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SPINNING KNOTS ABOUT SUBMANIFOLDS; SPINNING KNOTS ABOUT PROJECTIONS OF KNOTS

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A method is defined and discussed for constructing higher dimensional codimension two knots. These methods generalize the various known methods of spinning of knots. Although the focus is on knotted sphere pairs, the methods more generally provide a method of producing a variety of knottings from any given codimension two pair of manifolds.

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spun knots   twist-spun knots   projections of knots

Introduction

In this paper certain generalizations of the idea of spun knots are considered. Section 1 describes the first of these generalizations. In Section 2 of this paper we show that many of these knots are different from previously known knots. Section 3 describes a very general construction of spinning a knot about a projection of a knot, and Section 4 defines the notion of π-regular homotopy, a notion which can be used to analyze our construction and show equivalences of some of the knots we define.

Our constructions are quite general and will work in any of the well known categories such as differentiable, piece-wise linear or locally flat topological knottings. We note that, in the differentiable category, where higher codimension knottings of spheres is possible, these constructions could be clearly extended to these codimensions; we do not explore that possibility here.

1. Spinning a knot about a submanifold

We will write $S^{n+k+2}=\partial(D^{n+k+1}\times D^2)\approx(S^{n+k}\times D^2)+(D^{n+k+1}\times S^1)$ and consider the standard $S^{n+k}\subseteq S^{n+k+2}$ to be the set corresponding to $S^{n+k}\times\{0\}\subseteq S^{n+k}\times D^2$. Suppose $N^{n+k}\subseteq S^{n+k}$ where $N^{n+k}\approx M^k\times D^n$, M^k is a compact k-manifold without boundary. If α is a knotted n-disk in D^{n+2}, we may define a new embedding, ϕ, of N^{n+k} in S^{n+k+2} such that $\phi|\partial N^{n+k}=\mathrm{id}$, and $\phi(N^{n+k})\cap(S^{n+k}-N^{n+k})=\emptyset$ as follows: we require that $\phi\circ i_x=i'_x\circ\alpha$ for all $x\in M^k$, where $i_x:D^n\to M^k\times D^n$ and $i'_x:D^{n+2}\to M^k\times D^n\times D^2\approx M^k\times D^{n+2}$ are injections into the second factor. Now $(S^{n+k}-N^{n+k})+\phi(N^{n+k})$ is a new embedding of S^{n+k} in S^{n+k+2} which we will call the spinning of α about M^k and denote by $M^k\otimes\alpha$. This is not an entirely adequate notation, since it refers to M^k rather than N^{n+k}, but it does correspond more closely to other notations in the literature [4].

These knots are generalizations of previous notions of spinning. If M^1 is the standard circle in S^2 and N^2 the standard tubular neighborhood, then $S^1\otimes\alpha$ is the spinning of α [2]. If S^k is the standard k-sphere in S^{n+k}, N the standard trivialization of the tubular neighborhood of S^k in S^{n+k}, then $S^k\otimes\alpha$ is the k-spinning, or k-superspinning of α [4, 5, 1]. Furthermore, it is not hard to see that if $S^p\times S^{k-p}$ is standardly embedded in S^{n+k} with N^{n+k} being the standard trivialization of the tubular neighborhood of $S^p\times S^{k-p}$ in S^{n+k}, then $(S^p\times S^{k-p})\otimes\alpha=S^p\otimes(S^{k-p}\otimes\alpha)$; that is, $(S^p\times S^{k-p})\otimes\alpha$ is the knot obtained by p-spinning the spun knot $S^{k-p}\otimes\alpha$.

The knot $M^k\otimes\alpha$ depends on the trivialization of the tubular neighborhood of M^k in S^n. For example, if M^1 is the unknotted circle in S^3 and we choose the non-standard trivialization of the tubular neighborhood that corresponds to p twists as we go around M^1 in S^3, then if we spin α about M^1 using this trivialization, it is not hard to see that $M^1\otimes\alpha$ is the p-twist spinning of α (see [11]).

If $\alpha:(P,\partial P)\to(Q,\partial Q)$ is an embedding of a manifold P into Q with tubular neighborhood T then $\mathrm{Comp}(\alpha)$ will denote $\overline{Q-T}$, the closed complement of α.

We will first examine the closed complement, K, of the unknotted S^{n+k} in S^{n+k+2}. Let $D^2_{1/2}\subseteq D^2$ be the disk of radius $\frac{1}{2}$. Write

$$S^{n+k+2}=\partial(D^{n+k+1}\times D^2)\approx(S^{n+k}\times D^2)+(D^{n+k+1}\times S^1).$$

Then

$$\begin{aligned}K&=(S^{n+k}\times D^2-D^2_{1/2})+(D^{n+k+1}\times S^1)\\&\approx(S^{n+k}\times(D^1\times S^1))+(D^{n+k+1}\times S^1)\end{aligned}$$

where in this last formulation the identification of the two summands is $S^{n+k}\times\{1\}\times S^1=\partial(D^{n+k+1})\times S^1$. It will be useful to introduce the following notation: if $X\subseteq S^{n+k}$, $\nu(X)=X\times\overline{D^2-D^2_{1/2}}$ where this set is viewed as a subset of the first summand of K above; thus $K=\nu(S^{n+k})+(D^{n+k+1}\times S^1)$. If $N^{n+k}=M^k\times D^n\subseteq S^{n+k}$, where M^k is a closed k-manifold then $\nu(M)=N^{n+k}\times\overline{(D^2-D^2_{1/2})}\approx(M^k\times D^n)\times(D^1\times S^1)$.

To obtain $\mathrm{Comp}(M^k\otimes\alpha)$, we do the following: for each $x\in M^k$, we remove $\{x\}\times D^n\times D^1\times S^k$ (the closed complement of an unknotted D^n in D^{n+2}) and replace it by $\mathrm{Comp}(\alpha)$, thus replacing $\nu(M)$ by $M^k\times\mathrm{Comp}(\alpha)$. Let $Q=\overline{S^{n+k}-N^{n+k}}$. Now we may write: $\mathrm{Comp}(M^k\otimes\alpha)=[(M^k\times\mathrm{Comp}\ \alpha)+\nu(Q)]+[D^{n+k+1}\times S^1]$.

Remark. Our construction can be done more generally. For example, we point out the following three situations. First, suppose we have an orientable manifold Σ^{n+k}

in S^{n+k+2}. Then [10] Σ^{n+k} has trivial normal bundle. Suppose we have $N^{n+k} \subseteq \Sigma^{n+k}$ where $N^{n+k} \approx M^k \times D^n$, and we are given a knotted n-disk in D^{n+2}. Then we could re-embed N^{n+k} as above and use this together with the original embedding of $\Sigma^{n+k} - N^{n+k}$ to obtain a new embedding of Σ^{n+k}. Secondly, we could consider compact manifolds, Σ, with $\partial\Sigma^{n+k} \neq \emptyset$. In this case we would require M^k to be a proper submanifold of Σ^{n+k} with trivial tubular neighborhood N^{n+k}. Lastly, we note that we can easily generalize to the case where M is a submanifold of Σ with trivial normal bundle where the components of M have possibly different dimensions. For a component of dimension k_i we could take a knotted $(\dim \Sigma - k_i)$-dimensional disk in a $(\dim \Sigma - k_i = 2)$-dimensional disk and perform our construction on each component.

2. Fibering of knots spun about manifolds

A knotted S^n in S^{n+2} will be called a fibered knot if its closed complement fibers over a circle: it is called standardly fibered if the restriction of the fibration to the boundary yields the product fibration $S^n \times S^1 \to S^1$ (if $n \neq 2, 3$, then a fibered knot will be standardly fibered [4]). We will show the following lemma.

Lemma 1. *If α is a standardly fibered knot, then so is $M^k \otimes \alpha$.*

Proof. We begin by examining, in detail, the case when α is the trivial knot (and thus $M^k \otimes \alpha$ is trivial). Note that in K we may view $\{m\} \times D^n \times D^1 \times S^1$ as a fibration over S^1 of the complement of the unknotted ball pair (D^{n+2}, D^n), where $m \in M^k$. This gives rise to a (product) fibration of $\nu(M)$ with fiber $T_x = M^k \times D^n \times D^1 \times \{x\}$. We can extend this to a (product) fibration of $\nu(S^{n+k})$ where the fiber is $(T_x) + [(Q \times D^1) \times \{x\}]$. This then further extends to the (product) fibration $B^{n+k+1} \times S^1$ of K where $B^{n+k+1} \times \{x\}$ is written: $(T_x) + [(Q \times D^1) \times \{x\}] + [D^{n+k+1} \times \{x\}]$.

Now suppose α is a fibered knot with fiber $F(\alpha)$; then we may obtain a fibering, over S^1 of $M^k \times \mathrm{Comp}(\alpha)$ with fiber $F_x = M^k \times F(\alpha)$, where $x \in S^1$. If α is standardly fibered, we may extend this to a fibration of $\mathrm{Comp}(M^k \otimes \alpha)$ with fiber $(F_x) + [(Q \times D^1) \times \{x\}] + [D^{n+k+1} \times \{x\}]$. We may describe this fibration as follows: for each $x \in S^1$, we replace the standard fiber T_x of the trivial knot by F_x. We will examine the fiber of $\mathrm{Comp}(M^k \otimes \alpha)$ in more detail. To see how the first two summands are put together, we note that

$$\begin{aligned}\partial F_x = \partial T_x &= \partial[M^k \times D^n \times D^1 \times \{x\}] \\ &= [M^k \times \partial D^n \times D^1 \times \{x\}] + [M^k \times D^2 \times \partial D^1 \times \{x\}] \\ &= [\partial N^{n+k} \times D^1 \times \{x\}] + [N \times \partial D^1 \times \{x\}],\end{aligned}$$

and

$$\partial[(Q \times D^1) \times \{x\}] = [\partial Q \times D^1 \times \{x\}] + [Q \times \partial D^1 \times \{x\}].$$

The identifications of F_x and $(Q\times D^1)\times\{x\}$ are those identifying $\partial N^{n+k}\times D^1\times\{x\}$ with $\partial Q\times D^1\times\{x\}$. Finally, $D^{n+k+1}\times\{x\}$ fits onto $F_x+[(Q\times D^1)\times\{x\}]$ by identifying $\partial D^{n+k+1}\times\{x\}$ with the $(n+k)$-sphere $[N^{n+k}\times\{1\}\times\{x\}]+[Q\times\{1\}\times\{x\}]$. □

Next we will consider some particular examples of knotted 3-spheres. If N is a compact oriented 2-manifold in S^3 we will say it is standardly embedded if the components of S^3-N are homeomorphic. Some standard embeddings are illustrated in Fig. 1; these embeddings all have product neighborhoods with fiber dimension 1, thus we may spin knotted arcs about these submanifolds.

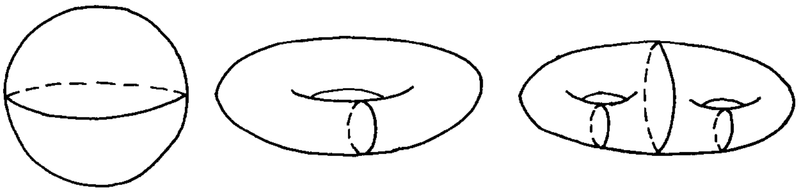


Fig. 1

Proposition 2. *If α is the trefoil knot, M_i the standardly embedded 2-manifold in S^3 of genus i, then the knots $M_i\otimes\alpha$, $i=0,1,2,\ldots$ are all distinct.*

Proof. Since α is a fibered knot (see [11]) it follows from the above lemma that all the knots $M_i\otimes\alpha$ are fibered. If any two of these knots had homeomorphic complements, the corresponding infinite cyclic covers would have homeomorphic total spaces. But the total spaces of those infinite cyclic covers have the homotopy types of the fibers, and the following calculation shows that the homotopy types of the fibers of the knots $M_i\otimes\alpha$ are all distinct. We wish to calculate the second homology group (with integer coefficients) of the fiber of the knot $M_i\otimes\alpha$. We first simplify our calculation by noting that, in the fiber, we may deform $[Q\times D^1]\times\{x\}$ to $([\partial Q\times D^1]\cup[Q\times 1])\times\{x\}$ for each $x\in S^1$. Thus the fiber of $M_i\otimes\alpha$ can be written, up to homotopy type, as $[M_i\times F(\alpha)]+D^{n+k+1}$. The Meyer–Vietoris sequence of the fiber then becomes:

$$\to H_2(M_i)\xrightarrow{\alpha} H_2(M_i\times F(\alpha))\oplus H_2(D^{n+k+1})\xrightarrow{\beta} H_2(F(M_i\otimes\alpha))$$

$$\xrightarrow{\gamma} H_1(M_i)\xrightarrow{\delta} H_1(M_i\times F(\alpha))\oplus H_1(D^{n+k+1})\to$$

Now using the Kunneth formula we can see that

$$H_2(M_i\times F(\alpha))\approx[H_2(M_i)\otimes H_0(F(\alpha))]$$
$$\oplus[H_0(M_i)\otimes H_2(F(\alpha))]\oplus[H_1(M_i)\otimes H_1(F(\alpha))];$$

we note that the middle term is zero since $H_2(F(\alpha))=0$ and also note that α maps $H_2(M_i)$ isomorphically onto $H_2(M_i)\otimes H_0(F(\alpha))$. Similarly we see that

$$H_1(M_i\times F(\alpha))\approx[H_1(M_i)\otimes H_0(F(\alpha))]\oplus[H_0(M_i)\otimes H_1(F(\alpha))],$$

where δ maps $H_1(M_i)$ isomorphically onto $H_1(M_i)\otimes H_0(F(\alpha))$. Using this information, we may obtain the following exact sequence from the Meyer–Vietoris sequence above:

$$0\to H_1(M_i)\otimes H_1(F(\alpha))\to H_2(F(M_i\otimes\alpha))\to 0$$

and thus $H_2(F(M_i\otimes\alpha))\approx H_1(M_i)\otimes H_1(F(\alpha))$. Now $H_1(F(\alpha))\approx Z\oplus Z$ and $H_1(M_i)$ is isomorphic to the direct sum of $2i$ copies of Z; thus $H_2(F(M_i\otimes\alpha))$ is isomorphic to $4i$ copies of Z. Therefore the fibers of the knots $M_i\otimes\alpha$ all have distinct homotopy types and thus the knots are all distinct. □

In particular, we note that the above result shows that spinning about a manifold will give different knots than the ordinary kinds of spinning. The knot $M_0\otimes\alpha$ is the 2-super-spinning of α; $M_1\otimes\alpha$ is the knotted 3-sphere obtained by spinning the spinning of α.

We note that, in the above formula for $\mathrm{Comp}(M^k\otimes\alpha)$, we may write $\nu(Q)+[D^{n+k+1}\times S^1]\approx D^{n+k+1}\times S^1$ by viewing $\nu(Q)$ as $Q\times S^1\times[\frac{1}{2},1]$ which we attach to $\partial(D^{n+k+1}\times S^1)$ along $Q\times S^1\times\{1\}$. We can then write:

$$\mathrm{Comp}(M^k\otimes\alpha)\approx[M^k\times\mathrm{Comp}\,\alpha]+[D^{n+k+1}\times S^1],$$

if X is the intersection of these two pieces then X corresponds to the subset of

$$\partial(M^k\times\mathrm{Comp}\,\alpha)\approx M^k\times\partial D^{n+1}\times S^1,$$

corresponding to

$$(M^k\times D^n\times S^1)+(M\times\partial D^n\times S^1)\approx M^k\times D^n\times S^1,$$

and X also corresponds to the subset of $\partial(D^{n+k+1}\times S^1)\approx S^{n+k}\times S^1$ which we can write $M^k\times D^n\times S^1$. Now by deforming $D^{n+k+1}\times S^1$ to $D^{k+1}\times S^1$, we can write $\mathrm{Comp}(M^k\otimes\alpha)$, up to homotopy, as $[M^k\times\mathrm{Comp}\,\alpha]+[D^{k+1}\times S^1]$ where the intersection is homeomorphic to $M^k\times S^1$. We may now analyze the homology of $\mathrm{Comp}(M^k\otimes\alpha)$ via the Meyer–Vietoris sequence as in [7, Section 4] and obtain results such as the following; here $\pi=\pi_1(\mathrm{Comp}\,\alpha)$; $F(\pi,a)$ is as defined in [7, Theorem 2.3].

Theorem 3. *If M^k is a homology sphere then as $Z[\pi]$ modules*

$$H_i(\mathrm{Comp}\,M^k\otimes\alpha)\approx\begin{cases}H_i(\mathrm{Comp}\,\alpha), & 2\leqslant i\leqslant k,\\ H_{k+1}(\mathrm{Comp}\,\alpha)\oplus F(\pi,a), & i=k+1,\\ H_i(\mathrm{Comp}\,\alpha)\oplus H_{i-k}(\mathrm{Comp}\,\alpha), & i\geqslant k+2.\end{cases}$$

3. Spinning a knot about a projection

We will next define a generalization of spinning a knot about a submanifold. We will first need several preliminary definitions.

A multiple fibering of D^n is a sequence of homeomorphisms, $f_i : D^{k_i} \times D^{n-k_i} \to D^n$ which send coordinate lines to coordinate lines with possibly reversed orientation; that is, if $\pi_j : D^{k_i} \times D^{n-k_i} \to D^1$ is projection on the jth coordinate, $j = 1, \ldots, n$, then for each j there is a unique j' such that $\pi'_j \circ f_i = \phi \circ \pi_j$ where $\phi : D^1 \to D^1$ is either the identity map or the map $\phi(t) = -t$. The map f_i is called the ith fibering; we will refer to $f_i(D^{k_i} \times \{0\})$ as the base of the ith fibering, and call $f_i(\{x\} \times D^{n-k_i})$ the ith fiber over x.

Next we will define a multiple knotting corresponding to a multiple fibering. Suppose $\{f_i\}_{i=1}^p$ is a multiple fibering of D^n, and we consider $D^n \subsetneq D^{n+2} = D^n \times D^2$. Suppose K_i is a collection of codimension two knotted $(n - k_i)$-disks with trivializations of tubular neighborhood given by maps

$$K_i : \; D^{n-k_i} \times D^2 \to D^{n-k_i} \times D^2 \quad \text{with} \quad K_i | \partial D^{n-k_i} \times D^2 = \mathrm{id}.$$

The knotted disk pair corresponding to K_i is then

$$\alpha_i = (D^{n-k_i} \times D^2, \, K_i(D^{n-k_i} \times \{0\})).$$

For each $i = 1, \ldots, p$, we define a map

$$\nu_i : \; D^{k_i} \times D^{n-k_i} \times D^2 \to D^{k_i} \times D^{n-k_i} \times D^2$$

by $\nu_i = (\mathrm{id} \times K_i) \circ (f_i \times \mathrm{id})$. We may describe ν_i as follows: choose a k_i-dimensional coordinate plane for a base and choose a direction for the fiber, then, over each point in the base knot the fiber according to K_i. A multiple knotting of D^n in D^{n+2}, with respect to $\{f_i\}_{i=1}^p$, by $\{K_i\}_{i=1}^p$, will be the map $\nu = \nu_p \circ \cdots \circ \nu_2 \circ \nu_1$.

We will illustrate this construction by considering a double fibering of D^2 where the knots α_1 and α_2 are both the trefoil knot. We will take $K_1 = K_2 : D^1 \times D^2 \to D^1 \times D^2$ to be a trivialization of the tubular neighborhood of the knot such that if $p : D^1 \times D^2 \to D^1 \times D^1$ is projection along the last coordinate and $p' : D^1 \times D^2 \to D \times D$ is projection along the middle coordinate, then $pK_i = K_i p'$, see Fig. 2. In order to depict the resulting multiply knotted 2-disk in D^4, we will look at the projection of this knotted disk in $D^2 \times D$; that is, we will look at $\pi\nu(D^2)$ where $\pi : D^2 \times D^2 \to D^2 \times D$ is projection along the last coordinate. Suppose f_1 and f_2 are as indicated in Fig. 3(a), in order to most clearly see the projection of ν it will be convenient to alter f_2 slightly to a map f'_2 so that f'_2 is as pictured in Fig. 3(b) (it is not hard to see that this change will not alter the ambient isotopy class of $\nu(D^2)$ in D^4). Fig. 4(a) shows the projection of $\nu_1(D^2 \times \{0\})$ and Fig. 4(b) shows the projection of $\nu_1(D^2 \times D^2)$.

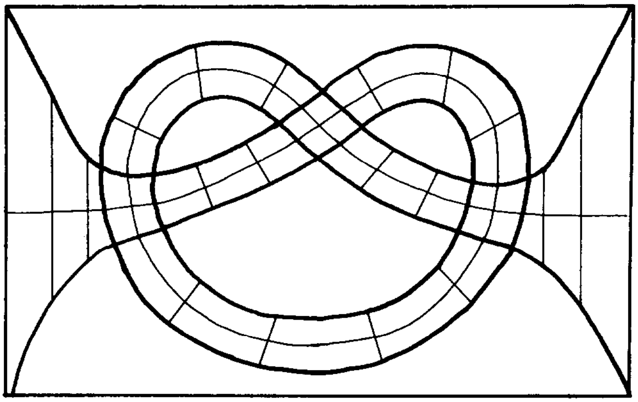


Fig. 2

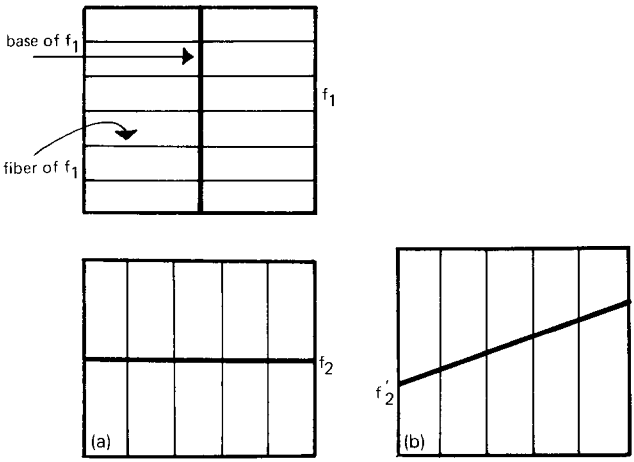


Fig. 3

The reason for altering f_2 to f'_2 may now be seen: $\pi\nu_1 f_2(D^1 \times \{0\})$ has self-intersections, see Fig. 5(a); whereas, $\pi\nu f'_2(D^1 \times \{0\})$ does not, see Fig. 5(b). The set $\pi\nu(D^2)$ can now be seen to be as shown in Fig. 6.

Now suppose M^k is a compact closed k-manifold embedded in $\Sigma^n \times D^1$ where Σ^n is an n-manifold. Let $\Sigma^n \times D^1 \to S^n$ be projection on the first factor. Suppose further $\pi | M^k$ is an immersion with trivial tubular neighborhood such that this immersion has normal crossings. (In [10] we show that if $k = n-2$, M^k orientable, and $\Sigma^n = S^n$, then after an isotopy of M^k, these conditions will be automatically satisfied.) Let $M^* = \pi(M^k)$, then by compactness of M^* and normality of the crossings, M^* will have a neighborhood Q in S^n such that we may write Q as the union of finitely many n-balls, Q_j, $j = 1, \ldots, q$, such that for each j, the pair

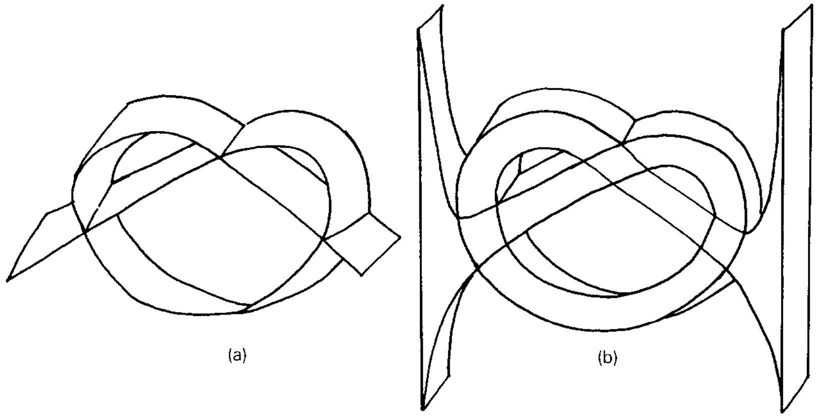


Fig. 4

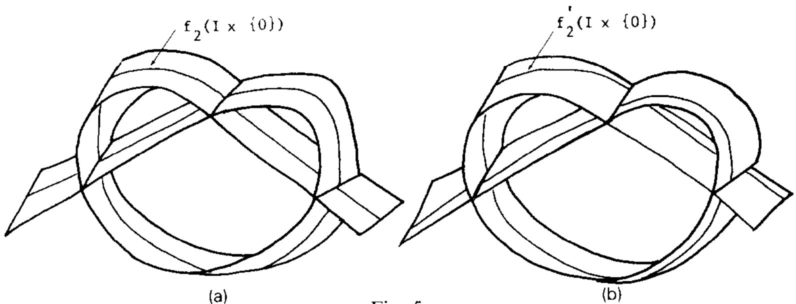


Fig. 5

$(Q_j, Q_j \cap M^*)$ is homeomorphic to a collection of coordinate hyperplanes (perhaps only one) in an n-ball, furthermore these hyperplanes will correspond to the bases of multiple fiberings of Q_j with orthogonal hyperplanes corresponding to fibers; we describe this multiple fibering in more detail in the next paragraph.

Let T be a tubular neighborhood of M^* in Σ^n; that is, T is a total space of a trivial disk bundle over M together with an immersion $\tilde{\phi}: T \to \Sigma^n$ which extends the immersion ϕ'. We will choose closed coordinate neighborhoods of the bundle so as to be compatible with $\tilde{\phi}$ and the structure of Q as follows. We will write T as the union of n-balls, T_i^j where $\tilde{\phi}|T_i^j$ maps T_i^j homeomorphically onto Q_j, and $T_i^j = M_i^j \times D^{n-r(i,j)}$ where M_i^j as an $r(i,j)$-dimensional subdisk of an $r(i,j)$-dimensional component of M and for each $x \in M_i^j$, $\{x\} \times D^{n-r}$ is a fiber of T. Let $x \in \bigcap_j \phi'(M_i^j)$, we will let $x_i = (\phi')^{-1}(x) \cap M_i^j$, where $i = 1, \ldots, i_j$. We will choose the index i so that if $i < i'$ if and only if $h(x_i) < h(x_{i'})$ where $h: \Sigma^n \times R^1 \to R^1$ is projection; connectedness of M_i^j and the continuity of h will imply that this is a well defined subscripting of M_i^j. More briefly, we could say that if $i < i'$, then $\phi(M_i^j)$ lies below $\phi(M_{i'}^j)$ with respect to π. We will also want to have any two T_i^j's to

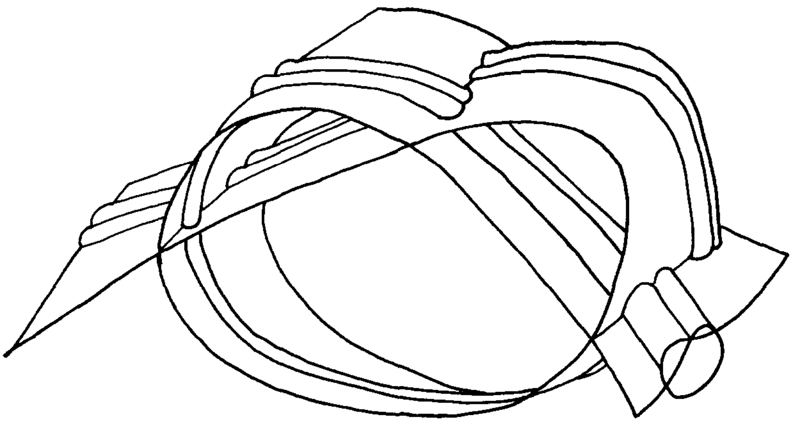


Fig. 6

meet, if at all, in a nice $(n-1)$-ball in the boundary of each as follows: for each i, j we will choose a homeomorphism (coordinate map) $\psi_i^j: D^{r(i,j)} \times D^{n-r(i,j)} \to T_i^j$ such that $\psi_i^j(D^{k_i} \times \{0\}) = M_i^j$ and such that if T_i^j and $T_{i'}^{j'}$ are distinct with non-empty intersection then it follows that $r(i,j) = r(i',j')$, and $(\psi_i^j)^{-1}(T_i^j \cap T_{i'}^{j'})$ is a face of $D^{r(i,j)} \times D^{n-r(i,j)}$ corresponding to a set $J^{r(i,j)-1} \times D^{n-r(i,j)}$ (where $J^{r(i,j)-1}$ is a face of $D^{r(i,j)}$), and $(\psi_{i'}^{j'})^{-1}(T_i^j \cap T_{i'}^{j'})$ is a face of $D^{r(i',j')} \times D^{n-r(i',j')}$ corresponding to a set $J'^{r(i',j')-1} \times D^{n-r(i',j')}$. ($J'^{r(i',j')-1}$ is a face of $D^{r(i',j')}$.) The map $(\psi_{i'}^{j'})^{-1} \circ \psi_i^j$ is to be a linear map of the form $h_{i,j}^{i',j'} \times g_{i,j}^{i',j'}$, where $h_{i,j}^{i',j'}$ is a linear map from $J^{r(i,j)-1}$ to $J'^{r(i',j')-1}$ and $g_{i,j}^{i',j'}$ is a linear map from $D^{n-r(i,j)}$ to itself. We may view the functions $g_{i,j}^{i',j'}$ as elements of the transformation group of the fiber of the $D^{n-r(i,j)}$ bundle corresponding to the component of T to which T_i^j and $T_{i'}^{j'}$ belong. Since T is assumed to be a trivial bundle, we may assume that all the $g_{i,j}^{i',j'}$ are identity maps. If $\partial M \neq \emptyset$ and ∂T denotes the restriction of the tubular neighborhood T to ∂M, we will further require $(\psi_i^j)^{-1}[T_i^j \cap \partial T]$ to correspond to a face of $D^{r(i,j)} \times D^{n-r(i,j)}$ if $T_i^j \cap \partial T \neq \emptyset$. We next use $\tilde{\phi}|T_1^j$ to identify Q_j with D^n. Let $\rho_j: Q_j \to I^n$ be defined by $\alpha \circ (\psi_1^j)^{-1} \circ (\tilde{\phi}|T_1^j)^{-1}$ where $\alpha: D^r \times D^{n-r} \to D^n$ is the canonical map. We can now define for each j, $j = 1, \ldots, q$, a multiple fibering $\{f_i^j\}_{i=1}^{l_j}$ by $f_i^j = \rho_j \circ \tilde{\phi} \circ \psi_i^j$. Note that $f_1^j = \alpha$, all j.

We next wish to show that these multiple fiberings agree, in the sense that if Q_j and $Q_{j'}$ are distinct with $Q_j \cap Q_{j'} \neq \emptyset$ then the multiple fiberings agree on the intersection. We first point out a certain notational difficulty pertaining to the index i. Let us consider the case where M^* is a 2-manifold immersed in S^3 and Q_j and $Q_{j'}$ are two distinct non-disjoint cubes; let $(M_i^j)_* = \phi'(M_i^j)$. We will be concerned with the two possible situations illustrated in Fig. 7. In the first, Fig. 7(a), we will wish to say that the fiberings f_i^j agree with the fiberings $f_i^{j'}$ for $i = 1, 2$; in the second, Fig. 7(b), we will wish to say that f_2^j agrees with $f_1^{j'}$ and f_3^j agrees with $f_2^{j'}$. This problem accounts for our introduction of the index $i(j, j')$ below. For each pair

(j, j') with $j \neq j'$, $Q_j \cap Q_{j'} \neq \emptyset$, let $H_i^{j,j'} = (\psi_i^j)^{-1}(Q_j \cap Q_{j'})$; then $H_i^{j,j'}$ will be the face of $D^{r(i,j)} \times D^{n-r(i,j)}$ corresponding to $Q_j \cap Q_{j'}$ via ψ_i^j. We will say that $H_i^{j,j'}$ is of class I if $H_i^{j,j'} = J^{r(i,j)-1} \times D^{n-r(i,j)}$ where $J^{r(i,j)-1}$ is a face of $D^{r(i,j)} \subseteq D^{r(i,j)} \times D^{n-r(i,j)}$; we will say that $H_i^{j,j'}$ is of class II if $H_i^{j,j'} = D^{r(i,j)} \times J^{n-r(i,j)-1}$ where $J^{n-r(i,j)-1}$ is a face of $D^{n-r(i,j)}$. Now if $j = j'$, $Q_j \cap Q_{j'} \neq \emptyset$ then i_j and $i_{j'}$ differ by at most one; that is, if $(Q_j, Q_j \cap M^*)$ looks like i_j intersecting hyperplanes in an n-cube, then $(Q_{j'}, Q_{j'} \cap M^*)$ looks like either $i_j - 1$, i_j, or $i_j + 1$ intersecting hyperplanes in an n-cube. Thus for each ordered pair (j, j') with $j \neq j'$, $Q_j \cap Q_{j'} \neq \emptyset$ there is at most one index, call it $i(j, j')$ such that $H_{i(j,j')}^{j,j'}$ is of class II. In Fig. 7(a), $i(j', j) = 3$; in Fig. 7(b), $i(j', j) = 1$. Now define $f_i^{j,j'} = f_i^j | H_i^{j,j'}$ if $i(j, j')$ does not exist; and, if $i(j, j')$ does exist, define

$$f_i^{j,j'} = \begin{cases} f_i^{j,j'} | H_i^{j,j'} & \text{if} \quad i < i(j, j'), \\ f_{i-1}^{j,j'} | H_{i-1}^{j,j'} & \text{if} \quad i > i(j, j'). \end{cases}$$

Now it is easy to check that the multiple fiberings agree on $Q_j \cap Q_{j'}$ by verifying that for all i, j, j' that $f_i^{j',j} = f_i^{j,j'} \circ (\psi_i^j \circ (\psi_i^{j'})^{-1})$. Also we note that, in the case $\partial M \neq \emptyset$, that if $T_i^j \cap \partial T \neq \emptyset$ we obtain a multiple fibering of I^{n-1} by restriction $\{f_i^j\}_{i=1}^{i_j}$ to the face of D^n corresponding to $(\psi_i^j)^{-1}(T_i^j \cap \partial T)$ and that if $T_{i'}^{j'} \cap \partial T \neq \emptyset$, $j \neq j'$, $(T_i^j \cap T_{i'}^{j'}) \cap \partial T \neq \emptyset$, then the two such multiple fiberings agree on $(Q_j \cap Q_{j'}) \cap \partial \Sigma$.

We are now ready to define our knot. Suppose we have an embedding $\phi : M \to \Sigma^n \times D^1$ with Q_j, f_i^j as described above. Let the components of M be denoted by M_p, $p = 1, \ldots, P$ and suppose p also denotes a function such that $M_i^j \subseteq M_{p(i,j)}$; let $k_p = \dim M_p$. Suppose we are given embeddings $K_p : D^{n-k_p} \times D^2 \to D^{n-k_p} \times D^2$, $p = 1, \ldots, P$, and let α_p denote the corresponding knotted disk pairs. Let ν_j be the multiple knotting, with respect to f_i^j of D^n in D^{n+2} by $\{K_{p(i,j)}\}_{i=1}^{i_j}$. Now we define an embedding, F, of Σ^n in $\Sigma^n \times I^2$ as follows: $F | \overline{\Sigma^n - \bigcup_{j=1}^q Q_j} = \mathrm{id}$; and for $j = 1, \ldots, q$ we require that $F \circ (\tilde{\phi} \circ \psi_1^j) = [(\pi \circ \psi_1^j) \times \mathrm{id}_{D^2}] \circ \nu_j$; more briefly, we use the multiple knotting ν_j to re-embed each Q_j. We will refer to F as the spinning of $\{\alpha_p\}_{p=1}^P$ by $\tilde{\phi}$. We will be particularly interested in examining the above construction for $\Sigma^n = S^n$. We may then obtain various knotted n-spheres in S^{n+2} by knotting S^n in $S^n \times D^2$ as above and then embedding $S^n \times D^2$ into S^{n+2} in the standard way.

With a fixed embedding of M in $S^n \times D^1$ there are two choices to make. The first is the choice of trivialization of the tubular neighborhood, T, of M^*, denoted by $\tilde{\phi}$

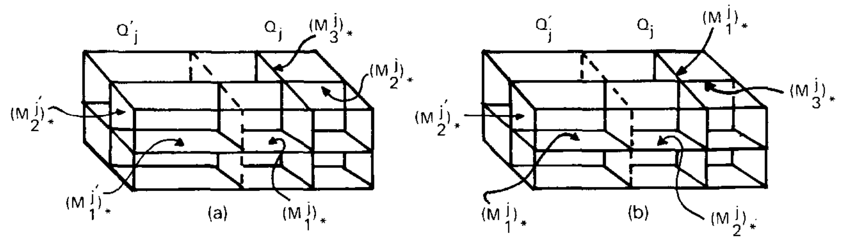


Fig. 7

above. The second is the choice of embedding of $S^n \times D^2$ into S^{n+2}. Suppose we choose the natural embedding of $S^n \times D^2$ into S^{n+2} via inclusions and standard identifications $S^n \times D^2 \subseteq S^n \times D^2 \cup D^{n+1} \times S^1 \approx S^{n+2}$. To define knotted n-spheres in S^{n+2} in this manner we need only choose $\tau : M^k \times D^{n-k} \times D^2 \to S^n \times D^2$ such that $\tau | M^k \times D^{n-k} \times \{0\} = \tilde{\phi}$ and $\tau(M^k \times \{0\} \times D^2) = \phi(M) \times D^2$. We will call such a map a spin trivialization. Given a knotted $(n-k)$-disk, α, and a trivialization, K, of the tubular neighborhood of α in D^{n-k+2} described by a map $K : D^{n-k} \times D^2 \to D^{n-k+2}$, we may use our construction to define a knotted n-sphere in S^{n+2} which we will denote by $\phi(M) \otimes_\tau K$. If M is not a connected manifold and has components $\{M_p\}$, $p = 1, \dots, P$, and we wish to spin a collection of knots, with trivializations $\{K_p\}$, then we will denote the spun knot by $\{\phi(M_p)\} \otimes_\tau \{K_p\}$. For simplicity we will mostly discuss the case where M is connected.

Next suppose that τ' corresponds to another choice of spin-trivialization, that is, τ' is a map $\tau' : M^k \times D^{n-k} \times D^2 \to S^2 \times D^2$ such that for each $x \in M^k$, $\tau'(\{x\} \times D^{n-k} \times \{0\}) = \tau(\{x\} \times D^{n-k} \times \{0\})$ and $\tau'(\{x\} \times \{0\} \times D^2) = \tau(\{x\} \times \{0\} \times D^2)$ such that $\tau^{-1}\tau' | \{x\} \times D^{n-k} \times D^2$ corresponds to a linear automorphism of $D^{n-k} \times D^2$. We may consider this to be an element of the product of special orthogonal groups $\mathrm{SO}(n-k) \times \mathrm{SO}(2)$; thus τ' gives rise to a map of M^k into $\mathrm{SO}(n-k) \times \mathrm{SO}(2)$. It is easily seen that the knot we obtain depends only on the homotopy class of this map. Let $[M, \mathrm{SO}(n-k) \times \mathrm{SO}(2)]$ denote the homotopy classes of such maps. If τ' gives rise to an element $\gamma \in [M^k, \mathrm{SO}(n-k) \times \mathrm{SO}(2)]$ we will denote the knot obtained by spinning α using τ' by $\phi(M) \otimes_\gamma \alpha$; we will say that γ is the twisting of τ' with respect to τ. (In the case that M is not a connected manifold with components M_p, $p = 1, \dots, P$, where the dimension of M_p is k_p, we may wish to spin a collection of knots $\{\alpha_p\}$ about M where α_p is a knotted $(n-k_p)$-disk which we spin along M_p. In this case we will denote the knot we obtain by $\{\phi(M_p)\} \otimes_\gamma \{\alpha_p\}$.)

In the case where $M^k = S^k$ and $\phi : S^k \to S^n$ is the standard embedding we may view an element of $[S^k, \mathrm{SO}(n-k) \times \mathrm{So}(2)]$ as an element of $\pi_k(\mathrm{SO}(n-k) \times \mathrm{SO}(2))$. If $\gamma \in \pi_k(\mathrm{SO}(n-k) \times \mathrm{SO}(2))$ then, since $\pi_k(\mathrm{SO}(n-k) \times \mathrm{SO}(2)) \approx \pi_k(\mathrm{SO}(n-k)) \oplus \pi_k(\mathrm{SO}(2))$ we may write $\gamma = \gamma_t \oplus \gamma_n$. Here γ_t corresponds to the tangential twisting discussed by Hsaing and Sanderson in [8]. The term γ_n describes the normal twisting. Now since $\pi_1(\mathrm{SO}(2)) \approx Z$ and $\pi_k(\mathrm{SO}(2)) = 0$ if $k > 1$ then we will only have normal twisting if we spin about a circle and in this case the integer γ_n corresponds to Zeeman's twisting operation [11].

Now SO(2) has the homotopy type of a circle and, as is shown by obstruction theory, the first cohomology of a complex with integer coefficients is in one-to-one correspondence with homotopy classes of maps from that complex to the circle. Thus there can be no normal twisting about a manifold with zero first cohomology.

Remark 1. This definition generalizes spinning about a submanifold as follows: given $M^k \times D^{n-k} \subseteq S^n$ and a knotted $(n-k)$-disk, α, we can consider $M^k \times D^{n-k}$ as $M^k \times D^{n-k} \times \{0\} \subseteq S^n \times I$; then writing M as the union of k-balls, M_j such that any two distinct M_j meet in a face of each, we can let $Q_j = M_j \times D^{n-k}$, identify T_1^j

with Q_j (we will have $i_j = 1$ for all j), define ψ_i^j appropriately and we will have $M \otimes \alpha = M^* \otimes \alpha$.

Remark 2. We note that the notation $M^* \otimes \{\alpha_p\}_{p=1}^P$ is not entirely an adequate notation, for it makes no explicit mention of the maps $\{\psi_i^j\}$ or the trivializations $\{K_p\}$, all of which are necessary for the construction of $M^* \otimes \{\alpha_p\}$; in those cases that explicit mention will be needed notationally, we will denote the knot by $\{M^* \otimes \{\alpha_p\}, \{K_p\}, \{T_{ij}^j\}, \{\psi_{ij}^j\}\}$.

Remark 3. In certain cases we can relax our requirement that T be a trivial bundle. If T is nontrivial, then the functions $g_{i,j}^{i',j'}$ will not all be identity maps. We will still be able to define knots $M^* \otimes \{\alpha_p\}$ and obtain a knotted embedding of Σ^n in $\Sigma^n \times D^2$ if we insist that the α_p are invariant under $\{g_{i,j}^{i',j'}\}$ in the following sense: we will require that for all i, j, i', j',

$$(g_{i,j}^{i',j'} \times \mathrm{id}_{D^2})\alpha_{p(i,j)} = \alpha_{p(i,j)} g_{i,j}^{i',j'} .$$

For example, if M^2 is a nonorientable surface in $S^3 \times D^1$ we would require that the knot be amphicheiral.

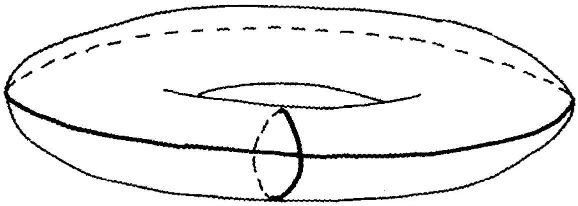
Remark 4. We may view the above construction as a way of knotting Σ in the trivial I^2 bundle, $\Sigma \times I^2$. Since this construction is local in nature, we may extend it to the case of knotting Σ in E where E is the total space of some I^2 bundle over Σ. In this case we will also need some special requirements for the knots $\{\alpha_p\}$. We will need to have

$$(g_{i,j}^{i',j'} \times f_{i,j}^{i',j'})\alpha_{p(i,j)} = \alpha_{p(i,j)} g_{i,j}^{i',j'}, \quad \text{where} \quad f_{i,j}^{i',j'} : I^2 \to I^2$$

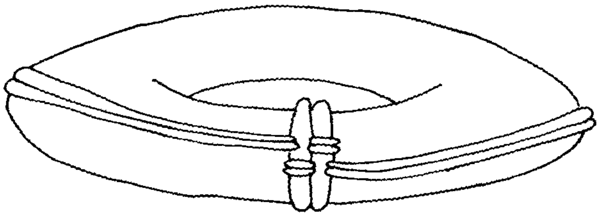
is the appropriate element in the transformation group of the I^2 bundle over Σ. An example of such a construction is found in [9] in obtaining knotted projective planes in S^4.

Remark 5. This construction may also be generalized to the situations such as discussed in the remark at the end of Section 1. For example, one could take the standard embedding of the torus, T, in S^4 and take the projection of the link as shown in Fig. 8(a). If we spin a trefoil knot about this, using the "obvious" trivializations of normal bundles, we obtain the knot whose projection in $\mathbb{R}^3$ is shown in Fig. 8(b). This knotted torus has been examined by Asano [3]; it is not the connected sum of an unknotted torus and a knotted 2-sphere.

Remark 6. If we take a knot with one crossing as in Fig. 9(a) and spin a trefoil about it, we may obtain a knotted sphere whose projection is shown in Fig. 9(b). By an isotopy we may obtain a slightly different projection of this knot as shown in Fig. 9(c). We note that we can now recognize this knot as that obtained by deform-spinning the "granny" knot, where the deformation is shown in Fig. 9(d).

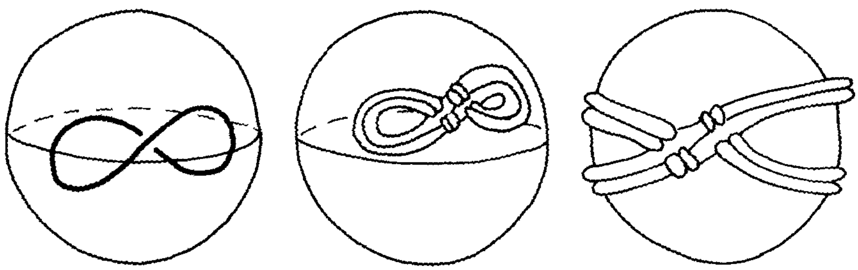


(a)

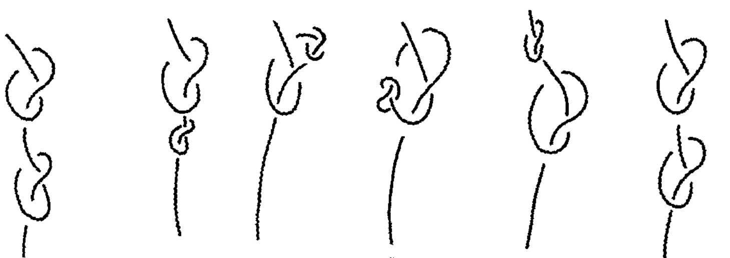


(b)

Fig. 8



(a) (b) (c)



(d)

Fig. 9

Remark 7. Finally, we note that we may further generalize our construction by the following basic change. Instead of defining ν_i as we did, we might, for each i, replace ν_i by an embedding

$$\delta_i : D^{k_i} \times D^{n-k_i} \times D^2 \to D^{k_i} \times D^{n-k_i} \times D^2,$$

which satisfies the following conditions for each $x \in D^{n-k_i}$:

(a) $\delta_i(\{x\}\times\{0\}\times\{0\})=\nu_i(\{x\}\times\{0\}\times\{0\})$,
(b) $\delta_i(\{x\}\times D^{n-k_i}\times D^2)=\nu_i(\{x\}\times D^{n-k_i}\times D^2)$
(c) $\delta_i|\partial(D^{k_i}\times D^{n-k_i})\times D^2=\nu_i|\partial(D^{k_i}\times D^{n-k_i})\times D^2$.

By replacing the $\{\nu_i\}$ by $\{\delta_i\}$ in the construction we will obtain a more general class of knots which we will call knots obtained by deform-spinning a knot about the projection of a knot. The essential difference between this deform-spinning and the spinning previously defined is that ν_i knots $\{x\}\times D^{n-k_i}\times D^2$ in $\{x\}\times D^{n-k_i}\times D^2$ in the same way for each $x\in D^{k_i}$ whereas if $y\in D^{n-k_i}$ then the embeddings $\delta_i|\{x\}\times D^{n-k_i}\times D^2$ and $\delta_i|\{y\}\times D^{n-k_i}\times D^2$ are only required to be isotopic embeddings.

4. π-regular homotopies

We will later see that $M^*\otimes\{\alpha_i\}$ is not independent of the ambient isotopy class of M in $\Sigma\times I$, however, it will be independent of the π-regular homotopy class of M in $\Sigma\times I$, which we now will define. Suppose f_0 and f_1 are two embeddings of $(M,\partial M)$ into $(\Sigma\times I,\partial\Sigma\times I)$ such that πf_1 and πf_2 are immersions where $\pi:\Sigma\times I\to\Sigma$ is projection. We will say that f_0 and f_1 are π-regularly homotopic if there is an ambient isotopy, $\{H_t\}$, of $\Sigma\times I$ such that $\{\pi\circ H_t\circ f_0\}$ is a regular homotopy of M in Σ and $H_1\circ f_0=f_1$. If $\{F_t\}$ is a regular homotopy of M in Σ, we will say that an ambient isotopy, $\{H_t\}$ of M in $\Sigma\times I$ is a lifting of $\{F_t\}$ to a π-regular homotopy if, for all t in I, $\pi H_t=F_t$.

Suppose $\{H_t\}$ is a π-regular homotopy between two immersions, f_0 and f_1, of M^k in Σ, suppose τ_0 is a spin-trivialization of M in $\Sigma\times D^2$ corresponding to the immersion f_0; we will consider τ_0 to be standard. Suppose also that τ_0' is another spin-trivialization which corresponds to an element $\gamma_0\in[M^k,\mathrm{SO}(n-k),\mathrm{SO}(2)]$. Then, by extending $\{H_t\}$ to a regular homotopy of a tubular neighborhood of f_0, we may obtain spin-trivializations τ and τ' of the immersion H' where $H':M\times I\to\Sigma\times I$ and is defined by $H'(x,t)=(H_tf_0(x),t)$, such that τ and τ' are level preserving extensions of τ_0 and τ_0'. By level preserving, we mean that τ and τ' are embeddings of $(M\times I)\times D^{n-k}\times D^2$ into $\Sigma\times I$ such that if τ_t and τ_t' denote the restrictions of τ and τ', respectively, to $(M\times\{t\})\times D^{n-k}\times D^2$, then each embeds $(M\times\{t\})\times D^{n-k}\times D^2$ into $\Sigma\times\{t\}$. Furthermore, γ will be the twisting of τ_t' with respect to τ_t. We may next define an isotopy $\{F_t\}$ of Σ in $\Sigma\times D^2$ by $F_t=H_tf_0(m)\otimes_\gamma\alpha$. Using the isotopy extension theorem we get a corresponding ambient isotopy. Thus we may obtain the following lemma.

Lemma 4. *If f_0 is π-regularly homotopic to f_1, then $f_0(m)\otimes_\gamma\alpha$ is ambiently isotopic to $f_1(m)\otimes_\gamma\alpha$.*

We next wish to show that, in a certain sense, multiple knotting is a commutative operation. Suppose we have an embedding, F, of $(D^{k_1}\times D^{n-k_1})\cup(D^{k_2}\times D^{n-k_2})$ into

$D^n \times I$ such that if $F_j = F|D^{k_j} \times D^{n-k_j}$, $j = 1, 2$, then $\{f_j\}$ is a multiple fibering of D^n where $f_j = pF_j$ and $p: D^n \times I \to D^n$ is projection. Let $\Phi: D^n \times I \to D^n \times I$ be defined by $\Phi(x, t) = (x, 1-t)$ and let $F' = \Phi \circ F$, $F'_j = \Phi \circ F_j$, $f'_j = p \circ F'_j$. Let ρ be the function $\rho(1) = 2$, $\rho(2) = 1$, then the multiple fiberings $\{f_j\}$ and $\{f'_j\}$ differ only in the order of the fibering; in other words, $\{f_j\}$ is the same as $\{f'_{\rho(j)}\}$. Now if we define ϕ_j, $\phi_j: D^{k_j} \to D^n \times I$ to be the restriction of F_j to $D^{k_j} \times \{0\}$ and similarly define ϕ'_j, then $\{\phi_j(D^{k_j})\} \otimes \{K_j\}$ can be identified with the multiple knotting of $\{K_j\}$ by $\{f_j\}$. Furthermore, one can easily show that there is a π-regular homotopy, $\{H_t\}$ of $D^{k_1} \cup D^{k_2}$,

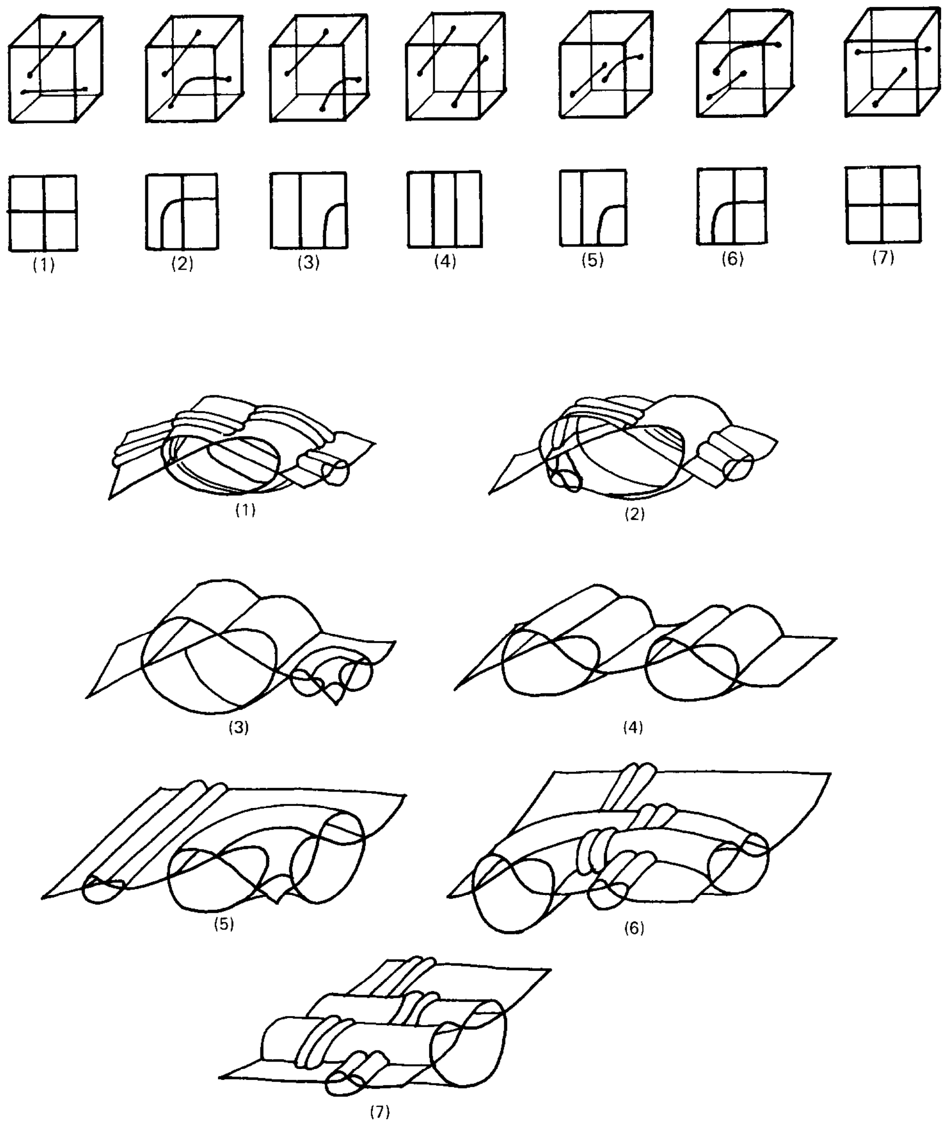


Fig. 10

such that $H_1 \circ \phi_j = \phi'_{p(j)}$. Thus we have $\{\phi_j(D^{k_j})\} \otimes \{K_j\}$ is ambiently isotopic to $\{\phi'_j(D^{k_j})\} \otimes \{K_j\}$ and thus, *as a disk pair*, the multiple knotting of $\{K_j\}$ by $\{f_j\}$ and the multiple knotting of $\{K_{p(j)}\}$ by $\{f'_j\}$ are the same. Figure 10 shows the projection of this ambient isotopy for the multiple knotting described in Figs. 2–6. Furthermore, we note that we can also find a π-regular homotopy $\{H'_t\}$ such that $H'_1(\phi_1(D^{k_1})) \cap H'_1(\phi_2(D^{k_2})) = \emptyset$; we can see this illustrated in Fig. 10(4). This is of interest since it then follows that the knot group of the multiple knotting is obtainable in a simple way from the knot groups of the knots $\phi^1(D^{k_1}) \otimes K_1$ and $\phi^2(D^{k_2}) \otimes K_2$ by taking the free product of these groups and then identifying a "meridian" of each. If $k_1 = k_2$ and K_j represents the knot α_j from $j = 1, 2$, then we may more simply say that the knot group of the multiple knotting is the same as the knot group of the connected sum $\alpha_1 \# \alpha_2$.

More generally, we have:

Lemma 5. *If ν is a multiple knotting of I^n in I^{n+2} by $\{k_j\}_{j=1}^{p}$, then the fundamental group of this knotting is isomorphic to the free product of the fundamental groups of the associated knots $\{\alpha_j\}_{j=1}^{p}$, with meridians amalgamated.*

The above lemma is useful for computing the knot groups of the knots we construct. For example, the knot shown in Fig. 9(b) can be shown to have the same group as the two-twist spun trefoil. (We conjecture that this knot may in fact be the two-twist spun trefoil.)

Consider the following situation. We wish to construct a knotted 3-sphere in 5-space by taking an embedded surface in S^3 and taking a knotted arc, and spinning the knot about this submanifold as discussed in Section 1. As long as the surface is not a two-sphere, there are a lot of embeddings of surfaces in S^3. One might wonder whether an interesting knot might be produced by spinning a knot around a knotted torus in S^3, rather than the standard one. However, this is not possible. The following lemma, together with the above discussion shows that any knot obtained by spinning a knot about a knotted manifold in S^3 is isotopic to a knot obtained by spinning that knot about the standard embedding of that surface.

Lemma 6. *If F_0 and F_1 are two submanifolds of S^3 of the same genus, then considering them to be submanifolds of $S^3 \times R^1$, F_0 and F_1 are π-regularly homotopic.*

Proof. Consider first a particular example as shown in Fig. 11(a). Here our surface F_1 is a torus in S^3 which is the boundary of the tubular neighborhood of a trefoil knot. Let B be the 3-ball as shown in Fig. 11(b) which intersects F_1 in two annuli A_1 and A_2. Now consider S^3 to be the subset $S^3 \times \{0\}$ of $S^3 \times \mathbb{R}^1$. Now we can find a smooth isotopy ϕ_t of F_1 which "raises A_2 up." That is, an isotopy fixed outside a neighborhood of A_2 which changes only the R^1-coordinates of points so that $\phi_1(A_2) \subseteq B \times \{1\}$ (and of course $\phi_1(A_1) \subseteq B \times \{0\}$). We may then follow ϕ_t by an isotopy ψ_t which is fixed outside a neighborhood of A_2 and such that $\psi_t | A_2$ is an

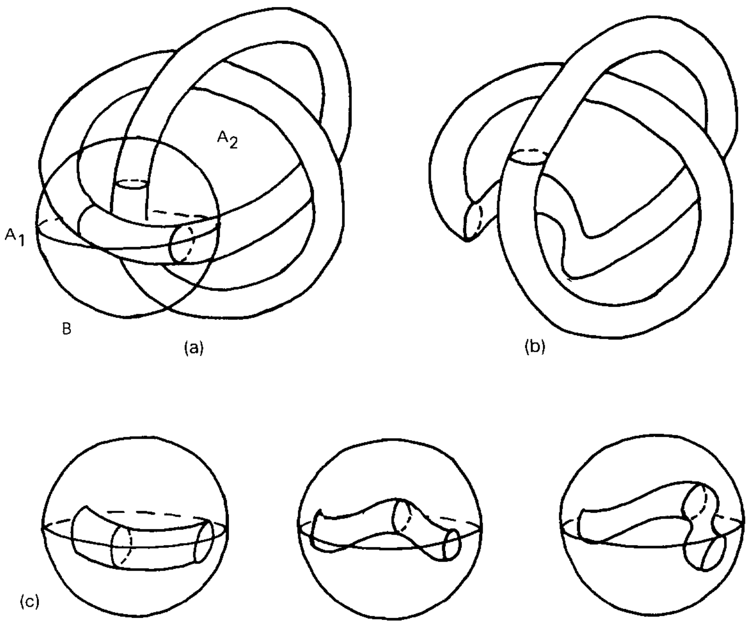


Fig. 11

isotopy of A_2 inside the 3-ball $B\times\{1\}$ as indicated in Fig. 11(c). The resulting embedding will have projection as shown in Fig. 11(b) and, as a set, this is a new embedding. Thus we have shown F_0 is π-regularly homotopic to F_1.

The proof in the general case uses this same technique. It is well known [6] that any surface in S^3 may be thought of as a cube with knotted holes and knotted handles. Our argument shows that we can unknot the holes and handles by a π-regular isotopy. □

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