DESIGN OF ELECTRICITY SUPPLY NETWORKS

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Received 1 April 1982
Revised 24 June 1982

The problem of designing a secure electricity supply network at minimal cost is formulated as a mathematical program. It is also shown how computationally convenient new constraints may be derived and these are added to the original set. The problem is dualized and solved approximately. It is indicated how this approach can be built into a Branch-and-Bound scheme for solving the original design problem, and an illustrative example is given.

The electricity transmission and distribution system in England is constantly being extended to meet increasing consumer demand [1] and the problem arises of finding a 'satisfactory' system of minimum cost. In 1972 Boardman and Hogg [1] noted that this was "largely achieved by trial-and-error techniques, the system designer having to rely on his expertise and experience". Since that time several heuristic methods and interactive methods have been devised but except for very small problems, exact methods have not been applied successfully.

Several load substations \( L_1, L_2, \ldots, L_n \) which may receive power from supply substations \( S_{n+1}, \ldots, S_{n+m} \) are given. Power lines may be placed between any supply substation and any load substation, and between any pair of load substations. Between specified substations, 'distance' \( d_{ij} \) apart, there may be \( z_{ij} = 0, 1, \ldots, p \) lines incurring a cost \( c(z_{ij})d_{ij} \). Clearly \( c(0) = 0 \) and there may be economies of scale with \( c(z_{ij}) < c(u)z_{ij}/u \) for \( 1 \leq u < z_{ij} \).

Each load substation \( L_i \) has a power requirement \( \omega_i \) which must be met. For each set \( X \subseteq \{L_1, \ldots, L_n\} \) there is a power requirement \( \omega = \omega(X) = \sum_{L_i \in X} \omega_i \) and, taking account of possible fluctuation in demand, this necessitates \( l(\omega) \) power lines to be incoming to \( X \) for security. (As a safeguard against the possibility of a line failure one may insist that \( l(\omega) \geq 2, \omega > 0 \).)

Thus \( l \), which is a step function, might be expected to show an element of 'concavity' in that the gaps \( \omega_{n+1} - \omega_n (\omega_n > 0) \) between successive jumps of 1 in \( l \) might be expected to increase (see Fig. 1 and [4]).

\[
\text{ESP} \quad \text{minimise } \sum_{i=1}^{n+m} d_{ij} c(z_{ij}),
\]
Fig. 1. Typical form of the security function 1.

\[
\sum_{i \in I, j \notin I} z_{ij} \geq I(\omega(T)) \quad \forall T \subseteq \{1, 2, \ldots, n\},
\]

(1)

\[ z_{ij} = z_{ji}, \]

(2)

\[ z_{ij} \in \{0, 1, \ldots, p\} \]

(3)

(Note that we have modified the notation somewhat. From now on the load substations \( L_i \) will usually be denoted simply by \( i \). Similarly \( S_j \) will be replaced by \( j \).)

This deceptively simple looking problem, is made difficult by the forms of the functions \( c \) and \( l \). Indeed the problem can be seen to be difficult, in the theoretical sense of being NP-hard, by noting that any Euclidean Travelling Salesman Problem (TSP) can be represented as a special case of ESP (with \( m = 1, p = 1 \) and \( \omega(T) = 2 \forall T \)). However, methods have been devised for solving quite large TSPs and the approach of the present paper bears some similarity to that adopted by Miliotis [6] in the context of TSPs.

In this paper we shall be concerned with the special case in which \( p = 2 \) and \( c(2)/c(1) = K = \frac{2}{3} \) which is appropriate if underground cables are being used [4]. This particular form of ESP is reformulated in Section 1. In section 2 a scheme for deriving appropriate new security constraints is described and in Section 3 a scheme for obtaining reasonably tight lower bounds is proposed. This allows error bounds to be determined on results obtained by heuristic methods, can provide feasible solutions as a 'byproduct', and provides a basis for the application of a Branch-and-Bound method if so desired. A numerical example is discussed in Section 4. The final section provides a discussion of the approach and gives some tentative conclusions.

1. Brief review of earlier work

Since we shall be allowing at most two lines per right-of-way (\( p = 2 \)) it is convenient to introduce binary variables \( x_{ij} \) and \( y_{ij} \) where \( x_{ij} \) specifies if the 'first' line between \( i \) and \( j \) is to be constructed and \( y_{ij} \) specifies if the 'second' line is to be constructed. ESP can now be reformulated as
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\[ \text{minimise } \varphi(x, y) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} (x_{ij} + \frac{1}{2} y_{ij}), \quad (4) \]

subject to \[ \sum_{\substack{i \in T \atop j \in T}} (x_{ij} + y_{ij}) \geq l(\omega(T)) \quad \forall T \subset \{1, 2, \ldots, n\}, \quad (5) \]

\[ x_{ij} \geq y_{ij} \quad \forall i, j, \quad (6) \]

\[ x_{ij} = x_{ji}, \ y_{ij} = y_{ji} \quad \forall i, j, \quad (7) \]

\[ x_{ij}, y_{ij} \in \{0, 1\} \quad \forall i, j. \quad (8) \]

There are \(2^n - 1\) constraints implied by (5) and, not surprisingly, a direct application of integer programming methods is unwieldy and is impractical for all but the smallest problems, say with \(n < 10\) (see Cory [3]). Consequently heuristic methods have been developed.

In a pioneering paper Burstall [2] used a cost function, relating to overhead lines, in which the 1st, 3rd, 5th... lines had the same cost \(C\) and the 2nd, 4th... all had the same cost \(KC\) with \(0 < K \leq 1\). An improvement heuristic was devised with, at each step, one or two lines being added to the network then excess lines being removed. Security checking was performed exactly and problems with \(n + m\) up to 16 were studied. (The problem of Section 4 is from Burstall [2] but with our cost function used.)

Richards and Boardman [7] in their \textit{minimax} algorithm started with a feasible solution in which security was assured by connecting load substations directly to supply substations in a minimal cost way. If there is no path between load substations \(i\) and \(j\) which does \textit{not} pass through a supply substations then a single line between \(i\) and \(j\) may be added provided loss of security cannot be detected when a single line from \(i\) and/or \(j\) to a supply substation is/are removed; the net cost is \(r_{ij}\). That change for which \(r_{ij}\) is maximal is chosen and performed before going to the next step. Final security is not guaranteed.

Green and Boardman [4] took this approach further with a more stringent security test at each step and a final complete security check. A set \(V\) of connected load substations is a \textit{weakest set containing \(i\) if} \(i \in V \subset \{1, \ldots, n\}\) and

\[ \sum_{\substack{i \in V \atop j \notin V}} (x_{ij} + y_{ij}) - l(\omega(V)) \]

is minimal. When the addition of a line between load substations \(i\) and \(j\) is being considered the security of the weakest sets containing \(i\) and \(j\) are tested. The method was quite quick requiring only a few seconds processing time on an ICL 1906S computer for the problem of Section 4 and generally seems to lead to high quality solutions — some justification for this is provided in Section 4.

Finally mention may be made of the \textit{maximin} constructive method of Richards and Boardman [7]. The network is built up such that at any step the line added is to a subset of load substations for which the minimal cost of attaining security is a
maximum. Since only a selection of subsets of load substations is considered, ultimate security is not guaranteed.

Generally, difficulty has been experienced with guaranteeing security. Often the security checks are heuristic and some authors have relied on statistical sampling. However, the assumption that $I$ is concave means that

$$I(\omega(A \cup B)) \leq I(\omega(A)) + I(\omega(B))$$

(9)

if $A \cap B = \emptyset$. Thus each component of the network that results from deleting supply substations can be tested separately this often leading to a considerable saving in computational effort.

Exact solution methods have been successfully applied to only very small problems. Heuristically obtained solutions are available for larger (though still fairly small) problems; the quality of these solutions is not known for certain and we look in the next two sections at how the possible error may be bounded by obtaining lower bounds to ESP'.

2. Obtaining lower bounds for ESP

As noted earlier, computational difficulties are experienced in guaranteeing security as expressed by the large number of constraints in (5). Consequently in looking for lower bounds, we would like to reduce the number of these security constraints to some small set which is critical in determining an optimal solution. Because of the concavity of the function $I$ the constraints of (5) with $T$ containing one (or only a few) elements are in a sense 'tighter' than those for which $T$ contains a larger number of elements. Thus we seek constraints based on small $T$. The following theorems allow appropriate derived constraints to be formed for $T$ containing more than a single element.

**Theorem 1.** If $|T| > 1$, then

$$\frac{1}{2} \sum_{i,j \in T} z_{ij} + \sum_{i \in T} z_{i0} \geq \bar{f}(T) = \frac{\sum_{k} f(T - \{k\}) + f(T)}{|T|}$$

(10)

where $z_{ij}$ is written in place of $x_{ij} + y_{ij}$ and $t(T)$ in place of $l(\omega(T))$ (the total number of incoming lines required by the set $T$). Also $\bar{f}(T) = t(T)$, by definition, for singleton sets $T$. ($\lceil \xi \rceil$ denotes the 'smallest integer not less than $\xi$.)

**Proof.** See appendix.

**Corollary.** If $|T| > 1$, then

$$\frac{1}{2} \sum_{i,j \in T} x_{ij} + \sum_{i \in T} x_{i0} \geq \bar{f}(T) = \frac{\sum_{k} f(T - \{k\}) + f(T)}{|T|}$$
where \( f(T) = \lceil \frac{1}{2} t(T) \rceil \) is the number of 'first' lines (i.e. first-laid with respect to pairs \( i, j \) of substations) required by set \( T \) and \( \overline{f}(T) = f(T) \), by definition, for singleton sets \( T \).

**Example 1.** Consider the network with four load stations 1, 2, 3, 4 whose requirements are

\[
\omega(\{1\}) = 0.9, \quad \omega(\{2\}) = 1.3, \quad \omega(\{3\}) = 1.5, \quad \omega(\{4\}) = 1.2
\]

and suppose the function \( I \) is specified by Table 1. (Note that this function is not quite concave in the sense used earlier. Meckiff et al. [5] using statistical principles derived a security function which is truly concave. However, for purposes of comparison we have preferred to use the form of Table 1 which has been much used [1,2,4,7]. Note also that, in general, the security of larger sets \( T \) must be tested as the security of subsets of \( T \) may be satisfied only with the aid of inter-load substation lines.)

<table>
<thead>
<tr>
<th>No. of lines</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Requirement met</td>
<td>1.0</td>
<td>1.5</td>
<td>2.5</td>
<td>3.0</td>
<td>3.5</td>
<td>4.5</td>
<td>5.0</td>
<td>6.0</td>
<td>7.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No. of lines</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Requirement met</td>
<td>7.5</td>
<td>8.5</td>
<td>9.5</td>
<td>10.0</td>
<td>11.0</td>
<td>12.0</td>
<td>13.0</td>
<td>14.0</td>
<td>15.0</td>
<td>16.0</td>
</tr>
</tbody>
</table>

**Solution.** Clearly, from Table 1,

\[
\begin{align*}
t(\{1\}) &= 2, \quad t(\{2\}) = 3, \quad t(\{3\}) = 3, \quad t(\{4\}) = 3, \\
t(\{1, 2\}) &= I(\omega(\{1, 2\})) = I(\omega(\{1\}) + \omega(\{2\})) = I(2.2) = 4, \\
t(\{1, 3\}) &= 4, \quad t(\{2, 3\}) = 5, \quad t(\{1, 2, 3\}) = 7, \text{ etc.}
\end{align*}
\]

Then

\[
\begin{align*}
\overline{t}(\{1, 2\}) &= \lceil \frac{1}{2}(\overline{t}(\{1\}) + \overline{t}(\{2\}) + t(\{1, 2\})) \rceil = \lceil \frac{1}{2}(2 + 3 + 4) \rceil = 5, \\
\overline{t}(\{1, 3\}) &= 5, \quad \overline{t}(\{2, 3\}) = 6, \\
\overline{t}(\{1, 2, 3\}) &= \lceil \frac{1}{2}(\overline{t}(\{1, 2\}) + \overline{t}(\{1, 3\}) + \overline{t}(\{2, 3\}) + t(\{1, 2, 3\})) \rceil \\
&= \lceil \frac{1}{2}(5 + 5 + 6 + 7) \rceil = 8.
\end{align*}
\]

Further results are given in Table 2.

The last three columns of Table 2 require explanation. The notation \([2, 4]\) is used to indicate that substations 2 and 4 are to be regarded as forming a single composite substation and in this context a line from 2 to 4 is irrelevant.

It may be observed that, for the tabulated values, \( t(T) \leq \overline{t}(T) \leq \sum_{i \in T} t(\{i\}) \); the following theorem shows this to be true generally.

**Theorem 2.** Suppose \( t \) is an increasing function such that \( t(A \cup B) \leq t(A) + t(B) \) for
Table 2

<table>
<thead>
<tr>
<th>$T$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${4}$</th>
<th>${1, 2}$</th>
<th>${1, 3}$</th>
<th>${1, 4}$</th>
<th>${2, 3}$</th>
<th>${2, 4}$</th>
<th>${3, 4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega(T)$</td>
<td>0.9</td>
<td>1.3</td>
<td>1.5</td>
<td>1.2</td>
<td>2.2</td>
<td>2.4</td>
<td>2.1</td>
<td>2.8</td>
<td>2.5</td>
<td>2.7</td>
</tr>
<tr>
<td>$t(T)$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$\tilde{t}(T)$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$f(T)$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$\tilde{f}(T)$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

$A$ and $B$ disjoint. Then, if $|T| > 1$

1. \[ t(T) \leq \frac{1}{(|T| - 1)} \left( \sum_{i \in T} t(T - \{i\}) \right), \]

2. \[ t(T) \leq \tilde{t}(T) \leq \sum_{i \in T} t(\{i\}). \]

**Proof.** A proof by induction is fairly straightforward and will not be given.

**Corollary.** Analogous results hold for the function $f$.

3. **Lower bounds for ESP**

ESP' of (4)–(8) has now been relaxed to

$$\text{ESP'} \quad \text{minimise} \quad \varphi(x, y) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} (x_{ij} + \frac{1}{2}y_{ij}),$$

subject to $x_{ij} \geq y_{ij}$, $V_{i,j} (g_{ij})$, $x_{ij} = x_{ji}$, $V_{i,j} (\theta_{ij}^{g})$, $y_{ij} = y_{ji}$, $V_{i,j} (\theta_{ij}^{g})$, $x_{ij}, y_{ij} \in \{0, 1\}$, $V_{i,j}$ (11)

and a selection of constraints of the form

$$\sum_{i \in T} \sum_{j \notin T} (x_{ij} + y_{ij}) \geq t(T), \quad (\lambda_T)$$

$$\sum_{i \in T} x_{ij} \geq f(T), \quad (\mu_T)$$

$$\frac{1}{2} \sum_{i,j \in T} (x_{ij} + y_{ij}) + \sum_{i \in T} (x_{il} + y_{il}) \geq \tilde{t}(T), \quad (\tilde{\lambda}_T)$$
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(Note that some of the constraints may refer to sets containing composite substations obtained by 'coalescing' several load substations. Note also that $x_{ii}$ and $y_{ii}$ are to be taken as zero in the summations.)

ESP is now further relaxed to $ESP(\varrho, \ldots, \lambda, \mu, \bar{\lambda}, \bar{\mu})$ by incorporating all constraints, other than (11), into the objective with multipliers indicated above.

$$ESP(\varrho, \ldots, \lambda, \mu, \bar{\lambda}, \bar{\mu})$$

$$\min \{ t(T)\bar{\lambda}_T + f(T)\mu_T + \bar{r}(T)\bar{\lambda}_T + \bar{f}(T)\bar{\mu}_T \}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} (A_{ij}x_{ij} + B_{ij}y_{ij}),$$

subject to $x_{ij}, y_{ij} \in \{0, 1\}$

Now much use of the constraint $x_{ij} \geq y_{ij}$ has been made through introduction of the constraints involving $f(T)$ and $\bar{f}(T)$, and we will set $\varrho = 0$. At any stage $\theta^{\lambda}$ and $\theta^{\mu}$ will be assumed chosen so that

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji} \quad \text{if } 1 \leq i, j \leq n,$$

$$A_{ij} = B_{ij} = 0 \quad \text{if } i > n, \quad j \leq n.$$

We choose $\bar{\mu} = \frac{1}{2}\bar{\lambda}$ corresponding to the different coefficients of the $d_{ij}$ terms in $A_{ij}$ and $B_{ij}$. Finally, since composite load substations are allowed the $\lambda_T$ and $\mu_T$ constraints are redundant and so we set $\lambda = \mu = 0$.)

With these simplifications the solution to $ESP(\varrho, \ldots)$ becomes

$$\Phi(\bar{\lambda}) = \sum_{T} \bar{h}(T)\bar{\lambda}_T + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \min(0, B_{ij})$$

where $\bar{h}(T)$ has been written in place of $\bar{r}(T) + \frac{1}{2}\bar{f}(T)$.

As usual we would like to find an optimal or near optimal solution to

$$\max \Phi(\bar{\lambda}),$$

subject to $\bar{\lambda} \geq 0$

in order to obtain a lower bound to $ESP^\varrho$.

**Example 2.** Obtain a good lower bound for $ESP^\varrho$ on the network, with four load substations 1, 2, 3 and 4 and two supply substations 5 and 6, whose distance matrix
is given by

$$
\frac{1}{4}(d_{ij}) = \begin{bmatrix}
14 & 11 & 17 & 9 & 18 \\
14 & 16 & 5 & 11 & 16 \\
11 & 16 & 16 & 18 & 9 \\
17 & 5 & 16 & 16 & 13 \\
9 & 11 & 18 & 16 & \cdot & \cdot \\
18 & 16 & 9 & 13 & \cdot & \cdot
\end{bmatrix}.
$$

**Solution.** We start with \( \bar{x} = 0 \), and with \( \theta^x \) and \( \theta^y \) chosen so as to clear the lower left submatrix. Then

$$
B = \begin{bmatrix}
14 & 11 & 17 & 18 & 36 \\
14 & 16 & 5 & 22 & 32 \\
11 & 16 & 36 & 18 \\
17 & 5 & 16 & 32 & 26 \\
\end{bmatrix},
$$

the last two rows being omitted as they are now superfluous. Set

$$
T \leftarrow \{4\}, \quad \bar{x}_T \leftarrow 4, \quad \Phi \leftarrow 0 + (4)4 = 16,
$$

$$
B \leftarrow \begin{bmatrix}
14 & 11 & 15 & 18 & 36 \\
14 & 16 & 3 & 22 & 32 \\
11 & 16 & 36 & 18 \\
15 & 3 & 14 & 28 & 22 \\
\end{bmatrix}.
$$

(Notice now the left hand submatrix is kept symmetric.)

$$
T \leftarrow \{1, 2, 3, 4\}, \quad \bar{x}_T \leftarrow 6, \quad \Phi \leftarrow 16 + (13)6 = 94,
$$

$$
B \leftarrow \begin{bmatrix}
11 & 8 & 12 & 12 & 30 \\
11 & 13 & 0 & 16 & 26 \\
8 & 13 & 11 & 30 & 12 \\
12 & 0 & 11 & 22 & 16 \\
\end{bmatrix}.
$$

$$
T \leftarrow \{2, 4\}, \quad \bar{x}_T \leftarrow 16, \quad \Phi \leftarrow 94 + (11\frac{1}{2})16 - \frac{5}{2}(-4 - 4) = 258,
$$

$$
B \leftarrow \begin{bmatrix}
3 & 0 & 4 & -4 & 14 \\
3 & 5 & 0 & 0 & 10 \\
0 & 5 & 3 & 14 & -4 \\
4 & 0 & 3 & 6 & 0 \\
\end{bmatrix}.
$$

(Note that the negative term arises since \( B_{15} = B_{36} = -4 \).)

It is readily checked that the solution in which for \( i < j \) all variables are zero except

$$
x_{15}, y_{15}; \quad x_{25}, y_{25}; \quad x_{36}, y_{36}; \quad x_{46}, y_{46}; \quad x_{13}; \quad x_{24},
$$

is secure. (This solution is formed by allowing \( x_{ij} (y_{ij}) \) to be nonzero only if \( B_{ij} \leq 0 \).)
Since the value of this solution is
\[ \frac{3}{2}(d_{15} + d_{25} + d_{36} + d_{46}) + (d_{13} + d_{24}) = 258 \]
equal to the lower bound above, it must be optimal.

The solution, and its relationship to the bounding process are illustrated in Fig. 2.

Fig. 2. The requirement \( \omega_i \) is shown beside each load substation \( i \). Contributions made by dual variables \( \lambda_T \) for \( T \) equal to \{4\}, \{1,2,3,4\} and \{2,4\}, 1,3\} are shown by straight, wavy and broken lines respectively. The double lines correspond to the fact that load substation — supply substation distance is 'used up at twice the rate' (see choice of variables \( \theta_i, \theta^\prime \)). The protuding lines from 5 and 6 correspond to the 'correction terms' \( \min(0, B_{ij}) \) in the expression for \( \Phi(\lambda) \).

4. A larger example

Fig. 3 represents a secure network for the problem for which the distance matrix \( (d_{ij}) \) is given in Table 3, the load requirements are as given in Table 4 and the function \( l \) is as specified in Table 1. This problem which is taken from Green and Boardman [4] is a modified version of one given by Burstall [2]. The solution given, of value 242 \( \frac{1}{2} \), is the one obtained by the method of Green and Boardman — it is in fact optimal as will be shown below.

A bound \( \Phi(\lambda) \) was found by hand calculation using sets \( T \) containing up to 4 load substations (though the possibility of some of these being composite substations was permitted.) Values for \( \lambda_T \) were determined by 'informed judgement' and so it is likely that the maximum value of \( \Phi(\lambda) \) would not be found even for the restricted range of \( T \) considered. In fact the best bound obtained using the above approach was some way short of 242 \( \frac{1}{2} \) and so it was decided to use Branch-and-Bound. Branching is on a selected pair of load substations \( i \) and \( j \) according as there are none, one or two lines between \( i \) and \( j \). In each case \( d_{ij} \), and hence \( B_{ij} \) is set to infinity. For exclusion nothing more is necessary. For inclusion of one (two) lines, \( d_{ij} (\frac{1}{2}d_{ij}) \) is added to the objective and \( r(T), \tilde{r}(T) \) and \( \tilde{f}(T) \) all reduced by 1 (2) for each set \( T \) which contains one but not both of \( i \) and \( j \).
The first branching (Fig. 4) was on $L_4L_6$. ($L_4$ and $L_6$ are relatively close but $L_4L_6$ seems unlikely to appear in an optimal solution; without restriction $B_{46}$ is likely to become negative.) It was found that two lines between $L_4$ and $L_5$ could not appear in an optimal solution but otherwise the situation was unresolved. Further branching, on $L_1L_3$ did resolve the situation and it was found that the solution of Fig. 3 is optimal.
Fig. 4. Branch-and-Bound search tree for the 15-substation problem.

Table 5
Bound when $L_4 L_6$ and $L_1 L_3$ are excluded

<table>
<thead>
<tr>
<th>Set $T$</th>
<th>Amount $h(T)$ (× 3)</th>
<th>Rate $h(T)$</th>
<th>Return (× 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{9}</td>
<td>4 (× 3)</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>{5, 7, 8, 9}</td>
<td>6</td>
<td>13</td>
<td>78</td>
</tr>
<tr>
<td>{6, 7, 9}</td>
<td>10</td>
<td>15.5</td>
<td>155</td>
</tr>
<tr>
<td>{1, 2, 5, 8}</td>
<td>6</td>
<td>13</td>
<td>78</td>
</tr>
<tr>
<td>{1, 2}</td>
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</tr>
<tr>
<td>{1, 2, {3, 4}}</td>
<td>2</td>
<td>11.5</td>
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</tr>
<tr>
<td>{1, 2, {3, 4}}</td>
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<td>10</td>
<td>120</td>
</tr>
<tr>
<td>{1, 2, 6}</td>
<td>2</td>
<td>10.5</td>
<td>21</td>
</tr>
<tr>
<td>{4}</td>
<td>26</td>
<td>4</td>
<td>104</td>
</tr>
<tr>
<td>{6}</td>
<td>14</td>
<td>4</td>
<td>56</td>
</tr>
<tr>
<td>{6, {4, 7, 9}}</td>
<td>6</td>
<td>13</td>
<td>78</td>
</tr>
</tbody>
</table>

Sum of negative elements of $B = -32$.

Bound $= \frac{1}{3}(807 - \frac{3}{2}(-32)) = 242\frac{1}{3}$.

(Note: The factors of 3 are introduced merely to avoid fractions.)

Values of the dual variables giving a lower bound of $242\frac{1}{3}$ (the optimal value) given for the case when $L_4 L_6$ and $L_1 L_3$ are excluded is given in Table 5.

5. Discussion

Problems of the size of the one in figure 3 have been solved heuristically in a few seconds computing time on an ICL 1906S computer using the Green and Boardman method [4]. However, as observed earlier, exact methods have previously only been successful for smaller problems. In Sections 2 and 3 a (lower) bounding scheme was
suggested and Section 4 indicates how this might be incorporated into a viable Branch and Bound scheme providing an exact method.

There are of course several points which need to be considered further in order to provide a practicable method. Firstly how are the dual variables \( \lambda_T \) to be selected? An attractive possibility is to approximate \( \max \Phi(\lambda) \) by using subgradient optimization provided a relatively small class of appropriate sets \( T \) has been identified. This class could be predetermined or built up by a ‘column generation’ scheme. Another approach would be to allow the computer to perform optimizations and the problem-solver interactively to provide sets \( T \) making use of geometric intuition. If Branch-and-Bound is being employed then there is the question of how suitable branching variables are to be chosen. Finally, there is the combinatorial problem of testing for security. We have given this little attention above, but it may be observed that with careful programming this should not cause difficulty for problems of a size encountered in practice.

These points will be the subject of further investigation.

Acknowledgements

The authors wish to express their gratitude for several valuable comments made by one of the referees.

Appendix

Proof of Theorem 1. First consider \( T = \{a, b\} \) consisting of two elements. Then the following security constraints must hold (where \( z_{ij} \) is written in place of \( x_{ij} + y_{ij} \)):

\[
\begin{align*}
  z_{ab} + \sum_{j \neq b} z_{aj} & \geq t(\{a\}), \\
  z_{ba} + \sum_{j \neq a} z_{bj} & \geq t(\{b\}), \\
  \sum_{j \neq a} z_{aj} + \sum_{j \neq b} z_{bj} & \geq t(\{a, b\}).
\end{align*}
\]

These may be added to obtain

\[
(z_{ab} + z_{ba}) + 2 \left( \sum_{j \neq b} z_{aj} + \sum_{j \neq a} z_{bj} \right) \geq t(\{a\}) + t(\{b\}) + t(\{a, b\}).
\]

Noting that \( z_{ab} = z_{ba} \) and that all variables are integers allows us to divide by 2 to get

\[
z_{ab} + \left( \sum_{j \neq b} z_{aj} + \sum_{j \neq a} z_{bj} \right) \geq \frac{1}{2}(t(\{a\}) + t(\{b\}) + t(\{a, b\}))
\]

(12)

which is just the required result with \( T = \{a, b\} \) since \( t(\{i\}) = f(\{i\}) \). Suppose now
that \(|T| = p\), and that the desired result holds if \(|T| = p - 1 \geq 2\), then for \(T - \{a\}\), each \(a \in T\), we have

\[
\frac{1}{2} \sum_{i,j \in T - \{a\}} z_{ij} + \sum_{i \in T - \{a\}} z_{iia} \geq \bar{t}(T - \{a\}),
\]

that is

\[
\frac{1}{2} \sum_{i,j \in T - \{a\}} z_{ij} + \sum_{i \in T - \{a\}} z_{iia} + \sum_{a \in T} z_{iit} \geq \bar{t}(T - \{a\}).
\]

(13)

As \(a\) varies over \(T\) any element \(\frac{1}{2} z_{ij}, i, j \in T\), will appear in the first term of (13) in \((p - 2)\) cases (when \(a \neq i, j\)), and \(z_{ij}\) will appear just once in the second term of (13). Thus the sum over \(a\) of the first two terms of (13) is

\[
\sum_{a \in T} \left( \frac{1}{2} \sum_{i,j \in T - \{a\}} z_{ij} + \sum_{i \in T - \{a\}} z_{iia} \right) = \frac{1}{2} p \sum_{i,j \in T} z_{ij}.
\]

(14)

Summing the third terms of (13) gives

\[
\sum_{a \in T} \sum_{i \in T - \{a\}} z_{iia} = (p - 1) \sum_{i \in T} z_{iia}.
\]

(15)

Also the security of \(T\) implies

\[
\sum_{i \in T} z_{iia} \geq t(T).
\]

(16)

Adding (16) to the sum of inequalities (13) and using (14) and (15) yields

\[
p \left( \frac{1}{2} \sum_{i,j \in T} z_{ij} + \sum_{i \in T} z_{iia} \right) \geq \sum_{a \in T} \bar{t}(T - \{a\}) + t(T)
\]

(17)

Since \(z_{ij} = z_{ji}\) dividing (17) by \(p\) leads to the desired result.

**Proof of the Corollary.** Since \(x_{ij} \geq y_{ij}\) it follows that

\[
\sum_{j \neq i} x_{ij} \geq \sum_{j \neq i} y_{ij} \quad \text{and} \quad \sum_{j \neq i} y_{ij} \geq \left\lceil \frac{1}{2} \sum_{j \neq i} y_{ij} \right\rceil
\]

where the summations are over the same, but arbitrary, index set. The desired result follows by analogy with the proof of the main theorem.

**References**
