Solution of Multicriteria Control Problems in Certain Types of Linear Distributed-Parameter Systems by a Multicriteria Simplex Method

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1. INTRODUCTION

Considerable attention has been given in the past decade to the optimal control problems by applying linear programming techniques and linear programming has proved to be a very efficient tool for linear lumped-parameter systems [8].

For linear distributed-parameter systems, there are only two known papers where particular examples of optimal control of distributed-parameter systems were solved by linear programming techniques [3, 6]. However, these papers only consider a single objective as a consequence of the application of traditional linear programming techniques.

On the other hand, in 1973, M. Zeleny introduced a Multicriteria Simplex Method, which is a simple generalization of the conventional single-objective simplex method [9, 10, 11]. He developed theory and algorithms which can be applied to linear programming problems involving multiple, noncommensurable objective functions. Nowadays, a multicriteria simplex method seems to be most powerful and attractive method for linear multiobjective programming problems.

In previous papers, we have considered multicriteria linear continuous optimal control problems of lumped-parameter systems through the application of multicriteria simplex method and indicated the efficiency of the proposed method [4, 5].

In this paper, we discuss the problems of optimal control of one dimensional linear stationary distributed parameter systems with several cost functionals by means of a multicriteria simplex method.

2. STATEMENT OF THE PROBLEM

Let one-dimensional linear stationary distributed-parameter systems be controlled by boundary control functions $u_i(t)$ ($i = 1, ..., m$) and the state of the
systems is assumed to be specified by a state function which can be expressed as

\[ q(t, x) = k(t, x) + \int_0^t G(t, \tau, x) u(\tau) \, d\tau \]  \hspace{1cm} (2.1)

where \( q(t, x) \) is the state function in dependence on the space coordinate \( x \) \((0 \leq x \leq L)\) and time \( t \) \((0 \leq t \leq T)\) and \( u(t) = (u_1(t), \ldots, u_m(t)) \). Furthermore, \( k(t, x) \) and \( G(t, \tau, x) = (g_1(t, \tau, x), \ldots, g_m(t, \tau, x))' \) are known Green functions which are determined corresponding to given differential equations and initial and boundary conditions. One-dimensional linear distributed-parameter systems controlled by boundary functions are usually representable in the form of (2.1).

The control functions are assumed to be constrained as

\[ |u_i(t)| \leq 1. \]  \hspace{1cm} (2.2)

Of particular interest are performance indices expressible in linear form for the obvious reasons that we consider the application of a multicriteria simplex method. Some of the most popularly used performance indices for linear distributed-parameter systems are as follows.

1. **Minimum fuel problem.**

\[ J_1 = \int_0^T \| u(t) \|_1 \, dt = \int_0^T \sum_{i=1}^m |u_i(t)| \, dt. \]  \hspace{1cm} (2.3)

2. **Minimum amplitude problem.**

\[ J_2 = \max_{0 \leq t \leq T} \| u(t) \|_1 = \max_{0 \leq t \leq T} \sum_{i=1}^m |u_i(t)|. \]  \hspace{1cm} (2.4)

3. **Minimum final distance problem.**

\[ J_3 = \int_0^L \| q^*(x) - q(T, x) \| \, dx, \]  \hspace{1cm} (2.5)

\[ J_4 = \max_{0 \leq x \leq L} |q^*(x) - q(T, x)|, \]  \hspace{1cm} (2.6)

where \( q^*(x) \) is a prescribed spatial distribution function of the state.

In this paper, we consider the following multicriteria linear optimal control problem of distributed-parameter systems as an illustration.

**Problem 2.1.** To find the optimal control \( u(t) \) which in a given time \( T \)
brings system (2.1) from the given initial state \( q(0, x) \) to the final state \( q(T, x) \) so that the multicriteria functional

\[
J = \left[ J_1, J_2, J_3, J_4 \right]
\]

is minimized under the constraints (2.2).

3. LINEAR MULTICRITERIA PROGRAMMING FORMULATION BY NUMERICAL INTEGRATION FORMULA

In order to reduce the problem under consideration to a linear multicriteria programming problem, we first decompose the control variable \( u(t) \) into a pair of nonnegative variables as follows

\[
u(t) = u^+(t) - u^-(t),
\]

\[
u^+(t) \geq 0, \quad u^-(t) \geq 0.
\]

Then, using a numerical integration formula—such as e.g. Simpson's composite formula, Newton-Cotes' formula or Gauss' formula—to the definite integrals in the right hand side of (2.5), yields

\[
J_3 = \int_0^L | q^*(x) - q(T, x) | \, dx
\]

\[
= \int_0^L \left[ q^*(x) - k(T, x) - \int_0^T G(T, \tau, x) \left( u^+(\tau) - u^-(\tau) \right) \, d\tau \right] \, dx
\]

\[
\approx L \sum_{j=0}^{N} c_j \left[ q^*(x_j) - k(T, x_j) - \int_0^T G(T, \tau, x_j) \left( u^+(\tau) - u^-(\tau) \right) \, d\tau \right]
\]

\[
\approx L \sum_{j=1}^{N} c_j \left[ q^*(x_j) - k(T, x_j) - T \sum_{k=0}^{K} d_k G(T, \tau_k, x_j) \left( u^+(\tau_k) - u^-(\tau_k) \right) \right]
\]

where \( c_j \)'s and \( d_k \)'s are the weights assigned to the values of the integrands at the points \( x_j \)'s and \( \tau_k \)'s respectively, and \( N \) and \( K \) are the numbers of terms. The values of \( x_j \)'s and \( \tau_k \)'s and the weights \( c_j \)'s and \( d_k \)'s are known in the integration formula.

We also express the argument of the absolute value in (3.3) as the difference of nonnegative quantities as follows.

\[
q^*(x_j) - k(T, x_j) - T \sum_{k=0}^{K} d_k G(T, \tau_k, x_j) \left( u^+(\tau_k) - u^-(\tau_k) \right) = g^+_{j} - g^-_{j}.
\]
Furthermore, we introduce auxiliary variables $U$ and $Q$ satisfying the inequality constraints

$$U \geq \sum_{i=1}^{m} (u_i^+(\tau_k) + u_i^-(\tau_k)), \quad (3.5)$$

$$Q \geq g_i^+ + g_i^- . \quad (3.6)$$

As the result of the above discussion, an approximate multicriteria linear programming problem may be stated as follows:

**Problem 3.1.** Minimize $J = [J_1, J_2, J_3, J_4]'$ subject to

$$q^*(x_j) - k(T, x_j) - T \sum_{k=0}^{K} d_k G(T, \tau_k, x_j) (u_i^+(\tau_k) - u_i^-(\tau_k)) = g_i^+ - g_i^- ,$$

$$1 \geq u_i^+(\tau_k) \geq 0 , \quad 1 \geq u_i^-(\tau_k) \geq 0 , \quad g_i^+ \geq 0 , \quad g_i^- \geq 0 ,$$

$$U \geq \sum_{i=1}^{m} (u_i^+(\tau_k) + u_i^-(\tau_k)) ,$$

$$Q \geq g_i^+ + g_i^- ,$$

where

$$J_1 = T \sum_{k=1}^{K} d_k \sum_{i=1}^{m} (u_i^+(\tau_k) + u_i^-(\tau_k)) ,$$

$$J_2 = U ,$$

$$J_3 = L \sum_{j=0}^{N} c_j (g_j^+ + g_j^-) ,$$

$$J_4 = Q .$$

**Remark.** In order to reduce the amount of computation time, there may be different formulations which lead Problem 2.1 to a different multicriteria programming problem. We shall not consider that problem here.

4. **ILLUSTRATIVE EXAMPLE**

To illustrate the method described in this paper, we give the following minimum final distance and minimum fuel problem of one-dimensional heat conduction system. This problem has been studied extensively from the single objective viewpoint (i.e. minimum final distance) by many authors [1, 2, 6, 7].
Example (minimum final distance and minimum fuel problem).

\[ \min \left( \int_0^1 |q^+(x) - q(x, T)| dx, \int_0^T u(t) dt \right) \]  
subject to

\[ \frac{\partial q(x, t)}{\partial t} = \frac{\partial^2 q(x, t)}{\partial x^2}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \]  

\[ q(x, 0) = 0, \]  

\[ \frac{\partial q(x, t)}{\partial x} \bigg|_{x=0} = -\alpha (m(t) - q(0, t)), \]  

\[ \frac{\partial q(x, t)}{\partial x} \bigg|_{x=1} = 0, \]  

\[ \gamma dm(t)/dt + m(t) = u(t), \]  

\[ 0 \leq u(t) \leq 1. \]  

Using the Laplace transform, the solution of the partial differential equation becomes [6]

\[ q(t, x) = \int_0^\tau G(t - \tau, x) u(\tau) d\tau \]  

where

\[ G(t, x) = \frac{\kappa^2 \cos(1 - x)}{\cos \kappa - (\kappa/\alpha) \sin \kappa} \exp(-\kappa^2 t) \]  

\[ + 2\kappa^2 \sum_{i=1}^{\infty} \frac{\cos(1 - x) \beta_i}{(\kappa^2 - \beta_i^2) \{(1/\alpha) + (1 + \alpha)\beta_i \}} \exp(-\beta_i^2 t), \]  

\[ \kappa = 1/\gamma^{1/2}, \]  

and \( \beta_i \)'s are the real roots of the transcendental equation

\[ \beta \tan \beta = \alpha. \]  

Applying the method described in this paper, the multicriteria linear programming formulation becomes as follows:

\[ \min \left( \sum_{j=0}^N c_j (g_j^+ + g_j^-), \sum_{k=0}^K d_k u(\tau_k) \right) = (J_1, J_2)' \]
subject to

\[
T \sum_{k=0}^{K} d_k G(T - \tau_k, x_j) u(\tau_k) = q^*(x_j) - (g^+_j + g^-_j),
\]

\[1 \geq u(\tau_k) \geq 0,
\]

\[g^+_j \geq 0, \quad g^-_j \geq 0.
\]

Numerical values of the parameters of the system were taken as \(\alpha = 10, \gamma = 0.04 (\kappa = 5), \quad q^*(x) = 0.2 \quad (0 \leq x \leq 1), \quad T = 0.2,\) and the number of division of the intervals were taken as \(N = 20, K = 20.\) The infinite series of (4.9) was approximated by the sum of the first eleven terms of (4.9), because the infinite series will converge rapidly with the increase of number \(t,\) provided that \(t \neq 0.\) When \(t = 0,\) using the initial value theorem in the Laplace transform, we obtain

\[G(0, x) = 0.
\]

We have coded a multicriteria simplex program in FORTRAN following Zeleny [11] in order to handle problem with up to 7 objectives, 63 constraints and 150 variables.

**TABLE I**

Thirty-nine Nondominated Extreme Solutions

<table>
<thead>
<tr>
<th>(J_1)</th>
<th>(J_2)</th>
<th>(J_3)</th>
<th>(J_4)</th>
<th>(J_5)</th>
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<td>1 0.3726621E-01 0.2558615E+00 21 0.8165085E-01 0.1486729E+00</td>
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<tr>
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</table>
By using Simpson's numerical integration formula, we get 39 different non-
dominated (Pareto optimal, noninferior, efficient) extreme solutions (see Table I). 
CPU time required for execution was about 5 minutes on the FACOM 230–75. 
Individual minimum of $J_1$ and $J_2$ are achieved in solutions (1) and (39) re-
spectively, and solution (1) coincides with the result of Sakawa [6].

5. Conclusion

In this paper multicriteria one-dimensional linear stationary distributed-
parameter systems controlled by boundary control functions have been investi-
gated through the application of a multicriteria simplex method. In order to 
reduce the control problem of this distributed-parameter system to approximate 
multicriteria linear programming problems, we used a numericcal integration 
formula and introduced the suitable auxiliary variables. Then applying the 
multicriteria simplex method, all the nondominated extreme solutions were 
obtained.

This knowledge of nondominated extreme solutions might be quite helpful, 
since any nondominated solutions is a convex combination of nondominated 
extreme solutions. However, the decision maker must choose a single non-
dominated solution as the final solution of the given problem. Of course, there 
exist many different approaches to achieve this, but our remaining problem is to 
provide an efficient and attractive method which enables the decision maker to 
obtain a final solution.

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