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AN EDGE-COLORATION THEOREM FOR BIPARTITE GRAPHS WITH APPLICATIONS

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An edge-coloration theorem for bipartite graphs, announced in [4], is proved from which some well-known theorems due to König [5] and the author [2, 3] are deduced. The theorem is further applied to prove the "dual" of a theorem due to Lovász [6].

1. Bipartite graphs

All graphs considered below are non-null, finite and have no loops. Multiple edges are permitted.

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. A chain in G is a sequence

$$\mu = [x_0, \lambda_1, x_1, \lambda_2, \ldots, x_{r-1}, \lambda_r, x_r] \tag{1.1}$$

where (i) $x_0, x_1, \ldots, x_r \in V(G)$, (ii) $\lambda_1, \lambda_2, \ldots, \lambda_r \in E(G)$, and (iii) λ_i joins x_{i-1} and x_i, $1 \leq i \leq r$. If $x_r = x_0$, then μ is called a cycle in G of length r. A graph is *bipartite* if it has no cycle of odd length.

Let G be any graph. Let k be any non-negative integer and let $C_k = \{\alpha, \beta, \ldots\}$ denote a set of k distinct elements called "colors". Any mapping

$$\sigma : E(G) \to C_k \tag{1.2}$$

is called a *k-coloration* of G. If $\lambda \in E(G)$ and $\sigma(\lambda) = \alpha$, then λ is called an α-edge.

Let σ be any k-coloration of G. For $x \in V(G)$, let $\nu(x, \sigma)$ denote the number of distinct colors α such that there is at least one α-edge incident with x. Obviously, $\nu(x, \sigma) \leq k$. Also, $\nu(x, \sigma) \leq d_G(x)$ where $d_G(x)$, called the degree of x, is the number of edges incident with x in G. Hence, we have

$$\nu(x, \sigma) \leq \min\{k, d_G(x)\} \quad \text{for all} \quad x \in V(G). \tag{1.3}$$

We shall prove the following

Theorem 1.1. *If G is a bipartite graph, then, for every non-negative integer k, there exists a k-coloration σ of G such that*

$$\nu(x, \sigma) = \min\{k, d_G(x)\} \quad \textit{for all} \quad x \in V(G). \tag{1.4}$$

Proof. If $k=0$ or 1, then any k-coloration σ of G satisfies (1.4). It is, therefore, enough to consider $k \geq 2$. Let σ be a k-coloration of G such that $\sum_{x \in V(G)} \nu(x, \sigma)$ is largest possible. Since G is finite, such a σ exists. We shall prove that σ satisfies (1.4).

Let, if possible, σ not satisfy (1.4). Then, there is a vertex $x_0 \in V(G)$ for which $\nu(x_0, \sigma) < \min\{k, d_G(x_0)\}$. Since $\nu(x_0, \sigma) < d_G(x_0)$, there is a color α such that there are at least two α-edges incident with x_0; also, since $\nu(x_0, \sigma) < k$, there is a color β such that there is no β-edge incident with x_0. Choose α, β as above and let

$$\mu = [x_0, \lambda_1, x_1, \lambda_2, \ldots, x_{r-1}, \lambda_r, x_r], \quad r \geq 1,$$

be an (α, β)-alternating chain where $x_0, x_1, \ldots, x_{r-1}$ are distinct vertices, λ_1, $\lambda_3, \ldots$ are α-edges, $\lambda_2, \lambda_4, \ldots$ are β-edges and which satisfies at least one of the following two conditions:

(i) λ_r is an α-edge [resp. β-edge] and there is no β-edge [resp. α-edge] incident with x_r;

(ii) λ_r is an α-edge [resp. β-edge] and there is another α-edge [resp. β-edge] incident with x_r.

Since G is finite, such a chain μ can always be found. Now, interchange colors α and β on all edges belonging to μ, leaving the colors of the rest of the edges unchanged and let ρ be the k-coloration of G so obtained. Since the chain μ satisfies (i) or (ii), it is easily seen that $\nu(x, \rho) \geq \nu(x, \sigma)$ for all vertices x except possibly when $x = x_0$. We now observe that x_r cannot coincide with x_0. In fact if $x_r = x_0$, then, since there was no β-edge incident with x_0 with respect to σ, μ would be a cycle of odd length in G contradicting the assumption that G is bipartite. Hence, since there were two α-edges incident with x_0 with respect to σ, $\nu(x_0, \rho) > \nu(x_0, \sigma)$. But, then $\sum_{x \in V(G)} \nu(x, \sigma) < \sum_{x \in V(G)} \nu(x, \rho)$ which is contradictory to the choice of σ. Hence, σ must satisfy (1.4) and the theorem is proved.

Let G be any graph and k be any non-negative integer. A k-coloration σ of G may be called "good" if $\nu(x, \sigma) = \min(k, d_G(x))$ for all $x \in V(G)$. Theorem 1.1, then, states that a bipartite graph always has good k-colorations for all $k \geq 0$.

The above theorem had been discovered by the author several years ago but was first announced in [4].

In the following sections, we apply Theorem 1.1 to derive some well-known results in the theory of graphs.

2. Theorems of König and Gupta

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. Let $F \subseteq E(G)$ F is called a *matching* [resp. *cover*] if for all $x \in V(G)$, F contains at most [resp. at least] one edge incident with x. The *chromatic index* of G, denoted $\chi_1(G)$, is the smallest number k such that the edge-set $E(G)$ can be partitioned into k

matchings. The *cover index* $\kappa(G)$ of G is the largest number k such that $E(G)$ can be partitioned into k covers. If $\Delta(G)=\max_{x\in V(G)} d_G(x)$ and $\delta(G)=\min_{x\in V(G)} d_G(x)$ are the maximum and minimum degrees in G respectively, then, clearly

$$\chi_1(G)\geq\Delta(G), \tag{2.1}$$

and

$$\kappa(G)\leq\delta(G). \tag{2.2}$$

Now, let G be a bipartite graph. Let $k=\Delta(G)$. By Theorem 1.1, there exists a k-coloration $\sigma: E(G)\to\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ of G such that $\nu(x,\sigma)=\min\{k, d_G(x)\}=d_G(x)$ for all $x\in V(G)$. Let E_i be the set of all α_i-edges, $1\leq i\leq k$. Then, each E_i must be a matching so that $E_1, E_2,\ldots,E_k$ form a partition of $E(G)$ into $k=\Delta(G)$ matchings. Hence, $\chi_1(G)\leq\Delta(G)$. Since $\chi_1(G)\geq\Delta(G)$ always, we obtain the following theorem due to König [5].

Theorem 2.1. *For any bipartite graph G,*

$$\chi_1(G)=\Delta(G).$$

Just as above, by taking $k=\delta(G)$, from Theorem 1.1 we obtain the following theorem due to the author [2, 3].

Theorem 2.2. *For any bipartite graph G,*

$$\kappa(G)=\delta(G).$$

3. Digraphs

We consider below digraphs which are non-null and finite. Parallel arcs and loops are to be permitted.

Let $D=(X,A)$ be a digraph with vertex-set X and arc-set A. Let $F\subseteq A$. For any vertex $x\in X$, the out-degree $d_F^+(x)$ of x in F is the number of arcs in F with initial vertex x and the in-degree $d_F^-(x)$ is the number of arcs in F with terminal vertex x. F is called a *matching* of D if $\max\{d_F^+(x), d_F^-(x)\}\leq 1$ for all $x\in X$; F is called a *cover* of D if $\min\{d_F^+(x), d_F^-(x)\}\geq 1$ for all $x\in X$. The chromatic index $\chi_1(D)$ of D is the minimum number k such that A can be partitioned into k matchings; the cover index $\kappa(D)$ of D is the maximum number k such that A can be partitioned into k covers. Let $\Delta(D)=\max_{x\in X}\max\{d_A^+(x), d_A^-(x)\}$ and $\delta(D)=\min_{x\in X}\min\{d_A^+(x), d_A^-(x)\}$. Then, obviously,

$$\chi_1(D)\geq\Delta(D), \tag{3.1}$$

and

$$\kappa(D)\leq\delta(D). \tag{3.2}$$

In fact, we shall prove that equalities in (3.1) and (3.2) always hold. To this end, we associate to the digraph $D=(X, A)$ a graph G as follows: $V(G)=\{x', x'': x\in X\}$ and for each arc in A with initial vertex x and terminal vertex y, there is an edge joining x' and y'' in $E(G)$. It is easy to see that G is a bipartite graph with $\Delta(G)=\Delta(D)$ and $\delta(G)=\delta(D)$. Also, a set of edges F of G is a matching [resp. cover] of G if and only if the corresponding set of arcs is a matching [resp. cover] of D. Hence, from Theorems 2.1 and 2.2, we easily obtain the following.

Theorem 3.1. *For any digraph D,*

$$\chi_1(D)=\Delta(D). \tag{3.3}$$

Theorem 3.2. *For any digraph D,*

$$\kappa(D)=\delta(D). \tag{3.4}$$

A digraph $D=(X, A)$ is said to be regular of degree n if $d_A^+(x)=d_A^-(x)=n$ for all $x\in X$. The following theorem is an immediate consequence of Theorem 3.1 or Theorem 3.2.

Theorem 3.3. *If $D=(X, A)$ is a digraph which is regular of degree n, then A can be partitioned into n sets $A_1, A_2, \ldots, A_n$ such that each of the digraphs $D_i=(X, A_i)$, $1\leq i\leq n$, is regular of degree* 1.

It may be noted (see, e.g. [1, p. 230]) that the above theorem implies the well-known theorem due to Petersen [8].

Theorems 3.2 and 2.2 were motivated by a problem suggested by Ore [7, Problem 4, p. 210]. The results contained in this section and the previous section were announced in [2] and are included in [3].

4. A theorem of Lovász and its dual

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. A graph H is called a factor (spanning subgraph) of G if $V(H)=V(G)$ and $E(H)\subseteq E(G)$. If H and K are factors of G such that $E(H)\cap E(K)=\emptyset$, $E(H)\cup E(K)=E(G)$, then we write $G=H+K$ and call it a decomposition of G.

The following theorem is due to Lovász [6].

Theorem 4.1. *Let G be any graph with maximum degree $\Delta(G)$. For any non-negative integers h and k with $h+k=\Delta(G)+1$, there exists a decomposition $G=H+K$ such that $\Delta(H)\leq h$ and $\Delta(K)\leq k$.*

The original proof of Theorem 4.1 seems complicated. A simpler proof, using the König's Theorem 2.1, was shown to the author by Berge. We shall apply Theorem 1.1 to prove the following "dual" of Theorem 4.1.

Theorem 4.2. *Let G be any graph with minimum degree $\delta(G)$. For any non-negative integers h and k with $h+k=\delta(G)-1$, there exists a decomposition $G=H+K$ such that $\delta(H)\geqslant h$ and $\delta(K)\geqslant k$.*

Proof. The theorem is proved by induction on h. If $h=0$, then, we may define H and K so that $E(H)=\emptyset$, $E(K)=E(G)$ and the theorem is true. Assume, as induction hypothesis, that the theorem holds for some integer $h=l$, $0\leqslant l<\delta(G)-1$. We shall prove the theorem for $h=l+1$.

Now, by hypothesis, there exists a decomposition $G=H+K$ such that $\delta(H)\geqslant l$, $\delta(K)\geqslant\delta(G)-l-1$. We choose the decomposition $G=H+K$ such that K has the smallest number of edges possible.

Let $S=\{x:d_H(x)=l\}$ and $\bar{S}=V(G)-S$. If $S=\emptyset$, then the present decomposition $G=H+K$ meets our requirement with $h=l+1$. Let $S\neq\emptyset$.

Now, define a graph B as follows: $V(B)=V(K)=V(G)$, $E(B)$ consists of precisely those edges of K which have exactly one endpoint in S and the other in $\bar{S}$. Clearly, B is a bipartite graph which is a factor of K. Now, we observe that, by our choice of K, S must be independent in K so that for all $x\in S$,

$$d_B(x)=d_K(x)=d_G(x)-d_H(x)\geqslant\delta(G)-l.$$

By Theorem 1.1, there exists a $(\delta(G)-l)$-coloration σ of B such that $\nu(x,\sigma)=\min\{\delta(G)-l, d_B(x)\}$ for all $x\in V(B)=V(G)$. Let E_1 be the set of edges of B (and hence of K) of one of the colors with respect to σ. Define factors H_1 and K_1 of G as follows: $E(H_1)=E(H)\cup E_1$, $E(K_1)=E(K)-E_1$. It is now easy to verify that the decomposition $G=H_1+K_1$ meets our requirement with $h=l+1$.

Hence, the theorem is proved.

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