J. Differential Equations 252 (2012) 2630–2647

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Bistable travelling waves in competitive recursion systems ☆

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ARTICLE INFO

Article history:
Received 6 July 2011
Available online 26 October 2011

MSC:
45G15
45M10
95D25

Keywords:
Upper and lower solutions
Bistable waves
Global stability
Monotone semiflows

ABSTRACT

This paper is devoted to the study of spatial dynamics for a class of discrete-time recursion systems, which describes the spatial propagation of two competitive invaders. The existence and global stability of bistable travelling waves are established for such systems under appropriate conditions. The methods involve the upper and lower solutions, spreading speeds of monostable systems, and the monotone semiflows approach.

1. Introduction

Population dispersal is a very important topic in spatial ecology. In order to consider the effects of a dispersal process on evolution dynamics, ordinary differential equations or difference equations with spatial structure are usually used. In this paper, we consider the following discrete-time two species competition model:

$$\begin{aligned} p_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1q_n(x-y))}k_1(y)\,dy, \\ q_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2p_n(x-y))}k_2(y)\,dy, \end{aligned} \tag{1.1}$$

☆ Research supported in part by the NSERC of Canada and the MITACS of Canada.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.005

where $p_n(x)$ and $q_n(x)$ denote the population densities of two species at time n and position x, respectively; $k_i(y)$ represents the dispersal kernel of two species and $\int_{\mathbb{R}} k_i(y)\,dy = 1$, $\int_{\mathbb{R}} e^{\alpha y} k_i(y)\,dy < \infty$, for all $\alpha \in \mathbb{R}$, $i = 1, 2$. We assume that all parameters are positive constants and the kernel k_i has the symmetric property $k_i(-y) = k_i(y)$, which implies that the dispersal is isotropic and that the growth and dispersal properties are the same at each point.

There have been extensive investigations on travelling wave solutions of monotone discrete-time recursion systems

$$u_{n+1} = Q[u_n], \quad n \geqslant 0, \tag{1.2}$$

where $u_n(x) = (u_n^1(x), \ldots, u_n^k(x))$ is a vector-valued function on $\mathbb{R}$, and Q is a translation invariant and order-preserving operator with monostable or bistable structure. We refer to [1,4,8,9,12,13] and references therein. It is well known that the change of variables

$$u_n = p_n, \qquad v_n = 1 - q_n$$

converts system (1.1) into the following cooperative system:

$$\begin{aligned} u_{n+1}(x) &= \int\limits_{\mathbb{R}} \frac{(1+r_1)u_n(x-y)}{1 + r_1(u_n(x-y) + a_1(1 - v_n(x-y)))} k_1(y)\,dy, \\ v_{n+1}(x) &= \int\limits_{\mathbb{R}} \frac{a_2 r_2 u_n(x-y) + v_n(x-y)}{1 + r_2((1 - v_n(x-y)) + a_2 u_n(x-y))} k_2(y)\,dy, \end{aligned} \tag{1.3}$$

which is order preserving with respect to the standard componentwise ordering in the relevant range $0 \leqslant u_n \leqslant 1$, $0 \leqslant v_n \leqslant 1$. Note that system (1.1) has four possible constant equilibria: $(0,0)$, $(0,1)$, $(1,0)$, and (p^*, q^*), where

$$p^* = \frac{1 - a_1}{1 - a_1 a_2}, \qquad q^* = \frac{1 - a_2}{1 - a_1 a_2},$$

and hence, system (1.3) has four equilibria: $E^0 = (0,1)$, $E^1 = (0,0)$, $E^2 = (1,1)$, and $E^3 = (u^*, v^*)$, where $u^* = p^*$, $v^* = 1 - q^*$. It is easy to see that the positive coexistence equilibrium exists if and only if $(1 - a_1)(1 - a_2) > 0$, and otherwise it is biologically irrelevant.

For the spatially homogeneous system associated with (1.1):

$$\begin{aligned} p_{n+1} &= \frac{(1+r_1)p_n}{1 + r_1(p_n + a_1 q_n)}, \\ q_{n+1} &= \frac{(1+r_2)q_n}{1 + r_2(q_n + a_2 p_n)}, \end{aligned} \tag{1.4}$$

Cushing et al. gave a complete classification of its global dynamics (see [2, Lemma 3]). Weinberger, Lewis and Li [13] obtained sufficient conditions for the linear determinacy of spreading speed of system (1.2) with the monostable structure, and applied their results to system (1.1) in a companion paper [5]. Recently, Lin, Li and Ruan [7] established the existence of monostable traveling waves connecting unstable equilibrium $(0,0)$ and stable equilibrium (p^*, q^*), and the spreading speed for system (1.1) with $a_1, a_2 \in (0,1)$. If $a_1, a_2 \in (1, +\infty)$, we know from [2, Lemma 3] that the equilibrium (p^*, q^*) is a saddle, $(0,1)$ and $(1,0)$ are stable, and $(0,0)$ is unstable for the spatially homogeneous system (1.4). Further, there exists a separatrix Γ such that all orbits of system (1.4) below Γ converge to $(1,0)$, while all orbits of system (1.4) above Γ converge to $(0,1)$. In the current paper, we are interested in the existence of bistable travelling waves connecting $(0,1)$ and $(1,0)$, and their global stability with phase shift. Clearly, it suffices to study travelling waves connecting E^1 to E^2 for system (1.3). In order to obtain bistable travelling waves, we appeal to the theory of bistable waves recently developed in [3] for monotone semiflows, which allow the existence of multiple intermediate unstable equilibria in between two stable ones. For the global stability of travelling waves, we use a dynamical system approach, as illustrated in [15, Theorem 10.2.1] and [14, Theorem 3.1].

The rest of this paper is organized as follows. In Section 2, we establish the existence of bistable travelling waves by verifying the abstract assumptions in [3]. In Section 3, we use a global convergence result for monotone systems (see [15, Theorem 2.2.4]) and the method of upper and lower solutions to prove the global stability of travelling waves and their uniqueness up to translation. In Section 4, we present some numerical simulations to illustrate our analytic results.

2. Existence of travelling waves

In this section, we establish the existence of bistable travelling waves for system (1.3). We start with some notations.

Let $\mathcal{C} := C(\mathbb{R}, \mathbb{R}^2)$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ equipped with the compact open topology, that is, a sequence ψ_n converges to ψ in $\mathcal{C}$ if and only if $\psi_n(x)$ converges to $\psi(x)$ in $\mathbb{R}^2$ uniformly for x in any compact subset of $\mathbb{R}$. Let $\mathcal{C}_+ = \{(\psi_1, \psi_2) \in \mathcal{C}\colon\ \psi_i(x) \geqslant 0,\ \forall x \in \mathbb{R},\ i = 1, 2\}$. It is easy to see that $\mathcal{C}_+$ is a nonempty closed cone of $\mathcal{C}$ and induces a partial order of $\mathcal{C}$. For any $\psi^1 = (\psi_1^1, \psi_2^1), \psi^2 = (\psi_1^2, \psi_2^2) \in \mathcal{C}$, we denote $\psi^2 \geqslant_{\mathcal{C}} \psi^1$ if $\psi^2 - \psi^1 \in \mathcal{C}_+$ and $\psi^2 >_{\mathcal{C}} \psi^1$ if $\psi^2 - \psi^1 \in \mathcal{C}_+ \setminus \{0\}$. For any vectors a, b in $\mathbb{R}^2$, we can define $a \geqslant (>) b$ similarly, and $a \gg b$ if $a - b \in int(\mathbb{R}^2_+)$. For any $a, b, r \in \mathbb{R}^2$ with $a \leqslant b$ and $r \gg 0$, we define $\mathcal{C}_r := \{\psi \in \mathcal{C}\colon\ r \geqslant \psi \geqslant 0\}$ and $\mathcal{C}_{[a,b]} := \{\psi \in \mathcal{C}\colon\ b \geqslant \psi \geqslant a\}$.

Since we are interested in bistable travelling waves, throughout this paper we assume that $a_1 > 1$ and $a_2 > 1$. It is easy to see that the existence of travelling waves connecting two stable equilibria $(0, 1)$ and $(1, 0)$ in system (1.1) is equivalent to that of travelling waves connecting two ordered stable equilibria E^1 and E^2 in system (1.3). Further, there are two unordered unstable equilibria E^0 and E^3 between these two stable ones.

Let $\beta \in \mathbb{R}^2_+$ and Q be a map from $\mathcal{C}_\beta$ to $\mathcal{C}_\beta$ with $Q(0) = 0$ and $Q(\beta) = \beta$. Let E be the set of all fixed points of Q restricted on $[0, \beta]_{\mathbb{R}^2}$. According to [3], we need the following assumptions:

(A1) (*Translation invariance*) $T_y \circ Q[\phi] = Q \circ T_y[\phi]$, $\forall \phi \in \mathcal{C}_\beta,\ y \in \mathbb{R}$, where T_y is defined by $T_y[\phi](x) = \phi(x - y)$.
(A2) (*Continuity*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology.
(A3) (*Monotonicity*) Q is order preserving in the sense that $Q[\phi] \geqslant Q[\psi]$ whenever $\phi \geqslant \psi$ in $\mathcal{C}_\beta$.
(A4) (*Compactness*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact with respect to the compact open topology.
(A5) (*Bistability*) Two fixed points 0 and β are strongly stable from above and below, respectively, for the map $Q : [0, \beta] \to [0, \beta]$, that is, there exist a number $\delta > 0$ and unit vectors e_1 and $e_2 \in int(\mathbb{R}^2_+)$ such that

$$Q[\eta e_1] \ll \eta e_1, \qquad Q[\beta - \eta e_2] \gg \beta - \eta e_2, \quad \forall \eta \in (0, \delta],$$

and the set $E \setminus \{0, \beta\}$ is totally unordered.
(A6) (*Counter-propagation*) For each $\alpha \in E \setminus \{0, \beta\}$, $c^*_-(\alpha, \beta) + c^*_+(0, \alpha) > 0$, where $c^*_-(\alpha, \beta)$ and $c^*_+(0, \alpha)$ represent the leftward and rightward spreading speeds of monostable subsystem $\{Q^n\}_{n \geqslant 0}$ restricted on $[\alpha, \beta]_{\mathcal{C}}$ and $[0, \alpha]_{\mathcal{C}}$, respectively.

By [3, Theorem 3.1], it follows that under assumptions (A1)–(A6), there exists $c \in \mathbb{R}$ such that the discrete semiflow $\{Q^n\}_{n \geqslant 0}$ admits a nondecreasing travelling wave with speed c and connecting 0 and β, that is, there exists a nondecreasing function $\varphi \in \mathcal{C}$ such that $Q^n[\varphi](x) = \varphi(x - cn)$, $\forall x \in \mathbb{R}$, $n \geqslant 0$, with $\varphi(-\infty) := \lim_{x \to -\infty} \varphi(x) = 0$ and $\varphi(+\infty) := \lim_{x \to +\infty} \varphi(x) = \beta$.

Define an operator $Q = (Q_1, Q_2)$ on $\mathcal{C}$ by

$$Q_1[u,v](x) = \int_{\mathbb{R}} \frac{(1+r_1)u(x-y)}{1+r_1(u(x-y)+a_1(1-v(x-y)))} k_1(y)\,dy,$$

$$Q_2[u,v](x) = \int_{\mathbb{R}} \frac{a_2r_2u(x-y)+v(x-y)}{1+r_2((1-v(x-y))+a_2u(x-y))} k_2(y)\,dy.$$

Then system (1.3) can be expressed as

$$U_{n+1}(x) = Q[U_n](x), \qquad U_n := (u_n, v_n), \quad n \geqslant 0.$$

Lemma 2.1. *The map Q satisfies* (A1)–(A6) *with $\beta = E^2$ and $E = \{E^0, E^1, E^2, E^3\}$.*

Proof. It is easy to verify Q satisfies (A1)–(A4). It remains to prove (A5) and (A6).

Let $\widehat{Q}$ be the restriction of Q to $[0,\beta]$, that is, $\widehat{Q} = (\widehat{Q}_1, \widehat{Q}_2)$ and

$$\widehat{Q}_1[u,v] = \frac{(1+r_1)u}{1+r_1(u+a_1(1-v))},$$

$$\widehat{Q}_2[u,v] = \frac{a_2r_2u+v}{1+r_2((1-v)+a_2u)}.$$

Then $\widehat{Q}$ has four fixed points E^i, $i = 0,1,2,3$, and we need to show that the fixed point $E^1 = (0,0)$ is stable from above and $E^2 = (1,1)$ is stable from below. The Jacobian matrices of $\widehat{Q}$ at E^1 and E^2 are

$$J_{E^1} = \begin{pmatrix} \frac{1+r_1}{1+a_1r_1} & 0 \\ \frac{a_2r_2}{1+r_2} & \frac{1}{1+r_2} \end{pmatrix}, \qquad J_{E^2} = \begin{pmatrix} \frac{1}{1+r_1} & \frac{a_1r_1}{1+r_1} \\ 0 & \frac{1+r_2}{1+a_2r_2} \end{pmatrix}.$$

It is obvious that J_{E^1} has two positive eigenvalues $\lambda_1 = \frac{1+r_1}{1+a_1r_1} < 1$ and $\lambda_2 = \frac{1}{1+r_2} < 1$. If $\lambda_1 > \lambda_2$, then J_{E^1} has a unit eigenvector $e_0 \gg 0$ associated with λ_1 such that

$$J_{E^1}(e_0) = \lambda_1 e_0 \ll e_0;$$

if $\lambda_1 \leqslant \lambda_2 < 1$, we take $k \in (\lambda_2, 1)$, $\varepsilon_0 \in (0, \frac{(k-\lambda_2)(1+r_2)}{a_2r_2})$ and unit vector $e_0 = (\frac{\varepsilon_0}{\sqrt{1+\varepsilon_0^2}}, \frac{1}{\sqrt{1+\varepsilon_0^2}})^T \gg 0$ such that

$$J_{E^1}(e_0) \ll ke_0 \ll e_0.$$

By the continuous differentiability of $\widehat{Q}$, it then follows that there exists $\delta > 0$ such that

$$\begin{aligned} \widehat{Q}(\eta e_0) &= \widehat{Q}(0) + \int_0^1 D\widehat{Q}(t\eta e_0)\eta e_0\,dt \\ &= \eta \int_0^1 D\widehat{Q}(t\eta e_0)e_0\,dt \\ &\leqslant \eta k e_0 \ll \eta e_0 \end{aligned}$$

for all $\eta \in (0,\delta]$, and hence, E^1 is strongly stable from above for the map $\widehat{Q}$. A similar argument shows that E^2 is strongly stable from below.

In order to calculate the spreading speed $c^*(E^0, E^1)$, we only need to consider the following one-dimensional monotone subsystem of (1.1):

$$q_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+r_2)q_n(x-y)}{1+r_2 q_n(x-y)} k_2(y)\,dy, \quad n \geqslant 0. \tag{2.1}$$

Let $h(q) = \frac{(1+r_2)q}{1+r_2 q}$, $\forall q \in [0,1]$. Then h satisfies the following two conditions:

(H1) $h \in C([0,1],[0,1])$, $h(0)=0$, $h'(0) = 1+r_2 > 1$, $h(1)=1$, and $|h(q_1)-h(q_2)| < (1+r_2)|q_1-q_2|$, $\forall q_1, q_2 \in [0,1]$.
(H2) $q < h(q) < h'(0)q$, $\forall q \in (0,1)$, and $h'(q) = \frac{1+r_2}{(1+r_2 q)^2} > 0$, $\forall q \in [0,1]$.

By [4, Theorem 2.1], (2.1) has a monostable travelling wave connecting 0 to 1 with the minimal wave speed c_h^*, where

$$c_h^* = \inf_{\mu>0} \frac{\ln(h'(0)\int_{\mathbb{R}} e^{\mu y} k_2(y)\,dy)}{\mu} = \inf_{\mu>0} \frac{\ln((1+r_2)\int_{\mathbb{R}} e^{\mu y} k_2(y)\,dy)}{\mu}$$

is the spreading speed, and $c^*(E^0, E^1) = c_h^*$.

For the computation of $c^*(E^0, E^2)$, we consider the following one-dimensional monotone system

$$p_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+r_1)p_n(x-y)}{1+r_1 p_n(x-y)} k_1(y)\,dy, \quad n \geqslant 0. \tag{2.2}$$

Using the similar analysis as we did for system (2.1), we get

$$c^*\big(E^0, E^2\big) = \inf_{\mu>0} \frac{\ln((1+r_1)\int_{\mathbb{R}} e^{\mu y} k_1(y)\,dy)}{\mu}.$$

Further, we have the following claim.

Claim 1. $c^*(E^0, E^1) + c^*(E^0, E^2) > 0$.

Since $k_i(-y) = k_i(y)$, $\forall y \in \mathbb{R}$, $i = 1, 2$, we have

$$\begin{aligned} K_i(\mu) &:= \int_{-\infty}^{\infty} e^{\mu y} k_i(y)\,dy = \int_{-\infty}^{\infty} e^{-\mu y} k_i(y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big(e^{\mu y} + e^{-\mu y}\big) k_i(y)\,dy \\ &> \int_{-\infty}^{\infty} k_i(y)\,dy = 1, \end{aligned}$$

and hence,

$$\ln\left((1+r_i)\int\limits_{\mathbb{R}} e^{\mu y}k_i(y)\,dy\right) > 0.$$

Thus, we obtain

$$\lim_{\mu\to 0^+}\frac{\ln((1+r_i)\int_{\mathbb{R}} e^{\mu y}k_i(y)\,dy)}{\mu} = \infty.$$

On the other hand, since $\int_{-\infty}^{\infty} k_i(y)\,dy = 2\int_0^{\infty} k_i(y)\,dy = 1$, there exists a sufficiently small number $y_0 > 0$ such that $\int_{y_0}^{\infty} k_i(y)\,dy > 0$, and

$$\int\limits_{-\infty}^{\infty} e^{\mu y}k_i(y)\,dy \geqslant \int\limits_{y_0}^{+\infty} e^{\mu y}k_i(y)\,dy \geqslant e^{\mu y_0}\int\limits_{y_0}^{+\infty} k_i(y)\,dy.$$

By L'Hôspital's rule, we have

$$\liminf_{\mu\to\infty}\frac{\ln((1+r_i)\int_{\mathbb{R}} e^{\mu y}k_i(y)\,dy)}{\mu} \geqslant \lim_{\mu\to\infty}\frac{\ln((1+r_i)e^{\mu y_0}\int_{y_0}^{\infty} k_i(y)\,dy)}{\mu} = y_0 > 0.$$

Therefore,

$$c^*\left(E^0, E^i\right) = \inf_{\mu>0}\frac{\ln((1+r_i)\int_{\mathbb{R}} e^{\mu y}k_i(y)\,dy)}{\mu} > 0, \quad i = 1, 2,$$

and hence, $c^*(E^0, E^1) + c^*(E^0, E^2) > 0$.

For the computation of $c^*(E^3, E^2)$, we let $x_n = u_n - u^*$ and $y_n = v_n - v^*$, then system (1.3) becomes

$$\begin{aligned}
x_{n+1}(x) &= -u^* + \int\limits_{\mathbb{R}} \frac{(1+r_1)(u^* + x_n(x-y))}{1 + r_1(u^* + x_n(x-y)) + a_1r_1(1 - (v^* + y_n(x-y)))}k_1(y)\,dy,\\
y_{n+1}(x) &= -v^* + \int\limits_{\mathbb{R}} \frac{a_2r_2(u^* + x_n(x-y)) + v^* + y_n(x-y)}{1 + r_2(1 - (v^* + y_n(x-y))) + a_2r_2(u^* + x_n(x-y))}k_2(y)\,dy.
\end{aligned} \tag{2.3}$$

It is easy to verify that system (2.3) is cooperative and positively invariant in $\mathcal{C}_{[0,\beta]} := \{\psi \in \mathcal{C}\colon\ 0 \leqslant \psi \leqslant \beta\}$, $\beta = (1-u^*, 1-v^*) \gg 0$. The spatially homogeneous system

$$\begin{aligned}
x_{n+1} &= -u^* + \frac{(1+r_1)(u^* + x_n)}{1 + r_1(u^* + x_n) + a_1r_1(1 - (v^* + y_n))},\\
y_{n+1} &= -v^* + \frac{a_2r_2(u^* + x_n) + v^* + y_n}{1 + r_2(1 - (v^* + y_n)) + a_2r_2(u^* + x_n)}
\end{aligned} \tag{2.4}$$

has stable equilibrium 0 and unstable one β in $[0, \beta] \subset \mathbb{R}^2$, and there are no other equilibria between these two equilibria.

In order to compute the spreading speed $c^*(0, \beta)$ of system (2.3), we consider the linearization of (2.3) at zero solution

$$x_{n+1}(x) = \int_{\mathbb{R}} \left(\frac{1 + a_1 r_1 (1 - v^*)}{1 + r_1} x_n(x - y) + \frac{a_1 r_1 u^*}{1 + r_1} y_n(x - y) \right) k_1(y)\, dy,$$

$$y_{n+1}(x) = \int_{\mathbb{R}} \left(\frac{a_2 r_2 (1 - v^*)}{1 + r_2} x_n(x - y) + \frac{1 + r_2 v^*}{1 + r_2} y_n(x - y) \right) k_2(y)\, dy. \tag{2.5}$$

For any $\mu \in \mathbb{R}_+$, let $x_n(x) = e^{-\mu x}\beta_n$, $y_n(x) = e^{-\mu x}\gamma_n$, $n \geqslant 0$. Then β_n, γ_n satisfy

$$\beta_{n+1} = \frac{1 + a_1 r_1 (1 - v^*)}{1 + r_1} K_1(\mu)\beta_n + \frac{a_1 r_1 u^*}{1 + r_1} K_2(\mu)\gamma_n,$$

$$\gamma_{n+1} = \frac{a_2 r_2 (1 - v^*)}{1 + r_2} K_1(\mu)\beta_n + \frac{1 + r_2 v^*}{1 + r_2} K_2(\mu)\gamma_n. \tag{2.6}$$

Define the matrix

$$B_\mu := \begin{pmatrix} \frac{1+a_1 r_1(1-v^*)}{1+r_1} K_1(\mu) & \frac{a_1 r_1 u^*}{1+r_1} K_2(\mu) \\ \frac{a_2 r_2 (1-v^*)}{1+r_2} K_1(\mu) & \frac{1+r_2 v^*}{1+r_2} K_2(\mu) \end{pmatrix}.$$

It is easy to see B_μ is positive for any $\mu \geqslant 0$, that is, each entry of B_μ is positive. Let $\lambda(\mu)$ be the principle eigenvalue of B_μ, then $\lambda(\mu)$ is positive with a strongly positive eigenvector (see [10, Theorem A.4]). In particular,

$$B_0 = \begin{pmatrix} \frac{1+a_1 r_1(1-v^*)}{1+r_1} & \frac{a_1 r_1 u^*}{1+r_1} \\ \frac{a_2 r_2 (1-v^*)}{1+r_2} & \frac{1+r_2 v^*}{1+r_2} \end{pmatrix}.$$

Simple calculation can show that B_0 is to be the Jacobian matrix of $\widetilde{Q}$ evaluated at 0. From the instability of 0, we know that $\lambda(0) > 1$. Since $K_i(\mu) > 1$, $\forall \mu > 0$, $i = 1, 2$, we have $B_\mu > B_0$, $\forall \mu > 0$. From the monotonicity of the principle eigenvalue with respect to the positive matrix [10, Theorem A.4], we know $\lambda(\mu) > \lambda(0) > 1$, $\forall \mu > 0$. Let $\Phi(\mu) := \frac{\ln \lambda(\mu)}{\mu}$, then $\Phi(\mu) > 0$, $\forall \mu > 0$ and $\lim_{\mu \to 0^+} \Phi(\mu) = \infty$. Further, we have

$$\begin{aligned}
\liminf_{\mu \to \infty} \Phi(\mu) &= \liminf_{\mu \to \infty} \frac{\ln \lambda(\mu)}{\mu} \\
&= \liminf_{\mu \to \infty} \frac{\ln \frac{\operatorname{tr} B_\mu + \sqrt{(\operatorname{tr} B_\mu)^2 - 4 \det B_\mu}}{2}}{\mu} \\
&\geqslant \liminf_{\mu \to \infty} \frac{\ln(\operatorname{tr} B_\mu)}{\mu} \\
&\geqslant \liminf_{\mu \to \infty} \frac{\ln \frac{1+a_1 r_1(1-v^*)}{1+r_1} K_1(\mu)}{\mu} \\
&\geqslant \lim_{\mu \to \infty} \frac{\ln \frac{1+a_1 r_1(1-v^*)}{1+r_1} e^{\mu y_0} \int_{y_0}^{\infty} k_1(y)\, dy}{\mu} \\
&= y_0 > 0,
\end{aligned}$$

where $\operatorname{tr} B_\mu$ is the trace of B_μ. It follows that $\bar{c} := \lim_{\mu > 0} \Phi(\mu) > 0$.

Let operator $\widetilde{Q}$ and M from $\mathcal{C}_{[0,\beta]}$ to $\mathcal{C}_{[0,\beta]}$ be defined by the right-hand side of systems (2.3) and (2.5), respectively. Since B_μ is positive, for any $\epsilon \in (0,1)$, we can choose $\vec{\delta} := (\delta, \delta)^T \gg 0$ in $\mathbb{R}^2$ sufficiently small such that

$$\widetilde{Q}[\psi] \geqslant (1-\epsilon)M[\psi], \quad \forall \psi \in \mathcal{C}_{[0,\vec{\delta}]}.$$

Let $M_\epsilon = (1-\epsilon)M$. Then M_ϵ is monotonic and satisfying $Q \geqslant M_\epsilon$, and $M_\epsilon \to M$ as $\epsilon \to 0$. By [6, Theorem 3.10], we know $c^*(0,\beta) \geqslant \bar{c}$. Then we have $c^*(E^3, E^2) = c^*(0,\beta) \geqslant \bar{c} > 0$.

In order to compute $c^*(E^3, E^1)$, let $x_n = -u_n + u^*$, $y_n = -v_n + v^*$. Then system (1.3) becomes

$$\begin{aligned} x_{n+1}(x) &= u^* - \int_{\mathbb{R}} \frac{(1+r_1)(u^* - x_n(x-y))}{1 + r_1(u^* - x_n(x-y)) + a_1 r_1(1-(v^* - y_n(x-y)))} k_1(y)\,dy, \\ y_{n+1}(x) &= v^* - \int_{\mathbb{R}} \frac{a_2 r_2(u^* - x_n(x-y)) + v^* - y_n(x-y)}{1 + r_2(1-(v^* - y_n(x-y))) + a_2 r_2(u^* - x_n(x-y))} k_2(y)\,dy. \end{aligned} \tag{2.7}$$

It is easy to verify that system (2.7) is cooperative and the spatially homogeneous system has unstable equilibrium 0 and stable equilibrium $\eta = (u^*, v^*) \gg 0$ in $[0,\eta] \subset \mathbb{R}^2$. Using a similar linearization argument as we did for system (2.3), we get the spreading speed $c^*(0,\eta)$ of (2.7), and $c^*(E^3, E^1) = c^*(0,\eta) > 0$. Therefore, $c^*(E^3, E^2) + c^*(E^3, E^1) > 0$. □

As a consequence of Lemma 2.1 and [3, Theorem 3.1], we have the following result.

Theorem 2.1. *Let all parameters be positive and $a_1, a_2 \in (1, \infty)$. Then there exists $c \in \mathbb{R}$ such that the cooperative system* (1.3)*, which is obtained by making substitution $u_n = p_n$, $v_n = 1 - q_n$ in model* (1.1)*, has a nondecreasing travelling wave $\varphi(x - cn) \in \mathcal{C}_{E^2}$ with speed c and connecting two stable equilibria $E^1 = (0,0)$ and $E^2 = (1,1)$.*

3. Global stability

In this section, we determine the global stability and uniqueness of bistable travelling waves for system (1.3).

Let $\varphi(x - cn) = (\varphi_1(x - cn), \varphi_2(x - cn))$ be a nondecreasing travelling wave solution of (1.3) connecting E^1 to E^2. Letting $z = x - c(n+1)$, we transform (1.3) into the following system

$$\bar{U}_{n+1}(z) = T_{-c} \circ Q[\bar{U}_n](z), \quad n \geqslant 0. \tag{3.1}$$

Thus, $\varphi(z)$ is an equilibrium solution of system (3.1), that is, $\varphi(z) = T_{-c} \circ Q[\varphi](z)$, $\forall z \in \mathbb{R}$. In what follows, we denote $\bar{U}_n(z, \psi)$ to be the solution of (3.1) with initial data $\bar{U}_0 = \psi$. Clearly, the solution $U_n(x, \psi)$ of (1.3) with initial data ψ is given by $U_n(x,\psi) = \bar{U}_n(x - cn, \psi)$. Then we have the following observation.

Lemma 3.1. *The following statements are valid*:

(i) *If $\psi \in \mathcal{C}_{[E^1, E^2]}$ is nondecreasing and satisfies*

$$\limsup_{\xi \to -\infty} \psi(\xi) \ll E^3 \ll \liminf_{\xi \to \infty} \psi(\xi), \tag{3.2}$$

then for any $\varepsilon > 0$, there exists $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ such that $\varphi(z - \tilde{z}) - \vec{\varepsilon} \leqslant \bar{U}_0(z, \psi) \leqslant \varphi(z + \tilde{z}) + \vec{\varepsilon}$.

(ii) *If the kernel* k_i, $i = 1, 2$, *has a compact support, then for any* $\varepsilon > 0$ *and* $\psi \in \mathcal{C}_{[E^1,E^2]}$ *with*

$$\limsup_{\xi\to-\infty} \psi(\xi) \ll E^3 \ll \liminf_{\xi\to\infty} \psi(\xi), \tag{3.3}$$

there exist $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ *and a large time* $n_0 \in \mathbb{N}^+$ *such that* $\varphi(z - \tilde{z}) - \vec{\varepsilon} \leqslant \bar{U}_{n_0}(z, \psi) \leqslant \varphi(z + \tilde{z}) + \vec{\varepsilon}$.

Proof. (i) It is easy to see that

$$\limsup_{\xi\to-\infty} \psi(\xi) \ll E^1 + \vec{\varepsilon} = \lim_{\xi\to-\infty} \varphi(\xi) + \vec{\varepsilon},$$

$$\limsup_{\xi\to\infty} \psi(\xi) \ll E^2 + \vec{\varepsilon} = \lim_{\xi\to\infty} \varphi(\xi) + \vec{\varepsilon},$$

$$\liminf_{\xi\to-\infty} \psi(\xi) \gg E^1 - \vec{\varepsilon} = \lim_{\xi\to-\infty} \varphi(\xi) - \vec{\varepsilon},$$

$$\liminf_{\xi\to\infty} \psi(\xi) \gg E^2 - \vec{\varepsilon} = \lim_{\xi\to\infty} \varphi(\xi) - \vec{\varepsilon}.$$

Then there exists $Z_0 > 0$ such that $\varphi(z) - \vec{\varepsilon} \leqslant \psi(z) \leqslant \varphi(z) + \vec{\varepsilon}$ holds for all $|z| \geqslant Z_0$. By the monotonicity of ψ and φ, there exists $\tilde{z} > 0$ such that $\varphi(z - \tilde{z}) - \vec{\varepsilon} \leqslant \bar{U}_0(z, \psi) \leqslant \varphi(z + \tilde{z}) + \vec{\varepsilon}$.

(ii) Let $L > 0$ be a sufficiently large number such that $\operatorname{supp} k_i \subseteq [-L, L]$, $i = 1, 2$, and $\limsup_{\xi\to-\infty} \psi(\xi) \ll E^3 \ll \liminf_{\xi\to\infty} \psi(\xi)$. Without loss of generality, we assume $\psi(\xi) \leqslant l_1$, $\forall \xi \in \mathbb{R}$, and $\psi(\xi) \leqslant l_2$, $\forall \xi \leqslant 0$, where $E^3 \ll l_1 \leqslant E^2$, $E^1 \leqslant l_2 \ll E^3$. Let $V_n^+ = \bar{U}_n(2l_1 - l_2)$, $V_n^- = \bar{U}_n(l_2)$ be the spatially homogeneous solutions of (3.1) with $V_0^+ = 2l_1 - l_2$ and $V_0^- = l_2$. Let $\bar{c} \in (c - L, c + L)$ and $\xi : \mathbb{R} \to [0, 1]$ be a nondecreasing functional satisfying $\xi(z) \equiv 1$, $\forall z \geqslant 1$, and $\xi(z) \equiv 0$, $\forall z \leqslant 0$. Define

$$V_n(z) = V_n^+ \xi(z + n\bar{c}) + V_n^- \bigl(1 - \xi(z + n\bar{c})\bigr).$$

Then it is easy to verify $V_0(z) \geqslant \psi(z)$, $\forall z \in \mathbb{R}$. We now claim that for any discrete time n, there exist $\tilde{z}_n \in \mathbb{R}$ such that

$$V_{n+1}(z) \geqslant Q[V_n](z - \tilde{z}_{n+1} + c), \quad \forall z \in \mathbb{R}.$$

We first prove that

$$V_{n+1}(z) \geqslant Q[V_n](z + c)$$

whenever $|z|$ is large enough.

For the sake of convenience, we define the nondecreasing operator $G = (G_1, G_2) : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$G_1(x_1, x_2) = \frac{(1 + r_1)x_1}{1 + a_1 r_1 + r_1 x_1 - a_1 r_1 x_2}, \qquad G_2(x_1, x_2) = \frac{a_2 r_2 x_1 + x_2}{1 + r_2 + a_2 r_2 x_1 - r_2 x_2}.$$

Then system (3.1) can be expressed as

$$\bar{U}_{n+1}(z) = \int_{\mathbb{R}} G\bigl(\bar{U}_n(z + c - y)\bigr) S(y)\, dy, \quad n \geqslant 0, \tag{3.4}$$

where $S(y) = \operatorname{diag}(k_1(y), k_2(y))$.

For fixed $n \in \mathbb{N}^+$, if $z > -n\bar{c} - c + L + 1$, then $z + (n+1)\bar{c} > z + n\bar{c} + c - L > 1$, and

$$\begin{aligned} V_{n+1}(z) - Q[V_n](z+c) &= V_{n+1}^+ \xi\big(z + (n+1)\bar{c}\big) + V_{n+1}^-\big(1 - \xi\big(z + (n+1)\bar{c}\big)\big) \\ &\quad - \int_{-L}^{L} G\big(V_n^+ \xi(z + n\bar{c} + c - y) + V_n^-\big(1 - \xi(z + n\bar{c} + c - y)\big)\big) S(y)\,dy \\ &= V_{n+1}^+ - \int_{-L}^{L} G\big(V_n^+\big) S(y)\,dy = 0. \end{aligned}$$

If $z < -n\bar{c} - c - L$, then $z + (n+1)\bar{c} < z + n\bar{c} + c + L < 0$. It follows that

$$\begin{aligned} V_{n+1}(z) - Q[V_n](z+c) &= V_{n+1}^+ \xi\big(z + (n+1)\bar{c}\big) + V_{n+1}^-\big(1 - \xi\big(z + (n+1)\bar{c}\big)\big) \\ &\quad - \int_{-L}^{L} G\big(V_n^+ \xi(z + n\bar{c} + c - y) + V_n^-\big(1 - \xi(z + n\bar{c} + c - y)\big)\big) S(y)\,dy \\ &= V_{n+1}^- - \int_{-L}^{L} G\big(V_n^-\big) S(y)\,dy = 0. \end{aligned}$$

Consequently, the above claim follows from the fact that $V_{n+1}(\cdot)$ and $Q[V_n](\cdot + c)$ are increasing due to the monotonicity of operator Q and $V_n(\cdot)$.

Since $V_0(z) \geqslant \psi(z)$, $\forall z \in \mathbb{R}$, combining the claim we have

$$\begin{aligned} V_1(z) &\geqslant Q[V_0](z - \tilde{z}_1 + c) \geqslant Q[\psi](z - \tilde{z}_1 + c) = \bar{U}_1(z - \tilde{z}_1, \psi), \\ V_2(z) &\geqslant Q[V_1](z - \tilde{z}_2 + c) \geqslant Q[\bar{U}_1](z - \tilde{z}_1 - \tilde{z}_2 + c) = \bar{U}_2(z - \tilde{z}_1 - \tilde{z}_2, \psi). \end{aligned}$$

By induction, we have

$$V_n(z) \geqslant \bar{U}_n(z - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_n, \psi), \quad n \geqslant 0.$$

Note that $\lim_{z\to-\infty} V_n(z) = V_n^-$, $\lim_{z\to\infty} V_n(z) = V_n^+$, $\lim_{n\to\infty} V_n^- = E^1$, and $\lim_{n\to\infty} V_n^+ = E^2$. It then follows that for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}^+$ such that

$$\begin{aligned} \lim_{z\to-\infty} V_{\tilde{n}}(z) &= V_{\tilde{n}}^- \leqslant E^1 + \frac{\vec{\varepsilon}}{2} \ll E^1 + \vec{\varepsilon} = \lim_{z\to-\infty} \varphi(z) + \vec{\varepsilon}, \quad \forall \tilde{n} \geqslant N_0, \\ \lim_{z\to\infty} V_{\tilde{n}}(z) &= V_{\tilde{n}}^+ \leqslant E^2 + \frac{\vec{\varepsilon}}{2} \ll E^2 + \vec{\varepsilon} = \lim_{z\to\infty} \varphi(z) + \vec{\varepsilon}, \quad \forall \tilde{n} \geqslant N_0. \end{aligned}$$

Thus, there exists $Z_1 > 0$ such that

$$V_{\tilde{n}}(z) \leqslant \varphi(z) + \vec{\varepsilon}, \quad \forall |z| \geqslant Z_1.$$

By the monotonicity of $V_{\tilde{n}}(\cdot)$ and $\varphi(\cdot)$, there exists $\tilde{z}_0 \in \mathbb{R}$ such that

$$V_{\tilde{n}(z)} \leqslant \varphi(z + \tilde{z}_0) + \vec{\varepsilon}, \quad \forall z \in \mathbb{R}.$$

Hence, we have

$$\bar{U}_{\tilde{n}}(z-\tilde{z}_1-\tilde{z}_2-\cdots-\tilde{z}_n,\psi)\leqslant V_{\tilde{n}}(z)\leqslant\varphi(z+\tilde{z}_0)+\vec{\varepsilon},\quad\forall z\in\mathbb{R}.$$

Let $\tilde{z}=\sum_{i=0}^{\tilde{n}}\tilde{z}_i$. It then follows that

$$\bar{U}_{\tilde{n}}(z,\psi)\leqslant\varphi(z+\tilde{z})+\vec{\varepsilon},\quad\forall z\in\mathbb{R}.$$

A similar argument on the lower bound of $U_{\tilde{n}}(z,\psi)$ completes the proof. $\square$

In order to use the method of upper and lower solutions, we first introduce the following concepts.

Definition 3.1. A function sequence $W_n^+(z)\in C(\mathbb{R},\mathbb{R}^2)$, $n\geqslant 0$, is an upper solution of (3.1) if $W_n^+(z)$ satisfies

$$W_{n+1}^+\geqslant Q\big[W_n^+\big](z+c),\quad n\geqslant 0.$$

A function sequence $W_n^-(z)\in C(\mathbb{R},\mathbb{R}^2)$, $n\geqslant 0$, is a lower solution of (3.1) if $W_n^-(z)$ satisfies

$$W_{n+1}^-\leqslant Q\big[W_n^-\big](z+c),\quad n\geqslant 0.$$

Note that the Fréchet derivatives of G at E^1 and E^2 are

$$DG(0,0)=\begin{pmatrix}\frac{1+r_1}{1+a_1r_1} & 0\\ \frac{a_2r_2}{1+r_2} & \frac{1}{1+r_2}\end{pmatrix},\qquad DG(1,1)=\begin{pmatrix}\frac{1}{1+r_1} & \frac{a_1r_1}{1+r_1}\\ 0 & \frac{1+r_2}{1+a_2r_2}\end{pmatrix}.$$

It is obvious that $DG(0,0)$ and $DG(1,1)$ are nonnegative with eigenvalues between 0 and 1. Choose $\epsilon_1>0$ small enough such that $DG(0,0)<A^-$, $DG(1,1)<A^+$, where

$$A^-=\begin{pmatrix}\frac{1+r_1}{1+a_1r_1} & \epsilon_1\\ \frac{a_2r_2}{1+r_2} & \frac{1}{1+r_2}\end{pmatrix},\qquad A^+=\begin{pmatrix}\frac{1}{1+r_1} & \frac{a_1r_1}{1+r_1}\\ \epsilon_1 & \frac{1+r_2}{1+a_2r_2}\end{pmatrix},$$

and the principle eigenvalues of $A^\pm$ are between 0 and 1. Since $A^\pm$ are positive, there exist strongly positive eigenvectors $\rho^\pm=(\rho_1^\pm,\rho_2^\pm)$ corresponding to the principle eigenvalues of $A^\pm$ satisfying $\vec{0}\ll\rho^-\leqslant\rho^+\leqslant\vec{1}$. Note that we can choose $\rho^\pm$ as close to the origin as we wish due to the fact that the eigenvector space is linearly closed. Let $\rho(z):\mathbb{R}\to\mathbb{R}^2$ be a positive nondecreasing map such that $\rho(z)=\rho^+$, $\forall z\geqslant z_1>0$, and $\rho(z)=\rho^-$, $\forall z\leqslant z_2<0$, where z_i, $i=1,2$, are two fixed real numbers. Motivated by [14], we have the following result on the upper and lower solutions for (1.3).

Lemma 3.2. *There exist positive number σ and $\varepsilon_0\in(0,1)$ such that for any $\hat{z}$ and $\varepsilon\in(0,\varepsilon_0)$,*

$$W_n^\pm=\varphi\big(z\pm\hat{z}\pm\varepsilon\big(1-e^{-\sigma n}\big)\big)\pm\varepsilon\rho(z\pm\hat{z})e^{-\sigma n},\quad\forall z\in\mathbb{R},\ n\geqslant 0,$$

are upper and lower solutions of system (3.1)*, respectively.*

Proof. Without loss of generality, we assume that $\hat{z}=0$. Let $z_n=\varepsilon(1-e^{\sigma n})$, $\forall n\geqslant 0$. Then $\{z_n\}_{n\geqslant 0}$ is increasing and between 0 and 1, where the positive number σ is to be determined. Denote $D_n^\pm(z):=W_{n+1}^\pm(z)-Q[W_n^\pm](z+c)$, and $G_j^i(u):=\frac{\partial}{\partial x_j}G^i(u)$, $B=\sup\{|G_j^i(u)|:\ u\in[E^1-\vec{1},E^2+\vec{1}]\}$, where $u=(x_1,x_2)\in\mathbb{R}^2$. It is obvious that there exist $\delta>0$, $k\in(0,1)$ such that $G_j^i(u)\leqslant A_{ij}^-$ for all $\|u-E^1\|\leqslant\delta$, $G_j^i(u)\leqslant A_{ij}^+$ for all $\|u-E^2\|\leqslant\delta$, and $A^\pm\rho\leqslant k\rho$ for all $\|\rho-\rho^\pm\|\leqslant\delta$, where $\rho:=(\rho_1,\rho_2)\in\mathbb{R}^2$. Since

$\varphi(-\infty) := \lim_{z\to-\infty}\varphi(z) = E^1$, $\varphi(\infty) := \lim_{z\to\infty}\varphi(z) = E^2$, and $\rho(z) \subseteq [\rho^-, \rho^+]$, $\forall z \in \mathbb{R}$, it follows that there exist $M > \max\{z_1 + 1, 1 - z_2\}$ and $\varepsilon_1 \in (0, 1)$ such that

$$\begin{gathered}\left\|\varphi(z) + \varepsilon\rho(\eta) - E^1\right\| \leqslant \delta, \quad \forall\varepsilon \in (0, \varepsilon_1],\ \eta \leqslant -M + 1,\ z \leqslant -M + 1,\\ \left\|\varphi(z) + \varepsilon\rho(\eta) - E^2\right\| \leqslant \delta, \quad \forall\varepsilon \in (0, \varepsilon_1],\ \eta \geqslant M - 1,\ z \geqslant M - 1.\end{gathered}$$

Then we have

$$\begin{aligned} D_n^+(z) &= W_{n+1}^+(z) - Q\left[W_n^+\right](z+c)\\ &= \varphi(z+z_{n+1}) + \varepsilon\rho(z)e^{-\sigma(n+1)}\\ &\quad - \int_{\mathbb{R}} G\big(\varphi(z+z_n+c-y) + \varepsilon\rho(z+c-y)e^{-\sigma n}\big)S(y)\,dy\\ &= \varphi(z+z_{n+1}) - \varphi(z+z_n) + \varepsilon\rho(z)e^{-\sigma(n+1)}\\ &\quad - \int_{\mathbb{R}} \big[G\big(\varphi(z+z_n+c-y) + \varepsilon\rho(z+c-y)e^{-\sigma n}\big) - G\big(\varphi(z+z_n+c-y)\big)\big]S(y)\,dy\\ &= \varphi(z+z_{n+1}) - \varphi(z+z_n) + \varepsilon\rho(z)e^{-\sigma(n+1)}\\ &\quad - \int_{\mathbb{R}} \big[G\big(\varphi(y+z_n) + \varepsilon\rho(y)e^{-\sigma n}\big) - G\big(\varphi(y+z_n)\big)\big]S(z+c-y)\,dy\\ &= \varphi(z+z_{n+1}) - \varphi(z+z_n) + \varepsilon\rho(z)e^{-\sigma(n+1)}\\ &\quad - \int_{\mathbb{R}} \left(\int_0^1 DG\big(\varphi(y+z_n) + s\varepsilon\rho(y)e^{-\sigma n}\big)\varepsilon\rho(y)e^{-\sigma n}\,ds\right)S(z+c-y)\,dy. \end{aligned} \tag{3.5}$$

Let I be the 2×2 identity matrix and $\Gamma_n = [-M - z_n - \eta, M - z_n + \eta]$, where $\eta > 1$ is large enough such that

$$\left(\int_{-\infty}^{-\eta+c} + \int_{\eta+c}^{\infty}\right)S(y)\,dy < \varepsilon I.$$

Now for any $n \geqslant 0$, we claim $D_n^+(z) \geqslant 0$. We consider three cases.

Case (i): $z > M - z_n + \eta$. It is clear that $z > M - 1$ and $z + c - y \geqslant \eta + c$ if $y \leqslant M - z_n$. By the monotonicity of φ, we have

$$\begin{aligned} &D_n^+(z)\\ &\quad\geqslant \varepsilon\rho(z)e^{-\sigma(n+1)} - \int_{\mathbb{R}}\left(\int_0^1 DG\big(\varphi(y+z_n) + s\varepsilon\rho(y)e^{-\sigma n}\big)\varepsilon\rho(y)e^{-\sigma n}\,ds\right)S(z+c-y)\,dy\\ &\quad= \varepsilon\rho(z)e^{-\sigma(n+1)}\\ &\qquad - \left(\int_{-\infty}^{-M-z_n} + \int_{-M-z_n}^{M-z_n} + \int_{M-z_n}^{\infty}\right)\left(\int_0^1 DG\big(\varphi(y+z_n) + s\varepsilon\rho(y)e^{-\sigma n}\big)\varepsilon\rho(y)e^{-\sigma n}\,ds\right)S(z+c-y)\,dy \end{aligned}$$

$$\begin{aligned}
&\geqslant \varepsilon\rho(z)e^{-\sigma(n+1)} - \left(\int_{-\infty}^{-M-z_n} + \int_{M-z_n}^{\infty}\right) k\rho(y)\varepsilon e^{-\sigma n} S(z+c-y)\,dy \\
&\quad - \int_{-M-z_n}^{M-z_n} \left(\int_0^1 DG\big(\varphi(y+z_n) + s\varepsilon\rho(y)e^{-\sigma n}\big)\varepsilon\rho(y)e^{-\sigma n}\,ds\right) S(z+c-y)\,dy \\
&\geqslant \varepsilon\rho^+ e^{-\sigma(n+1)} - k\rho^+\varepsilon e^{-\sigma n}\varepsilon - k\rho^+\varepsilon e^{-\sigma n} - 2B\varepsilon\big\|\rho^+\big\|e^{-\sigma n}\varepsilon\vec{e} \\
&= \varepsilon e^{-\sigma n}\big(\rho^+\big(e^{-\sigma} - k\big) - k\rho^+\varepsilon - 2B\big\|\rho^+\big\|\varepsilon\vec{e}\big) \geqslant 0
\end{aligned}$$

provided $\sigma \in (0, -\ln k)$, and ε is small enough.

Case (ii): $z < -M - z_n - \eta$. Clearly, $z < -M + 1$, and $z + c - y < -\eta + c$ if $y > -M - z_n$. Then

$$\begin{aligned}
D_n^+(z) &\geqslant \varepsilon\rho(z)e^{-\sigma(n+1)} - \left(\int_{-\infty}^{-M-z_n} + \int_{M-z_n}^{\infty}\right) k\rho(y)\varepsilon e^{-\sigma n} S(z+c-y)\,dy \\
&\quad - \int_{-M-z_n}^{M-z_n} \left(\int_0^1 DG\big(\varphi(y+z_n) + s\varepsilon\rho(y)e^{-\sigma n}\big)\varepsilon\rho(y)e^{-\sigma n}\,ds\right) S(z+c-y)\,dy \\
&\geqslant \varepsilon\rho^- e^{-\sigma(n+1)} - \int_{-\infty}^{-M-z_n} k\rho(y)\varepsilon e^{-\sigma n} S(z+c-y)\,dy - k\varepsilon e^{-\sigma n}\rho^+\varepsilon - 2B\varepsilon\big\|\rho^+\big\|e^{-\sigma n}\varepsilon\vec{e} \\
&\geqslant \varepsilon\rho^- e^{-\sigma(n+1)} - k\rho^-\varepsilon e^{-\sigma n} - k\varepsilon e^{-\sigma n}\rho^+\varepsilon - 2B\varepsilon\big\|\rho^+\big\|e^{-\sigma n}\varepsilon\vec{e} \\
&= \varepsilon e^{-\sigma n}\big(\rho^-\big(e^{-\sigma} - k\big) - k\rho^+\varepsilon - 2B\big\|\rho^+\big\|\varepsilon\vec{e}\big) \geqslant 0,
\end{aligned}$$

provided that $\sigma \in (0, -\ln k)$, and ε is small enough.

Case (iii): $z \in \Gamma_n = [-M - z_n - \eta, M - z_n + \eta]$, that is, $z + z_n \in [-M - \eta, M + \eta]$. The uniform continuity of φ and [8, Lemma 5] imply that $\varphi \in C^1(\mathbb{R}, \mathbb{R}^2)$, $\varphi'(z)$ is uniformly continuous, and

$$\varphi'(z) = \int_{\mathbb{R}} DG\big(\varphi(y)\big)\varphi'(y)S(z+c-y)\,dy \gg 0.$$

Since φ is strictly increasing in compact set $[-M - 2\eta, M + 2\eta]$, there exists $\vec{\theta} = (\theta, \theta) \gg 0$ such that

$$\varphi(y) - \varphi(x) \geqslant \vec{\theta}(y - x), \quad y > x, \quad \forall x, y \in [-M - 2\eta, M + 2\eta]. \tag{3.6}$$

It is obvious that

$$0 < d_n = (z + z_{n+1}) - (z + z_n) = z_{n+1} - z_n = \varepsilon e^{-\sigma n}\big(1 - e^{-\sigma}\big) < 1 < \eta,$$

and hence, $z + z_{n+1} < z + z_n + 1 < M + \eta + 1 < M + 2\eta$. By (3.6), we have

$$\varphi(z + z_{n+1}) - \varphi(z + z_n) \geqslant \vec{\theta}(z_{n+1} - z_n). \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\begin{aligned} D_n^+(z) &\geqslant \vec{\theta}(z_{n+1} - z_n) + \varepsilon\rho^- e^{-\sigma(n+1)} - 2B\rho^+ e^{-\sigma n}\varepsilon\vec{e} \\ &= \vec{\theta}\varepsilon e^{-\sigma n}\big(1 - e^{-\sigma}\big) + \varepsilon\rho^- e^{-\sigma(n+1)} - 2B\rho^+ e^{-\sigma n}\varepsilon\vec{e} \\ &= \varepsilon e^{-\sigma n}\big(\vec{\theta}\big(1 - e^{-\sigma}\big) + \rho^- e^{-\sigma} - 2B\big\|\rho^+\big\|\vec{e}\big) \geqslant 0, \end{aligned}$$

provided $\|\rho^+\| \leqslant \theta(1 - e^{-\sigma})/2B$.

Combining cases (i)–(iii), we see that there exist $\sigma > 0$ and sufficiently small number $\varepsilon_0 \in (0,1)$ such that $D_n^+(z) \geqslant 0$, $n \geqslant 0$, $z \in \mathbb{R}$. Thus, $W_n^+(z)$ is an upper solution of system (3.1). By a similar argument, we can prove $W_n^-(z)$ is a lower solution of (3.1). $\square$

Lemma 3.3. *The wave profile φ is a Lyapunov stable equilibrium of* (3.1)*.*

Proof. Let ε_0 and $W_n^\pm(z)$ be given in Lemma 3.2 with $\hat{z} = 0$. By the uniform continuity of φ and the boundedness of $\rho(z)$, it follows that there exists $K > 0$, independent of ε, such that $\|W_n^\pm(z,\varepsilon) - \varphi(z)\| < K\varepsilon$, $\forall z \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0)$. For any $\varepsilon \in (0, \varepsilon_0)$, let $\delta = \varepsilon \min\{\rho_1^-, \rho_2^-\} > 0$, then $\varepsilon\rho(z) \geqslant \vec{\delta}$. Thus, for any given ψ satisfying $\|\psi - \varphi\| < \delta$, we have

$$W_0^-(z,\varepsilon) = \varphi(z) - \varepsilon\rho(z) \leqslant \psi \leqslant \varphi(z) + \varepsilon\rho(z) = W_0^+(z,\varepsilon).$$

Then the comparison principle implies that

$$W_n^-(z,\varepsilon) \leqslant \bar{U}_n(z,\psi) \leqslant W_n^+(z,\varepsilon), \quad \forall z \in \mathbb{R},$$

and hence, $\|\bar{U}_n(\cdot,\psi) - \varphi(\cdot)\| \leqslant K\varepsilon$, $n \geqslant 0$, which completes the proof. $\square$

To prove the stability and uniqueness of travelling waves, we first recall a global convergence result for discrete-time monotone semiflows (see [15, Theorem 2.2.4]).

Lemma 3.4. *Let U be a closed and order convex subset of an ordered Banach space $\mathcal{X}$ with nonempty positive cone, and $f : U \to U$ continuous and monotone. Assume that there exists a monotone homeomorphism h from* $[0,1]$ *onto a subset of U such that*

(1) *for each $s \in [0,1]$, $h(s)$ is a stable fixed point for $f : U \to U$;*
(2) *each forward orbit of f on $[h(0), h(1)]_{\mathcal{X}}$ is precompact;*
(3) *if $\omega(x) > h(s_0)$ for some $s_0 \in [0,1)$ and $x \in [h(0), h(1)]_{\mathcal{X}}$, then there exists $s_1 \in (s_0, 1)$ such that $\omega(x) \geqslant h(s_1)$.*

Then for any precompact orbit $\gamma^+(y)$ of f in U with $\omega(y) \cap [h(0), h(1)]_{\mathcal{X}} \neq \emptyset$, there exists $s^* \in [0,1]$ such that $\omega(y) = h(s^*)$.

Let $\mathcal{X} = BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual supreme norm. Let $\mathcal{X}_+ = \{(\psi_1, \psi_2) \in \mathcal{X}\colon\ \psi_i(x) \geqslant 0,\ \forall x \in \mathbb{R},\ i = 1,2\}$. Then $\mathcal{X}_+$ is a closed cone of $\mathcal{X}$ and its induced partial ordering makes $\mathcal{X}$ into a Banach lattice.

Now we are in the position to prove the main result of this section.

Theorem 3.1. *Let $\varphi(x - cn)$ be a monotone travelling wave solution of system* (1.3) *and $U_n(x,\psi)$ be the solution of* (1.3) *with $U_0(\cdot,\psi) = \psi(\cdot) \in \mathcal{X}_{[E^1,E^2]}$. Then the following statements are valid:*

(i) *For any nondecreasing $\psi \in \mathcal{X}_{[E^1,E^2]}$ satisfying* (3.2)*, there exists $s_\psi \in \mathbb{R}$ such that $\lim_{n\to\infty} \|U_n(x,t,\psi) - \varphi(x - cn + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$, and any monotone travelling wave solution of system* (1.3) *connecting E^1 to E^2 is a translation of φ.*

(ii) *If k_i, $i = 1, 2$, has a compact support, then for any $\psi \in \mathcal{X}_{[E^1,E^2]}$ satisfying* (3.3)*, there exists $s_\psi \in \mathbb{R}$ such that* $\lim_{n\to\infty} \|U_n(x,t,\psi) - \varphi(x - cn + s_\psi)\| = 0$ *uniformly for $x \in \mathbb{R}$, and any travelling wave solution of system* (1.3) *connecting E^1 to E^2 is a translation of φ.*

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ be given as in Lemma 3.2. From (i) and (ii) in Lemma 3.1, we see that for $\varepsilon\rho^- \gg 0$ and any $\psi \in \mathcal{X}_{[E^1,E^2]}$ satisfying (3.2) in case (i), or satisfying (3.3) in case (ii), there exist n_0 and $\tilde{z}$ such that for any $z \in \mathbb{R}$, we have

$$\bar{U}_{n_0}(z,\psi) \leqslant \varphi(z+\tilde{z}) + \varepsilon\rho^- \leqslant \varphi(z+\tilde{z}) + \varepsilon\rho(z+\tilde{z}) = W_0^+$$

and

$$\bar{U}_{n_0}(z,\psi) \geqslant \varphi(z-\tilde{z}) - \varepsilon\rho^- \geqslant \varphi(z-\tilde{z}) - \varepsilon\rho(z-\tilde{z}) = W_0^-.$$

Then the comparison principle and the construction of $W_n^{\pm}(z)$ imply that

$$W_n^-(z) \leqslant \bar{U}_n\big(z, \bar{U}_{n_0}(\cdot)\big) \leqslant W_n^+(z), \quad \forall z \in \mathbb{R},\ n \in \mathbb{N}^+.$$

Since $\bar{U}_n(z, \bar{U}_{n_0}(\cdot)) = \bar{U}_{n+n_0}(z,\psi)$, $\forall z \in \mathbb{R}$, $n \in \mathbb{N}^+$, we have

$$\varphi(z-\tilde{z}-\varepsilon_0) - \varepsilon\rho(z-\tilde{z})e^{-\sigma n} \leqslant \bar{U}_{n+n_0}(z,\psi) \leqslant \varphi(z+\tilde{z}+\varepsilon_0) - \varepsilon\rho(z+\tilde{z})e^{-\sigma n}. \tag{3.8}$$

Let $\Phi_n(\psi) := \bar{U}_n(\cdot,\psi)$, $\forall \psi \in \mathcal{X}$, $n \in \mathbb{N}^+$, be the solution semiflow determined by (3.1). By (3.8), the forward orbit $\gamma^+(\psi) := \{\Phi_n(\psi)\colon\ n \geqslant 0\}$ is bounded in $\mathcal{X}$. Note that $\lim_{z\to-\infty}\varphi(z) = E^1$, $\lim_{z\to\infty}\varphi(z) = E^2$. By Ascoli–Arzelà theorem, it then follows that $\gamma^+(\psi)$ is precompact in $\mathcal{X}$, and hence, the omega limit set $\omega(\psi)$ is nonempty, compact and invariant.

Let $z_0 = \tilde{z} + \varepsilon_0$, and $n \to \infty$ in (3.8); we have the omega limit set $\omega(\psi) \subset I := [\varphi(\cdot - z_0), \varphi(\cdot + z_0)]_{\mathcal{X}}$. Let $h(s) = \varphi(\cdot + s)$, $\forall s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset $\hat{I} \subset I$. Let $V = \mathcal{X}_{[E^1,E^2]}$. Then $\Phi_n : V \to V$ is a monotone autonomous semiflow. By Lemma 3.3, each $h(s)$ is a stable equilibrium for Φ_n. Clearly, each $\phi \in \hat{I}$ is increasing and satisfies (3.2) and (3.3), and hence, $\gamma^+(\phi)$ is precompact. By Lemma 3.4, it is suffice to verify the condition (3) to obtain the convergence of $\gamma^+(\psi)$.

Assume that for some $s_0 \in [-z_0, z_0)$, $\phi_0 \in \hat{I}$ and $\varphi(\cdot + s_0) <_{\mathcal{X}} \phi(\cdot)$ for all $\phi \in \omega(\phi_0)$, that is, $\varphi(\cdot + s_0) <_{\mathcal{X}} \omega(\phi_0)$. By the strong monotonicity of Q, we know $\varphi(z + s_0) \ll \Phi_n(\phi)(z)$, $\forall z \in \mathbb{R}$, $n \in \mathbb{N}$. By the invariance of $\omega(\phi_0)$, we get $\varphi(z + s_0) \ll \phi(z)$, $\forall \phi \in \omega(\phi_0)$, $z \in \mathbb{R}$.

By the uniform continuity of φ' and [11, Corollary A.19], it follows that $\lim_{z\to\infty}\varphi'(z) = \vec{0}$, and hence, we can choose a large positive number $z_1 \in (z_0, \infty)$ such that $\delta := \sup_{|z|\geqslant z_1 - z_0} \|\varphi'(z)\| \leqslant \frac{1}{4}\min\{\rho_1^-, \rho_2^-\}$. By the compactness of $\omega(\phi_0)$, there exists $s_1 \in (s_0, z_0)$ such that $s_1 - s_0 < \varepsilon_0$, and

$$\varphi(z + s_1) \ll \phi(z), \quad \forall z \in [-z_1, z_1],\ \phi \in \omega(\phi_0).$$

For any fixed $\phi \in \omega(\phi_0)$, there exists a sequence $\{n_j\}$ such that $n_j \to \infty$ as $j \to \infty$, and $\lim_{j\to\infty}\Phi_{n_j}(\phi_0) = \phi$. Fix an n_j such that

$$\big\|\Phi_{n_j}(\phi_0) - \phi\big\| \leqslant \delta(s_1 - s_0).$$

Since $\varphi(z + s_1) \ll \phi(z)$, $\forall z \in [-z_1, z_1]$, and

$$\varphi(z + s_0) - \varphi(z + s_1) \ll \phi(z) - \varphi(z + s_1), \quad \forall z \in \mathbb{R},$$

we have

$$\begin{aligned}\Phi_{n_j}(\phi_0)(z) - \varphi(z+s_1) &= \Phi_{n_j}(\phi_0)(z) - \phi(z) + \phi(z) - \varphi(z+s_1)\\ &\geqslant -(s_1-s_0)\vec{\delta} - \sup_{|z|\geqslant z_1} \big\|\varphi(z+s_0) - \varphi(z+s_1)\big\|\vec{e}\\ &\geqslant -(s_1-s_0)\vec{\delta} - (s_1-s_0) \sup_{|z|\geqslant z_1} \big\|\varphi'(z)\big\|\vec{e}\\ &\geqslant -(s_1-s_0)\vec{\delta} - (s_1-s_0)\vec{\delta}\\ &= -2(s_1-s_0)\vec{\delta}\\ &\geqslant -\varepsilon_1\rho(z+s_1),\end{aligned}$$

where $\varepsilon_1 = \frac{s_1-s_0}{2} < \varepsilon_0$. By the construction of $W_n^-(z)$, we get

$$\Phi_{n_j}(\phi_0)(z) \geqslant \varphi(z+s_1) - \varepsilon_1\rho(z+s_1) = W_0^-(z).$$

It follows that

$$\begin{aligned}\Phi_n\big(\Phi_{n_j}(\phi_0)(z)\big) \geqslant W_n^-(z) &= \varphi\big(z+s_1-\varepsilon_1\big(1-e^{-\sigma n}\big)\big) - \varepsilon_1\rho(z+s_1)e^{-\sigma n}\\ &\geqslant \varphi(z+s_1-\varepsilon_1) - \varepsilon_1\rho(z+s_1)e^{-\sigma n}\\ &= \varphi\left(z+s_1-\frac{s_1-s_0}{2}\right) - \varepsilon_1\rho(z+s_1)e^{-\sigma n}\\ &= \varphi\left(z+\frac{s_1+s_0}{2}\right) - \varepsilon_1\rho(z+s_1)e^{-\sigma n}, \quad \forall z\in\mathbb{R},\ n\in\mathbb{N}^+.\end{aligned}$$

Let $n = n_i - n_j$, and $n_i \to \infty$; we obtain $\phi(\cdot) \geqslant \varphi(z+\frac{s_1+s_0}{2})$. Denote $s_2 = \frac{s_1+s_0}{2}$, then $s_2 \in (s_0, s_1) \subseteq [s_0, z_0]$, and $\varphi(\cdot + s_2) \leqslant_{\mathcal{X}} \phi(\cdot)$. By the arbitrariness of $\phi \in \omega(\phi_0)$, we have $\phi(\cdot + s_2) \leqslant_{\mathcal{X}} \omega(\phi_0)$.

By Lemma 3.4, there exists $s_\psi \in [-z_0, z_0]$ such that $\omega(\psi) = h(s_\psi) = \varphi(\cdot + s_\psi)$. Then $\lim_{n\to\infty} \Phi_n(\psi) = \varphi(\cdot + s_\psi)$. Since $U_n(x,\psi) = \bar{U}_n(x-cn,\psi) = \Phi_n(\psi)(x-cn)$, we have $\lim_{n\to\infty} \|U_n(x,\psi) - \varphi(x-cn+s_\psi)\| = 0$ uniformly for $x\in\mathbb{R}$.

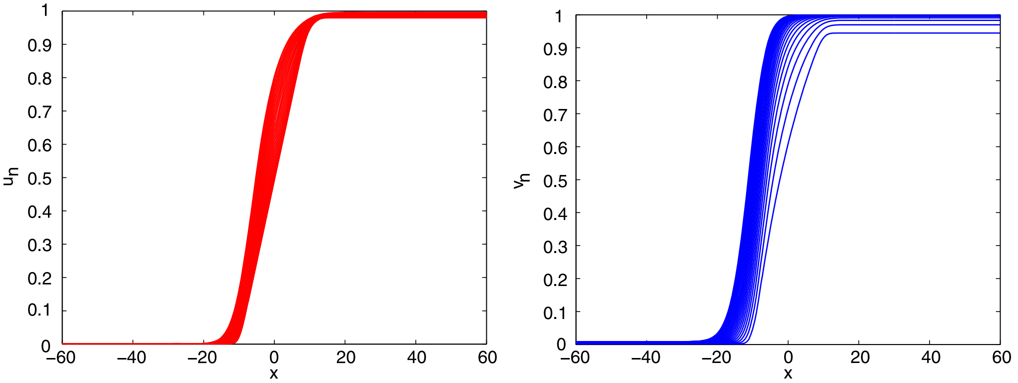
Let $\tilde{\varphi}(x-\tilde{c}n)$ be a travelling wave solution (or monotone travelling wave solution) of system (1.3) connecting E^1 to E^2 in case (ii) (or (i)). Clearly, $\tilde{\varphi}$ satisfies (3.3) (or (3.2)) in Lemma 3.1. By what we have proved above, there exists $\tilde{s}_\psi \in \mathbb{R}$ such that $\lim_{n\to\infty} \|\tilde{\varphi}(\cdot - \tilde{c}n) - \varphi(\cdot - cn + \tilde{s}_\psi)\| = 0$. By change of variable $\tilde{x} = x - cn$, we have $\lim_{n\to\infty} \|\tilde{\varphi}(\cdot + (c-\tilde{c})n) - \varphi(\cdot + \tilde{s}_\psi)\| = 0$. Since $\tilde{\varphi}(-\infty) = E^1$, $\tilde{\varphi}(\infty) = E^2$ and $\varphi(\cdot)$ is strictly increasing on $\mathbb{R}$, we then obtain $\tilde{c} = c$, and hence, $\tilde{\varphi}(\cdot) = \varphi(\cdot + \tilde{s}_\psi)$. $\square$

4. Numerical simulations

Now we present some simulation results for the main results in Section 3.

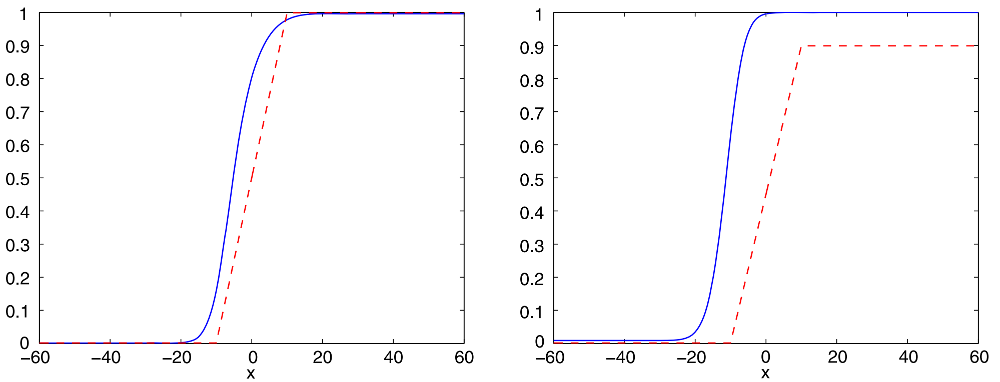
By Theorem 3.1, system (1.3) admits a unique monotone bistable travelling wave up to translation, which is globally stable with phase shift. In order to simulate this result, we truncate the infinite domain $\mathbb{R}$ to finite domain $[-L, L]$, where L is sufficiently large. Let $a_1 = 6/5$, $a_2 = 10$, $r_1 = 1/9$, $r_2 = 1/10$, $k_1(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)$, and $k_2(y) = \frac{1}{\sqrt{4\pi}} \exp(-y^2/4)$. The evolution of the solution is shown in Fig. 1 for $L = 60$ with the initial condition

$$u_0(x) = \begin{cases} 1/800, & -60 \leqslant x \leqslant -10;\\ 799/800 + 798(x-10)/16\,000, & -10 \leqslant x \leqslant 10;\\ 799/800, & 10 \leqslant x \leqslant 60,\end{cases}$$



(a) The evolution of u_n. (b) The evolution of v_n.

Fig. 1. The evolution of u_n and v_n when $n = 1, 2, \ldots, 20$.



(a) u components. (b) v components.

Fig. 2. The initial condition and numerical wave profile.

$$v_0(x) = \begin{cases} 1/1000, & -60 \leqslant x \leqslant -10; \\ 899/1000 + 898(x-10)/20\,000, & -10 \leqslant x \leqslant 10; \\ 899/1000, & 10 \leqslant x \leqslant 60. \end{cases}$$

The numerical wave profile and the initial condition are plotted by solid and dashed lines in Fig. 2, respectively. We can see, under the given parameters and kernel functions, that the solution rapidly converges to the numerical wave profile, and the sign of the wave speed is negative.

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