Iterative roots with circuits for piecewise continuous and globally periodic maps

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\textbf{ABSTRACT}

Given a set $D$ of real numbers and a piecewise continuous and globally periodic map $F : D \rightarrow D$, we provide a description of all piecewise continuous iterative roots with circuits. We give some applications to real Möbius transformations as well as some examples for iterative roots with one single circuit and two different circuits.

\section{Introduction}

Regarded as a weak version of the embedding flow problem [9,10,18], the problem of iterative roots is interesting in both functional equations and dynamical systems (see e.g. [2,4,15,19]). A map $f : D \rightarrow D$ is called an $n$-th iterative root of a self-map $F : D \rightarrow D$ if $n$ is a nonnegative integer such that

$$f^n(x) = F(x), \quad \forall x \in D. \quad (1.1)$$

At the beginning of 19th century, Ch. Babbage [1] investigated the problem of iterative roots of the identity map, i.e. the Babbage equation $f^n = \text{id}$. A map $f : D \rightarrow D$ is said to be globally periodic with the prime period $m$ (cf. [5,8,13,16]) if $m$ is the smallest positive integer such that $f^m = \text{id}$. It follows from [11, Theorem 15.2, Theorem 15.3] that:

\textbf{Lemma 1.1.} Every continuous solution of the Babbage equation $f^n = \text{id}$ on a real interval is

(i) the identity map when $n$ is odd;
(ii) the identity map or an involution, i.e. $f^2 = \text{id}$, when $n$ is even.

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In 1998, Cheng et al. [6] discussed square iterative roots of the map $1/x$. In order to construct piecewise continuous iterative roots on subsets of the real line, they proposed an interesting definition of circuit:

**Definition 1.1.** A finite sequence $\{I_1, I_2, \ldots, I_k\}$ of pairwise disjoint intervals is called a $k$-circuit of a map $f : D \to D$ if

$$
\bigcup_{j=1}^k I_j \subseteq D, \ k \geq 2, \ f|_{I_j} \text{ is continuous such that } f(I_j) = I_{j+1}, \ j = 1, \ldots, k-1, \quad \text{and} \quad f(I_k) = I_1.
$$

For convenience, we say a self-map on the interval $I$ has a 1-circuit $\{I\}$.

Their result on piecewise continuous square iterative roots was generalized to the $n$-th iterative roots for any integer $n \geq 2$ in [7]. Given a Möbius transformation, the paper [17] presented all meromorphic iterative roots.

In 1998, Cheng et al. [6] discussed square iterative roots of the map $1/x$. Let $\gcd(n, k)$ denote the greatest common divisor of the nonzero integers $n$ and $k$.

**Lemma 2.1.** Suppose $F \in \mathcal{E}_m(\bigcup J_i)$. Then $F$ has an $m$-circuit. Furthermore, if $F$ has another $m_1$-circuit, then $m_1$ is a divisor of $m$.

Let $\gcd(n, k)$ denote the greatest common divisor of the nonzero integers $n$ and $k$.

**Lemma 2.2.** A map $f$ has a $k$-circuit if and only if $f^n$ has an $m$-circuit with $m = k/\gcd(n, k)$ for all $n \in \mathbb{N}$.

For any $m, n \in \mathbb{N}$ let $K(m, n) = \{k \in \mathbb{N} : k = m \cdot \gcd(n, k)\}$. It is easy to calculate $K(2, 1) = \{2\}$, $K(1, 2) = \{1, 2\}$, $K(2, 2) = \{4\}$, and $mn \in K(m, n)$.

**Lemma 2.3.** Suppose $F \in \mathcal{E}_m(\bigcup J_i)$. Then Eq. (1.1) restricted to an $m$-circuit of $F$ is equivalent to

$$
\begin{align*}
  f^{\frac{m}{k}} &= F^s, & \text{if} \ gcd(n, k) \text{ is odd}, & (2.1) \\
  f^{\frac{2m}{k}} &= F^{2s}, & \text{if} \ gcd(n, k) \text{ is even}, & (2.2)
\end{align*}
$$

where $k \in K(m, n)$, $s$ and $t$ are integers with $gcd(n, k) = sn + tk$.

**Proof.** By Lemma 2.1, $F$ has an $m$-circuit. By Lemma 2.2, every piecewise continuous solution of Eq. (1.1) has a $k$-circuit where $k \in K(m, n)$.

Put $d = \gcd(n, k)$. Let $k = k_1d$ and $n = n_1d$ where the integers $k_1$ and $n_1$ are coprime. The fact $k \in K(m, n)$ implies $n_1 = mm/k$ and $k_1 = m$. We shall consider the following two cases.

Case (a): $n_1$ is odd. Let $I$ be a $k$-circuit of $f$. Let $\varphi := f^k|_I$. Then $\varphi$ is continuous and strictly monotone from $I$ onto itself. Suppose that Eq. (1.1) holds. Then

$$
\varphi^{n_1} = \varphi^{\frac{mn}{k}} = f^{k} \varphi^{\frac{mn}{k}} = f^{mn} = F^m = \text{id}.
$$

By Lemma 1.1, $\varphi = f^k = \text{id}$. It follows that

$$
f^{\frac{m}{k}} = f^d = f^{sn+tk} = F^s \circ f^{tk} = F^s.
$$

Therefore Eq. (2.1) holds.

For any $k \in K(m, n)$, $s$ and $t$ are integers with $gcd(n, k) = sn + tk$.
Conversely, suppose that Eq. (2.1) holds. According to $F^m = \text{id}$, $m = k_1$ and $sn_1 + tk_1 = 1$, we have

$$f^n = f^{n_1} = f^{n_1} = F^{sn_1 + tm} = F^{sn_1 + tk_1} = F.$$ 

Therefore Eq. (1.1) holds.

Case (b): $n_1$ is even. Suppose that Eq. (1.1) holds. With the similar argument as case (a), we have $\varphi^{n_1} = \text{id}$, which implies that $\varphi^2 = f^{2n_1} = \text{id}$ by Lemma 1.1. It follows that

$$f^{2n} = f^{2d} = f^{2m+2tk} = F^{2s} \circ f^{2tk} = F^{2s}.$$ 

Therefore Eq. (2.2) holds.

Conversely, suppose that Eq. (2.2) holds. According to $F^m = \text{id}$, $m = k_1$ and $sn_1 + tk_1 = 1$, we have

$$f^n = f^{2d{n_1}} = f^{2d{n_1}} = F^{sn_1 + tm} = F^{sn_1 + tk_1} = F.$$ 

Therefore Eq. (1.1) holds. This completes the proof. □

**Remark 2.1.** We claim that Lemma 2.3 reduces the original equation (1.1) to the simplest form (2.1) (resp. (2.2)) when $n_1$ is odd (resp. even). In fact, the iteration index $k/m$ (resp. $2k/m$) in Eq. (1.1) (resp. (2.2)) is equal to or smaller than the index $n$ in Eq. (1.1). Moreover, the iteration index $k/m = d$ (resp. $2k/m$) is the smallest positive integer which can be written in the form $sn + tk$ (resp. $sn + 2tk$).

### 3. Construction of iterative roots with circuits

In this section, we shall construct all piecewise continuous solutions with circuits of Eq. (1.1) when $F \in \mathcal{E}_m(\bigcup J_i)$. Simply put a piecewise continuous solution is a union of piecewise continuous solutions on disjoint circuits. We only need to describe those solutions defined over a single $k$-circuit.

**Theorem 3.1.** Suppose $F \in \mathcal{E}_m(\bigcup J_i)$. Let $k \in K(m, n)$ and $I_1, \ldots, I_k$ be arbitrary pairwise disjoint intervals such that

$$I_{j+k\over m^i} = F^s(I_j), \quad j = 1, \ldots, k/m \quad \text{if } {mn \over k} \text{ is odd}, \quad (3.1)$$

$$I_{j+k\over m^{i+2k}} = F^{2s}(I_j), \quad j = 1, \ldots, 2k/m \quad \text{if } {mn \over k} \text{ is even}. \quad (3.2)$$

Further, if $mn/k$ is odd (resp. even), let $f_1, \ldots, f_{k/m-1}$ (resp. $f_{2k/m-1}$) be arbitrary continuous and strictly monotonic maps on intervals $I_1, \ldots, I_{k/m-1}$ (resp. $I_{2k/m-1}$) respectively, and such that

$$f_i(I_1) = I_{i+1}, \quad i = 1, 2, \ldots, k/m - 1 \quad (\text{resp. } 2k/m - 1). \quad (3.3)$$

Define $f_{i+k/m-1}$ (resp. $f_{i+2k/m-1}$) on interval $I_{i+k/m-1}$ (resp. $I_{i+2k/m-1}$) by

$$f_{i+k\over m^i} = F^s \circ f_{i+1}^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+1}^{-1} \circ f_{i+1}^{-1}, \quad i = 1, 2, \ldots, k \quad (3.4)$$

$$f_{i+k\over m^{i+2k}} = F^{2s} \circ f_{i+1}^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+1}^{-1} \circ f_{i+1}^{-1}, \quad i = 1, 2, \ldots, k. \quad (3.5)$$

Then $f : \bigcup_{i=1}^k I_i \to \bigcup_{i=1}^k I_i$ defined by

$$f(x) = f_i(x), \quad x \in I_i, \quad i = 1, 2, \ldots, k, \quad (3.6)$$

is a $k$-circuit solution of Eq. (1.1). Furthermore, every piecewise continuous $k$-circuit solution is constructed in this way.

**Proof.** We only consider the case $mn/k$ is odd. The similar discussion can be proceeded when $mn/k$ is even.

First, we need to construct $k$ disjoint intervals $I_1, \ldots, I_k$ such that condition (3.1) holds. Choose arbitrarily initial subintervals $I_1, \ldots, I_{k/m}$ in any interval of an $m$-circuit of $F$. By Lemma 2.3, the following $k$ intervals

$$I_1, \ldots, I_{k/m}, \quad F^s(I_1), \ldots, \quad F^s(I_{k/m}), \ldots, \quad F^{m-1}s(I_1), \ldots, \quad F^{m-1}s(I_{k/m}) \quad (3.7)$$

are disjoint and fulfill (3.1).

Second, we show that the finite sequence consisting of (3.7) is a $k$-circuit of $f$. According to (3.1) and $k \in K(m, n)$, we have

$$I_{i+{k\over m^i}} = F^{si}(I_j) \quad \text{for } i = 1, 2, \ldots, m-1, \quad j = 1, 2, \ldots, {k\over m}. \quad (3.8)$$
By (3.3), (3.4) and $F^m = \text{id}$, we have
\[ f(I_j) = I_{j+1} \quad \text{for} \quad j = 1, 2, \ldots, k - 1, \quad f(I_k) = f^k(I_1) = f^{k-m}(I_1) = F^{s-m}(I_1) = I_1. \]

Third, we claim that $f$ defined by (3.6) is an $n$-th iterative root of $F$. In fact, $f|_{I_i} = f_i$. By (3.3) and (3.4), we have
\[
\begin{align*}
F(x) &= f_{i+1}^k \circ f_{i+2}^j \circ \cdots \circ f_{i+1} \circ f_i(x) \\
&= (F^k \circ f_{i+1}^{-1} \circ f_{i+2}^{-1} \circ \cdots \circ f_{i+1}^{-1}) \circ f_{i+2} \circ \cdots \circ f_{i+1} \circ f_i(x) \\
&= F^k(x), \quad \forall x \in \bigcup_{i=1}^k I_i.
\end{align*}
\]

By Lemma 2.3, $f$ is a $k$-circuit solution of Eq. (1.1).

Finally, we show that every $n$-th iterative root with a $k$-circuit for $F$ can be obtained in that manner. It is known from [12, Theorem 11.1] that an iterative root of $F : D \to D$ is bijective if and only if $F$ is bijective. Then the global periodicity of $F$ implies that $F$ is bijective, so is every iterative root of $F$. It follows from (3.6) that maps $f_1, f_2, \ldots, f_{k/m-1}$ are continuous, strictly monotonic on intervals $I_1, I_2, \ldots, I_{k/m-1}$ respectively, and satisfy condition (3.3). In view of (2.1), (3.1) and (3.6), the relationship (3.4) holds. \( \square \)

Note that these piecewise continuous iterative roots on the whole domain may have either finite or infinitely many discontinuities. It is determined by the total number of circuits and the values of initial maps at their interval endpoints (cf. [6, Examples 4.5–4.6]).

4. Applications to real Möbius transformations

In this section, we first give a sufficient and necessary condition for global periodicity of real Möbius transformations.

A Möbius transformation on the complex plane is given by
\[
F(z) = \frac{az + b}{z + d},
\]
where $a, b, d$ are any complex numbers satisfying $ad - b \neq 0$. This definition is extended to the whole Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by defining $F(-d) = \infty$ and $F(\infty) = a$. This turns $F$ into a bijective holomorphic function from $\hat{\mathbb{C}}$ to itself.

Lemma 4.1. ([14, Theorem 3]) A Möbius transformation $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is globally periodic with the prime period $m > 1$ if and only if
\[
\lambda = \exp\left(\frac{2j\pi}{m}\right)
\]
where $\lambda := (d + \beta)/(d + \alpha)$, $\alpha$ and $\beta$ are two fixed points of $F$, and $\gcd(j, m) = 1$.

Lemma 4.2. Let $m > 1$ be an integer. A Möbius transformation $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is globally periodic with the prime period $m$ if and only if $F$ has an $m$-periodic point in $\mathbb{C}$.

Proof. The necessity is obvious. Now we shall prove the sufficiency. Suppose that $F$ has an $m$-periodic point $x^* \in \mathbb{C}$ with $m > 1$. Let $\alpha$ and $\beta$ be two fixed points of $F$, i.e., solutions of $x^2 + (d - a)x - b = 0$. We claim that $\alpha \neq \beta$. Assume that $\alpha = \beta$. According to [14, Theorem 2(a)], for any initial value $x_0 \in \mathbb{C} \setminus \{\alpha\}$, $F^n(x_0)$ approaches $\alpha$ as $n$ tends to infinity. This contradicts the assumption that $x^* \in \mathbb{C}$ is a point with the period $m > 1$.

By [14, p. 335, Formula (7)], we have for any positive integer $l$
\[
\frac{F^l(x^*) - \alpha}{F^l(x^*) - \beta} = \lambda^l \frac{x^* - \alpha}{x^* - \beta},
\]
where $\lambda := (d + \beta)/(d + \alpha)$, $\alpha$ and $\beta$ are two different fixed points of $F$. Since $x^* = F^m(x^*)$ and $x^* \neq F^l(x^*)$ for $1 \leq l \leq m - 1$, we obtain that $\lambda$ satisfies the condition (4.1). By Lemma 4.1, $F$ is globally periodic with the prime period $m$. \( \square \)

A real Möbius transformation $F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is defined by
\[
F(x) = \frac{ax + b}{x + d}, \quad ad - b \neq 0, \quad a, b, d \in \mathbb{R}.
\]

Lemma 4.3. A real Möbius transformation $F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is globally periodic with the prime period $m > 1$ if and only if one of the two following alternatives occurs
(i) $m = 2$, $F(x) = \frac{ax + b}{x - a}$.
(ii) $m \geq 3$,

$$F(x) = \frac{\alpha - \beta \lambda}{1 - \lambda} x - \frac{a\alpha \lambda - \beta}{1 - \lambda}$$

where $\lambda = \exp(\frac{2\pi i}{m})$ with $\gcd(j, m) = 1$, $\alpha$ is an arbitrary complex number with nonzero imaginary part.

**Proof.** Case (i) is trivial. Consider case (ii): let $F$ be a globally periodic Möbius transformation (4.2) with the prime period $m \geq 3$. Let $\alpha$ and $\beta$ be two fixed points of $F$, and put $\lambda = (d + \beta)/(d + \alpha)$. By Lemma 4.1 and $m \geq 3$, we have $\lambda \neq 1$ which leads to $\beta \neq \alpha$. Then

$$(d - a)^2 + 4b \neq 0.$$  

(4.4)

Solving the following system of equations for $a, b$ and $d$

$$\lambda = \frac{d + \beta}{d + \alpha}, \quad \alpha + \beta = a - d, \quad \alpha \beta = -b,$$

we have

$$a = \frac{(\alpha - \beta \lambda)}{1 - \lambda}, \quad b = -\alpha \beta, \quad d = \frac{(\alpha \lambda - \beta)}{1 - \lambda}.$$  

(4.5)

If $(d - a)^2 + 4b > 0$, then both $\alpha$ and $\beta$ are real but neither of $a$ and $d$ in (4.5) is real. It follows from (4.4) that $(d - a)^2 + 4b < 0$ which implies that $\beta = \alpha$ with nonzero imaginary part.

On the other hand, suppose that (4.3) holds. By Lemma 4.1, $F$ is globally periodic with the prime period $m$. Moreover, let $\alpha = \mu + iv, \beta = \mu - iv$ for any real $\mu$ and any nonzero real $v$. One can check that $a, b, d$ in the form (4.3) are all real.  

The following lemma gives the relationship between circuit and global periodicity of a real Möbius transformation.

**Lemma 4.4.** A real Möbius transformation $F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is globally periodic with the prime period $m > 1$ if and only if $F$ has an $m$-circuit.

**Proof.** Let $F$ be of the form (4.2).

**Sufficiency.** If $F$ has an $m$-circuit

$$\{I, F(I), \ldots, F^{m-1}(I)\}, \quad m \geq 2,$$

then $F^m(I) = I$ and the restriction $F^m$ to the interval $I$ is continuous. It follows that $F^m(I) = \text{closure}(I)$. So $\{I, F(I), \ldots, F^{m-1}(I)\}$ is a primitive cycle of length $m$ [cf. [3, p. 8]]. By [3, p. 9, Lemma 6], there exists an $m$-periodic point of $F$. By Lemma 4.2, $F$ is globally periodic with the prime period $m$.

**Necessity.** If a real Möbius transformation $F$ is globally periodic with the prime period $2$, it follows from Lemma 4.3 that $a^2 + b \neq 0$ and $a = -d$. When $a^2 + b > 0$, $F$ has two different real fixed points $\alpha, \beta$, and $\{(\infty, \alpha), (\alpha, -d)\}$ is a 2-circuit of $F$. When $a^2 + b < 0$, $F$ has no any real fixed point and $\{((-\infty, d), (-d, \infty))\}$ is a 2-circuit of $F$.

If $F$ is globally periodic with the prime period $m > 2$, then it follows from [14, Theorem 1] and Lemma 4.3 that

$$F^l(x) = \frac{\alpha - \beta \lambda}{1 - \lambda} \frac{x - \alpha \tilde{x}}{x + \alpha \lambda - \beta}, \quad \text{for } l = 1, 2, \ldots, \text{where } \lambda = \exp\left(\frac{2j\pi}{m}\right).$$

Thus $F^l$ has one singular point $x_l := \frac{(\alpha^l - \beta)}{1 - \lambda}$ (maybe $\infty$). One can easily check that $x_l$ is real, $x_{l_1} \neq x_{l_2}$ for $1 \leq l_1 < l_2 \leq m$ and $x_l = x_{m+1}$ for each $l = 1, 2, \ldots$. Therefore, all iterates of $F$ produce $m$ real distinct singular points $x_l, l = 1, 2, \ldots, m$, including the point $\infty$, which divide the real line into pairwise disjoint $m$ intervals $I_1, I_2, \ldots, I_m$. By Lemma 4.3, we have

$$F(x_l) = \frac{(\alpha - \tilde{x} \lambda)}{1 - \lambda} \cdot \frac{\alpha^l - \beta}{\lambda - 1} - \alpha \lambda (1 - \lambda) = x_{l-1}, \quad \text{for } l = 2, 3, \ldots, m.$$

It follows that $(I_1, I_2, \ldots, I_m)$ is an $m$-circuit of $F$. Furthermore, since $\bigcup_{j=1}^m I_j = \mathbb{R} \cup \{\infty\}$, $F$ has only $m$-circuits. The proof is complete.  

Remark that a real Möbius transformation $F$ is not necessarily globally periodic with the prime period 1 if $F$ has a 1-circuit. For instance, $[1, 2]$ is a 1-circuit of $F$ given by $(3x - 2)/x$ but $F$ is not globally periodic.
We end with two examples. Suppose hereafter $F : \mathbb{R} \setminus \{ -1, 2 \} \to \mathbb{R} \setminus \{ -1, 2 \}$ given by

$$f(x) = \frac{2x - 7}{x + 1}.$$ 

Note that $F$ is globally periodic with the prime period 3.

**Example 4.1.** Construct an $n$-th iterative root with a 3-circuit for $F$.

The relation $k = m \cdot \gcd(n, k)$ with $m = k = 3$ implies $3 \mid n$. So we consider the following four cases.

Case (i): $n = 6t - 1, t = 1, 2, \ldots$. By (2.1) and (3.1), Eq. (1.1) has a unique 3-circuit solution $f = f^2$ up to its 3-circuit $(I_1, F(I_1), F(I_1))$, where $I_1 = (1/2, 1, 2, \infty)$ or any subinterval of these three.

Case (ii): $n = 6t - 2, t = 1, 2, \ldots$. By (2.1), Eq. (1.1) is equivalent to the simplest form $f^2 = F^2$. By (2.2), (3.2) and Theorem 3.1, choose the initial map $f_1(x) = \frac{2x + 1}{x + 1}$ from $(1/2, 1, 2, \infty)$ and define $f_{j+1} := F^2 \circ f_j^{-1}$ for $j = 1, 2$. Then we obtain a 3-circuit iterative root on $\mathbb{R} \setminus \{ -1, 2 \}$

$$f(x) = \begin{cases} 
2x + 1, & x \in (1/2, 1, 2) \\
\frac{28x - 55}{x + 1}, & x \in (1, 2) \\
\frac{6x - 13}{3x - 5}, & x \in (2, \infty)
\end{cases}$$

Case (iii): $n = 6t - 4, t = 1, 2, \ldots$. Similarly to case (ii), an iterative root with a 3-circuit is given by

$$f(x) = \begin{cases} 
-x^2 + 9, & x \in \{ -1, 1 \}, \\
-x + 25, & x \in (1, 2) \\
-x + 19, & x \in (2, \infty)
\end{cases}$$

Case (iv): $n = 6t - 5, t = 1, 2, \ldots$. By (2.1) and (3.1), Eq. (1.1) is equivalent to the simplest form $f = F$.

**Example 4.2.** Construct a square iterative root with multi-circuits for $F$.

Since $K(3, 2) = \{ 3, 6 \}$, every square iterative root of $F$ has only 3-circuits or 6-circuits. By Lemma 2.3, $f^2 = F$ itself is the simplest.

First, we construct a 3-circuit iterative root. According to (2.2), (3.2) and Theorem 3.1, choose an initial interval $I_1 = (1/2, 1, 2, \infty)$ and an initial map $f_1(x) = \frac{2x + 1}{x + 1}$ from $I_1$ onto $I_2 := F(I_1)$, then define $f_{j+1} = F \circ f_j^{-1}$ for $j = 1, 2$. Then we obtain a square iterative root with a 3-circuit

$$\{ (-\infty, -3), (-1, 1/2), (2, 13/5) \}.$$ 

Next, we construct a 6-circuit iterative root on the remaining subintervals of the domain. According to (2.1), (3.1) and Theorem 3.1, choose two initial intervals $J_1 := [-3, -2]$ and $J_2 := [-2, -1]$ and an initial map $g_1(x) = x + 1$ from $J_1$ onto $J_2$, then define $g_{j+1} = F \circ g_j^{-1}$ for $i = 1, 2, 3, 4, 5$. Then we obtain a square iterative root with a 6-circuit

$$\{ (1, 2), [3/4, 11], (11, \infty), [5/4, 3], [3/5, 2] \}. $$

Combining the above two iterative roots into one, a square iterative root with a 3-circuit and a 6-circuit on $\mathbb{R} \setminus \{ -1, 2 \}$ is given by

$$f(x) = \begin{cases} 
\frac{4x + 27}{5x}, & x \in (1/2, 1, 2) \\
x + 1, & x \in [-3, -2) \\
2x - 9, & x \in [-2, -1) \\
-\frac{35x + 82}{5x + 23}, & x \in (-1, 4/5) \\
-\frac{10x - 1}{x - 8}, & x \in [4/5, 3/2) \\
x / (x + 1), & x \in [5/4, 2) \\
2x + 9, & x \in (2, 12/7) \\
-7x - 4, & x \in [12/7, 11) \\
5x - 19, & x \in (11, \infty)
\end{cases}$$

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