The Extended Centroid and $X$-Inner Automorphisms of Ore Extensions

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Communicated by Nathan Jacobson

Received August 15, 1989

For the Ore extension $R[t; S, D]$, where $R$ is a prime ring, we determine the center, the extended centroid, and the $X$-inner automorphisms. The results depend on the structure of ideals of $R[t; S, D]$. © 1992 Academic Press, Inc.

INTRODUCTION

Let $R$ and $Q_s(R)$ denote a prime ring and its symmetric Martindale quotient ring, respectively. The center of $Q_s(R)$ is called the extended centroid of $R$. Recall that an automorphism $\sigma$ of $R$ is $X$-inner if $\sigma$ becomes inner when extended to $Q_s(R)$. In recent years, the $X$-inner automorphisms and the extended centroid were computed for several kinds of rings extensions such as crossed products [21, 22], enveloping rings [24], and coproducts [15, 16]. The aim of this paper is to deal with similar questions for Ore extensions $R[t; S, D]$, where $S$ is an automorphism and $D$ an $S$-derivation of $R$. In particular the results obtained are generalizations of [20, 17].

It is well known that Ore extensions play an important role while investigating cyclic algebras, enveloping rings of solvable Lie algebras,

* The paper was partially written while the second named author was supported by a fellowship awarded by Institut d'Estudes Catalanes, Spain.
group rings, crossed products.... Moreover they give a natural method of constructing examples and counter-examples (cf., e.g., [3, 7]). In case $R$ is a simple ring the properties of Ore extensions were studied, in particular, by S. A. Amitsur [1, 2], P. M. Cohn [6, 7], and G. Cauchon [4].

In order to get the main results we investigate, in the second section of the paper, the structure of ideals of $R[t; S, D]$. To do this it is necessary to extend $R$ to $Q_s(R)$ and to study the overring $Q_s(R)[t; S, D]$. The main result in this section gives necessary and sufficient conditions, in terms of properties of $S$ and $D$, for $R[t; S, D]$ to have $R$-disjoint ideals (Theorem 2.6). The methods we use are based on [4, 12].

The above considerations enable us to describe, in Section 3, the center of $R[t; S, D]$ (Theorem 3.7) and to prove that the extended centroid of $R[t; S, D]$ is isomorphic to $ZZ^{-1}$, the field of quotients of the center $Z$ of $Q_s(R)[t; S, D]$.

In the final part, we describe $X$-inner automorphisms $\sigma$ of $R[t; S, D]$ stabilizing $R$. For doing this we consider overrings $R[t; S, D] \subset Q_s(R)[t; S, D] \subset Q_s(Q_s(R)[t; S, D])$.

In this setting we show that $\sigma$ can be written as a product $\sigma = \sigma_1 \sigma_2$ where $\sigma_1$ is induced by a conjugation by a unit $u$ of $Q_s(R)$ and $\sigma_2$ is a conjugation by $P$, a monic invariant polynomial of $Q_s(R)[t; S, D]$.

1. Preliminaries

Throughout the paper, $R$ will denote a prime ring, $T = Q_s(R)$ (resp. $Q_l$) its symmetric (resp. left) Martindale ring of quotients, $Z(R)$ will stand for the center of $R$, and $C(R)$ for its extended centroid equal, by definition, to $Z(T)$. The unspecified term "ideal" will mean "two-sided ideal."

Recall that $Q_l = \lim_{\text{lim}} \text{Hom}(R^1, R)$, where $\mathcal{F}$ is the filter of all non-zero two-sided ideals of $R$. $T$ is the subring of $Q_l$ consisting of elements $q$ for which there exists a non-zero two-sided ideal $I$ of $R$ such that $qI \subseteq R$.

In the following lemma we collect basic properties of $T$ and $Q_l$ which will be used frequently.

**Lemma 1.1** (see [9, 25, 18]). (1) For any $q_1, \ldots, q_n$ in $Q_l$ (resp. in $T$) there exists a non-zero ideal $I$ of $R$ such that $Iq_i \subseteq R$ (resp. and $q_iI \subseteq R$) for all $i \in \{1, 2, \ldots, n\}$.

(2) If $Iq = 0$ or $qI = 0$ for some $q \in Q_l$ and non-zero ideal $I$ of $R$ then $q = 0$.

(3) If $I$ is a non-zero ideal of $R$ and $x: I \to R$ is a homomorphism of left $R$-modules, then there exists $q \in Q_l$ such that $a(x) = xq$ for all $x \in I$. 

(4) $C(R)$ is a field.

(5) Let $0 \neq v \in Q_i$. If $Rv = vR$, then $v$ belongs to $T$ and is invertible in $T$.

Let us recall that an element $v \in T$ such that $Rv = vR$ is called a normal element in $T$.

For any invertible element $v \in R$, $I_v$ will stand for the inner automorphism induced by $v$, i.e., $I_v(x) = vxv^{-1}$ for all $x \in R$.

Throughout the paper $S$ will denote an automorphism of $R$ and $D$ a $S$-derivation of $R$. Recall that the $S$-derivation $D$ is an endomorphism of the additive group of $R$ which satisfies

$$D(ab) = D(a)b + S(a)D(b)$$

for all $a, b \in R$.

For $w \in R$, $D_{w,S}$ will stand for the inner $S$-derivation determined by $w$, i.e., $D_{w,S}(x) = wx - S(x)w$ for all $x \in R$.

The following property of $S$-derivations of prime rings is similar to the one of ordinary derivation and we leave its proof as an easy exercise.

**Lemma 1.2.** If $D$ is a non-zero $S$-derivation of $R$ then the left annihilator of $D(R)$ is equal to zero.

Let us recall that the Ore extension $R[t; S, D]$ is a free left $R$-module with the basis $1, t, t^2, \ldots$ and multiplication defined according to the rule $tr = S(r)t + D(r)$ for any $r \in R$. For $f(t) \in R[t; S, D]$, $deg f(t)$ will denote the degree of $f(t)$.

Clearly if $R$ is a prime ring, then $R[t; S, D]$ is a prime ring as well, and it is well known that both $S$ and $D$ can be uniquely extended to $T = Q_i(R)$, the symmetric Martindale ring of quotient of $R$, and to $Q_t$, the left Martindale ring of quotient of $R$. Thus we can consider the overrings $T[t; S, D], Q_t[t; S, D], Q_i(R[t; S, D]),$ and $Q_i(T[t; S, D])$ of $R[t; S, D]$.

We close this section with the following two technical results.

**Lemma 1.3.** Let $f(t) \in Q_i[t; S, D]$. Suppose that there exists an automorphism $\varphi$ of $R$ such that $f(t)x = \varphi(x)f(t)$ for all $x \in R$. Then

1. $f(t) \in T[t; S, D]$.
2. The leading coefficient of $f(t)$ is invertible in $T$.
3. $f(t)x = \varphi(x)f(t)$ for all $x \in T$.

**Proof.** Let $f(t) = \sum_{m=0}^{\infty} a_m t^m \in Q_i[t; S, D]$ with $a_m \neq 0$.

(2) Let $x \in R$. Then the leading coefficient of $f(t)x$ is equal to $a_m S^m(x)$. Hence, because $f(t)x = \varphi(x)f(t)$, $a_m S^m(x) = \varphi(x)a_m$ for all $x \in R$. Now Lemma 1.1(5) implies that $a_m \in T$ is invertible.
(1) By the above $a_m \in T$. Suppose that $a_m$, $a_{m-1}$, ..., $a_{m-k+1} \in T$ for some $0 < k \leq m$. Since $\varphi \in \text{Aut}(R)$ and $a_i \in Q_i$, $0 \leq i \leq m$, Lemma 1.1(1) can be used to find a non-zero ideal $J$ of $R$ such that

$$a_iJ \subseteq R \quad \text{for } m-k+1 \leq i \leq m \quad \text{and} \quad \varphi(J)a_j \subseteq R \quad \text{for } 0 \leq j \leq m.$$ 

Let $\mathcal{P}$ denote the set of endomorphisms of the additive group of $R$ consisting of all finite sums of products of length not greater than $m$ of maps $S$ and $D$.

Since $S(S^{-1}(J)J) \subseteq J$ and $D(S^{-1}(J)J) \subseteq J$ an easy inductive argument and primeness of $R$ enable us to find a non-zero ideal $I$ of $R$ such that $I \subseteq J$ and $p(I) \subseteq J$ for all $p \in \mathcal{P}$.

Let $x \in I$. Then, by assumption, $f(t)x = \varphi(x)f(t)$; looking at coefficients of $t^{m-k}$ on both sides of this equation we get

$$a_m p_m(x) + \cdots + a_{m-k+1} p_{m-k+1}(x) + a_{m-k} S^{m-k}(x) = \varphi(x)a_{m-k}$$

for some suitable $p_m$, ..., $p_{m-k+1} \in \mathcal{P}$.

The choice of $I$ implies that $\varphi(x)a_{m-k}$ and $a_i p_i(x)$ belong to $R$ for $m-k+1 \leq i \leq m$. Therefore $a_{m-k} S^{m-k}(I) \subseteq R$ and $a_{m-k} \in T$. This establishes property (1).

(3) Let $x \in T$ and $I$ be a non-zero ideal of $R$ such that $Ix \subseteq R$. Then for any $y \in I$ we have $f(t)y = \varphi(y)f(t)x$ and $f(t)y = \varphi(y)f(t)$. This means that $\varphi(I)(f(t)x - \varphi(x)f(t)) = 0$. Therefore, because $\varphi(I)$ is a non-zero ideal of $R$, $f(t)x = \varphi(x)f(t)$ for all $x \in T$.

**Lemma 1.4.** Let $q$ be an element in $Q(t; S, D)$ and $I$ be a non-zero ideal of $R$. If $Iq = 0$ then $q = 0$.

**Proof.** Suppose that $Iq = 0$ and let $J$ be a non-zero ideal of $T[t; S, D]$ such that $qJ \subseteq T[t; S, D]$ then $0 = (Iq)J = I(qJ)$. Hence $qJ = 0$ and $q - 0$.

2. **Ideals in $R[t; S, D]$**

Let $p(t) \in R[t; S, D]$. It is not hard to see that $R[t; S, D]p(t)$ is a right $R$-module (resp. a two-sided ideal of $R[t; S, D]$) if for every $x \in R$ there is a $y \in R$ such that $p(t)x = yp(t)$ (resp. and additionally $p(t)t = (bt + a) p(t)$ for some $a, b \in R$). Thus the following definitions arise naturally:

**Definitions.** (1) Let $p(t) \in R[t; S, D]$, we say that $p(t)$ is a right semi-invariant polynomial if for every $x \in R$ there exists $y \in R$ such that
$p(t)x = yp(t)$. If additionally $p(t)x = (bt + a)p(t)$ for some $a, b \in R$ then we say that the polynomial $p(t)$ is right invariant.

In the sequel, we will drop the adjective right and speak about semi-invariant and invariant polynomials.

(2) A $S$-derivation $D$ of $R$ is called quasi-algebraic if there exist an endomorphism $\theta$ of $R$ and elements $0 \neq a_n, a_{n-1}, ..., a_0, b \in R$ with $n > 0$, such that

$$\sum_{i=1}^{n} a_i D^i(x) + b D_{a_0,0}(x) = 0 \quad \text{for all} \quad x \in R.$$

We will see later that these definitions are strongly related. In particular quasi-algebraicness of $D$ implies the existence of semi-invariant polynomials. We have seen that the existence of invariant polynomials yields the existence of ideals of $R[t; S, D]$. The following proposition, crucial in our considerations, shows that the converse holds.

**Proposition 2.1.** For any non-zero ideal $I$ of $R[t; S, D]$ there exists a unique monic invariant polynomial $f_1(t) \in T[t; S, D]$ having the following properties

1. $\deg f_1(t) = \min \{\deg f(t) | f(t) \in I \setminus \{0\} \} = n$ and every polynomial $g(t) \in I$ of degree $n$ can be written in the form $af_1(t)$ for some $a \in R$.

2. $I \subset T[t; S, D] f_1(t)$.

**Proof.** Let $I$ be a non-zero ideal of $R[t; S, D]$. Define $n = \min \{\deg f(t) | f(t) \in I \setminus \{0\} \}$, $E = \{f(t) \in I | \deg f(t) \leq n \}$, and $J$ the set of leading coefficients of polynomials from $E$. Clearly $E$ is a left ideal and using the fact that $S$ is an automorphism of $R$, it is easy to see that $J$ is a two-sided ideal of $R$. For any $a \in J$, there exist elements $a_n = a$, $a_{n-1}, ..., a_0 \in R$ such that $\sum_{i=0}^{n} a_i t^i \in E$. The choice of $n$ implies that the elements $a_{n-1}, ..., a_0$ are uniquely determined by $a$. This means that the maps $\alpha_i: J \to R$, $0 \leq i \leq n$, given by $\alpha_i(a) = a_i$ ($a_n = a$) are well defined. Clearly, these maps are homomorphisms of left $R$-modules. Therefore there exist elements $q_n = 1, ..., q_0 \in Q$ such that $a_i = \alpha_i(a) = a q_i$ for $0 \leq i \leq n$ and $a \in J$.

Define $f_1(t) = \sum_{i=0}^{n} q_i t^i \in Q_I[t; S, D]$. The above yields that every element from $E$ has a unique presentation of the form $af_1(t)$ for some $a \in I$. This gives uniqueness of $f_1(t)$ and proves the statement (1) provided $f_1(t)$ is an invariant polynomial in $T[t; S, D]$.

Let us show that $f_1(t)$ belongs to $T[t; S, D]$ and is semi-invariant.

Since $f_1(t)$ is monic, for any $x \in R$ we can divide $f_1(t)x$ on the right by $f_1(t)$ getting $f_1(t)x = S''(x)f_1(t) + r(t)$ for some polynomial $r(t)$ in
$Q_r[t; S, D]$ such that $\deg r(t) < \deg f_i(t) = n$. For any $a \in I$, $ar(t) = af_i(t)x - aS^n(x)f_i(t) \in I$ and $\deg ar(t) < n$. The choice of $n$ implies that $Jr(t) = 0$. Therefore $r(t) = 0$ and $f_i(t)x = S^n(x)f_i(t)$ for all $x \in R$. Now Lemma 1.3 implies that in fact $f_i(t)$ belongs to $T[t; S, D]$ and is semi-invariant in $T[t; S, D]$.

Before proving invariance of the polynomial $f_i(t)$, we will show that the statement (2) holds.

Let $g(t) \in I$. Dividing $g(t)$ by $f_i(t)$ on the right we get $g(t) = h(t)f_i(t) + r(t)$ for some $h(t), r(t) \in T[t; S, D]$, where $\deg r(t) < n$.

Let $B$ be a non-zero ideal of $R$ such that $Bh(t) \subset R[t; S, D]$. Then for any $b \in B$ and $a \in S^{-n}(J)$, we have

$$br(t)a = bg(t)a - bh(t)f_i(t)a = bg(t)a - bh(t)S^n(a)f_i(t) \in I.$$ 

Therefore $Br(t)J \subset I$ and the choice of $n$ implies that $Br(t)J = 0$. This shows that $r(t) = 0$ and $g(t) = h(t)f_i(t) \in T[t; S, D]f_i(t)$. This establishes the property (2).

To finish the proof, we have to show that the semi-invariant polynomial $f_i(t)$ is in fact invariant.

For $a \in J$, $af_i(t) \in I$. Property (2) then shows $af_i(t)t = (at + q)f_i(t)$ for some $q \in T$. On the other hand, $f_i(t)t = (t + q')f_i(t) + r(t)$ for some $q' \in T$ and $r(t) \in T[t; S, D]$ such that $\deg r(t) < \deg f_i(t) = n$.

Hence $ar(t) = (at + q)f_i(t) - a(t + q')f_i(t) = (q - aq')f_i(t)$. Since $\deg ar(t) < \deg f_i(t)$ and $f_i(t)$ is monic one gets $q = aq'$ and $ar(t) = 0$. This yields $Jr(t) = 0$. Therefore $r(t) = 0$ and $f_i(t)t = (t + q')f_i(t)$. It means that $f_i(t)$ is invariant. 

**Corollary 2.2.** For any two-sided ideal $I$ of $T[t; S, D]$ there exists a monic invariant polynomial $f(t)$ in $T[t; S, D]$ such that 

1. $\deg f(t) = \min\{\deg h(t) | h(t) \in I \setminus \{0\} \}$.
2. $I \subset T[t; S, D]f(t)$.

**Proof.** It is straightforward to verify that $I' = I \cap R[t; S, D]$ is a non-zero ideal of $R[t; S, D]$ satisfying $\min\{\deg h | h \in I \setminus \{0\}\} = \min\{\deg h | h \in I' \setminus \{0\}\}$ and the polynomial $f(t) = f_r(t)$ has the desired properties.

Now we will look more carefully at relations between quasi-algebraicness of $D$ and existence of monic semi-invariant polynomials.

**Lemma 2.3.** Let $p(t) = \sum_{i=0}^{n} q_i t^i$, $n \geq 1$, be a monic semi-invariant polynomial in $R[t; S, D]$. Then $\sum_{i=1}^{n} q_i D^i(x) + D_{q_0S}(x) = 0$ for all $x \in R$ and $D$ is a quasi-algebraic $S$-derivation of $R$. 


Proof. By the definition of semi-invariance, for any \( x \in R \) there exists \( y \in R \) such that \( p(t)x = yp(t) \). Since \( p(t) \) is monic we obtain \( y = S^n(x) \). Now one can verify that comparison of the coefficients of degree zero in the equation \( p(t)x = S^n(x) \) \( p(t) \) gives precisely the thesis of the lemma.

The following proposition shows that a partial converse of the above lemma holds.

**Proposition 2.4.** Suppose \( D \) is a quasi-algebraic \( S \)-derivation of \( R \) then there exists a monic semi-invariant polynomial \( p(t) = \sum_{i=0}^{n} q_i t^i \in T[t; S, D] \) such that \( \deg p(t) \geq 1 \) and

\[
\sum_{i=1}^{n} q_i D^i(x) + D_{q_0, S^n}(x) = 0 \quad \text{for all } x \in T.
\]

**Proof.** Throughout the proof \( \text{End}(R, +) \) will stand for the ring of all endomorphisms of the additive group of \( R \). Let \( n \geq 1 \) be the minimal number such that there exist an endomorphism \( \theta \) of \( R \) and elements \( a_n \neq 0, a_{n-1}, \ldots, a_0, b \in R \) such that

\[
\sum_{i=1}^{n} a_i D^i + b D_{a_0, \theta} = 0 \quad \text{in } \text{End}(R, +).
\]

Let \( I = \{ c = c_n \in R | \exists c_{n-1}, \ldots, c_0 \in R: \sum_{i=1}^{n} c_i D^i + c_0 D_{a_0, \theta} = 0 \} \).

It is clear that \( I \) is a non-zero left ideal of \( R \). We will show that \( I \) is in fact a two-sided ideal of \( R \).

For \( x \in R \), \( L_x \in \text{End}(R, +) \) will denote the left multiplication by \( x \). It is easy to verify that the identity

\[
D \cdot L_x = L_{S^n(x)} \cdot D + L_{D(x)}
\]

holds in \( \text{End}(R, +) \) for all \( x \in R \).

Let \( c \in I \) and \( x \in R \). Then by definition of \( I \), \( \sum_{i=1}^{n} c_i D^i + c_0 D_{a_0, \theta} = 0 \) for some \( c = c_n, \ldots, c_0 \in R \). Thus also \( \sum_{i=1}^{n} c_i D^i L_x + c_0 D_{a_0, \theta} L_x = 0 \). By making use of (1), the last equation can be written in the form

\[
cS^n(x) D^n + d_{n-1} D^{n-1} + \cdots + d_1 D + L_{\sum_{i=1}^{n} c_i D^i} + c_0 D_{a_0, \theta} L_x = 0
\]

for some \( d_{n-1}, \ldots, d_0 \in R \). Hence

\[
cS^n(x) D^n + d_{n-1} D^{n-1} + \cdots + d_1 D
\]

\[
+ L_{\sum_{i=1}^{n} c_i D^i(x)} + c_0 D_{a_0, \theta}(x) + c_0 \theta(x) D_{a_0, \theta} = 0
\]

and

\[
cS^n(x) D^n + d_{n-1} D^{n-1} + \cdots + d_1 D + c_0 \theta(x) D_{a_0, \theta} = 0.
\]
This shows that \( cS^n(x) \in I \). Because \( c \in I \) and \( x \in R \) were arbitrary and \( S \) is an automorphism of \( R \), the above yields that \( I \) is a two-sided ideal of \( R \).

Now we are in position to associate to \( D \) a monic semi-invariant polynomial. The choice of \( n \) implies that for \( c \in I \), the elements \( c = c_0, c_n-1, \ldots, c_1 \), such that \( \sum_{i=1}^{n-1} c_i D_i' + c_0 D_{a_0,0} = 0 \) for some \( c_0 \in R \) are uniquely determined by \( c \). Moreover if \( \sum_{i=1}^{n-1} c_i D_i' + c_0 D_{a_0,0} = 0 \) then \( (c_0 - c) D_{a_0,0} = 0 \). Hence by Lemma 1.2, \( c_0 = c' \) provided \( D_{a_0,0} \neq 0 \). The above shows that the maps \( \alpha_i : I \to R, \ 0 \leq i \leq n \), given by \( \alpha_i(c) = c_i \) for \( 1 \leq i \leq n \) and \( \alpha_0(c) = c_0 \) if \( D_{a_0,0} \neq 0 \) or \( c_0 = 0 \) if \( D_{a_0,0} = 0 \) are well defined. Clearly these maps are homomorphisms of left \( R \)-modules. Thus there are elements \( q_n = 1, q_{n-1}, \ldots, q_1, q_0 \in Q_i \) such that for all \( c \in I \), \( c_i = \alpha_i(c) = cq_i \), \( 1 \leq i \leq n \), and \( c_0 = \alpha_0(c) = cq_0 \). Hence for any \( c \in I \) and \( y \in R \) we have \( c(D^n + q_{n-1} D^{n-1} + \cdots + q_1 D + q_0 D_{a_0,0})(y) = 0 \). Therefore

\[
(D^n + q_{n-1} D^{n-1} + \cdots + q_1 D + q_0 D_{a_0,0})(y) = 0 \quad \text{for every} \quad y \in R. \quad (2)
\]

Let \( p(t) = \sum_{i=0}^{n-q} q_i t^i \in Q_i[t; S, D] \), where \( q_0 = q_0 a_0 \). We will show that the monic polynomial \( p(t) \) satisfies the conditions of the proposition.

For any \( k \geq 0 \) and \( 0 \leq i \leq k \), define \( f^k_i \in \text{End}(R, +) \) as the sum of all products of length \( k \) with \( i \) letters \( S \) and \( (k-i) \) letters \( D \). It is straightforward to verify that the following generalized Leibniz's formula holds:

\[
D^k(xy) = \sum_{i=0}^{k} f^k_i(x) D^i(y) \quad \text{for every} \quad x, y \in R. \quad (3)
\]

Substituting \( y \) by \( xy \) in (2) and using (3) we get, for any \( x, y \in R \)

\[
0 = \sum_{k=1}^{n} q_k \left( \sum_{i=0}^{k} f^k_i(x) D^i(y) \right) + q_0 \theta(x) D_{a_0,0}(y) + q_0 D_{a_0,0}(x) y
\]

\[
= \sum_{i=1}^{n} \left( \sum_{k=i}^{n} q_k f^k_i(x) \right) D^i(y) + \sum_{k=1}^{n} q_k D^k(x) y
\]

\[
+ q_0 \theta(x) D_{a_0,0}(y) + q_0 D_{a_0,0}(x) y = 0.
\]

Hence, because of (2),

\[
\sum_{i=1}^{n} \left( \sum_{k=i}^{n} q_k f^k_i(x) \right) D^i(y) + q_0 \theta(x) D_{a_0,0}(y) = 0 \quad \text{for all} \quad x, y \in R. \quad (4)
\]

On the other hand, multiplying the formula (2) by \( S^n(x) \) on the left we get

\[
S^n(x) \sum_{i=1}^{n} q_i D^i(y) + S^n(x) q_0 D_{a_0,0}(y) = 0 \quad \text{for all} \quad x, y \in R.
\]
Taking the difference of the last equation and (4), we obtain
\[
\sum_{i=1}^{n-1} \left( \sum_{k=1}^{n} q_k f_i^k(x) - S^n(x) q_i \right) D^i(y) + (q_0 \theta(x) - S^n(x) q_0) D_{a_0, \theta}(y) = 0 \quad \text{for all } x, y \in R.
\]
For \( x \) fixed, one can multiply this equation on the left by an appropriate ideal \( \mathcal{I}(x) \) of \( R \) and obtain similar equations but with coefficients in \( R \). From these formulas, the choice of \( n \), and Lemma 1.1(2), it follows easily that for any \( x \in R \):

(i) \( \sum_{i=1}^{n-1} q_k f_i^k(x) = S^n(x) q_i \) for \( 1 \leq i \leq n - 1 \);
(ii) \( (q_0 \theta(x) - S^n(x) q_0) D_{a_0, \theta}(y) = 0 \) for all \( y \in R \).

If \( D_{a_0, \theta} = 0 \) then \( q_0 \theta(x) = S^n(x) q_0 \) for all \( x \in R \).

Suppose \( D_{a_0, \theta} \neq 0 \). Then by Lemma 1.2 and (ii), \( q_0 \theta(x) = S^n(x) q_0 \) for all \( x \in R \). Hence, for all \( x \in R \),
\[
q_0 D_{a_0, \theta}(x) = q_0 a_0 x - q_0 \theta(x) a_0 = q_0 a_0 x - S^n(x) q_0 a_0 = D_{a_0, \theta}(x).
\]
Therefore making use of the identity (2) we have \( D^n(x) + q_{n-1} D^{n-1}(x) + \cdots + q_1 D(x) + D_{q_0 a_0, S^n(x)} = 0 \) for all \( x \in R \). This property together with (i) says exactly that \( p(t)x = S^n(x) p(t) \) for all \( x \in R \). Now the fact that \( p(t) \) belongs to \( T[t; S, D] \) and is semi-invariant is a consequence of Lemma 1.3. The last statement of the proposition follows directly from Lemma 2.3.

In the case when \( R \) is a simple ring, the above proposition was proved in [13, Chap. 1]. The proof presented here is an adaptation of the original argumentation.

The monic semi-invariant polynomial constructed in the proposition is not uniquely determined by \( D \). In fact, let \( p(t) \) be a monic semi-invariant polynomial of degree \( n \geq 1 \). Suppose that \( S^n \) is an inner automorphism determined by an invertible element \( a \in T \). Then \( p(t) + a \) is also a monic semi-invariant polynomial of degree \( n \) (cf. also Proposition 2.10 in [11]).

In the sequel, we will need some additional properties of monic semi-invariant polynomials.

**Lemma 2.5.** Suppose that \( p(t) = \sum_{i=0}^{n} a_i t^i \in T[t; S, D] \) is a monic semi-invariant polynomial. Let \( c = a_{n-1} - S(a_{n-1}) \) and \( d = - D(a_0) - ca_0 \). Then

1. \( S^n D S^{-n} - D = D_{c, S} \)
2. \( g(t) = p(t) t - tp(t) - cp(t) \) is a semi-invariant polynomial. Moreover if \( g(t) \neq 0 \) then the leading coefficient of \( g(t) \) is a unit in \( T \).
If \( p(t) \) is of minimal degree among the monic semi-invariant polynomials of non-zero degree, then:

3) For any \( l \geq 1 \)

\[
p(t)^l t - tp(t)^l = T_{l,S^n}(c) p(t)^l + T_{l,S^n}(d) p(t)^{l-1},
\]

where for \( x \in R \), \( T_{l,S^n}(x) = x + S^n(x) + S^{2n}(x) + \cdots + S^{(l-1)n}(x) \)

4) \( dx = S^{n+1}(x) \) for any \( x \in R \).

**Proof.** Notice that because the semi-invariant polynomial \( p(t) \) is monic \( p(t)x = S^n(x) p(t) \) for all \( x \in T \).

1) A direct computation shows that

\[
p(t)^2 t - tp(t)^2 = (a_{n-1} - S(a_{n-1})) t^n + \cdots + (a_0 - S(a_0) - D(a_1)) t - D(a_0)
= ct^n + \cdots - D(a_0).
\]

For \( a \in T \) one has \( p(t)a = S^n(a) p(t) \), hence

\[
(p(t) t - tp(t))a = p(t) ta - tS^n(a) p(t) = p(t)(S(a) t + D(a)) - tS^n(a) p(t).
\]

By making use of (1), we obtain

\[
(ct^n + \cdots - D(a_0))a = S^{n+1}(a) p(t) t + S^n(D(a)) p(t) - tS^n(a) p(t),
\]
i.e.,

\[
cS^n(a) t^n + \cdots = S^{n+1}(a) p(t) t + S^n(D(a)) p(t)
- S^{n+1}(a) tp(t) - D(S^n(a)) p(t)
\]
and so,

\[
cS^n(a) t^n + \cdots = S^{n+1}(a)(p(t) t - tp(t)) + (S^n D(a) - DS^n(a)) p(t).
\]

Using once more the equation (1) one gets

\[
cS^n(a) t^n + \cdots = S^{n+1}(a)(ct^n + \cdots - D(a_0))
+ (S^n D(a) - DS^n(a))(t^n + a_{n-1} t^{n-1} + \cdots).
\]

A comparison of the leading coefficients gives us

\[
cS^n(a) = S^{n+1}(a)c + S^n D(a) - DS^n(a)
\text{ for all } a \in T.
\]
Since $S$ is an automorphism of $T$ one can rewrite this equation in the form
\[ cu - S(a)c = S'' DS^{-n}(a) - D(a) \quad \text{for all} \quad a \in T, \]
i.e., $S'' DS^{-n} - D = D_{c,S}$.

(2) Equation (2) can now be written in the form
\[
(p(t)t - tp(t))a = S''(a)(p(t)t - tp(t))
+ (cS''(a) - S''(a)c) p(t) \quad \text{for every} \quad a \in T
\]
i.e.,
\[
(p(t)t - tp(t))a = S''(a)(p(t)t - tp(t) - cp(t))
+ cS''(a) p(t) \quad \text{for every} \quad a \in T.
\]

But $cS''(a) p(t) = cp(t)a$, hence
\[
(p(t)t - tp(t) - cp(t))a = S''(a)(p(t)t - tp(t) - cp(t)) \quad \text{for every} \quad a \in T.
\]

Therefore the polynomial $g(t) = p(t)t - tp(t) - cp(t)$ is semi-invariant and satisfies $g(t)a = S''(a)g(t)$ for all $a \in T$. Now the fact that the leading coefficient of $g(t)$ is invertible in $T$, provided $g(t) \neq 0$, follows from Lemma 1.3. This establishes the statement (2).

Now assume that $p(t)$ is of minimal degree among the monic non-constant semi-invariant polynomials.

(3) Let $g(t)$ denote the semi-invariant polynomial constructed in (2). Suppose that $g(t) \neq 0$ and let $\lambda$ denote the leading coefficient of $g(t)$. It is straightforward to verify that $\deg g(t) < \deg p(t) = n$ and $\lambda^{-1}g(t)$ is a monic semi-invariant polynomial. The choice of $n$ then implies that $\lambda^{-1}g(t) = 1$, i.e., $g(t) \in T$. Thus Eq. (1) in statement (1) above shows that the constant term of $g(t) = p(t)t - tp(t) - cp(t)$ is equal to $d = -D(a_0) - ca_0$. Hence $\lambda = d = -D(a_0) - ca_0 = g(t)$ and $p(t)t - tp(t) = g(t) + cp(t) = cp(t) + d = T_{1,S}(c)p(t) + T_{1,S}(d)$. This provides the proof of the statement (3) for $l = 1$.

Suppose that the statement (3) holds for some $l \geq 1$. Then $p(t)^{l+1} t = p(t)(p(t)^l t) = p(t)(tp(t)^l + T_{l,S}(c)p(t)^l + T_{l,S}(d)p(t)^{l-1})$. Therefore $p(t)^{l+1} t = tp(t)^{l+1} + (cp(t) + d)p(t)^l + S''(T_{l,S}(c))p(t)^{l+1} + S''(T_{l,S}(d))p(t)^{l-1}$. And so, $p(t)^{l+1} t = (tp(t)^{l+1} + (S''(T_{l,S}(c)) + c)p(t)^l + (S''(T_{l,S}(d)) + d)p(t)^l$. It means that $p(t)^{l+1} t = p(t)^{l+1} - T_{l+1,S}(c)p(t)^l + T_{l+1,S}(d)p(t)^l$. This establishes the proof of (3).

(4) We have shown in (3) that $g(t) = d$. The statement (4) is now a direct consequence of (2).
We will say that an ideal of $R[t; S, D]$ is $R$-disjoint if $I \cap R = \{0\}$. Now we are ready to prove the main result of the section.

**Theorem 2.6.** For the ring $R[t; S, D]$, the following conditions are equivalent:

1. There exists a non-zero $R$-disjoint ideal of $R[t; S, D]$.
2. There exists a non-zero $T$-disjoint ideal of $T[t; S, D]$.
3. There exists a monic, non-constant invariant polynomial in $T[t; S, D]$.
4. There exists a monic, non-constant semi-invariant polynomial in $T[t; S, D]$.
5. $D$ is a quasi-algebraic $S$-derivation of $T$.

**Proof:** The implication (1) $\implies$ (3) is given by Proposition 2.1. Implications (3) $\implies$ (2), (2) $\implies$ (1), and (3) $\implies$ (4) are clear. Therefore in order to establish the equivalence of statements (1) $\iff$ (4), it is enough to show that (4) implies (2). Let $p(t)$ be a monic semi-invariant polynomial of minimal non-zero degree, say $\deg p(t) = n$. Consider $I = \{ h(t) \in T[t; S, D] \mid h(t) T[t; S, D] \subseteq T[t; S, D] p(t) \}$. Clearly $I$ is a $T$-disjoint two-sided ideal of $T[t; S, D]$. Moreover, since $p(t)$ is monic, $I$ is $T$-disjoint. We will show that $p(t)^n \in I$. Lemma 2.5(3) implies that $p(t)^n t \in T[t; S, D] p(t)^n$. Furthermore, $p(t)^n t \in T[t; S, D] p(t)^n$. Since $p(t)$ is semi-invariant, the above shows that $p(t)^n g(t) \in T[t; S, D] p(t)$ for all $g(t) \in T[t; S, D]$ of degree not greater than $n - 1$.

Suppose that $g(t) \in T[t; S, D]$ and $\deg g(t) \geq n$. Since $p(t)$ is monic we can divide $g(t)$ on the right by $p(t)$ getting $g(t) = h(t) p(t) + r(t)$ for some $h(t)$, $r(t) \in T[t; S, D]$ with $\deg r(t) < n$.

Since $\deg r(t) < n$, we know that $p(t)^n r(t) \in T[t; S, D] p(t)$, so $p(t)^n g(t) = p(t)^n h(t) p(t) + p(t)^n r(t) \in T[t; S, D] p(t)$. This yields that $p(t)^n \in I$. Now the fact that $p(t)^n$ is a monic polynomial implies that $I$ is a non-zero $T$-disjoint ideal. This establishes the proof of implication (4) $\implies$ (2) and gives equivalence of conditions (1) $\iff$ (4).

By Lemma 2.3, (4) implies (5). Thus in order to finish the proof it is enough to show that (5) implies one of the equivalent conditions (1) $\iff$ (4). Suppose that $D$ is a quasi-algebraic $S$-derivation of $T$. Then in virtue of Proposition 2.4 there is a monic semi-invariant polynomial of non-zero degree $p(t) \in Q_s(T)[t; S, D]$. (Let us recall that for a prime ring $A$ we denote by $Q_s(A)$ the symmetric Martindale ring of quotient of $A$. So $Q_s(T) = Q_s(Q_s(R))$.) Now applying the equivalence (4) $\iff$ (1) for $R$ replaced by $T$, we get that there is a non-zero $T$-disjoint ideal of $T[t; S, D]$. This provides the proof of the theorem.
If \( R \) is a simple unital ring then \( T = R \). Therefore the above theorem gives us immediately the following.

**Corollary 2.7.** Suppose that \( R \) is a simple ring. The following conditions are equivalent:

1. \( R[t; S, D] \) is a simple ring.
2. \( D \) is not a quasi-algebraic \( S \)-derivation of \( R \).

This corollary was proved by Lemonnier [13]. The proof he used was quite involved and used different notions such as triangular extensions, Krull dimension, ... The proof we presented is elementary and is a simple adaptation of the one given in [12].

The above theorem shows the importance of semi-invariant polynomials while investigating ideals of \( R[t; S, D] \). The following proposition gives a complete description of such polynomials. Recall that for an invertible element \( c \in T \), \( I_c \) stands for the inner automorphism determined by \( c \).

**Proposition 2.8.** Let \( p(t) \in T[t; S, D] \) be a monic semi-invariant polynomial of minimal non-zero degree (we assume that there exists such a polynomial). Then:

1. If \( g(t) \in T[t; S, D] \) is a monic semi-invariant polynomial, then there exist elements \( c_0, \ldots, c_r \in T \) such that \( g(t) = \sum_{i=0}^r c_i p(t)^i \). Moreover if \( c_i \neq 0 \) for some \( 0 \leq i \leq r \) then \( c_i \) is invertible in \( T \) and \( S^{m-i} = I_{c_i} \), where \( m = \deg g(t) \), \( n = \deg p(t) \).
2. If additionally the automorphism \( S \) commutes with \( D \) then
   
   (i) If \( \text{char } T - p > 0 \) then \( p(t) \) has the form \( \sum_{i=0}^k c_i t^i a_0 + a_0 \), \( c_i \in T \).
   (ii) If \( \text{char } T = 0 \) then \( p(t) = t + a_0 \) for some \( a_0 \in T \).

**Proof.** Since \( g(t) \) is monic semi-invariant we have \( g(t)x = S^m(x) g(t) \), \( \forall x \in T \).

(1) We will prove statement (1) by induction on \( m = \deg g(t) \). If \( m = 0 \) then \( g(t) = 1 = c_0 \) and there is nothing to prove. Assume now that \( \deg g(t) = m > 0 \) and that statement (1) holds for any monic semi-invariant polynomial \( h(t) \in T[t; S, D] \) such that \( \deg h(t) < m \). Since \( p(t) \) is a monic polynomial, we can divide \( g(t) \) by \( p(t) \) on the right getting \( g(t) = q(t) p(t) + r(t) \) for some \( q(t), r(t) \in T[t; S, D] \) with \( \deg r(t) < n \). Moreover since \( g(t) \) and \( p(t) \) are monic, so is \( q(t) \). Let \( x \in T \), one has \( S^m(x) g(t) = g(t)x = q(t)(p(t)x + r(t)x) = q(t) S^m(x) p(t) + r(t)x \) and \( S^m(x) g(t) = S^m(x) q(t) p(t) + S^m(x) r(t) \). Therefore, by comparing these
two expressions of \( S^m(x) q(t) \) one obtains \( (q(t) S^n(x) - S^m(x) q(t)) p(t) = S^m(x) r(t) - r(t) x \) for every \( x \in T \).

Considerations of degrees imply

\[
q(t) x = S^{m-n}(x) q(t) \tag{1}
\]
\[
r(t) x = S^m(x) r(t). \tag{2}
\]

Equation (1) shows that the monic polynomial \( q(t) \) is semi-invariant. So by inductive hypothesis, \( q(t) = \sum_{i=1}^{r} c_i p(t)^{i-1} \) for some \( c_i \in T \).

Equation (2) and Lemma 1.3 imply that the leading coefficient, say \( c_0 \), of \( r(t) \) is invertible in \( T \). Hence \( c_0^{-1} r(t) \) is a monic semi-invariant polynomial in \( T[t; S, D] \) and \( \deg c_0^{-1} r(t) < n = \deg p(t) \). We conclude that \( c_0^{-1} r(t) \) is a constant and, in fact, we have \( c_0^{-1} r(t) = 1 \), i.e., \( r(t) = c_0 \in T \). Now, \( g(t) = q(t) p(t) + r(t) = \sum_{i=1}^{r} c_i p(t)^i + c_0 = \sum_{i=0}^{r} c_i p(t)^i \). It remains to prove that the non-zero coefficients \( c_i \) are invertible and satisfy the relations \( S^{m-in} = I_{c_i} \).

We have shown that \( g(t) = \sum_{i=0}^{r} c_i p(t)^i \) and we know that for any \( x \in T \),
\[ g(t) x = S^m(x) g(t) \text{ and } p(t) x = S^n(x) p(t). \]
Then
\[
\sum_{i=0}^{r} (S^m(x)c_i) p(t)^i = S^m(x) g(t) = \sum_{i=0}^{r} c_i p(t)^i x = \sum_{i=0}^{r} c_i S^n(x) p(t)^i.
\]

Hence \( S^m(x)c_i = c_i S^n(x) \) for any \( 0 \leq i \leq r \) and \( x \in T \). Since \( S \) is an automorphism of \( T \) this can be rewritten in the form \( S^{m-in}(x) c_i = c_i x \) for all \( x \in T \) and any \( 0 \leq i \leq r \). Now, Lemma 1.1(5) shows that if \( c_i \neq 0 \) then \( c_i \) is invertible and we have \( S^{m-in}(x) = c_i x c_i^{-1} = I_{c_i}(x) \) for all \( x \in T \).

(2) Since \( S \circ D = D \circ S \), one easily verifies that
\[
t^n a = \sum_{i=0}^{n} \binom{n}{i} D^{n-i} S^i(a) t^i \quad \text{for } a \in T \text{ and } n \in \mathbb{N}. \tag{3}
\]

Let \( p(t) = \sum_{i=0}^{n} a_i t^i \), \( a_i \in T \), \( a_n = 1 \). By making use of (3) and comparing coefficients of degree \( j \), \( 0 \leq j \leq n \), of both sides of the equation \( p(t) a = S^n(a) p(t) \) one gets
\[
S^n(a) a_j = \sum_{i=-j}^{n} \binom{i}{j} a_i D^{i-j} S^i(a), \quad 0 \leq j \leq n, \text{ for any } a \in T. \tag{4}
\]

Let us introduce the polynomials
\[
f_j(t) := \sum_{i=0}^{n-j} \binom{i+j}{j} a_{i+j} t^i, \quad 0 \leq j \leq n
\]
deg \( f_j(t) = n - j \) and we will show that \( f_j(t) \) is semi-invariant. For any \( a \in T \) one has

\[
f_j(t) a = \sum_{i=0}^{n-j} \binom{i+j}{j} a_{i+j} \left( \sum_{k=0}^{i} \binom{i}{k} S^k D^{i-k}(a) t^k \right)
\]

But \( (\binom{n}{j})(\binom{n-j}{i}) = \binom{n}{j}(\binom{n-j}{i-j}) \) and so

\[
f_j(t) a = \sum_{k=j}^{n} \binom{k}{j} \left( \sum_{i=k-j}^{n} \binom{i-j}{i} a_i S^{k-j} D^{i-k}(a) t^{k-j} \right)
\]

This shows that \( h(t) \) is semi-invariant. The minimality of \( \deg p(t) = n \) implies that \( f_j(t) = a_j \in T \) and

\[
\binom{k}{j} a_k = 0 \quad \text{for every } k, j \text{ such that } 1 \leq j < k \leq n.
\] (5)

Now we separate the proof in two parts:

(i) \( \text{char } T = p > 0 \). Then in virtue of (5), \( a_j \neq 0 \) \( (j \geq 1) \) can occur only when \( j \) is a power of \( p \). Therefore \( p(t) \) has the form \( \sum_{i=0}^{k} c_i t^p + a_0 \).

(ii) \( \text{char } T = 0 \), (5) implies in particular \( (\binom{n}{j}) = (\binom{n}{j}) a_n = 0 \). This forces \( n = 1 \). Thus \( p(t) = t + a_0 \), as we had to prove.

The above proposition was widely inspired by [2, 11].
3. CENTER AND EXTENDED CENTROID OF $R[t; S, D]$

For a prime ring $A$, $Z(A)$ and $C(A)$ will stand for the center and the extended centroid of $A$, respectively. As we noticed earlier if $R$ is a prime ring then $R[t; S, D]$ is also a prime ring and we will describe in this paragraph both $Z(R[t; S, D])$ and $C(R[t; S, D])$. We will show that while investigating either of these it is enough to look at $Z(T[t; S, D])$ and $C(T[t; S, D])$ where, as usual, $T$ denotes the symmetric Martindale ring of quotients of $R$.

LEMMA 3.1 (G. Cauchon [4]). Suppose that $S^n = I_w$ for some $n \geq 1$. Then there exist $m \geq 1$ and an invertible element $v \in T$ such that $S(v) = v$ and $S^m = I_v$.

Proof. Let $v = wS(w) \cdots S^{n-1}(w)$. Since $S^n(w) = w$, we have $S(v) = S(w) \cdots S^n(w) = S(S^n(w)) \cdots S^n(S^n(w)) = S^n(w) \cdots S^n(w)$. Hence $S(v) = wS(w) \cdots S^n(w)w^{-1} = vS^n(w)w^{-1} = v$.

On the other hand, for any $l \geq 0$, $S^l(w)$ is an invertible element in $T$ and it is easy to observe that $I_{S^l(w)} = S^n$. This shows that the inner automorphism induced by $v = wS(w) \cdots S^{n-1}(w)$ is $S^n$. Since $S(v) = v$ the proof is complete. 

The proof of the following lemma already appeared in Lemma 2.1 of [17]. We will present a sketch of it for the convenience of the reader.

LEMMA 3.2. (1) $Z(R[t; S, D]) = Z(T[t; S, D]) \cap R[t; S, D]$.

(2) $C(R[t; S, D])$ is isomorphic to $C(T[t; S, D])$.

Proof. The statement (1) is a consequence of Lemma 1.1.

(2) Let $c \in C(R[t; S, D])$ and $I$ be a non-zero ideal of $R[t; S, D]$ such that $Ic \subset R[t; S, D]$. Define $\alpha: T[t; S, D] \rightarrow T[t; S, D]$ by the rule $\alpha(\sum g_ia_if_i) = \sum g_ica_if_i$, where $g_i, f_i \in T[t; S, D]$, $a_i \in I$. One can check that $\alpha$ is a well-defined homomorphism of $(T[t; S, D] - T[t; S, D])$ bimodules. Thus $\alpha$ defines an element $c_\alpha$ in $C(T[t; S, D])$. Now it is easy to check that the map $L: C(R[t; S, D]) \rightarrow C(T[t; S, D])$ given by $L(c) = c_\alpha$ is an isomorphism with the inverse $L^{-1}$ described as follows: for any $c \in C(T[t; S, D])$, $I(c) = \{f(t) \in R[t; S, D] \mid cf(t) \in R[t; S, D]\}$ is a non-zero ideal of $R[t; S, D]$, and the restriction of the multiplication by $c$ to $I(c)$ determines the element $L^{-1}(c) \in C(R[t; S, D])$.

In order to describe the center of $R[t; S, D]$ some more information about invariant polynomials is needed. Propositions 3.4 and 3.5 are adaptations of Theorem 5.1.4 of [4] and Proposition 2.3 of [14], respectively.
LEMMA 3.3. Suppose that the polynomial \( f(t) = t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \in T[t; S, D] \), \( n \geq 1 \), is invariant. Then \( f(t)t = (t + c)f(t) \) where \( c = a_{n-1} - S(a_{n-1}) \).

Proof. Since \( f(t) \) is a monic invariant polynomial of degree \( n \), there exists \( c \in T \) such that \( f(t)t - (t + c)f(t) \). By comparing the coefficients of \( t^n \) on both sides of this equation one obtains that \( c = a_{n-1} - S(a_{n-1}) \).

PROPOSITION 3.4. Let \( M(t) \in T[t; S, D] \) be a monic invariant polynomial of minimal non-zero degree. (We assume that such a polynomial exists.) Then every monic invariant polynomial in \( T[t; S, D] \) can be written in the form \( \alpha \omega(t) M(t)' \) where \( l \in \mathbb{N} \), \( \alpha \) is an invertible element in \( T \), and \( \omega(t) \) belongs to the center of \( T[t; S, D] \).

Proof. Let \( h(t) \in T[t; S, D] \) be a non-constant, monic invariant polynomial. We can write \( h(t) = h_1(t) M(t)' \) where \( l \in \mathbb{N} = \{0, 1, \ldots\} \) and \( M(t) \) does not divide \( h_1(t) \) on the right. Obviously \( h_1(t) \) is still invariant and monic. Let us put \( h_1(t) = t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \). We will divide the proof in two cases but let us first point out the following remark: for \( u \) invertible in \( T \) and \( v \in T \), we have \( T[t; S, D] = T[t'; S', D'] \) where \( t' = ut + v \), \( S' = I_u \circ S \) and \( D' = D_{u^{-1}v} + uD \). Moreover there is a ring isomorphism \( \sigma: T[t'; S', D'] \rightarrow T[t; S, D] \) such that \( \sigma|_T = \text{id.} \) and \( \sigma(t') = ut + v \) and it is easy to verify that there exists a monic invariant polynomial of degree \( n \) in \( T[t; S, D] \) if and only if there exists a monic invariant polynomial of degree \( n \) in \( T[t'; S', D'] \).

Case 1. \( D = 0 \) then \( M(t) = t + u \) for some \( u \in T \).

Assume \( u = 0 \). Since \( M(t) = t \) does not divide \( h_1(t) \) we know that \( a_0 \neq 0 \). On the other hand since \( h_1(t) \) is invariant and monic we have \( h_1(t)a = S'(a) h_1(t) \) for every \( a \in T \) and a comparison of independent terms on both sides of this equation yields \( a_0 a = S'(a)a_0 \) for any \( a \in T \). Lemma 1.1(5) shows that \( a_0 \) is invertible in \( T \) and one can verify that \( a_0^{-1}h_1(t) = \omega(t) \) is central. Moreover \( h(t) = a_0 \omega(t) M(t)' \) as required.

Assume \( u \neq 0 \). Semi-invariance of \( M(t) \) implies \( ua = S(a)u \) for all \( a \in T \). Thus by Lemma 1.1(5), \( u \) is invertible and \( S = I_u \). Hence \( T[t; S'] = T[u^{-1}t] \) is an ordinary polynomial ring and the thesis is clear.

Case 2. \( D \neq 0 \). Let us write \( h_1(t) = q(t) M(t) + r(t) \) where \( \deg r(t) < \deg M(t) \) and, by the construction of \( h_1(t) \), \( r(t) \neq 0 \). Since \( h_1(t) \) and \( M(t) \) are invariant, there exist \( \alpha, \beta \in T \) such that

\[
h_1(t)t = (t + \beta) h_1(t) \quad \text{and} \quad M(t)t = (t + \alpha) M(t).
\] (1)

Now, for any \( a \in T \),

\[
h_1(t)a = S'(a) h_1(t) = S'(a) q(t) M(t) + S'(a) r(t)
\]
and also \( h_1(t)a = q(t) M(t)a + r(t)a = q(t) S^m(a) M(t) + r(t)a \). Therefore
\[
(S^n(a) q(t) - q(t) S^m(a)) M(t) = r(t)a - S^n(a) r(t).
\]
A comparison of degrees on both sides of this last equation yields that
\[
r(t)a = S^n(a) r(t) \quad \text{for all} \quad a \in T. \tag{2}
\]
Hence \( r(t) \neq 0 \) is a semi-invariant polynomial and, by Lemma 1.3(2), its leading coefficient \( s \) is invertible in \( T \).

We now consider two cases.

(a) \( \deg r(t) < \deg M(t) - 1 \).

One easily gets from (1) that \((t+\beta) q(t) - q(t)(t+\alpha)) M(t) = r(t)t - (t+\beta) r(t)\) and since \( \deg r(t) < \deg M(t) - 1 \) one obtains
\[
r(t)t = (t+\beta) r(t). \tag{3}
\]
Equations (2) and (3) show that \( r(t) \) is invariant. It means that \( s^{-1} r(t) \) is a monic invariant polynomial of degree smaller than \( \deg M(t) = m \) and the minimality of \( m \) forces \( s^{-1} r(t) \) and hence \( r(t) \) to belong to \( T \). Equation (2) and Lemma 1.1(5) show that \( r = r(t) \) is invertible in \( T \). Now \( w(t) = r^{-1} h_1(t) \) is easily seen (using (1), (2), (3) above) to be central. Hence \( h(t) = r w(t) M(t) \) is of the desired form.

(b) \( \deg r(t) = \deg M(t) - 1 \).

Since the leading coefficient of \( r(t) \) is invertible we can divide \( M(t) \) by \( r(t) \) on the right: \( M(t) = (ut + v) r(t) + s(t) \) where \( u, v \in T \) and \( s(t) \in T[t; S, D] \) is such that \( \deg s(t) < \deg r(t) \). Moreover \( u \) is invertible in \( T \) and for any \( a \in T \), \( M(t)a = S^m(a) M(t) = S^m(a) Yr(t) + S^m(a) s(t) \) where \( Y := ut + v \). On the other hand, one also has
\[
M(t)a = Yr(t)a + s(t)a = YS^n(a) r(t) + s(t)a.
\]
Hence \( (S^m(a) Y - YS^n(a)) r(t) = s(t)a - S^m(a) s(t) \). And finally \( Ya = S^{m-n}(a) Y \) for any \( a \in T \). But we know that \( Y = ut + v \) and \( u \) is invertible in \( T \), one thus obtains \( T[t; S, D] = T[Y; S^{m-n}] \) and we are back in the Case 1.

Let us remark that there can exist more than one monic polynomial of minimal non-zero degree (e.g., if \( S = \text{id.} \) then every monic invariant polynomial \( f \) is central and in this case \( f + u \) is also central for any \( u \in Z(T) \) such that \( D(u) = 0 \).

**Proposition 3.5.** Suppose that a non-zero power of \( S \) is an inner automorphism of \( T \). Then for every monic invariant polynomial \( f(t) \in T[t, S, D] \) there exists an invertible element \( v \in T \) and \( m \geq 1 \) such that \( S(v) = v \) and \( v^{-1} f(t)^m \) is central in \( T[t; S, D] \).
Proof. Let \( f(t) \in \mathbb{T}[t; S, D] \) be a monic invariant polynomial of degree \( n \geq 1 \).

**Case 1.** \( S \) is an inner automorphism of \( T \). Say \( S = I_v \) for some invertible element \( v \) in \( T \). Let \( y := v^{-1}t \), then we have \( \mathbb{T}[t; S, D] = \mathbb{T}[y; id, v^{-1}D] \) and \( v^{-n}f(t) = g(y) \) where \( g(y) \) is a monic invariant polynomial in \( \mathbb{T}[y; id, v^{-1}D] \). It is easy to verify that for any \( a \in T \) \( v^{-n}f(t)a = av^{-n}f(t) \), i.e., \( g(y)a = ag(y) \) and Lemma 3.3 shows that in fact \( g(y) = v^{-n}f(t) \) is central.

**Case 2.** \( S \) is not inner but some non-zero power of \( S \) is an inner automorphism of \( T \).

By Lemma 3.1, there exist \( m \geq 1 \) and an invertible element \( u \in T \) such that \( S^m = I_u \) and \( S(u) = u \). Define \( v = u^n \) and \( l = m \cdot n \). The polynomial \( f(t)^m \) is invariant and \( \deg(f(t)^m) = l \). Thus by Lemma 2.5(1),

\[
S' DS^{-l} - D = D_{c, S},
\]

where \( c \in T \) satisfies

\[
(t + c) f(t)^m = f(t)^m t.
\]

By making use of (1) and the facts that \( S^{-l} = I_{v^{-1}} \) and \( S(v) = v \) we get for any \( x \in T \),

\[
Cx - S(x)c = D_{c, S}(x)
= (S' DS^{-l} - D)(x)
= vD(v^{-1}xv) v^{-1} - D(x)
= -D(v) v^{-1}x + S(x) D(v) v^{-1}.
\]

Therefore \( (c + D(v) v^{-1})x = S(x)(c + D(v) v^{-1}) \) for all \( x \in T \). If \( c + D(v) v^{-1} \neq 0 \) then the above together with Lemma 1.1(5) shows that \( S \) is an inner automorphism of \( T \). This contradicts our assumption. Hence we have \( c + D(v) v^{-1} = 0 \). Using this and the facts that \( D(v^{-1}) = -S(v^{-1}) D(v) v^{-1} \) and \( S(v) = v \) we get \( tv^{-1} = S(v^{-1}) t + D(v^{-1}) = v^{-1}(t - D(v) v^{-1}) = v^{-1}(t + c) \). This together with (2) shows that \( tv^{-1}f(t)^m = v^{-1}(t + c) f(t)^m = v^{-1}f(t)^m t \). This means that \( v^{-1}f(t)^m \) commutes with \( t \). It is straightforward to verify that every element from \( T \) commutes with \( v^{-1}f(t)^m \). This shows that the element \( v^{-1}f(t)^m \) is central in \( \mathbb{T}[t; S, D] \). \( \square \)

We say that the center of \( T[t; S, D] \) is non-trivial if it is not included in \( T \).

In the following two theorems we give a precise description of the center.
of \( T[t; S, D] \). This together with Lemma 3.2(1) gives the description of the center of \( R[t; S, D] \). However, as we will see in Example 3.8, the center of \( R[t; S; D] \) can be trivial even in the case the center of \( T[t; S, D] \) is non-trivial.

**Theorem 3.6.** For the ring \( T[t; S, D] \) the following conditions are equivalent:

1. The center of \( T[t, S, D] \) is non-trivial.
2. A non-zero power of \( S \) is an inner automorphism of \( T \) and there exists a non-zero \( T \)-disjoint ideal of \( T[t; S, D] \).
3. A non-zero power of \( S \) is an inner automorphism of \( T \) and there exists a nonconstant monic invariant polynomial in \( T[t; S, D] \).
4. A non-zero power of \( S \) is an inner automorphism of \( T \) and there exists a nonconstant monic semi-invariant polynomial in \( T[t; S, D] \).
5. A non-zero power of \( S \) is an inner automorphism of \( T \) and \( D \) is a quasi-algebraic \( S \)-derivation of \( T \).

**Proof.** The equivalence of conditions (2) \( \div \) (5) is given by Theorem 2.6. Proposition 3.5 establishes the implication (3) \( \rightarrow \) (1). It remains to show that statement (1) implies one of the equivalent statement (2) \( \div \) (5).

Let \( f(t) \in Z(T[t; S, D]) \), \( \deg f(t) = r \geq 1 \), and let \( a \) denote the leading coefficient of \( f(t) \). Lemma 1.3 applied to the equation \( f(t)x = xf(t) \) shows that \( a \) is an invertible element in \( T \) and \( S' \) is an inner automorphism of \( T \). Of course \( I = T[t; S, D] f(t) \) is a non-zero \( T \)-disjoint ideal of \( T[t; S, D] \). We have thus shown that (1) \( \rightarrow \) (2).

**Theorem 3.7.** (1) If \( Z(T[t; S, D]) \) is trivial then \( Z(T[t; S, D]) = \{a \in Z(T) | S(a) = a \text{ and } D(a) = 0\} = Z(T)_{S,D} \).

(2) If \( Z(T[t; S, D]) \) is non-trivial then

(i) every element \( f(t) \in Z[t; S, D] \) can be presented in a form \( \sum_{i=0}^{r} c_i p(t)^i \) where \( c_i \in T \), \( p(t) \) is a monic semi-invariant polynomial of minimal non-zero degree. Moreover if \( c_i \neq 0 \) for some \( 0 \leq i \leq r \) then \( c_i \) is invertible in \( T \) and \( S^{-in} = I_c \) where \( n = \deg p(t) \).

(ii) \( Z(T[t; S, D]) = Z(T)_{S,D}[h(t)] \) where \( h(t) \) is a central polynomial of minimal non-zero degree.

(iii) Let \( h(t) \in T[t; S, D] \) be a central polynomial of minimal non-zero degree. Then there exists an invertible element \( \lambda \in T \) such that \( \lambda R = R\lambda \) (i.e., \( \lambda \) is normal in \( T \)), \( c \in Z(T)_{S,D} \), and a natural number \( l \) such that \( h(t) = \lambda M(t)^l + c \), where \( M(t) \) denotes a monic invariant polynomial of minimal non-zero degree. In particular \( Z(T[t; S, D]) = Z(T)_{S,D}[\lambda M(t)^l] \).
Proof. We have seen in the proof of Theorem 3.6 that if \( 0 \neq f(t) \in Z(T[t; S, D]) \) then the leading coefficient \( a \) of \( f(t) \) is a normal element in \( T \) and \( S' = I_a \) where \( r = \deg f(t) \). We will frequently use this observation in the following proof.

(1) This is left to the reader.

(2) (i) Theorem 3.6 shows that there exist non-constant semi-invariant polynomials in \( T[t; S, D] \), hence the polynomial \( p(t) \) exists. Let \( a \) be the leading coefficient of \( f(t) \in Z(T[t; S, D]) \) and let \( m = \deg f(t) \). Then \( a \) is invertible and \( S^m = I_a^{-1} \). This implies that the monic polynomial \( a^{-1}f(t) \) is semi-invariant. Therefore by Proposition 2.8, \( a^{-1}f(t) = \sum_{i=0}^{m-1} d_i p(t)^i \) where the non-zero \( d_i \)'s satisfy \( S^m - m = I_{d_i} \). Consequently \( f(t) = \sum_{i=0}^{m-1} c_i p(t)^i \) where \( c_i = a^i d_i \) and the non-zero \( c_i \)'s satisfy \( S^m = I_{c_i} \).

(ii) It suffices to show that \( Z(T[t; S, D]) \subset Z(T, h(t)) \). Let \( q(t) \) be a non-zero central element in \( T[t; S, D] \). If \( \deg q(t) = 0 \) then \( q(t) \in Z(T[t; S, D]) \cap T = Z(T, h(t)) \). Assume \( \deg q(t) = n > 0 \) and that any central polynomial of degree less than \( n \) belongs to \( Z(T, h(t)) \).

We know that the leading coefficient of \( h(t) \) is invertible and we can divide \( q(t) \) by \( h(t) \) getting \( q(t) = b(t) h(t) + c(t) \), \( \deg c(t) < \deg h(t) \). We easily obtain from this that \( b(t)a = ab(t) \) and \( c(t)a = ac(t) \) for any \( a \in T \).

Case 1. \( \deg c(t) < \deg h(t) - 1 \). The reader can check that in this case \( b(t) \) and \( c(t) \in Z(T[t; S, D]) \). Therefore the minimality of \( \deg h(t) \) implies \( c(t) \in Z(T, h(t)) \) and if \( b(t) \neq 0 \), \( \deg b(t) < n \) and the induction hypothesis shows that \( b(t) \in Z(T, h(t)) \) and hence \( q(t) \in Z(T, h(t)) \).

Case 2. \( \deg c(t) = \deg h(t) - 1 \). From the fact that \( c(t)a = ac(t) \) one concludes that the leading coefficient of \( c(t) \) is invertible. Hence we can divide \( h(t) \) by \( c(t) \) obtaining \( h(t) = (ut + v) c(t) + d(t) \) where \( \deg d(t) < \deg c(t) \) and \( u, v \in T \), \( u \) invertible. We deduce from \( h(t)a = ah(t) \) and \( c(t)a = ac(t) \) that \( ut + v \) commutes with elements from \( T \) and thus \( T[t; S, D] = T[t; u + v] \). The thesis is then obvious.

(iii) Let \( h(t), M(t) \) be as in (iii). First we will show that there exist \( l \geq 1 \) and \( \gamma \in T \) such that \( \gamma M(t) \) is a central polynomial of minimal non-zero degree.

Case 1. \( D = 0 \). Then \( M(t) = t + u \) for some \( u \in T \). If \( u \neq 0 \), Lemma 1.3 implies that \( u \) is invertible and in this case \( u^{-1} M(t) \) is central.

Assume \( M(t) = t \). Let \( \gamma \) be the leading coefficient of \( h(t) \). Since \( h(t) \) is central, \( \gamma \) is invertible and \( S' = I_{\gamma^{-1}} \). Hence \( (\gamma t')a = a(\gamma t') \) for any \( a \in T \). Moreover, since \( h(t) \) commutes with \( t \), we obtain \( S(\gamma) = \gamma \). Hence \( \gamma t' \) commutes with \( t \) and \( \gamma M(t) = \gamma t' \) is central of minimal non-zero degree.
Case 2. \( D \neq 0 \). Let us divide \( h(t) \) by \( M(t) \),

\[
h(t) = b(t) M(t) + c(t),
\]

where \( b(t), c(t) \in T[t; S, D] \) and \( \deg c(t) < \deg M(t) = m \).

\( M(t) \) is invariant and monic, hence for any \( a \in T \) one has

\[
M(t) a = S^m(a) M(t)
\]

and there exists \( \gamma \in T \) such that

\[
M(t) t = (t + \gamma) M(t).
\]

Equations (1) and (2) imply that for any \( a \in T \)

\[
c(t) a = \alpha c(t) \quad \text{and} \quad b(t) a - S^{-m}(a) b(t).
\]

Let us first assume that \( \deg c(t) < \deg M(t) - 1 \). Then (1) and (3) give us

\[
c(t) t = \omega(t) \quad \text{and} \quad b(t)(t + \gamma) = d(t).
\]

From (4) and (5), it results that \( c(t) \) is central. Since \( \deg c(t) < \deg h(t) \) one concludes that \( c = c(t) \in Z(T)_{S,D} \). Formulas (4) and (5) also show that \( b(t) \) is invariant. Moreover the leading coefficient of \( b(t) \), say \( \beta \), must be invertible. Hence \( \beta^{-1} h(t) \) is a monic invariant polynomial and Proposition 3.4 allows us to write \( \beta^{-1} h(t) = \alpha \omega(t) M(t)^{l-1} \) for some invertible \( \alpha \in T, \, \omega(t) \in Z(T[t; S, D]) \) and \( l \geq 1 \). Then \( b(t) = \omega(t) M(t)^{l-1} \) where \( u = \beta \alpha \) is invertible in \( T \). Hence \( \deg \omega(t) < \deg b(t) < \deg h(t) \). The minimality of \( \deg h(t) \) then implies that \( \deg \omega(t) = 0 \) and so \( \omega(t) \in Z(T)_{S,D} \).

Thus \( b(t) = \gamma M(t)^{l-1} \) with \( \gamma = \omega(t) \in T \) and (1) shows that \( h(t) - c(t) = b(t) M(t) = \gamma M(t)^{l} \) is central, and since \( \deg(h(t)) = \deg(\gamma M(t)^{l}) \), \( \gamma M(t)^{l} \) is of minimal degree among central non-constant polynomials.

Let us now examine the case when \( \deg c(t) = \deg M(t) - 1 \). Equation (4) shows in particular that the leading coefficient of \( c(t) \) is invertible.

Let us divide \( M(t) \) by \( c(t) : M(t) = (ut + v) c(t) + d(t) \) where \( u, v \in T \) and \( u \) is invertible. Put \( y = ut + v \). Equations (2) and (4) imply that

\[
ya = S^m(a) y \quad \text{for every} \quad a \in T.
\]

Hence \( T[t; S, D] = T[y; S^m] \) and we are back to the case when \( D = 0 \).

Therefore in any case, we can find \( 0 \neq \gamma \in T \) and \( l \geq 1 \) such that \( \gamma M(t)^{l} \) is central of minimal non-zero degree. Now the statement (iii) is an easy consequence of (ii).

The following example shows that the center of \( R[t; S, D] \) can be trivial although the center of \( T[t; S, D] \) is not.
Example 3.8. Let $F$ be a field of characteristic zero and $R$ be a ring of infinite matrices of the form

$$
\begin{pmatrix}
M & 0 \\
O & \text{diag}(k, \ldots, k, \ldots)
\end{pmatrix},
$$

where $k \in F$, $M \in M_n(F)$ for some $n \geq 1$.

Then $T$, the symmetric Martindale ring of quotients of $R$, is the ring of infinite matrices over $F$ having in each row and column all but a finite number of zero entries from $F$ (cf. Proposition 3.3 in [25]).

We set $S = \text{identity}$ and $D = D_{a,S}$ where $a = \text{diag}(1, 0, 1, 0, \ldots) \in T$. Observe that $D_{R}$ is a derivation of $R$. Now, by Theorem 3.7, $Z(T[t; id, D]) = F[t - a]$ and we will show that $Z(R[t; id, D]) = F$. By Lemma 3.2, we know that $Z(R[t; id, D]) = F[t - a] \cap R[t; S, D]$.

Let $f(t) = \sum_{n=0}^{\infty} k_n(t - a)^n \in Z(R[t; S, D])$, $k_n \neq 0$. If $n > 0$ then the coefficient of $t^{n-1}$ in the polynomial $f(t)$ is equal to $-k_n a + k_{n-1} \notin R$. This shows that $n = 0$ and $Z(R[t; id, D]) = F$.

In case $S = id$, Theorem 3.6 says that the center of $T[t; S, D]$ is non-trivial if there are non-zero $T$-disjoint ideals in $T[t; S, D]$. Clearly if $I$ is a non-zero $T$-disjoint ideal of $T[t; S, D]$ then $I \cap R[t; S, D]$ is a non-zero $R$-disjoint ideal of $R[t; S, D]$. Thus Example 3.8 shows also that the above equivalence does not hold for $R[t; S, D]$.

Before giving the main theorem of this section, let us prove the following technical lemma which will be also useful in Section 4. Recall that if $A$ is a prime ring $Q_s(A)$ denotes the symmetric Martindale ring of quotient of $A$.

Lemma 3.9. Let $q$ be a unit in $Q_s(T[t; S, D])$ such that $Rq = qR$. If $I$ is a non-zero ideal of $R[t; S, D]$ such that $Iq \subset R[t; S, D]$, then there exists a monic invariant polynomial $f(t) \in T[t; S, D]$ such that $fq \in T[t; S, D]$.

Proof. Let $I$ be a non-zero ideal of $R[t; S, D]$ such that $Iq \subset R[t; S, D]$ and let $f(t) \in T[t; S, D]$ be the monic invariant polynomial associated to $I$ defined in Proposition 2.1. Then there exists a non-zero ideal $J$ of $R$ such that $Jf(t) \subset I$. Hence $Jfq \subset R[t; S, D]$. Let $0 \neq a \in J$, since $q$ is a unit $afq$ is a non-zero polynomial, say of degree $n$. Let $\bar{J} = \{b \in J| \deg bfq \leq n\}$. Clearly $\bar{J}$ is a left ideal of $R$. In fact $\bar{J}$ is a two-sided ideal. To see this put $\varphi = S^k \circ I_{q^R}$ where $k = \deg f(t)$. Notice that, because of our assumption on $q$, $\varphi$ is an automorphism of $R$. For any $r \in R$ and $b \in \bar{J}$ we have $fqr = \varphi(r)fq$ and $\deg(\text{brfq}) = \deg(\text{bfqf}^{-1}(r)) \leq \deg bfq \leq n$. Therefore $br \in \bar{J}$, i.e., $\bar{J}$ is a two-sided ideal of $R$. For $b \in \bar{J}$, there exist $b_0, \ldots, b_n \in R$ such that $bfq = \sum_{i=0}^{n} b_i t^i$. Clearly the maps $\alpha_i: \bar{J} \rightarrow R$ given by $\alpha_i(b) = b_i$, $i = 0, 1, \ldots, n$, are well defined homomorphisms of left $R$-modules. Thus there are elements $q_n, \ldots, q_0 \in Q_s(R)$ such that $\alpha_i(b) = b_i = bq_i$, $i = 0, 1, \ldots, n$. 


Let $\omega = \sum_{i=0}^{n} q_i t^i \in Q_i(R)[t; S, D]$. By the above, the following identity holds in $Q_i(R)[t; S, D]$:

$$bfq = b \omega \quad \text{for all} \quad b \in \bar{J}. \quad (1)$$

We want to know that the identity (1) is satisfied in $Q_i(T[t; S, D])$. A little care is needed here because $\omega \in Q_i(R)[t; S, D]$ and this ring is not included in $Q_i(T[t; S, D])$. We will show that $\omega \in T[t; S, D]$. Let $b \in \bar{J}$, $r \in R$. Then by (1), $b \omega r = b \varphi(r) \omega$. It means that $J((\omega r - \varphi(r) \omega)) = 0$ holds in $Q_i[R][t; S, D]$ for all $r \in R$. Therefore $\omega r = \varphi(r) \omega$ for all $r \in R$. Since $\varphi$ is an automorphism of $R$, Lemma 1.3 implies that $\omega \in T[t; S, D]$.

Thus we can look at (1) as an equation in $Q_i(T[t; S, D])$ and rewrite it in the form $\bar{J}(f q - \omega) = 0$. Hence by Lemma 1.4, $f q = \omega \in T[t; S, D]$. 

The description of the center of $T[t; S, D]$ enables us to prove the following.

**THEOREM 3.10.** The extended centroid of $R[t; S, D]$ is isomorphic to the field of quotient $ZZ^{-1}$, where $Z$ is the center of $T[t; S, D]$.

**Proof.** Let $C$ denote the extended centroid of $T[t; S, D]$. In virtue of Lemma 3.2 it is enough to show that $C = ZZ^{-1}$, where $Z = Z(T[t; S, D])$. Clearly $ZZ^{-1} \subseteq C$, and in order to establish the theorem it is enough to prove the reverse inclusion.

Let $c \in C \setminus \{0\}$ and consider $I = \{f(t) \in R[t; S, D] | f(t) c \in R[t; S, D]\}$. Obviously $I$ is a non-zero ideal of $R[t; S, D]$, and it is standard to check that $c$ and $I$ satisfy assumptions of Lemma 3.9. Therefore, there exists a monic invariant polynomial $f(t) \in T[t; S, D]$ such that $f(t) c \in T[t; S, D]$. If $f(t) = 1$, then $c \in Z$. So we may assume that $\deg f(t) = n \geq 1$.

**Case 1.** No power of $S$ is an inner automorphism of $T$. Let us set $f(t) c = \sum_{i=0}^{l} a_i t^i \in T[t; S, D]$, $a_i \neq 0$. Since $c$ commutes with elements from $T[t; S, D]$, $f(t) c$ is an invariant polynomial and

$$S^n(x) f(t) c = f(t) xc = f(t) cx = \left( \sum_{i=0}^{l} a_i t^i \right) x = a_i S^l(x) t^l + \cdots, \quad \text{for all} \quad x \in T.$$ 

Comparing the leading coefficients on both sides of this equation we get $a_i S^l(x) = S^n(x) a_i$ for all $x \in T$ and since no power of $S$ is an inner automorphism of $T$, Lemma 1.1(5) implies that $n = l$ and so $a : = a_l$ is central. Thus $a^{-1} f(t) c$ is a monic semi-invariant polynomial as well as $f(t)$. Because of our assumption on $S$, Proposition 2.8 yields that $f(t) = p(t)^k$ and $a^{-1} f(t) c = p(t)^l$ for some $k, l \in \mathbb{N}$ where $p(t)$ denotes the monic
semi-invariant polynomial of minimal nonzero degree. Observe that 
k = 1 because \( f(t)x = S^n(x)f(t) \) and \( a^{-1}f(t)cx = S^n(x)a^{-1}f(t)c \) for all 
\( x \in T \). Therefore \( 0 = f(t) - a^{-1}f(t)c = (1 - a^{-1}c)f(t) \). This shows that 
c = a \in T[t; S, D] \cap C = Z.

**Case 2.** A non-zero power of \( S \) is an inner automorphism of \( T \). By 
Proposition 3.5, there exist \( m \geq 1 \) and an invertible element \( v \in T \) that 
v \( ^{-1}f(t)^m \in Z \). Then also \( v^{-1}f(t)^m \in Z \) and \( c \in ZZ^{-1} \).}

In the special case when either \( D = 0 \) or \( S = \text{id} \), the above theorem was 
proved in [17].

It is well known that if \( I \) is a non-zero ideal of \( R \), then the centroid of 
\( I \) is a subring of \( C(R) \) in a natural way. The above theorem enables us to 
show that there exists a prime \( R \) such that the extended centroid \( C(R) \) is 
not generated as a field by centroids of ideals of \( R \). This answers Krempa's 
question (cf. [10]). The example was suggested to us by W. S. Martindale.

**Example 3.11.** Let \( F \) be a field and \( S \) an automorphism of \( F \) of infinite 
order. Let \( K \) denote the subfield of invariant element under the action 
of \( S \). Consider the ring \( R = F[X; S][Y; S] \). Then \( R \) has the following 
properties

1. \( C(R) = K(X^{-1}Y) \).
2. For every non-zero ideal \( I \) of \( R \), \( Z(I) = K \), where \( Z(I) \) denotes the 
centroid of \( I \).

For a prime ring \( A \), \( Q_s(A) \) will denote the symmetric Martindale's ring 
of quotients.

1. Since \( S \) is of infinite order, Theorems 3.6 and 3.7 imply that 
\( Z(F[X; S]) = K \) and Theorem 3.10 shows that 
\( Z(Q_s(F[X; S])) = K \). (i)

Because \( X \) is an invertible element in \( Q_s(F[X; S]) \), \( S \) becomes an inner 
automorphism of \( Q_s(F[X; S]) \) and Theorem 3.7 together with (i) yields that 

\[ Z = Z(Q_s(F[X; S])[Y; S]) = K[X^{-1}Y]. \]

Therefore by Theorem 3.10, \( C(R) = ZZ^{-1} = K(X^{-1}Y) \).

2. Let \( I \) be a non-zero ideal of \( R \) and \( 0 \neq c = f(t)/g(t) \in K(X^{-1}Y) \) 
where \( f(t) \) and \( 0 \neq g(t) \) are polynomials in \( t = X^{-1}Y \). First let us show that 
if \( c \in Z(I) \) then \( \deg_t f(t) = \deg_t g(t) \).

Let \( w, v \in I \) be non-zero polynomials of minimal degree with respect to \( X \)
and \( Y \), respectively. Suppose that \( c \in Z(I) \). Then \( f(X^{-1}Y)I \subset g(X^{-1}Y)I \) and the choice of \( w, v \) implies that

\[
\deg_X(w) - \deg, f(t) \geq \deg_X(w) - \deg, g(t) \\
\deg, (v) + \deg, f(t) \geq \deg, (\omega) + \deg, g(t).
\]

Hence \( \deg, f(t) = \deg, f(t) \).

Let \( 0 \neq c = f(t)/g(t) \in Z(I) \). The equality \( \deg, f(t) = \deg, g(t) \) implies that there exist \( k \in K \) and \( \bar{f}(t) \in K[t] \), \( \deg, \bar{f}(t) < \deg, g(t) \), such that \( c = k + \bar{f}(t)/g(t) \). Clearly \( k \in Z(I) \) and consequently \( \bar{f}(t)/g(t) \in Z(I) \) where \( \deg, \bar{f}(t) < \deg, g(t) \). But as we have just proved this is impossible unless \( \bar{f}(t)/g(t) = 0 \); i.e., \( c = k \in K \). This shows that \( Z(I) = K \).

4. \( X \)-INNER AUTOMORPHISMS OF \( R[t; S, D] \)

In this final section we determine the \( X \)-inner automorphisms \( \sigma \) of \( R[t; S, D] \) stabilizing \( R \), i.e. \( X \)-inner automorphisms such that \( \sigma(R) = R \). Throughout the section \( \sigma \) will denote such an automorphism of \( R[t; S, D] \).

In order to describe \( X \)-inner automorphisms we will have to work in various overrings of \( R[t; S, D] \). This forces us to extend \( \sigma \) to these overrings.

**Lemma 4.1.**

(i) \( \sigma|_R \) has a unique extension to an automorphism \( \sigma' \) of \( T \).

(ii) \( \sigma(t) = at + b \) where \( a, b \in R \) and \( a \) is invertible in \( R \).

(iii) \( \sigma \) has a unique extension to an automorphism of \( T[t; S, D] \), this extension preserves \( T \).

(iv) \( \sigma \) has unique extensions to \( Q_s(R[t; S, D]) \) and to \( Q_s(T[t; S, D]) \), the symmetric Martindale ring of quotients of \( R[t; S, D] \) and \( T[t; S, D] \), respectively.

**Proof.**

(i) This is well known as follows obviously from Lemma 2.1(iv) in [22].

(ii) Let us write \( \sigma(t) = a_it^i + \cdots + a_0 \), \( a_i \in R \), \( a_i \neq 0 \). Since \( \sigma \) is an automorphism, \( l \) must be \( > 0 \) and for any \( x \in R \) we have \( tx = S(x)t + D(x) \) and, by applying \( \sigma \) to this equality, we get

\[
\sigma(t) \sigma(x) = \sigma(S(x)) \sigma(t) + \sigma(D(x)) \quad \text{for all} \quad x \in R.
\]

Since \( \sigma \) preserves \( R \) we have \( \sigma(x) \in R \) and a comparison of the leading coefficients on both sides of the above equation yields

\[
a_i S^i(\sigma(x)) = \sigma(S(x)) a_i \quad \text{for all} \quad x \in R.
\]
In particular $a, R = Ra$, and Lemma 1.1(5) implies that $a, R = Ra$. Similarly, if $\sigma^{-1}(t) = c, s + \cdots + c, s$, $c, s \neq 0$, we conclude that $c, s$ is invertible in $T$. Since $\sigma = \sigma^{-1}(\sigma(t)) = \sigma^{-1}(\sigma(t))$, we obtain that $l = s = 1$, $\sigma(c, s) a, s = 1$, and $\sigma^{-1}(a, s) c, s = 1$. From this, (ii) follows easily.

(iii) Both $\sigma$ and $\sigma^{-1}$ preserve $R$ and $\sigma|_R$, $\sigma^{-1}|_R$ can be extended to automorphisms of $T$. Let $\sigma'$ and $\sigma'^{-1}$ be these extensions. For $f(t) = \sum q, t \in T[t; S, D]$ set $\sigma(f) = \sigma'(q, ) \sigma(t)$ and $\sigma^{-1}(f) = \sigma'^{-1}(q, ) \sigma^{-1}(t)$. It is easy to observe that $\sigma$ and $\sigma^{-1}$ are well defined homomorphisms of the additive structure of $T[t; S, D]$ and that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = 1$. In particular $\sigma$ is $1 - 1$ and onto.

We will now show that $\sigma$ is multiplicative.

Let $q \in T$ and $I$ a non-zero ideal of $R$ such that $Iq \subset R$. Consider $J = S^{-1}(I)I$. Then for $b \in J$ we have $D(b) \in I$. Let $n \geq 1$ then $S(b) qt^n = tbqt^n - D(b) qt^n$.

Applying $\sigma$ to this equality and using the fact that $bq$ and $D(b)q$ belong to $R$, we get

$$S(b)\sigma(qt^n)\sigma = t\sigma(bq)\sigma(t^n)\sigma - (D(b)q)\sigma(t^n)\sigma = t\sigma(b)\sigma(q^n)\sigma - (D(b)\sigma(q^n))\sigma = (S(b)\sigma)(t^n)\sigma - D(b)\sigma(q^n)\sigma = S(b)\sigma(t^n)\sigma - D(b)\sigma(q^n)\sigma = S(b)\sigma(q^n)\sigma.$$ 

Therefore $\sigma(S(J))(qt^n)\sigma - t\sigma(q^n)\sigma = 0$ for any $n \geq 1$. Since $\sigma$ and $S$ are automorphisms of $R$, $\sigma S(J)$ is a non-zero ideal of $R$, and the above equality yields $(qt^n)\sigma = t\sigma(q^n)\sigma$ for any $q \in T$ and $n \in N$. Now an easy inductive argument shows that $(t^mqt^n)\sigma = (t^m)\sigma(q^n)\sigma$ for any $q \in T$ and $m, n \in N$. This means that $\sigma$ is multiplicative.

Obviously the extension of $\sigma$ constructed above preserves $T$. On the other hand, it is easy to see that any extension of $\sigma \in \text{Aut}(R[t; S, D])$ to an automorphism of $T[t; S, D]$ must preserve $T$ and from this unicity of the extension is obvious.

(iv) As mentioned above this is easy and well known (loc. cit. [22]).

In the next lemma we will use the well-known (cf. [22]) internal characterization of $X$-inner automorphism. Recall [24] that an automorphism $\phi$ of a prime ring $R$ is $X$-inner if and only if there exist non-zero elements $a, b, c, d \in R$ which $arb = c\phi(r)d$ for all $r \in R$. 
Lemma 4.2. Let $\sigma$ be a $R$-stabilizing automorphism of $R[t; S, D]$. The following conditions are equivalent

1. $\sigma$ is an $X$-inner automorphism of $R[t; S, D]$.
2. $\sigma$ is an inner $X$-inner automorphism of $T[t; S, D]$.

Moreover if one of these conditions holds and $q \in Q_+(T[t; S, D])$ is a unit such that $qwq^{-1} = w^\sigma$ for all $w \in T[t; S, D]$ then there is an ideal $0 \neq I$ of $R[t; S, D]$ such that $Iq$, $Iq^{-1}$, $qI$, $q^{-1}I$ are contained in $R[t; S, D]$.

Proof. Implication $(1) \rightarrow (2)$ and the last statement follow from Lemma 2.5 of [24]. It remains to show that $(2) \rightarrow (1)$. Suppose that $\sigma$ is $X$-inner on $T[t; S, D]$. Thus there are non-zero $a, b, c, d$ in $T[t; S, D]$ such that $awb = cw^\sigma d$ for all $w \in T[t; S, D]$. Thus there are $a', b' \in R$ such that $0 \neq a'a$, $0 \neq b'b$, $a'c$ and $\sigma(b')d$ belong to $R[t; S, D]$. Then

$$a'awb'b = a'cw^\sigma(b')d \quad \text{for all} \quad w \in R[t; S, D].$$

Primeness of $R[t; S, D]$ implies $a'c$, $\sigma(b')d \neq 0$ and the above internal characterization of $X$-inner automorphism shows that $\sigma$ is an $X$-inner automorphism of $R[t; S, D]$. \[\qed\]

The above two lemmas say that while investigating $R$-stabilizing $X$-inner automorphisms of $R[t; S, D]$ it is possible to work inside $Q_+(T[t; S, D])$ instead of $Q_+(R[t; S, D])$ although, usually the former ring is not contained in the second one. Moreover Lemma 4.2 provides a natural method of constructing $X$-inner automorphisms $\sigma$ of two types:

Type I. Take $c \in T$ such that $cR = Rc$. Then $c$ is invertible in $T$. If $c\sigma^{-1} \in R[t; S, D]$, i.e., when $cS(c^{-1})$, $cD(c^{-1}) \in R$ then define $\sigma = _{R[t; S, D]}^c I_{R[t; S, D]}^c$.

Type II. Let $f(t) \in T[t; S, D]$ be a monic invariant polynomial. Then $f(t)$ is invertible in $Q_+(T[t; S, D])$. If $f(t)$ satisfies $R[t; S, D] \sigma f(t) = f(t) R[t; S, D]$, then take $\sigma = _{R[t; S, D]}^{f(t)} I_{R[t; S, D]}^{f(t)}$. Notice that in this case $\sigma|_R = S^n$ where $n = \deg f(t)$.

The results and techniques developed in previous sections enable us to show that, roughly speaking, the above mentioned $X$-inner automorphisms generate the group of all $X$-inner automorphisms of $R[t; S, D]$.

In the following theorem $M(t)$ will denote the monic invariant polynomial of minimal non-zero degree in $T[t; S, D]$ if such a polynomial exists, $M(t) = 1$ otherwise.

Theorem 4.3. Suppose that $\sigma$ is an $X$-inner automorphism of $R[t; S, D]$.
stabilizing $R$. Then there exist a unit $c \in T$ and an integer $r$ such that $\sigma = I_{c M(t)}$. Moreover:

1. The element $c$ satisfies the following conditions:
   
   (i) $cR = Rc$.
   
   (ii) $cS(c^{-1}) + cD(c^{-1}) + c\alpha c^{-1} \in R$, where $\alpha \in T$ is defined by the condition $M(t)'t = (t + \alpha) M(t)'$.

2. $\sigma(t) = cS(c^{-1}) t + cD(c^{-1}) + c\alpha c^{-1}$, where $\sigma \in T$ is as above.

3. $\sigma/R = (I_{c R}) \circ S^{mr}$ where $m = \deg M(t)$.

Proof. By Lemma 4.2, $\sigma$ is an $X$-inner automorphism of $T[t; S, D]$ and there exist a unit $u$ in $Q_s(T[t; S, D])$ and a non-zero ideal $I$ of $R[t; S, D]$ such that $u w u^{-1} = w^\sigma$ for all $w \in T[t; S, D]$ and $u, u^{-1} \in R[t; S, D]$. Since $\sigma$ stabilizes $R$ we have $u R = Ru$, hence Lemma 3.9 shows that there exists a manic invariant polynomial $f(t) \in T[t; S, D]$ such that $f u \in T[t, S, D]$. For any $r \in R$, $f u r = f \sigma^r u = S^k(\sigma(r)) f u$ where $k = \deg f$. This shows that $f u$ is semi-invariant and Lemma 1.3 implies that its leading coefficient is invertible in $T$. On the other hand, $f u t = f t^\sigma u$ and since $f$ is invariant one obtains, by making use of Lemma 4.1(ii), that there exist $c, d \in T$, $c$ invertible in $T$, such that $f u t = (ct + d) f u$. Hence $f u$ is invariant and its leading coefficient is invertible in $T$. Therefore there exist an invertible element $a \in T$ and a monic invariant polynomial $g$ in $T[t; S, D]$ such that

$$f u = ag.$$  \hfill (1)

Let $M(t)$ be the monic polynomial above. Since $f(t)$ and $g(t)$ are monic invariant polynomials, Proposition 3.4 and the definition of $M(t)$ yield that $f(t) = \alpha w(t) M(t)^l$ for some $l \in \mathbb{N}$, $w(t) \in Z(T[t; S, D])$ and $\alpha$ invertible in $T$. $g(t) = \alpha' w'(t) M(t)^k$ for some $s \in \mathbb{N}$, $w'(t) \in Z(T[t; S, D])$ and $\alpha'$ invertible in $T$.

The invariant polynomials $f(t)$, $g(t)$, $w'(t)$, as normal elements, become invertible in $Q_s(T[t; S, D])$ and formula (1) shows that

$$u = f^{-1} a g = S^{-k}(a) f^{-1} g = w^{-1} w'(a) S^{-ml}(\alpha^{-1} \alpha') M^{s-l}.$$  

Since $w$, $w'$ are central polynomials one gets $\sigma = I_u = I_{c M'(t)}$ where $c = S^{-k}(a) S^{-ml}(\alpha^{-1} \alpha')$ is invertible in $T$ and $r = s - l \in \mathbb{Z}$.

This completes the proof of the first part of the theorem.

For any $x \in R$ one has $\sigma(x) = u x u^{-1} = c M'(t) x M^{-r} c^{-1} = c S^{mr}(x) c^{-1}$ where $m = \deg M(t)$.

This proves statements (1)(i) and (3).
Since either $M(t)^-$ or $M(t)^+$ is a monic invariant polynomial in $T[t; S, D]$, there is $\alpha \in T$ such that $M(t)^+ t = (t + \alpha) M(t)^-$. Now we have

$$
\sigma(t) = I_{c, M}(t) = c M'(t) M^{-1} c^{-1} = c(t + \alpha) c^{-1}
$$

$$
= c S(c^{-1}) t + c D(c^{-1}) + c x c^{-1}.
$$

Since $\sigma(t) \in R[t; S, D]$, this proves (1)(ii) and (2).

The above theorem enables us to decompose $\sigma$ as $\sigma_1 \sigma_2$ where $\sigma_1$ is a conjugation by a unit $c \in T$ and $\sigma_2$ is a conjugation by a monic invariant polynomial. If $c x c^{-1} \in R$ then $\sigma_1$ and $\sigma_2$ are $X$-inner automorphisms of $R[t; S, D]$ of type I and II, respectively. As we will see this is always the case when $S$ and $D$ commute.

**Proposition 4.4.** Let $\sigma$ be an $X$-inner automorphism of $R[t; S, D]$ stabilizing $R$. Assume that $S$ is an $X$-inner automorphism of $R$. Then $\sigma$ is of type I, i.e., $\sigma = I_d$ where $d \in T$ is a normal element such that $d S(d^{-1}), d D(d^{-1}) \in R$. In particular $\sigma(t) = d S(d^{-1}) t + d D(d^{-1}).$

**Proof.** Let $v \in T$ be such that $S = I_v$, then $T[t; S, D] = T[v^{-1}; id, v^{-1} D] = T[y; id, v^{-1} D]$ and $M(t) = v^m M'(y)$ where $M(t)$ is the polynomial from $T[t; S, D]$ defined before Theorem 4.3, $m = \deg M(t)$, and $M'(y)$ is a monic invariant polynomial in $T[y; id, v^{-1} D]$. Hence Lemma 3.3 shows that $M'(y)$ is central in $T[y; id, v^{-1} D] = T[t; S, D]$. Theorem 4.3 implies that $\sigma = I_{c, M}$ for some $c \in T$ and $r \in Z$. By the above $M' = v^{m r} (M')'$ and since $(M')'$ is central one gets $\sigma = I_d$ where $d = c v^{m r}$. In particular $\sigma(t) = dt d^{-1} = d S(d^{-1}) t + d D(d^{-1}) \in R[t; S, D]$, and $d S(d^{-1}) \in R$, $d D(d^{-1}) \in R$. Since $\sigma = I_d$ stabilizes $R$, $d$ is a normal element.

The above proposition generalizes a result of S. Montgomery [20] who showed that $X$-inner automorphisms of $R[t; id, D]$ stabilizing $R$ are determined by normal elements $d \in T$ such that $d D(d^{-1}) \in R$.

In the sequel, we will need the following additional property of invariant polynomials.

**Lemma 4.5.** Let $f(t) = t^n + a t^{n-1} + \cdots \in T[t; S, D]$ be a monic invariant polynomial of nonzero degree. If $S$ is not an $X$-inner automorphism of $R$ and $S \circ D = D \circ S$, then $f(t)$ commutes with $t$.

**Proof.** We can apply Lemma 2.5(1) to the invariant polynomial $f(t)$ getting $S'' D S'' - D = D x S$ where $x = a - S(a)$. This yields $D_x S = 0$, since $S$ commutes with $D$. It means that for every $x \in R$, $x x = S(x) x$ and Lemma 1.1(5) implies that $S$ is an $X$-inner automorphism of $R$ if $x \neq 0$. Therefore $x = 0$ and Lemma 3.3 shows that $f(t)$ commutes with $t$. □
The following theorem shows that if $S$ and $D$ commute then $\sigma$ can be described without referring to an invariant polynomial.

**Theorem 4.6.** Let $\sigma$ be an $X$-inner automorphism of $R[t; S, D]$ stabilizing $R$. Assume that $S \circ D = D \circ S$. Then:

1. $S$ can be extended to $R[t; S, D]$ by setting $S(t) = t$.
2. $\sigma$ can be presented in a form $\sigma = I_c \circ S^l$ where $l \in \mathbb{Z}$, $d \in T$ is a normal element such that $dS(d^{-1}), dD(d^{-1}) \in R$. In particular $\sigma(t) = dS(d^{-1}) t + dD(d^{-1})$.

**Proof.** The statement (1) is easy and well known.

(2) If $S$ is an $X$-inner automorphism of $R$, the Proposition 4.4 gives the thesis with $l = 0$.

Assume that $S$ is not $X$-inner. Let $M(t)$ denote the polynomial defined before Theorem 4.3. By Theorem 4.3, $\sigma = I_{c_{M(t)}}$ for some $c \in T$ and $r \in \mathbb{Z}$. If $M(t) = 1$, then the thesis is clear. So suppose $\deg M(t) \geq 1$. Thus by Lemma 4.5, $M(t)$ commutes with $t$. Hence we have

$$\sigma(t) = cM(t)^r t M(t)^{-r} c^{-1} = cS(c^{-1}) t + cD(c^{-1}) = I_c \circ S^l(t) \in R[t, S, D]$$

and

$$\sigma(x) = cM(t)^{-r} x M(t)^r c^{-1} = cS^l(x) c^{-1} = I_c \circ S^l(x)$$

for $x \in R$, where $l = r \deg M(t)$.

This provides the proof of the theorem.

As a direct consequence of Theorem 4.3, Proposition 4.4, and Lemma 4.5 we get the following

**Theorem 4.7.** Suppose that $S \circ D = D \circ S$. Let $\sigma$ be an $X$-inner automorphism of $R[t; S, D]$ stabilizing $R$. Then $\sigma = \sigma_1 \sigma_2$ where $\sigma_1, \sigma_2$ are $X$-inner automorphisms of $R[t; S, D]$ of type I and II, respectively.

Let $\sigma$ be an $X$-inner automorphism of $R[t; S, D]$ stabilizing $R$. As we have seen in Proposition 4.4, $\sigma$ is of type I if $S$ is an $X$-inner automorphism of $R$. It is interesting to know in general when $\sigma$ is of type I. Since in this case $\sigma|_R$ has to be an $X$-inner automorphism of $R$, the following result offers a necessary condition for $\sigma$ to be of type I.

**Proposition 4.8.** Let $\sigma$, $M$, $c$, $r$ be as in Theorem 4.3, then some power of $\sigma|_R$ is $X$-inner if and only if some power of $S$ is $X$-inner.

**Proof.** Using notations from Theorem 4.3, we have $\sigma_R = (I_c|_R) \circ S^{mr}$. Let us put $n = mr$ and $\hat{I}_c : I_c|_R$. Then $\sigma|_R = \hat{I}_c \circ S^n$ and for any $s \in \mathbb{N}$

$$(\sigma|_R)^s = (\hat{I}_c \circ S^n)^s = I_u \circ S^{ns},$$
where $u = cS^n(c), \ldots, S^{n(l-1)}(c)$ is a normal element in $T$ and $\hat{I}_u = I_u|_R$. The equivalence stated in the proposition is now obvious.

The obtained results enable us to determine the group of all $X$-inner automorphisms of $R[t; S, D]$ in the case when $S$ commutes with $D$ and $R$ is a domain, then any $X$-inner automorphism of $R[t; S, D]$ stabilizes $R$ (Lemma 2 of [20]).

Recall that the center of $T$ is denoted by $C$ and $C_{S, D} = \{c \in C \mid S(c) = c, D(c) = 0 \} = Z(T[t; S, D]) \cap T$.

The group of all inner, $X$-inner automorphisms of $A = R[t; S, D]$ will be denoted by inn $A$, $X$-inn $A$, respectively.

Define $N = \{c \in T \mid cR = Rc, cS(c^{-1}) \in R$ and $cD(c^{-1}) \in R \}$. Since $N$ is exactly the set of all elements from $T$ which determine $X$-inner automorphisms of $R[t; S, D]$ of type I and since composition of such automorphisms is again an automorphism of type I, $N$ is a subgroup of units of $T$. In fact it is easy to check that the subgroup of all $X$-inner automorphisms of type I is isomorphic to $N/\langle Z(T[t; S, D]) \cap T \rangle$.

We will continue to denote $M(t)$ the monic invariant polynomial of minimal non-zero degree in $T[t; S, D]$ if such a polynomial exists and we put $M(t) = 1$ otherwise. Recall that, by Theorem 2.6, $M(t) = 1$ if and only if $R[t; S, D]$ has a non-zero $R$-disjoint ideals.

We have to consider composition of automorphisms of the form $I_{cM(t)^m} \cdot I_{dM(t)^l} = I_{g(t)}$ with $g(t) = cM(t)^m dM(t)^l = cS^{m(r)}(d) M(t)^{r+1}$ where $m = \deg M(t)$ and $r, l \in \mathbb{Z}$. Therefore we introduce the group $N \times \mathbb{Z}$ where $\mathbb{Z}$ is acting on $N$ and $S^m$, i.e., $(v, l) \cdot (v', l') = (vS^{m(r)}(v'), l + l')$ for $l, l' \in \mathbb{Z}$ and $v, v' \in N$. We also introduce $N_{S, D} = \{x \in N \mid S(x) = x, D(x) = 0 \}$ a multiplicative subgroup of $N$.

For a ring $B$, $U(B)$ will denote the multiplicative group of units of $B$.

**THEOREM 4.9.** Suppose that $R$ is a domain and that $S \circ D = D \circ S$, then for $A = R[t; S, D]$ we have:

(a) inn $A \cong U(R)/C_{S, D}/C_{S, D}$.

(b) If either $S$ is an $X$-inner automorphism of $R$ or $M(t) = 1$, then $X - \text{inn } A \cong N/C_{S, D}$ and $X - \text{inn } A/\text{inn } A \cong N/\langle U(R) \rangle C_{S, D}$.

(c) Assume $\deg M(t) = m \geq 1$ and $S$ is not an $X$-inner automorphism of $R$. Let $\mathbf{Z}$ act on $N$ via $S^m$ as above, then

(i) If no non-zero power of $S$ is $X$-inner then $X - \text{inn } A \cong (N/C_{S, D}) \times \mathbb{Z}$ and $X - \text{inn } A/\text{inn } A \cong (N/\langle U(R) \rangle C_{S, D}) \times \mathbb{Z}$.

(ii) If a non-zero power of $S$ is $X$-inner then

(1) There exists $\lambda \in N_{S, D}$, $r \in \mathbb{N}$, such that $Z[T; S, D] = C_{S, D}[\lambda M(t)^r]$. 


(2) \( X - \text{inn} A \cong ((N/C_{S,D}) \times \mathbb{Z})/\langle \lambda, r \rangle \) where \( \langle \lambda, r \rangle \) stands for the cyclic subgroup of \((N/C_{S,D}) \times \mathbb{Z}\) generated by \((\lambda, r)\).

(3) \( X - \text{inn} A/\text{inn} A \cong ((N/U(R) C_{S,D})) \times \mathbb{Z}/\langle \lambda, r \rangle. \)

**Proof.** Let us recall that if \( R \) is a domain, then any \( X \)-inner automorphism of \( A \) stabilizes \( R \) (Lemma 2 of [20]).

(a) Clearly \( \text{inn} A \cong U(A)/U(A) \cap Z(A) \). Since \( R \) is a domain we have \( U(A) = U(R) \). On the other hand, it is easy to check that \( U(A) \cap Z(A) = U(R) \cap C_{S,D} \). Therefore \( \text{inn} A \cong U(R)/U(R) \cap C_{S,D} \cong U(R) C_{S,D}/C_{S,D} \).

(b) Suppose that either \( S \) is \( X \)-inner or \( M(t) = 1 \). Then using either Proposition 4.4 or Theorem 4.3, respectively, we get that every \( X \)-inner automorphism of \( A \) is of type 1. Now it is standard to verify \( X - \text{inn} A \cong N/C_{S,D} \) and \( X - \text{inn} A/\text{inn} A \cong N/U(R) C_{S,D} \).

(c) Suppose that \( S \) is not \( X \)-inner and \( M(t) \neq 1 \). Then by Lemma 4.5, \( M(t) \) commutes with \( t \). Now using Theorem 4.3, it is straightforward to verify that the map \( \varphi: N \times \mathbb{Z} \to X - \text{inn} A \) given by \( \varphi((v, l)) = I_{\varphi M(t)^{l}} \) is an epimorphism of groups and \( \ker \varphi = \{ (v, l) \in N \times \mathbb{Z} \mid v \in C_{S,D} \text{ and } l = 0 \} \) and \( X - \text{inn} A \cong (N/C_{S,D}) \times \mathbb{Z} \).

By making use of (a) one easily gets \( X - \text{inn} A/\text{inn} A = (N/U(R) C_{S,D}) \times \mathbb{Z} \).

(ii) Assume that a non-zero power of \( S \) is an \( X \)-inner automorphism of \( R \). Then by Theorems 3.6 and 3.7, we have \( Z(T[t; S, D]) = C_{S,D}[\lambda M(t)^{r}] \) for some normal element \( \lambda \in T \) and \( r \in \mathbb{N} \). Since both \( M(t) \) and \( \lambda M(t)^{r} \) commute with \( t \), we get

\[
t = \lambda M(t)^{r} t M(t)^{-r} \lambda^{-1} = \lambda t \lambda^{-1} = \lambda S(\lambda^{-1}) t + \lambda D(\lambda^{-1}).
\]

It means that \( S(\lambda) = \lambda \) and \( D(\lambda) = 0 \), i.e., \( \lambda \in N_{S,D} \). This proves (1) of (ii).

Now let \( (v, l) \in \ker \varphi \), then \( (v, l)^{-1} = (S^{-m}(v^{-1}), -l) \in \ker \varphi \) so we can assume that \( l \in \mathbb{N} \).

Since \( (v, l) \in \ker \varphi \), we have \( v M(t)^{l} \in Z(T[t; S, D]) = C_{S,D}[\lambda M(t)^{r}] \). Hence there are \( \alpha_{0}, ..., \alpha_{s} \in C_{S,D} \) such that

\[
v M(t)^{l} = \alpha_{s}(\lambda M(t)^{r})^{s} + \alpha_{s-1}(\lambda M(t)^{r})^{s-1} + \cdots + \alpha_{0}
\]

\[
= \alpha_{s} \lambda^{s} M(t)^{rs} + \cdots + \alpha_{0},
\]
where the last equality is due to the fact that \( M(t) \) is invariant and \( S(\lambda) = \lambda \). This shows that \((v, l) = (x_\alpha, 0)(\lambda, r)'\). This means that
\[
\ker \varphi = C_{S,D} \langle (\lambda, r) \rangle \quad \text{and}
\]
\[
X - \text{inn } A \cong N \times \mathbb{Z} / \ker \varphi \cong ((N/C_{S,D}) \times \mathbb{Z}) / \langle (\lambda, r) \rangle.
\]

This proves (2) of (ii) and, by making use of (a), also yields (3) of (ii).

Let us remark that in virtue of Theorems 2.6 and (3.6) the technical assumptions of the above theorem can be written as:

(b) \( M(t) = 1 \) iff \( R[t; S, D] \) has no \( R \)-disjoint ideals.

(c)(i) \( R[t; S, D] \) has non-zero \( R \)-disjoint ideals but the center of \( R[t; S, D] \) is trivial.

(c)(ii) The center of \( R[t; S, D] \) is non-trivial.

**ACKNOWLEDGMENTS**

We thank T. Y. Lam and the referee for helpful comments and suggestions.

**REFERENCES**