal{D} = \emptyset\}$. In the special case $|\mathcal{D}| = 1$, we have that $C_{Q,\mathbb{I}} = \emptyset$ and so assertion 2 holds (e.g., by Remark 7.1, in which condition D is vacuous in this case).

Next, we establish assertion 2 under the assumption that $|\mathcal{D}| > 1$. In this case $\bigcap \mathcal{D}$ is a proper face of Q, so it follows from Proposition A.1 in the Appendix that the set $C_{Q,\mathbb{I}} = \bigcup\{F \in \mathbf{faces}(Q) \mid F \cap \bigcap \mathcal{D} = \emptyset\}$ is contractible. Now let $\{i_1, \ldots, i_r\}$ be any nonempty proper subset of $\{1, \ldots, |\mathcal{D}|-1\}$. Since $\bigcap \mathcal{D} \neq \emptyset$, the set $X_Q^{\mathcal{D},i_1} \cap \cdots \cap X_Q^{\mathcal{D},i_r} = Q \cap T_Q^{\mathcal{D},i_1} \cap \cdots \cap T_Q^{\mathcal{D},i_r}$ is a nonempty polytope, and is therefore contractible. We claim that $C_{Q,\mathbb{I}} \cap X_Q^{\mathcal{D},i_1} \cap \cdots \cap X_Q^{\mathcal{D},i_r}$ is contractible as well. Indeed, since $C_{Q,\mathbb{I}} \subsetneq Q$, we have that $C_{Q,\mathbb{I}} \cap X_Q^{\mathcal{D},i_1} \cap \cdots \cap X_Q^{\mathcal{D},i_r} = C_{Q,\mathbb{I}} \cap T_Q^{\mathcal{D},i_1} \cap \cdots \cap T_Q^{\mathcal{D},i_r} = \bigcup\{F \cap T_Q^{\mathcal{D},i_1} \cap \cdots \cap T_Q^{\mathcal{D},i_r} \mid F \in \mathbf{faces}(Q) \text{ and } F \cap \bigcap \mathcal{D} = \emptyset\} = \bigcup\{G \in \mathbf{faces}(Q \cap T_Q^{\mathcal{D},i_1} \cap \cdots \cap T_Q^{\mathcal{D},i_r}) \mid G \cap \bigcap \mathcal{D} = \emptyset\}$. The last set is contractible, by Proposition A.1, because the condition $\neg(\exists P \in \mathcal{D})[\bigcap(\mathcal{D} \setminus \{P\}) = \bigcap \mathcal{D}]$ implies that $\bigcap \mathcal{D}$ is a proper face of $Q \cap T_Q^{\mathcal{D},i_1} \cap \cdots \cap T_Q^{\mathcal{D},i_r}$. This justifies our claim.

It follows from the above that $\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$. Hence, by assertion 2 of Lemma 6.2, $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, and so condition D of Remark 7.1 is satisfied, as the hypotheses on the set $\mathcal{D}$ imply no two of the Xs are equal.

But $\bigcap\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\} = C_{Q,\mathbb{I}} \cap Q \cap T_Q^{\mathcal{D},1} \cap \cdots \cap T_Q^{\mathcal{D},|\mathcal{D}|-1} = C_{Q,\mathbb{I}} \cap \bigcap \mathcal{D} = \emptyset$. From this it follows that $\bigcup\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$ is not contractible. (For otherwise we would have that $\bigcup \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, whereas $\bigcap\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\} = \emptyset$ is not contractible, contrary to assertion 1 of Lemma 6.2.) Hence condition C of Remark 7.1 is satisfied too. This establishes that $\mathcal{D}$ is a minimal non-codeletable set of $\mathbb{I}$.

It remains to establish assertion 3. Suppose $|\mathcal{D}| \leq \dim(\mathbf{K})$. Since $\bigcap\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\} = \emptyset$ but $\bigcap \mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subsetneq \{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, and $|\mathcal{D}| \leq \dim(\mathbf{K}) = \dim(Q)$, it follows from Fact 6.3 that $\bigcup\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\} \neq \partial Q$. Let F be any proper face of the grid cell Q such that $F \not\subseteq \bigcup\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$. It follows from basic properties [33, Thm. 2.7] of the face lattices of polytopes that any proper face G of a polytope P is contained in a $(\dim(P)-1)$-dimensional face of P. Let F' be a $(\dim(\mathbf{K})-1)$-dimensional face of Q that contains F. Then $F' \not\subseteq \bigcup\{C_{Q,\mathbb{I}}, X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$.

By condition **X1** in the definition of a convex xel complex, there is a grid cell Q' of $\mathbf{K}$ such that $F' = Q \cap Q'$. As $F' \not\subseteq \bigcup\{X_Q^{\mathcal{D},1}, \ldots, X_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, we have that $Q' \notin \{T_Q^{\mathcal{D},1}, \ldots, T_Q^{\mathcal{D},|\mathcal{D}|-1}\}$, and hence that $Q' \notin \mathcal{D}$. As $F' \not\subseteq C_{Q,\mathbb{I}} = \bigcup \mathbf{Coattach}(Q, \mathbb{I})$, and $Q \cap Q' = F'$, the grid cell Q' cannot be a 0 of $\mathbb{I}$. Thus Q' is a 1 of $\mathbb{I}$ that is not in $\mathcal{D}$, and so we have established assertion 3. $\square$

9. A generalization of the Main Theorem to convex xel complexes of arbitrary dimensionality

A polyhedron X is said to be *acyclic* if it is nonempty and connected, and its homology groups $H_k(X)$ are trivial for all $k \geq 1$. (Equivalently, X is acyclic if and only if its reduced homology groups are all trivial.) It follows from [20, Thm. 8.3.10] that a polyhedron X is contractible if and only if X is acyclic and simply connected. As we observed in footnote 9, if X is a finite polyhedron in $\mathbb{R}^3$ or in the boundary of a 4D polytope, then X is contractible if and only if X is acyclic.

Now let X and Y be polyhedra such that $X \subseteq Y$. We will say that X is a *homology-retract* of Y if the inclusion map $i : X \to Y$ induces an isomorphism of $H_k(X)$ onto $H_k(Y)$, for all $k \geq 0$. If X is a deformation retract of Y, then X is a homology-retract of Y. If X and Y are simply connected, then the converse is also true (by [20, Thm. 8.3.10] and [10, Prop. 0.16, Cor. 0.20]), but in general the converse is false,[13] even for finite polyhedra in $\mathbb{R}^3$.

Let $\mathbb{I}$ be a binary image on a polytopal complex $\mathbf{K}$, and let $\mathcal{D}$ be a set of 1s of $\mathbb{I}$. Then the foreground polyhedra of $\mathbb{I}$ and $\mathbb{I}-\mathcal{D}$ are respectively $\bigcup \mathbb{I}^{-1}[\{1\}]$ and $\bigcup(\mathbb{I}^{-1}[\{1\}]\setminus\mathcal{D})$. We say $\mathcal{D}$ is a *homology-deletable* set of $\mathbb{I}$ if $\mathcal{D}$ is finite and $\bigcup(\mathbb{I}^{-1}[\{1\}]\setminus\mathcal{D})$ is a homology-retract of $\bigcup \mathbb{I}^{-1}[\{1\}]$. Since a deformation retract of $\bigcup \mathbb{I}^{-1}[\{1\}]$ is a homology-retract of $\bigcup \mathbb{I}^{-1}[\{1\}]$, every deletable set is homology-deletable; but it is possible for a set of 1s to be homology-deletable without being deletable, even when $\mathbf{K}$ is the 3D Cartesian grid complex.

We say $\mathcal{D}$ is a *homology-codeletable* set of $\mathbb{I}$ if $\mathcal{D}$ is a homology-deletable set of the binary image $(\mathbb{I}-\mathcal{D})^{\mathbf{c}}$. If P is any 1 of $\mathbb{I}$, then we say P is a *homology-simple* 1 of $\mathbb{I}$ if $\{P\}$ is a homology-deletable set of $\mathbb{I}$; similarly, we say P is a *homology-cosimple* 1 of $\mathbb{I}$ if $\{P\}$ is a homology-codeletable set of $\mathbb{I}$. We say that a set $\mathcal{D}$ of grid cells is a *homology-simple set* (*homology-cosimple set*) of $\mathbb{I}$ if $\mathcal{D}$ is a finite set and it is possible to arrange the elements of $\mathcal{D}$ into a sequence $(d_i \mid 0 \leq i < |\mathcal{D}|)$ in which each element d_i is a homology-simple (homology-cosimple) 1 of the image $\mathbb{I}-\{d_j \mid 0 \leq j < i\}$.

The definition of a homology-simple 1 can be regarded as an obvious generalization of Definition 3 in Niethammer et al.'s paper [23] from nD Cartesian grid complexes to arbitrary polytopal complexes. Some years ago, a homology-based definition of simple 1s that is mathematically equivalent to Niethammer et al.'s Definition 3 was suggested by the present author in [12]: If P is a 1 of a binary image $\mathbb{I}$ on the nD Cartesian grid complex, then P is "$(3^n-1, 2n)$-simple" in the sense of [12, Def. 13] if $\bigcup \mathbf{Attach}(P, \mathbb{I})$ is an acyclic polyhedron. The equivalence of this notion of a simple 1 to the concept of a homology-simple 1 follows from the argument outlined in the second and third sentences of footnote 10. But this argument applies to any polytopal complex, not just Cartesian grid complexes. So we have the following "homology version" of Theorem 4.5:

Theorem 9.1. *Let* $\mathbf{K}$ *be a polytopal complex and let* $P \in \mathcal{G}(\mathbf{K})$ *be a* 1 *of a binary image* $\mathbb{I}$ *on* $\mathbf{K}$. *Then:*

1. *P is a homology-simple* 1 *of* $\mathbb{I}$ *if and only if* $\bigcup \mathbf{Attach}(P, \mathbb{I})$ *is acyclic.*
2. *P is a homology-cosimple* 1 *of* $\mathbb{I}$ *if and only if* $\bigcup \mathbf{Coattach}(P, \mathbb{I})$ *is acyclic.*

When $\dim(\mathbf{K}) \leq 4$, the condition in Theorem 9.1 that $\bigcup \mathbf{Attach}(P, \mathbb{I})$ is acyclic is equivalent to the condition in Theorem 4.5 that $\bigcup \mathbf{Attach}(P, \mathbb{I})$ is contractible. It follows that, in a binary image on a polytopal complex of dimensionality $\leq$4, a 1 is simple (cosimple) if and only if it is homology-simple (homology-cosimple), and so a set of 1s is simple (cosimple) if and only if it is homology-simple (homology-cosimple), and is minimal non-simple (minimal non-cosimple) if and only if it is minimal non-homology-simple (minimal non-homology-cosimple).

Just as we modified Theorem 4.5 to produce Theorem 9.1, we obtain homology versions of Theorems 5.1, 5.3 and 5.4, Proposition 5.5, and Remark 7.1 by replacing "deletable", "codeletable", "simple", and "cosimple", with "homology-deletable", "homology-codeletable", "homology-simple", and "homology-cosimple", and replacing "deformation retract" and "contractible" with "homology-retract" and "acyclic". We claim that the homology versions of Theorems 5.1 and 5.3 are true, and that the homology versions of Theorem 5.4, Proposition 5.5, and Remark 7.1 hold not only when $\dim(\mathbf{K}) \leq 4$, but also when $\dim(\mathbf{K}) > 4$. This is because the homology versions of Facts 1.1 and 5.2 that are obtained by making the above-mentioned changes are all true, and Theorem 9.1 (unlike Theorem 4.5) holds

[13] For a counterexample, let P be a 2D polytope in $\mathbb{R}^3$, and let T be a polyhedral solid torus whose spine is ∂P. Let K be a polyhedral solid torus, in the interior of T, such that the spine of K is a knotted curve that has a Seifert surface [1] which includes $P \setminus T$ and lies entirely in the union of P with the interior of T. Let Q be a cube that contains $P \cup T$ in its interior. Let X and Y be the closures of $Q \setminus T$ and $Q \setminus K$, respectively. Then X is a homology-retract of Y, but X and Y have different fundamental groups (since the spine of K is knotted) and so X is not a deformation retract of Y.

for polytopal complexes of any dimensionality. Similarly, the homology version of Fact 6.1 (obtained by replacing "contractible" with "acyclic") is true even if A and B do not lie in $\mathbb{R}^3$ or the boundary of a 4D polytope, and so the same modification produces a homology version of Lemma 6.2 that is true even if the polyhedra in $\mathcal{S}$ do not lie in $\mathbb{R}^3$ or the boundary of a 4D polytope. A homology version of Fact 6.3 (again obtained by replacing "contractible" with "acyclic") is also valid; this follows from the argument of footnote 12 if we appeal to a homology version of the nerve theorem [21, Thm. 2.1] instead of the "usual" nerve theorem [5, Thm. 10.6(i)].

It follows from the above observations that, if we change "deletable" and "codeletable" to "homology-deletable" and "homology-codeletable" in the statements of the lemmas in Sections 7 and 8, then the resulting homology versions of those lemmas will all be true for convex xel complexes $\mathbf{K}$ of any dimensionality. This is because, as mentioned in Sections 7 and 8, the proofs of those lemmas use the hypothesis that $\dim(\mathbf{K}) \leq 4$ only when appealing to Proposition 5.5, Fact 6.1, Lemma 6.2, or Remark 7.1. When $\dim(\mathbf{K}) > 4$ we can instead appeal to the homology versions of those facts, and use the homology version of Fact 6.3, to establish the homology versions of the lemmas. We can therefore conclude that, when "deletable" and "codeletable" are changed to "homology-deletable" and "homology-codeletable", the Main Theorem is true for convex xel complexes $\mathbf{K}$ of any dimensionality.

10. Concluding remarks

The concepts of minimal non-deletable and minimal non-codeletable set provide the basis for a fairly general method (described in Section 3) for establishing that a proposed parallel thinning algorithm "preserves topology". For binary images on the grid cells of a polytopal complex $\mathbf{K}$, the method depends on knowing the answers to the following questions:

1. For algorithms that are expected to preserve weakly connected foreground and strongly connected background components:
   - Which sets of grid cells can be minimal non-deletable on $\mathbf{K}$?
   - Which sets of grid cells can be minimal non-deletable on $\mathbf{K}$ while being a *proper* subset of a weakly connected foreground component?
2. For algorithms that are expected to preserve strongly connected foreground and weakly connected background components:
   - Which sets of grid cells can be minimal non-codeletable on $\mathbf{K}$?
   - Which sets of grid cells can be minimal non-codeletable on $\mathbf{K}$ while being a *proper* subset of a strongly connected foreground component?

Over the past two decades, these questions have essentially been answered in the literature for the 2D, 3D, and 4D Cartesian grid complexes, the 2D hexagonal grid complex, and the 3D face-centered cubic grid complex [7–9,13, 15,18,28]. Our Main Theorem generalizes and unifies this earlier work, by answering the questions for almost any polytopal complex whose union is $\mathbb{R}^n$, when $n \leq 4$.

Two of the above-mentioned references [7,18] also answered the same questions for other concepts of deletable sets on the 3D Cartesian grid complex and the 3D face-centered cubic grid complex—concepts of deletability that can be used to define topology preservation by thinning and shrinking algorithms that preserve 18-connected foreground and 6-connected background components (or vice versa) in the 3D Cartesian grid complex, and algorithms that preserve both strongly connected foreground and strongly connected background components in the 3D face-centered cubic grid complex. (For examples of such algorithms, see [19,32].) This raises the interesting question of how the general theory developed in the present paper can be extended to deal with such alternative concepts of deletability.

Appendix. A property of polytopes

The purpose of this appendix is to establish the following result, which is used in the proof of Lemma 8.3:

Proposition A.1. *Let P be a polytope of dimensionality ≥ 1, and let P' be any proper face of P. Then the polyhedron $\bigcup\{D \in \mathbf{faces}(P) \mid D \cap P' = \emptyset\}$ is contractible.*

We will deduce this proposition from the following lemma:

Lemma A.2. *Let H be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, where $n \geq 1$, let $H_{\geq}$ be one of the two closed halfspaces of $\mathbb{R}^n$ that are bounded by H, and let $\mathbf{K}$ be a finite polytopal complex in $H_{\geq}$ such that all polytopes $F \in \mathbf{K}$ which intersect H satisfy the following conditions:*

1. *If $F \subset H$, then there is some $F' \in \mathbf{K} \setminus \{F\}$ for which $F' \cap H = F$.*
2. *If $F \not\subset H$, then there is no $R \in \mathbf{K}$ such that $R \not\subset H$ and $\emptyset \neq R \cap F \subset H$.*

Then the polyhedron $\bigcup\{Q \in \mathbf{K} \mid Q \cap H = \emptyset\}$ is a deformation retract of the polyhedron $\bigcup \mathbf{K}$.

Proof. If A and B are polytopes, then we write $A < B$ and $A \leq B$ to mean "A is a proper face of B" and "A is a face of B", respectively. Since $\mathbf{K}$ is a polytopal complex in $H_{\geq}$, $F \cap H \leq F$ for all $F \in \mathbf{K}$.

Suppose $\mathbf{K}$ is a minimal counterexample to the lemma. Then the collection of polytopes $\{Q \in \mathbf{K} \mid Q \cap H \neq \emptyset\}$ is nonempty. Let R be a polytope of maximal dimensionality in this collection. Then $R \not\subset H$, by condition 1 and the fact that there is no $R' \in \mathbf{K} \setminus \{R\}$ for which $R < R'$.

Suppose $\dim(R \cap H) < \dim(R) - 1$. Basic properties [33, Thm. 2.7] of the face lattices of polytopes imply that any face G of a polytope P is such that $\dim(G) < \dim(P) - 1$ is the intersection of two $(\dim(G) + 1)$-dimensional faces of P. It follows that $R \cap H$ is the intersection of two $(\dim(R \cap H) + 1)$-dimensional faces of R (and neither face is a subset of H, as each strictly contains $R \cap H$). This contradicts condition 2. Hence $\dim(R \cap H) = \dim(R) - 1$.

We claim that there is no $R' \in \mathbf{K} \setminus \{R\}$ for which $R \cap H < R'$. Indeed, suppose such an R' exists. Then $R \cap H = R \cap R'$, and so (on putting $F = R'$ in condition 2) we see that $R' \subset H$, and hence (by condition 1) that R' is a proper face of a polytope $R'' \in \mathbf{K}$. But now $\dim(R'') > \dim(R') > \dim(R \cap H) = \dim(R) - 1$, which implies $\dim(R'') > \dim(R)$, and so contradicts the fact that R is a polytope of maximal dimensionality in $\{Q \in \mathbf{K} \mid Q \cap H \neq \emptyset\}$. This justifies the claim.

Let $\mathbf{K}^* = \mathbf{K} \setminus \{R, R \cap H\}$. It follows from the above claim that $\mathbf{K}^*$ is a polytopal complex. Moreover, the hypotheses of the lemma hold when $\mathbf{K}$ is replaced with $\mathbf{K}^*$. So, since $\mathbf{K}$ is a minimal counterexample to the lemma, the polyhedron $\bigcup\{Q \in \mathbf{K} \mid Q \cap H = \emptyset\} = \bigcup\{Q \in \mathbf{K}^* \mid Q \cap H = \emptyset\}$ is a deformation retract of the polyhedron $\bigcup \mathbf{K}^*$.

But we can also show that $\bigcup \mathbf{K}^*$ is a deformation retract of $\bigcup \mathbf{K}$. Indeed, if p is a point in the complement of $H_{\geq}$ that lies in the affine hull of R and is sufficiently close to the centroid of $R \cap H$, then for each $x \in R$ there is a unique point x_p in $\bigcup(\mathbf{faces}(R) \setminus \{R, R \cap H\})$ that is collinear with x and p. (This depends on the fact that, since $\dim(R \cap H) = \dim(R) - 1$, no member of $\mathbf{faces}(R) \setminus \{R, R \cap H\}$ can have $R \cap H$ as a face, and so the centroid of $R \cap H$ does not lie in the affine hull of any member of $\mathbf{faces}(R) \setminus \{R, R \cap H\}$.) For some such point p, let $f_1 : R \to \bigcup \mathbf{K}^*$ be the function defined by $f_1(x) = x_p$ for all $x \in R$. Let f_2 be the identity map on $\bigcup \mathbf{K}^*$. Note that each of f_1 and f_2 is a continuous map whose domain is closed in $\mathbb{R}^n$, that $f_1 = f_2$ on the intersection of the domains of f_1 and f_2 (i.e., on $\bigcup(\mathbf{faces}(R) \setminus \{R, R \cap H\})$), and that the union of the domains of f_1 and f_2 is $\bigcup \mathbf{K}$. Hence the union of f_1 and f_2 is a continuous function from $\bigcup \mathbf{K}$ to $\bigcup \mathbf{K}^*$. If f is that function, then a deformation retraction $h : \bigcup \mathbf{K} \times [0, 1] \to \bigcup \mathbf{K}^*$ is given by $h(x, t) = tf(x) + (1 - t)x$.

Since $\bigcup \mathbf{K}^*$ is a deformation retract of $\bigcup \mathbf{K}$, and $\bigcup\{Q \in \mathbf{K} \mid Q \cap H = \emptyset\}$ is a deformation retract of $\bigcup \mathbf{K}^*$, it follows from Fact 5.2 that $\bigcup\{Q \in \mathbf{K} \mid Q \cap H = \emptyset\}$ is a deformation retract of $\bigcup \mathbf{K}$, and so $\mathbf{K}$ is not in fact a counterexample to the lemma. This contradiction completes the proof. □

Proof of Proposition A.1. Assume, as we may, that $P \subset \mathbb{R}^n$, where $n = \dim(P)$. Write x_1 to denote the first coordinate of x. For $t \in \mathbb{R}$, let H_t, $H_{\geq t}$, $H_{\leq t}$, $H_{>t}$, and $H_{<t}$ respectively denote the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = t\}$, the closed halfspaces $\{x \in \mathbb{R}^n \mid x_1 \geq t\}$ and $\{x \in \mathbb{R}^n \mid x_1 \leq t\}$, and the open halfspaces $\{x \in \mathbb{R}^n \mid x_1 > t\}$ and $\{x \in \mathbb{R}^n \mid x_1 < t\}$. Assume, as we may, that $P \subset H_{\geq 0}$ and that $P' = P \cap H_0$. Let ϵ be a positive real value such that, for all $D \in \mathbf{faces}(P)$, D intersects $H_{\leq \epsilon}$ only if D intersects H_0. This condition implies that no nonempty face of P can be a subset of H_ϵ, and also implies that any vertex of P which does not lie in H_0 must lie in $H_{>\epsilon}$.

Let $\mathbf{K} = \bigcup \left\{\{D \cap H_{\geq \epsilon}, D \cap H_\epsilon\} \mid D \in \mathbf{faces}(P)\right\}$. Then $\mathbf{K}$ is a finite collection of polytopes, and it is not hard to show that $\mathbf{K}$ also satisfies conditions **P1** and **P2** of the definition of a polytopal complex. Hence $\mathbf{K}$ is a finite polytopal complex. Let $H = H_\epsilon$ and $H_{\geq} = H_{\geq \epsilon}$. We now show that conditions 1 and 2 in the hypotheses of Lemma A.2 are satisfied for all polytopes $F \in \mathbf{K}$ which intersect H.

We first claim that no $D \in \mathbf{faces}(P)$ can satisfy $\emptyset \neq D \cap H_{\geq \epsilon} \subset H_\epsilon$. Indeed, if such a D existed it would have a vertex that is not in H_0, and (as we observed above) any such vertex must lie in $H_{>\epsilon}$. This contradiction justifies the claim. Since $H = H_\epsilon$ and $H_{\geq} = H_{\geq \epsilon}$, condition 1 of the lemma follows from this claim and the definition of $\mathbf{K}$. Regarding condition 2, note that if $F, R \in \mathbf{K}$, $R \not\subset H_\epsilon$, and $F \not\subset H_\epsilon$, then there exist $D_1, D_2 \in \mathbf{faces}(P)$ for

which $R = D_1 \cap H_{\geq\epsilon}$ and $F = D_2 \cap H_{\geq\epsilon}$, whence $\emptyset \neq R \cap F \subset H_\epsilon$ is impossible (by the above claim, on putting $D = D_1 \cap D_2$). Thus condition 2 holds.

As $P' = P \cap H_0$, we have that $\{D \in \mathbf{faces}(P) \mid D \cap P' = \emptyset\} = \{D \in \mathbf{faces}(P) \mid D \cap H_0 = \emptyset\} = \{D \in \mathbf{faces}(P) \mid D \subset H_{>\epsilon}\} = \{Q \in \mathbf{K} \mid Q \cap H_\epsilon = \emptyset\}$. So it follows from the lemma that $\bigcup\{D \in \mathbf{faces}(P) \mid D \cap P' = \emptyset\}$ is a deformation retract of $\bigcup \mathbf{K} = P \cap H_{\geq\epsilon}$. Now $P \cap H_{\geq\epsilon}$ is a polytope, and it is nonempty because P has a vertex that is not in H_0 (otherwise $P \subset H_0$, contrary to $\dim(P) = n$) which must lie in $H_{>\epsilon}$. Therefore $P \cap H_{\geq\epsilon}$ is contractible, and since any deformation retract of a contractible polyhedron is contractible (e.g., by Facts 4.2 and 5.2), the proposition is proved. □

References

[1] M.A. Armstrong, Basic Topology, Springer, New York, 1983.
[2] G. Bertrand, Simple points, topological numbers, and geodesic neighborhoods in cubic grids, Pattern Recognition Letters 15 (1994) 1003–1011.
[3] G. Bertrand, On P-simple points, Comptes-Rendus de l'Académie des Sciences de Paris, Série I 321 (1995) 1077–1084.
[4] G. Bertrand, A Boolean characterization of three-dimensional simple points, Pattern Recognition Letters 17 (1996) 115–124.
[5] A. Björner, Topological methods, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, vol. II, MIT Press, 1995, pp. 1819–1872.
[6] D. Brunner, G. Brunnett, R. Strand, A High-Performance Parallel Thinning Approach Using a Non-Cubic Grid Structure. Chemnitzer Informatik-Berichte CSR-06-08, Faculty of Computer Science, Chemnitz University of Technology, 2006.
[7] C.J. Gau, T.Y. Kong, Minimal nonsimple sets of voxels in binary images on a face-centered cubic grid, International Journal of Pattern Recognition and Artificial Intelligence 13 (1999) 485–502.
[8] C.J. Gau, T.Y. Kong, Minimal nonsimple sets in 4D binary images, Graphical Models 65 (2003) 112–130.
[9] R.W. Hall, Tests for connectivity preservation for parallel reduction operators, Topology and its Applications 46 (1992) 199–217.
[10] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[11] G.T. Herman, Geometry of Digital Spaces, Birkhäuser, 1998.
[12] T.Y. Kong, On the problem of determining whether a parallel reduction operator for n-dimensional binary images always preserves topology, in: R.A. Melter, A.Y. Wu (Eds.), Vision Geometry II (Boston, USA, September 1993), Proceedings, Proc. SPIE 2060, 1993, pp. 69–77.
[13] T.Y. Kong, On topology preservation in 2D and 3D thinning, International Journal of Pattern Recognition and Artificial Intelligence 9 (1995) 813–844.
[14] T.Y. Kong, Minimal non-simple and minimal non-cosimple sets in binary images on cell complexes, in: A. Kuba, L.G. Nyúl, K. Palágyi (Eds.), Discrete Geometry for Computer Imagery: 13th International Conference (DGCI 2006, Szeged, Hungary, October 2006), Proceedings, Springer, 2006, pp. 169–188.
[15] T.Y. Kong, C.J. Gau, Minimal nonsimple sets in 4-dimensional binary images with (8, 80)-adjacency, in: R. Klette, J. Žunić (Eds.), Combinatorial Image Analysis: 10th International Workshop (IWCIA 2004, Auckland, New Zealand, December 2004), Proceedings, Springer, 2004, pp. 318–333.
[16] T.Y. Kong, A.W. Roscoe, Characterizations of simply-connected finite polyhedra in 3-space, Bulletin of the London Mathematical Society 17 (1985) 575–578.
[17] T.Y. Kong, P.K. Saha, A. Rosenfeld, Strongly normal sets of contractible tiles in N dimensions, Pattern Recognition 40 (2007) 530–543.
[18] C.M. Ma, On topology preservation in 3D thinning, CVGIP: Image Understanding 59 (1994) 328–339.
[19] C.M. Ma, S.Y. Wan, Parallel thinning algorithms on 3D (18, 6) binary images, Computer Vision and Image Understanding 80 (2000) 364–378.
[20] C.R.F. Maunder, Algebraic Topology, Dover Publications, 1996.
[21] R. Meshulam, The clique complex and hypergraph matching, Combinatorica 21 (2001) 89–94.
[22] E.E. Moise, Geometric Topology in Dimensions 2 and 3, Springer, 1977.
[23] M. Niethammer, W.D. Kalies, K. Mischaikow, A. Tannenbaum, On the detection of simple points in higher dimensions using cubical homology, IEEE Transactions on Image Processing 15 (2006) 2462–2469.
[24] K. Palágyi, A. Kuba, A 3D 6-subiteration thinning algorithm for extracting medial lines, Pattern Recognition Letters 19 (1998) 613–627.
[25] K. Palágyi, A. Kuba, A parallel 3D 12-subiteration thinning algorithm, Graphical Models and Image Processing 61 (1999) 199–221.
[26] N. Passat, M. Couprie, G. Bertrand, Minimal simple pairs in the 3-D cubic grid. Rapports Internes IGM 2007-04, Institut Gaspard Monge, 2007.
[27] C. Ronse, A topological characterization of thinning, Theoretical Computer Science 43 (1986) 31–41.
[28] C. Ronse, Minimal test patterns for connectivity preservation in parallel thinning algorithms for binary digital images, Discrete Applied Mathematics 21 (1988) 67–79.
[29] A. Rosenfeld, A characterization of parallel thinning algorithms, Information and Control 29 (1975) 286–291.
[30] A. Rosenfeld, A.C. Kak, Digital Picture Processing, 2nd ed., vol. 2, Academic Press, 1982.
[31] P.K. Saha, B.B. Chaudhuri, Detection of 3D simple points for topology preserving transformation with applications to thinning, IEEE Transactions on Pattern Analysis and Machine Intelligence 16 (1994) 1028–1032.
[32] R. Strand, Surface skeletons in grids with non-cubic voxels, in: Proceedings of the 17th International Conference on Pattern Recognition (ICPR 2004, Cambridge, UK, August 2004), vol. 1, IEEE Computer Society, 2004, pp. 548–551.
[33] G.M. Ziegler, Lectures on Polytopes, Springer, 1995.