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Polygonal Approximations of Solutions of the Initial Value Problem for a Conservation Law*

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1. INTRODUCTION

In this article we introduce a new method for the study of the initial value problem

$$u_t + f(u)_x = 0, \qquad (x, t) \in (-\infty, \infty) \times [0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in (-\infty, \infty), \tag{1.2}$$

where $f(u)$ is locally Lipschitz continuous and $u_0(x)$ is bounded and of locally bounded variation on $(-\infty, \infty)$.

In general no classical solution of (1.1) and (1.2) exists even if f and u_0 are smooth. It has been demonstrated that it is possible to establish the existence of weak solutions by the methods of vanishing viscosity [1] finite differences [2] and smoothing [3].[1]

The weak solution is not necessarily unique. To attain uniqueness one usually imposes additional restrictions which are motivated by stability arguments or by physical considerations (whenever (1.1) is studied in connection with a physical model).

Here we establish the existence of a solution which satisfies the condition proposed by Hopf [5].[2] Such a solution is constructed first in the special case where $u_0(x)$ is a step function and $f(u)$ is piecewise linear. For $u_0(x)$ a step function, a local solution can be constructed as a superposition of solutions of the Riemann problem. In general, the solution of the Riemann

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[1] For recent more general results see [7, 8].

[2] The same condition is satisfied by the solution constructed using the method of vanishing viscosity.

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problem for (1.1) consists of constant states separated by shocks and/or simple centered waves. If, however, $f(u)$ is piecewise linear, then the constant states of the solution of the Riemann problem are separated exclusively by shocks (some of which may be contact discontinuities). This observation provided the motivation for developing the method which is presented in the present paper. Since simple waves are eliminated, the only possible interactions involve shock waves and lead to new Riemann problems. We show that by a solution of Riemann problems, the local solution of (1.1) and sucessive (1.2) for $u_0(x)$ a step function and piecewise linear $f(u)$ can be extended onto a global solution.

In the general case, we approximate $f(u)$ by a sequence of piecewise linear functions and $u_0(x)$ by a sequence of step functions. We then establish existence through a compactness argument suggested by Oleinik [2].

It is conceivable that a similar approach may be proved fruitful in the study of the initial value problem for systems of conservation laws. Certain results in this direction for the second-order wave equation

$$u_{tt} = \sigma(u_x)_x \tag{1.3}$$

have been obtained by L. Leibovich (Ph.D. thesis, Cornell University, 1971).

2. Admissible Solutions

Following Hopf [5], we state the following:

Definition 2.1. A locally bounded and measurable function $u(x, t)$ on $(-\infty, \infty) \times [0, \infty)$ is called an *admissible weak solution* of (1.1) and (1.2), if for any nondecreasing function $h(u)$ and any smooth nonnegative function $\phi(x, t)$ with compact support in $(-\infty, \infty) \times [0, \infty)$,

$$\int_0^\infty \int_{-\infty}^\infty [I(u)\,\phi_t + F(u)\,\phi_x]\,dx\,dt + \int_{-\infty}^\infty I(u_0)\,\phi(x, 0)\,dx \geqslant 0, \tag{2.1}$$

where

$$I(u) \equiv \int^u h(\xi)\,d\xi, \qquad F(u) \equiv \int^u h(\xi)\,df(\xi).^3 \tag{2.2}$$

Remark. An application of (2.1) for $h(u) \equiv 1$ and $h(u) \equiv -1$ yields

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t + f(u)\,\phi_x]\,dx\,dt + \int_{-\infty}^\infty u_0(x)\,\phi(x, 0)\,dx = 0, \tag{2.3}$$

[3] If $u(x, t)$ is defined on $(-\infty, \infty) \times [0, T)$ and (2.1) is satisfied for every nonnegative $\phi(x, t)$ with compact support in $(-\infty, \infty) \times [0, T)$, then $u(x, t)$ is called a local admissible weak solution of (1.1), (1.2) on $(-\infty, \infty) \times [0, T)$.

which is the standard condition satisfied by any weak solution of (1.1) and (1.2).

In the case of piecewise constant solutions with smooth shocks, (2.1) can be reduced to a local form:

PROPOSITION 2.1. *A piecewise constant function $u(x, t)$ with smooth line discontinuities which satisfies* (1.2) *is an admissible weak solution of* (1.1) *and* (1.2) *if and only if the following condition is satisfied: Suppose that $x = \bar{x}(t)$, $t \in (a, b)$, is any line of discontinuity of $u(x, t)$ and let*

$$u^- \equiv \lim_{x \to \bar{x}(t)^-} u(x, t), \qquad u^+ \equiv \lim_{x \to \bar{x}(t)^+} u(x, t).$$

Then,

(i) *the curve $\bar{x}(t)$ is a straight line with slope*

$$\frac{d\bar{x}}{dt} = \frac{f(u^+) - f(u^-)}{u^+ - u^-}; \tag{2.4}$$

(ii) *for any u between u^- and u^+,*

$$\frac{f(u^+) - f(u)}{u^+ - u} \leqslant \frac{f(u^+) - f(u^-)}{u^+ - u^-}. \tag{2.5}$$

The argument of Hopf [5] shows that, in the case considered here, weak solutions satisfy (i) and (ii); the proof of the converse is not difficult and is omitted.

Remark. Equation (2.4) is the classical Rankine–Hugoniot jump condition while (2.5) is the E-condition proposed by Oleinik [6]. If (1.1) is genuinely nonlinear, (2.5) reduces to $u^+ < u^-$ if f is strictly convex, $u^- < u^+$ if f is strictly concave (compare with Lax [4]).

3. EXISTENCE OF ADMISSIBLE SOLUTIONS

The main result of this work is contained in the following:

PROPOSITION 3.1. *Assume that $u_0(\cdot)$ is continuous on the left, of locally bounded variation on $(-\infty, \infty)$ and*

$$m \leqslant u_0(x) \leqslant M, \qquad x \in (-\infty, \infty). \tag{3.1}$$

Furthermore, let $f(u)$ be locally Lipschitz continuous and

$$|f(u) - f(u')| \leqslant K\,|u - u'|, \qquad \textit{for all } u, u' \in [m, M]. \tag{3.2}$$

Then there exists an admissible weak solution $u(x, t)$ of (1.1) *and* (1.2) *which has the following properties: For every fixed $t \in [0, \infty)$, the function $u(\cdot, t)$ is continuous on the left, bounded from above by M, from below by m, and is of locally bounded variation on $(-\infty, \infty)$. Moreover, the restriction of $u(\cdot, t)$ on any interval $[x_1, x_2]$ is solely determined by the restriction of $u_0(\cdot)$ on the interval $[x_1 - Kt, x_2 + Kt]$ (finite domain of dependence) and*

$$\operatorname*{Var}_{[x_1,x_2]} u(\cdot, t) \leqslant \operatorname*{Var}_{[x_1-Kt,x_2+Kt]} u_0(\cdot). \tag{3.3}$$

The proof will be given first in two special cases:

LEMMA 3.1 (The Riemann problem for the polygonal approximation). *Assume that f is piecewise linear, satisfies* (3.2), *and*

$$\begin{aligned} u_0(x) &\equiv u_l, && \text{for } -\infty < x \leqslant 0, \\ &\equiv u_r, && \text{for } 0 < x < \infty, \end{aligned} \tag{3.4}$$

where u_l and u_r are constants in $[m, M]$. Then there exists an admissible weak solution of (1.1) *and* (1.2) *which consists of a finite number of constant states separated by shocks centered at the origin.*

Proof. Suppose first that $u_l < u_r$. The boundary of the convex hull of the set $\{(u, v) \mid u_l \leqslant u \leqslant u_r, v \geqslant f(u)\}$ is a polygonal line with vertices at points $(u_l, f(u_l))$, $(u^1, f(u^1)), \ldots, (u^k, f(u^k))$, $(u_r, f(u_r))$, $u_l < u^1 < \cdots < u^k < u_r$, *where $(u^1, f(u^1)), \ldots, (u^k, f(u^k))$ are also vertices of the graph of f. Note that*

$$\begin{aligned} -K \leqslant \frac{f(u^1) - f(u_l)}{u^1 - u_l} < \frac{f(u^2) - f(u^1)}{u^2 - u^1} < \cdots < \frac{f(u^k) - f(u^{k-1})}{u^k - u^{k-1}} \\ < \frac{f(u_r) - f(u^k)}{u_r - u^k} \leqslant K. \end{aligned} \tag{3.5}$$

We set

$$\begin{aligned} u(x, t) &\equiv u_l, && \text{for } -\infty < \frac{x}{t} \leqslant \frac{f(u^1) - f(u_l)}{u^1 - u_l}, \\ &\equiv u^1, && \text{for } \frac{f(u^1) - f(u_l)}{u^1 - u_l} < \frac{x}{t} \leqslant \frac{f(u^2) - f(u^1)}{u^2 - u^1}, \\ &\ \vdots \\ &\equiv u^k, && \text{for } \frac{f(u^k) - f(u^{k-1})}{u^k - u^{k-1}} < \frac{x}{t} \leqslant \frac{f(u_r) - f(u^k)}{u_r - u^k}, \\ &\equiv u_r, && \text{for } \frac{f(u_r) - f(u^k)}{u_r - u^k} < \frac{x}{t} < \infty. \end{aligned} \tag{3.6}$$

It is clear that $u(x, t)$ satisfies (2.4) and (2.5) and hence it is an admissible solution of (1.1) and (1.2).

In the case $u_l > u_r$, let $(u_r, f(u_r))$, $(u^k, f(u^k)),\ldots, (u^1, f(u^1))$, $(u_l, f(u_l))$, $u_r < u^k < \cdots < u^1 < u_l$, be the vertices of the boundary of the convex hull of the set $\{(u, v) \mid u_r \leqslant u \leqslant u_l, v \leqslant f(u)\}$. Then it is easy to verify that the function $u(x, t)$ given by (3.6) is again an admissible weak solution of (1.1) and (1.2). Q.E.D.

LEMMA 3.2. *The assertion of Proposition 3.1 is true iff* $f(u)$ *is piecewise linear, satisfies* (3.2), *and*

$$\begin{aligned} u_0(x) &\equiv v_1, && \text{for } -\infty < x \leqslant x_1, \\ &\equiv v_2, && \text{for } \quad x_1 < x \leqslant x_2, \\ &\;\vdots \\ &\equiv v_n, && \text{for } x_{n-1} < x < \infty, \end{aligned} \tag{3.7}$$

where $v_1, \ldots, v_n$ *are constants in* $[m, M]$.

Proof. Let $\{(u^1, f(u^1)),\ldots, (u^s, f(u^s))\}$ be the set of all vertices of the graph of f with ordinates in $[m, M]$. We set

$$J \equiv \{u^1,\ldots, u^s\} \cup \{v_1, \ldots, v_n\}.$$

We will say that a function $u(x, t)$ on $(-\infty, \infty) \times [0, T)$ is of class D_T if the following conditions are satisfied:

(i) $u(x, t)$ is an admissible local weak solution of (1.1) and (1.2) on $(-\infty, \infty) \times [0, T)$.

(ii) For any fixed $t \in [0, T)$, $u(\cdot, t)$ is a step function with values in J, continuous on the left and of bounded variation on $(-\infty, \infty)$. Moreover,

$$\operatorname*{Var}_{(-\infty,\infty)} u(\cdot, t) \leqslant \operatorname*{Var}_{(-\infty,\infty)} u_0(\cdot). \tag{3.8}$$

(iii) For any $t, t' \in [0, T)$,

$$\int_{-\infty}^{\infty} | u(x, t) - u(x, t')| \, dx \leqslant K \mid t - t' \mid \operatorname*{Var}_{(-\infty,\infty)} u_0(\cdot). \tag{3.9}$$

We will prove that there exists a function $u(x, t)$ of class D_∞. To this end it is sufficient to show that first for some $\tau > 0$ there is a function in D_τ and second that if $u(x, t)$ is of class D_T for some $T > 0$, then there exists $T' > T$ and an extension of $u(x, t)$ on $(-\infty, \infty) \times [0, T')$ which is of class $D_{T'}$.

Since $u_0(x)$ is a step function, by superimposing solutions of the Riemann problem (Lemma 3.1) one obtains an admissible local solution $u(x, t)$ of (1.1) and (1.2) on $(-\infty, \infty) \times [0, \tau)$ provided that τ is small enough so that no interactions occur. It is clear that (3.8) holds as an equality for any $t \in [0, \tau)$. Also (3.9) is satisfied since K is a bound of the slope of every shock of $u(x, t)$. Therefore, $u(x, t)$ is of class D_τ.

Suppose now that $u(x, t)$ is of class D_T. On account of (3.9) there exists a function in $L^1_{\text{loc}}(-\infty, \infty)$ which will be denoted by $u(x, T)$ such that

$$u(x, t) \xrightarrow{L^1_{\text{loc}}(-\infty,\infty)} u(x, T), \qquad t \to T, \tag{3.10}$$

and

$$\int_{-\infty}^{\infty} | u(x, T) - u(x, t)| \, dx \leqslant K \mid T - t \mid \operatorname*{Var}_{(-\infty,\infty)} u_0(\cdot), \qquad \text{for any } t \in [0, T]. \tag{3.11}$$

Furthermore, in virtue of (3.8) and by Helly's selection principle, $u(x, T)$ can be identified with a function of bounded variation, continuous on the left, in which case

$$\operatorname*{Var}_{(-\infty,\infty)} u(\cdot, T) \leqslant \operatorname*{Var}_{(-\infty,\infty)} u_0(\cdot), \tag{3.12}$$

$$u(x, t) \to u(x, T), \; t \to T, \; \text{essentially on } (-\infty, \infty). \tag{3.13}$$

In particular, (3.12) and (3.13) imply that $u(x, T)$ is piecewise constant with values in J and has a finite number of discontinuities. Hence, by superimposing solutions of the Riemann problem, it is possible to construct an admissible local solution $u(x, t)$ of (1.1) on $(-\infty, \infty) \times [T, T')$ which admits initial conditions $u(x, T)$. As before, (3.9) will be satisfied for any $t, t' \in [T, T')$. Moreover,

$$\operatorname*{Var}_{(-\infty,\infty)} u(x, t) = \operatorname*{Var}_{(-\infty,\infty)} u(x, T), \qquad \text{for all } t \in [T, T'). \tag{3.14}$$

Recalling (3.11) and (3.12), we conclude that the extended u is of class $D_{T'}$.

The existence of some solution of (1.1) and (1.2) in D_∞ has thus been established. Note that the method of construction guarantees that the solution has the finite domain of dependence property so that (3.3) follows from (3.8).

Q.E.D.

Remark. The proof of Lemma 3.2 suggests the following numerical method of construction of a solution of (1.1) and (1.2) for piecewise linear $f(u)$ and $u_0(x)$ a step function: By superimposing solutions of the Riemann problem we construct a local solution on $(-\infty, \infty) \times [0, T_1)$, T_1 being the time where the first shock interaction will occur. We then repeat the process using $u(x, T_1)$ as new initial conditions (which is again a step function) thus extending the solution onto $(-\infty, \infty) \times [0, T_2)$, where T_2 is the time the first new shock interaction will occur and so on. Unfortunately there is no guarantee that one can reach by this procedure every point $t \in [0, \infty)$ in a finite number of steps. Suppose however that $f(u)$ is convex or concave. In this case, it is seen easily that the interaction of two (or more) shocks always produces a single shock so that the global solution can be constructed in a finite number of steps.

Proof of Proposition 3.1. Assume first that $u_0(x)$ is of bounded variation on $(-\infty, \infty)$. Then there exists a sequence $\{u_0^{(n)}(x)\}$ of step functions of the form (3.7) such that

$$u_0^{(n)}(x) \to u_0(x), \qquad n \to \infty, \qquad \text{in } L^1_{\text{loc}}(-\infty, \infty), \tag{3.15}$$

$$\operatorname*{Var}_{(-\infty,\infty)} u_0^{(n)}(x) \leqslant \operatorname*{Var}_{(-\infty,\infty)} u_0(x), \qquad n = 1, 2, \ldots . \tag{3.16}$$

Consider also a sequence $\{f_n(u)\}$ of piecewise linear functions with vertices at the points $(u^0, f(u^0)), \ldots, (u^n, f(u^n))$, where $u^i = m + i(M - m)/n$, $i = 0, \ldots, n$. Note that on account of (3.2),

$$|f_n(u) - f_n(u')| \leqslant K\,|u - u'|, \qquad \text{for all } u, u' \in [m, M], \qquad n = 1, 2, \ldots, \tag{3.17}$$

$$|f(u) - f_n(u)| \leqslant K\frac{M - m}{2n}, \qquad \text{for all } u \in [m, M], \qquad n = 1, 2, \ldots . \tag{3.18}$$

Let $u_n(x, t)$ be the admissible weak solution of the initial value problem

$$u_t + f_n(u)_x = 0, \qquad (x, t) \in (-\infty, \infty) \times [0, \infty), \tag{3.19}$$

$$u(x, 0) = u_0^{(n)}(x), \qquad x \in (-\infty, \infty), \tag{3.20}$$

whose existence has been established by Lemma 3.2. For any fixed $t \in [0, \infty)$ and in virtue of (3.8) and (3.16), the sequence $\{u_n(\cdot, t)\}$ is uniformly bounded and of uniformly bounded variation on $(-\infty, \infty)$. Therefore, by Helly's selection principle and Cantor's diagonalization process, there exists a subsequence $\{u_{n_k}(x, t)\}$ of $\{u_n(x, t)\}$ such that for every rational t' in $[0, \infty)$, $\{u_{n_k}(\cdot, t')\}$ is convergent pointwise as well as in $L^1_{\text{loc}}(-\infty, \infty)$. For any $t \in [0, \infty)$, any rational $t' \in [0, \infty)$ and any $-\infty < x_1 < x_2 < \infty$, we have

$$\begin{aligned}\int_{x_1}^{x_2} |\, u_{n_k}(x, t) - u_{n_l}(x, t)|\, dx \\ \leqslant \int_{-\infty}^{\infty} |\, u_{n_k}(x, t) - u_{n_k}(x, t')|\, dx + \int_{x_1}^{x_2} |\, u_{n_k}(x, t') - u_{n_l}(x, t')|\, dx \\ + \int_{-\infty}^{\infty} |\, u_{n_l}(x, t') - u_{n_l}(x, t)|\, dx.\end{aligned} \tag{3.21}$$

From (3.21), (3.9), and (3.16) we deduce that $\{u_{n_k}(\cdot, t)\}$ is convergent in $L^1_{\text{loc}}(-\infty, \infty)$ uniformly in t for t in compact sets. In particular, $\{u_{n_k}(x, t)\}$ is convergent in $L^1_{\text{loc}}((-\infty, \infty) \times [0, \infty))$ and let

$$u_{n_k}(x, t) \xrightarrow{L^1_{\text{loc}}((-\infty,\infty)\times[0,\infty))} u(x, t), \qquad k \to \infty. \tag{3.22}$$

For any fixed $t \in [0, \infty)$ and on account of Helly's selection principle, $u(\cdot, t)$ (modified if necessary on a set of measure zero) is bounded from above by M, from below by m, is continuous on the left and of bounded variation on $(-\infty, \infty)$. Moreover,

$$\operatorname*{Var}_{(-\infty,\infty)} u(\cdot, t) \leqslant \operatorname*{Var}_{(-\infty,\infty)} u_0(\cdot), \qquad t \in [0, \infty). \tag{3.23}$$

We claim that $u(x, t)$ is the desired weak admissible solution of (1.1) and (1.2). Indeed, fix an increasing function $h(u)$ and a nonnegative function $\phi(x, t)$ with compact support in $(-\infty, \infty) \times [0, \infty)$. From (2.1),

$$\int_0^\infty \int_{-\infty}^\infty [I(u_{n_k}) \phi_t + F_{n_k}(u_{n_k}) \phi_x] \, dx \, dt + \int_{-\infty}^\infty I(u_0^{(n_k)}) \phi(x, 0) \, dx \geqslant 0, \tag{3.24}$$

where

$$F_{n_k}(u) \equiv \int_m^u h(\xi) \, df_{n_k}(\xi). \tag{3.25}$$

Thus,

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty [I(u) \phi_t + F(u) \phi_x] \, dx \, dt + \int_{-\infty}^\infty I(u_0) \phi(x, 0) \, dx \\ &\qquad \geqslant \int_0^\infty \int_{-\infty}^\infty \{[I(u) - I(u_{n_k})] \phi_t + [F(u) - F_{n_k}(u)] \phi_x \\ &\qquad + [F_{n_k}(u) - F_{n_k}(u_{n_k})] \phi_x\} \, dx \, dt + \int_{-\infty}^\infty [I(u_0) - I(u_0^{(n_k)})] \phi(x, 0) \, dx. \end{aligned} \tag{3.26}$$

We set

$$H \equiv \max_{[m,M]} | h(\xi)| = \max\{| h(m)| , | h(M)|\}.$$

Using (2.2),[4] (3.25), (3.17), and (3.18) one obtains easily the estimates

$$|I(u) - I(u_{n_k})| = \left| \int_u^{u_{n_k}} h(\xi)\, d\xi \right| \leqslant H \mid u_{n_k} - u \mid, \tag{3.27}$$

$$|I(u_0) - I(u_0^{(n_k)})| \leqslant H \mid u_0 - u_0^{(n_k)} \mid, \tag{3.28}$$

$$|F_{n_k}(u) - F_{n_k}(u_{n_k})| = \left| \int_u^{u_{n_k}} h(\xi)\, df_{n_k}(\xi) \right| \leqslant KH \mid u_{n_k} - u \mid, \tag{3.29}$$

$$\begin{aligned} |F(u) - F_{n_k}(u)| &= \left| \int_m^u h(\xi)\, d(f - f_{n_k})(\xi) \right| \\ &= \left| h(u)\,[f(u) - f_{n_k}(u)] - \int_m^u (f(\xi) - f_{n_k}(\xi))\, dh(\xi) \right| \\ &\leqslant K[H + h(M) - h(m)]\, \frac{M - m}{2n_k}. \end{aligned} \tag{3.30}$$

Recalling (3.22) and (3.15) we conclude that the right side of (3.26) tends to zero as $k \to \infty$ so that (2.1) is satisfied. Hence $u(x, t)$ is an admissible weak solution of (1.1) and (1.2).

For $k = 1, 2, \ldots$, $u_{n_k}(x, t)$ has the finite domain of dependence property and this endows $u(x, t)$ with the same property. Therefore, we can relax the assumption that $u_0(x)$ is of bounded variation on $(-\infty, \infty)$ and replace it by the condition that $u_0(x)$ is of locally bounded variation. Furthermore, (3.3) follows from (3.23).[5] Q.E.D.

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[4] Here we pick m as the lower limit of integration in (2.2)

[5] For differentiable $f(u)$, Vol'pert [7] establishes the uniqueness of admissible weak solutions in which case $\{u_n\}$ itself converges to u.