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Finite element approximations of stochastic optimal control problems constrained by stochastic elliptic PDEs

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ABSTRACT

In this paper we study mathematically and computationally optimal control problems for stochastic elliptic partial differential equations. The control objective is to minimize the expectation of a tracking cost functional, and the control is of the deterministic, distributed type. The main analytical tool is the Wiener–Itô chaos or the Karhunen–Loève expansion. Mathematically, we prove the existence of an optimal solution; we establish the validity of the Lagrange multiplier rule and obtain a stochastic optimality system of equations; we represent the input data in their Wiener–Itô chaos expansions and deduce the deterministic optimality system of equations. Computationally, we approximate the optimality system through the discretizations of the probability space and the spatial space by the finite element method; we also derive error estimates in terms of both types of discretizations.

1. Introduction

Many physical and engineering models involve uncertain data or uncertain parameters; i.e., many realistic models are stochastic partial differential equations (SPDEs). In fact, deterministic elliptic partial differential equations (PDEs) have been reformulated into stochastic elliptic PDEs based on the Karhunen–Loève (K-L) expansion and there has been progress in both the analysis and the finite element approximations for stochastic elliptic PDEs in the last decade; see e.g., [9,2,3,24,4,5,11]. Also, it is well known that this elliptic SPDE is used to model fluid flows in porous media that is used in many areas of applied science and engineering (e.g., transport of pollutants in groundwater and oil recovery processes).

In this paper we consider the stochastic optimal control problems constrained by stochastic elliptic PDEs based on the Wiener–Itô (W-I) chaos expansion. In stochastic optimal control problems, we derive the optimality system of equations and solve that system to find the optimal solution for minimization problems. Usually, the optimality system of equations arising from an optimal control problem are coupled, which is the main difficulty in solving the optimization problems. Here, in order to solve stochastic optimal control problems, we apply the theory of Brezzi–Rappaz–Raviart (B-R-R), which plays an important role in uncoupling the optimality system of equations. For an abstract framework by using B-R-R theory in deterministic optimal control problems, we refer the reader to [17] and for applications [15,16,19].

The plan of the paper is as follows. In Section 2, we shall first mention the existence and uniqueness of stochastic elliptic PDEs and then show that there is an optimal solution for a stochastic optimal control problem constrained by stochastic elliptic PDEs. In Section 3, we are going to show the existence of a Lagrange multiplier to derive the optimality system of equations. In Section 4, we shall talk about the W-I chaos expansions and K-L expansions for our input data and compare

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them. Then in Section 5, we shall estimate the error for the finite element solution of stochastic elliptic PDEs with a stochastic input data and a deterministic input data under slightly weaker regularity requirements in the spatial domain compared to similar results in the literature. In Section 6, we use the B-R-R theory to derive discrete error estimates for the finite element approximations of the optimality system of equations.

2. Optimization of functionals involving stochastic functions

2.1. The model problem and stochastic functional

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure.

We consider the following stochastic elliptic PDE with the Dirichlet boundary condition: find a stochastic function $u : \overline{D} \times \Omega \to \mathbb{R}$ such that the following equation holds for almost every $\omega \in \Omega$:

$$\begin{aligned} &-\operatorname{div}\big[a(x,\omega)\nabla u(x,\omega)\big] = f(x) \quad \text{in } D,\\ &u(x,\omega) = 0 \quad \text{on } \partial D, \end{aligned} \tag{2.1}$$

where $D \subset \mathbb{R}^d$ is a convex bounded polygonal domain, $a : D \times \Omega \to \mathbb{R}$ is a stochastic function with a continuous and bounded covariance function, and $f \in L^2(D)$ that will be used as a distributed deterministic control. Note that in the paper, ∇ means differentiation with respect to $x \in D$ only. For the existence and uniqueness of the solution of our SPDE, we assume that there are positive m and M such that

$$m \leqslant a(x,\omega) \leqslant M \quad \text{a.e. } (x,\omega) \in D \times \Omega. \tag{2.2}$$

We now introduce a functional that involves stochastic functions. For a target solution $U : \overline{D} \times \Omega \to \mathbb{R}$, we consider

$$\begin{aligned} \mathcal{J}_\beta(u, f) &= E\left(\frac{1}{2}\int_D |u-U|^2\,dx + \frac{\beta}{2}\int_D |f|^2\,dx\right)\\ &= \frac{1}{2}\int_\Omega\int_D |u-U|^2\,dx\,dP + \frac{\beta}{2}\int_\Omega\int_D |f|^2\,dx\,dP, \end{aligned} \tag{2.3}$$

where β is a positive constant.

As you might see, we are using the cost functional whose notation is the same as that in [20] or in most papers of optimal control problems. However, we point out that the functional in the paper has the expectation.

We are going to minimize the stochastic functional (2.3) with suitable deterministic function f subject to the stochastic PDE (2.1) by solving the optimality system of stochastic equations.

2.2. Function spaces and notation

Throughout this paper, we use standard notations (e.g., see [1]) for the Sobolev spaces $H^r(D)$ for all real numbers r with norms $\|\cdot\|_{H^r(D)}$. We use $H^r_0(D)$ as the subspace of $H^r(D)$ whose function value is zero on the boundary of D with the norm $\|u\|^2_{H^1_0(D)} = \int_D |\nabla u|^2\,dx$ as usual. For a Hilbert space H^r, $(\cdot,\cdot)_{H^r}$ denotes the H^r inner product.

We define stochastic Sobolev spaces:

$$L^2\big(\Omega; H^r(D)\big) = \big\{v : D \times \Omega \to \mathbb{R} \;\big|\; \|v\|_{L^2(\Omega; H^r(D))} < \infty\big\},$$

where

$$\|v\|^2_{L^2(\Omega;H^r(D))} = \int_\Omega \|v\|^2_{H^r(D)}\,dP = E\|v\|^2_{H^r(D)}.$$

Note that the stochastic Sobolev space $L^2(\Omega; H^r(D))$ is a Hilbert space. For simplicity, we put $\mathcal{H}^r(D) = L^2(\Omega; H^r(D))$ and $\mathcal{L}^2(D) = L^2(\Omega; L^2(D))$.

We also introduce notation:

$$b[u, v] = E\int_D a\nabla u \cdot \nabla v\,dx \quad \text{and} \quad [u, v] = E\int_D uv\,dx.$$

2.3. The existence of an optimal solution

We have a weak formulation of (2.1) as follows: seek $u \in \mathcal{H}_0^1(D)$ such that

$$b[u, v] = [f, v] \quad \forall v \in \mathcal{H}_0^1(D). \tag{2.4}$$

From assumption (2.2) and the Lax–Milgram lemma (cf. [6]), we have the unique existence of the solution for (2.4).

We now examine the existence of an optimal solution that minimizes $\mathcal{J}_\beta(\cdot,\cdot)$. Let

$$\mathcal{U}_{ad} = \big\{(u, f) \in \mathcal{H}_0^1 \times L^2 \text{ such that (2.4) satisfied and } \mathcal{J}_\beta(u, f) < \infty\big\} \tag{2.5}$$

be the admissibility set.

We say that $(\hat{u}, \hat{f}) \in \mathcal{U}_{ad}$ is an *optimal solution* of $\mathcal{J}_\beta(u, f)$ if for all $(u, f) \in \mathcal{U}_{ad}$ satisfying that $\|u - \hat{u}\|_{\mathcal{H}_0^1(D)} + \|f - \hat{f}\|_{L^2(D)} \leqslant \epsilon$ for some $\epsilon > 0$,

$$\mathcal{J}_\beta(\hat{u}, \hat{f}) \leqslant \mathcal{J}_\beta(u, f). \tag{2.6}$$

We then prove the existence of an optimal solution.

Theorem 1. *There is an optimal solution $(\hat{u}, \hat{f}) \in \mathcal{U}_{ad}$ of $\mathcal{J}_\beta(u, f)$.*

Proof. Note that $\mathcal{U}_{ad}$ is not empty. Let $\{(u^{(n)}, f^{(n)})\}$ be a minimizing sequence in $\mathcal{U}_{ad}$, that is,

$$\lim_{n\to\infty} \mathcal{J}_\beta\big(u^{(n)}, f^{(n)}\big) = \inf_{(u,f)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u, f). \tag{2.7}$$

Then, we have that $\|f^{(n)}\|_{L^2(D)} \leqslant C$ for some $C > 0$. That is, the sequence $\{f^{(n)}\}$ is uniformly bounded in $L^2(D)$. Note that for some $C > 0$, we have $\|u^{(n)}\|_{\mathcal{H}_0^1(D)} \leqslant C\|f^{(n)}\|_{L^2(D)}$. Thus, $\{u^{(n)}\}$ is a uniformly bounded sequence in $\mathcal{H}_0^1(D)$. As a result, there is a convergent subsequence $\{(u^{(n_i)}, f^{(n_i)})\}$ such that

$$u^{(n_i)} \rightharpoonup \hat{u} \quad \text{weakly in } \mathcal{H}_0^1(D) \quad \text{and} \quad f^{(n_i)} \rightharpoonup \hat{f} \quad \text{weakly in } L^2(D) \tag{2.8}$$

for some $(\hat{u}, \hat{f}) \in \mathcal{H}_0^1(D) \times L^2(D)$.

Note that $\|f^{(n)}\|_{\mathcal{L}^2(D)} \leqslant C$. Hence, we also see that

$$f^{(n_i)} \rightharpoonup \hat{f} \quad \text{weakly in } \mathcal{L}^2(D). \tag{2.9}$$

This implies that

$$\big[f^{(n_i)}, v\big] \to [\hat{f}, v] \quad \forall v \in \mathcal{L}^2(D). \tag{2.10}$$

Because $\|\nabla u^{(n)}\|_{\mathcal{L}^2(D)} \leqslant \|u^{(n)}\|_{\mathcal{H}_0^1(D)} \leqslant C$, we have

$$\nabla u^{(n_i)} \rightharpoonup \nabla\hat{u} \quad \text{weakly in } \mathcal{L}^2(D).$$

This yields

$$\big[\nabla u^{(n_i)}, w\big] \to [\nabla\hat{u}, w] \quad \forall w \in \mathcal{L}^2(D).$$

The fact that $\nabla v \in \mathcal{L}^2(D)$ for $v \in \mathcal{H}_0^1(D)$ lead us to

$$\big[\nabla u^{(n_i)}, \nabla v\big] \to [\nabla\hat{u}, \nabla v] \quad \forall v \in \mathcal{H}_0^1(D).$$

Thus, we obtain

$$b\big[u^{(n_i)}, v\big] \to b[\hat{u}, v] \quad \forall v \in \mathcal{H}_0^1(D) \tag{2.11}$$

because $a\nabla v \in \mathcal{L}^2(D)$ for $v \in \mathcal{H}_0^1(D)$.

With the help of (2.10) and (2.11), we can show that

$$b[\hat{u}, v] = \lim_{n_i\to\infty} b\big[u^{(n_i)}, v\big] = \lim_{n_i\to\infty}\big[f^{(n_i)}, v\big] = [\hat{f}, v] \quad \forall v \in \mathcal{H}_0^1(D). \tag{2.12}$$

That is, $(\hat{u}, \hat{f})$ satisfies (2.4) and hence $(\hat{u}, \hat{f}) \in \mathcal{U}_{ad}$. Using the weak convergence (2.8) and the weak lower continuity of the functional $\mathcal{J}_\beta(\cdot,\cdot)$ we arrive at

$$\mathcal{J}_\beta(\hat{u}, \hat{f}) \leqslant \lim_{n_i\to\infty} \inf \mathcal{J}_\beta\big(u^{(n_i)}, f^{(n_i)}\big) = \inf_{(u,f)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u, f). \tag{2.13}$$

Therefore, $(\hat{u}, \hat{f})$ is an optimal solution. $\square$

3. The optimality system of stochastic equations

We will derive an optimality system of stochastic equations by using the Lagrange multiplier rule for the constrained minimization problem:

$$\min_{(u,f)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u,f) \quad \text{subject to (2.4)}. \tag{3.1}$$

For the deterministic case, we know that there exists a Lagrange multiplier; see e.g., [10]. Thus, without proving the existence of a Lagrange multiplier, we could derive an optimality system of the minimization problem in the deterministic problem; e.g., [20]. In the stochastic case, however, we first need to show that there is a Lagrange multiplier before using the Lagrange multiplier rule to derive an optimality system of stochastic equations. To show the existence of a Lagrange multiplier, we follow the method given in [17].

3.1. The abstract minimization problem

We begin with the definition of the abstract class of minimization problems. Let G, X, and Y be reflexive Banach spaces whose norms are denoted by $\|\cdot\|_G$, $\|\cdot\|_X$, and $\|\cdot\|_Y$ and whose dual spaces are denoted by G^*, X^*, and Y^*, respectively. Let Θ be the control set that is a closed convex subset of G.

We assume that the functional to be minimized takes the form

$$\mathcal{J}(v,z) = \lambda\mathcal{F}(v) + \lambda\mathcal{E}(z) \quad \forall (v,z)\in X\times\Theta, \tag{3.2}$$

where $\mathcal{F}$ is a functional on X, $\mathcal{E}$ is a functional on Θ, and λ is a given parameter that is assumed to belong to a compact interval $\Lambda\subset\mathbb{R}_+$.

We define the function $M: X\times\Theta\to X$ for the constraint equation $M(v,z)=0$ as follows:

$$M(v,z) = v + \lambda T N(v) + \lambda T K(z) \quad \forall (v,z)\in X\times\Theta, \tag{3.3}$$

where $N: X\to Y$ is a differentiable map, $K:\Theta\to Y$ is a bounded linear operator, $T: Y\to X$ is a bounded linear operator, and $\lambda\in\Lambda$.

With these definitions, we now consider the following constrained minimization problem:

$$\min_{(v,z)\in X\times\Theta} \mathcal{J}(v,z) \quad \text{subject to } M(v,z)=0. \tag{3.4}$$

3.2. Hypotheses concerning the abstract minimization problem

The set of hypotheses needed to justify the use of the Lagrange multiplier rule and to derive an optimality system can be determined is given by

(HE1) for each $z\in\Theta$, $v\mapsto\mathcal{J}(v,z)$ and $v\mapsto M(v,z)$ are Fréchet differentiable;
(HE2) $z\mapsto\mathcal{E}(z)$ is convex;
(HE3) for $v\in X$, $N'(v)$ maps from X into $Z\hookrightarrow\hookrightarrow Y$, where N' denotes the Fréchet derivative of N.

3.3. Results concerning the existence of Lagrange multipliers

In this section we give the following useful theorems (proofs can be found in [17]) about the abstract Lagrange multiplier rule.

Theorem 2. *Let $\lambda\in\Lambda$ be given. Assume that there exists an optimal solution (u,f) of (3.4) in $X\times\Theta$ and that (HE1)–(HE3) hold. Then there exists a $k\in\mathbb{R}$ and a $\mu\in X^*$ that are not both equal to zero such that*

$$k\langle\mathcal{J}_u(u,f),w\rangle - \langle\mu, M_u(u,f)\cdot w\rangle = 0 \quad \forall w\in X \quad \textit{and}$$
$$\min_{z\in\Theta}\mathcal{L}(u,z,\mu,k) = \mathcal{L}(u,f,\mu,k).$$

Theorem 3. *Let $\lambda\in\Lambda$ be given. Assume that there exists an optimal solution (u,f) of (3.4) in $X\times G$, that (HE1)–(HE3) hold, and that the mapping $z\mapsto\mathcal{E}(z)$ is Fréchet differentiable on G. Then there exists a $k\in\mathbb{R}$ and a $\mu\in X^*$, not both equal to zero, such that*

$$k\langle\mathcal{J}_u(u,f),w\rangle - \langle\mu, (I+\lambda T N'(u))\cdot w\rangle = 0 \quad \forall w\in X \quad \textit{and}$$
$$k\langle\mathcal{E}'(f),z\rangle - \langle\mu, TKz\rangle = 0 \quad \forall z\in G.$$

Remark 1. For two Theorems 2 and 3, if $1/\lambda$ is not in $\sigma(-TN'(u))$, we may choose $k=1$; see [17].

3.4. The existence of Lagrange multipliers and the optimality system of stochastic equations

We are now ready to prove the existence of a Lagrange multiplier for our minimization problem (3.1). The Lagrange multiplier rule may be used to convert the constrained minimization problem into an unconstrained one. Then we find the optimality system of stochastic equations.

Note that since our stochastic elliptic PDE has a unique solution regardless of the choice of λ, a parameter in the abstract setting, we take $\lambda = 1$.

Recall the stochastic optimal control problem:

$$\min \mathcal{J}_\beta(u, f) \quad \text{subject to } M(u, f) = 0\ \forall v \in \mathcal{H}_0^1(D), \tag{3.5}$$

where $M(u, f) = b[u, v] - [f, v]$.

We define $X = \mathcal{H}_0^1(D)$, $Y = \mathcal{H}^{-1}(D)$, $G = L^2(D)$, and $Z = \{0\}$. Then clearly we have $Z \hookrightarrow\hookrightarrow Y$. For the time being, we assume that the admissible set Θ for the control f is a closed, convex subset of G. We define the continuous linear operator $T \in \mathcal{L}(Y; X)$ as follows. For $g \in Y$, $Tg = u \in X$ is the unique solution of

$$b[u, v] = [g, v] \quad \forall v \in X. \tag{3.6}$$

We define the (differentiable) mapping $N : X \to Y$ by

$$\big\langle N(u), v\big\rangle = 0 \quad \forall v \in X \tag{3.7}$$

and define $K : G \to Y$ by

$$\langle Kf, \eta\rangle = -\langle f, \eta\rangle \quad \forall \eta \in X. \tag{3.8}$$

Then the constraint equation (2.4) can be expressed by $u + TKf = 0$. We note that

$$\mathcal{F}(u) = E\left(\frac{1}{2}\int_D |u - U|^2\, dx\right) \quad \text{and} \quad \mathcal{E}(f) = E\left(\frac{\beta}{2}\int_D |f|^2\, dx\right). \tag{3.9}$$

Next, we verify the hypotheses for the existence of Lagrange multipliers. First, notice that (HE1) is obvious. Second, (HE2) holds because $f \mapsto \mathcal{E}(f) = \frac{\beta}{2}\|f\|^2_{\mathcal{L}^2(D)}$ is convex. Third, because for $\forall u \in X$, $N'(u) \cdot v = 0 \in Z \hookrightarrow\hookrightarrow Y$ for $\forall v \in X$, (HE3) holds.

The Lagrangian is given by

$$\mathcal{L}(u, f, \xi, k) = k\mathcal{J}(u, f) - b[u, \xi] + [f, \xi]$$

for $\forall (u, f, \xi, k) \in X \times G \times X \times \mathbb{R}$.

By Theorem 2, there exists $\xi = T^*\mu \in X$ such that

$$\xi - kT^*\mathcal{F}'(u) = 0, \tag{3.10}$$

$$\mathcal{L}(u, f, \xi, k) \leqslant \mathcal{L}(u, z, \xi, k) \quad \forall z \in \Theta. \tag{3.11}$$

With $k = 1$, (3.10) becomes

$$b[\xi, \zeta] = [u - U, \zeta] \quad \forall \zeta \in X \tag{3.12}$$

and (3.11) implies that

$$\frac{\beta}{2}[z, z] + [z, \xi] - \frac{\beta}{2}[f, f] + [f, \xi] \geqslant 0 \quad \forall z \in \Theta \subseteq G. \tag{3.13}$$

For each $\epsilon \in (0, 1)$ and each $t \in \Theta$, set $z = \epsilon t + (1 - \epsilon) f \in \Theta$. Then from (3.13), we have

$$\frac{\beta\epsilon}{2}[t - f, t - f] + \beta[t - f, f] + [t - f, \xi] \geqslant 0 \quad \forall t \in \Theta. \tag{3.14}$$

By letting $\epsilon \to 0+$ in the above inequality, we have

$$[t - f, \beta f + \xi] \geqslant 0 \quad \forall t \in \Theta. \tag{3.15}$$

We now consider the case $\Theta = G$. Note that the mapping $z \mapsto \mathcal{E}(z)$ is Fréchet differentiable on G. Hence, by Theorem 3, (3.15) becomes an equality and by letting $z = t - f$ we obtain

$$[\beta f + \xi, z] = 0 \quad \forall z \in G. \tag{3.16}$$

The system formed by Eqs. (2.4), (3.12), and (3.16), which are necessary conditions for an optimum, is called an optimality system. We now conclude this section with the following theorem.

Theorem 4. *Let $(u, f) \in \mathcal{H}_0^1(D) \times L^2(D)$ be an optimal solution of* (3.1)*. Then there exists $\xi \in \mathcal{H}_0^1(D)$ such that* (3.12) *and* (3.16) *hold.*

4. Representation of stochastic functions

In this section, we represent stochastic functions by a countable set of deterministic functions and random variables. We first construct the Wiener–Itô (W-I) chaos expansions of stochastic functions. We then define the Karhunen–Loève (K-L) expansion of stochastic functions. Finally, we compare these two expansions.

4.1. Wiener–Itô chaos expansions

We first discuss the classical W-I chaos expansions of elements of the space of square-integrable functions defined on the space of tempered distributions in terms of stochastic Hermite polynomials; see [18]. Then we represent stochastic functions by a countably infinite sum of the products of a deterministic function and a random variable.

Let d be a fixed positive number. Consider a space of rapidly decreasing smooth real valued functions on $\mathbb{R}^d$, which we denote by $S(\mathbb{R}^d)$. We consider the dual space $S^*(\mathbb{R}^d)$, which is called the space of tempered distributions. We let $\Omega := S^*(\mathbb{R}^d)$ and denote the Borel σ-algebra over Ω by $\mathcal{F}$. Then there is a unique probability measure P over $\mathcal{F}$ such that

$$Ee^{i\langle\omega,\phi\rangle} := \int_\Omega e^{i\langle\omega,\phi\rangle}\,dP(\omega) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d)}\right) \quad \forall\phi \in S\big(\mathbb{R}^d\big), \tag{4.1}$$

where $\langle\omega,\phi\rangle = \omega(\phi)$ is the action of $\omega \in \Omega$ on $\phi \in S(\mathbb{R}^d)$. Here, P is called the white noise measure or the Gaussian measure on $S(\mathbb{R}^d)$.

Recall that the Hermite polynomials $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\big(e^{-x^2/2}\big); \quad n = 0, 1, 2, \ldots \tag{4.2}$$

and note that an $h_n(x)$ is a polynomial of degree n.

Define the Hermite functions $\xi_n(x)$ as follows:

$$\xi_n(x) = \pi^{-1/4}\big((n-1)!\big)^{-1/2}e^{-x^2/2}h_{n-1}(x); \quad n = 1, 2, 3, \ldots. \tag{4.3}$$

Observe that the Hermite functions are orthogonal with the weight $e^{-x^2/2}$ and in $S(\mathbb{R})$ for all n. Note that $\{\xi_n\}_{n=1}^\infty$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

Let $\delta^j = (\delta_1^j, \delta_2^j, \ldots, \delta_d^j)$, where $\delta_i^j \in \mathbb{N}$, and define the tensor products $\xi_{\delta^j} := \xi_{\delta_1^j} \otimes \xi_{\delta_2^j} \otimes \cdots \otimes \xi_{\delta_d^j}$, where

$$i < j \quad \Rightarrow \quad \delta_1^i + \delta_2^i + \cdots + \delta_d^i \leqslant \delta_1^j + \delta_2^j + \cdots + \delta_d^j. \tag{4.4}$$

It follows that the family of tensor products $\{\xi_{\delta^j}\}_{j=1}^\infty$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

Let us define

$$\mathcal{I} = \big\{\alpha = (\alpha_1, \alpha_2, \ldots) \,\big|\, a_i \in \mathbb{N} \cup \{0\} \text{ and there are only finitely many } \alpha_i \neq 0\big\}. \tag{4.5}$$

Note that the index set $\mathcal{I}$ is countable.

We define stochastic Hermite polynomials $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ by

$$H_\alpha(\omega) = \prod_{i=1}^\infty h_{\alpha_i}\big(\langle\omega, \xi_{\delta^i}\rangle\big); \quad \omega \in \Omega. \tag{4.6}$$

It follows that $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ forms an orthogonal basis for $L^2(\Omega)$.

The norm $\|H_\alpha\|_{L^2(\Omega)}$ satisfies $\|H_\alpha\|^2_{L^2(\Omega)} = \alpha! = \alpha_1!\alpha_2!\cdots$. Now we redefine H_α by $\frac{1}{\sqrt{\alpha!}}H_\alpha$. Then $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ forms an orthonormal basis for $L^2(\Omega)$. Hence every $f(\omega) \in L^2(\Omega)$ has a unique expansion

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega),$$

where

$$c_\alpha = E\big(f(\omega)H_\alpha(\omega)\big) = \int_\Omega f(\omega)H_\alpha(\omega)\,dP(\omega). \tag{4.7}$$

We call this expansion the W-I chaos expansion of a function $f \in L^2(\Omega)$ in terms of stochastic Hermite polynomials.

From the above argument, for a stochastic function $f(x,\omega)$, we have

$$f(x,\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha(x)H_\alpha(\omega), \tag{4.8}$$

where

$$c_\alpha(x) = E\big(f(x,\omega)H_\alpha(\omega)\big).$$

This expansion is called the W-I chaos expansion of a stochastic function $f(x,\omega)$.

Note that if $f(x,\omega)$ is known, then we know exactly what $c_\alpha(x)$'s are. Because our index set $\mathcal{I}$ is countable, for $n \in \mathbb{N}$, we may rewrite $f(x,\omega)$ as

$$f(x,\omega) = \sum_{n\geqslant 1} c_n(x)H_n(\omega), \tag{4.9}$$

where

$$c_n(x) = E\big(f(x,\omega)H_n(\omega)\big).$$

4.2. Karhunen–Loève expansions

In this section, we introduce the K-L expansions, which is well known as a theoretical tool for approximating stochastic functions; see [13,9,2–5,11,23]. If $a(x,\omega)$ is a stochastic function that has a continuous and bounded covariance function, it can be represented by

$$a(x,\omega) = \bar{a}(x) + \sum_{n\geqslant 0} \sqrt{\lambda_n}\phi_n(x)X_n(\omega), \tag{4.10}$$

where $\bar{a}(x) = Ea(x,\omega)$, $EX_n(\omega) = 0$, $E(X_n(\omega)X_m(\omega)) = \delta_{nm}$, and $(\lambda_n, \phi_n(x))$ are solutions to the eigenvalue problem:

$$\int_D C(x_1,x_2)\phi_n(x_1)\,dx_1 = \lambda_n\phi_n(x_2), \tag{4.11}$$

where $C(x_1,x_2) = E(a(x_1,\omega)a(x_2,\omega)) - Ea(x_1,\omega)Ea(x_2,\omega)$. We call this expansion the K-L expansion of $a(x,\omega)$.

4.3. Wiener–Itô chaos expansion versus Karhunen–Loève expansions

From the orthogonality relations for Hermite polynomials with respect to the Gaussian measure, for H_α's, we see that

$$E(H_\alpha H_\beta) = \delta_{\alpha\beta}$$

for any multi-indices α and β; see [18].

Also, by using the property of an even function $e^{-\frac{1}{2}x^2}$ with respect to the Gaussian measure, we have

$$EH_\alpha = 0$$

for any multi-index α.

Thus, the variance of H_α equals 1 and the covariance between H_α and H_β is equal to zero; i.e., for any $\alpha \neq \beta$, we obtain

$$E(H_\alpha H_\beta) = EH_\alpha EH_\beta.$$

If we choose the orthonormal eigenfunctions in the K-L expansion as a basis for $L^2(D)$ and assume that $c_n(x) \in L^2(D)$ in (4.9), then $c_n(x)$ can be represented by the infinite sum of orthonormal basis functions for $L^2(D)$. Thus, from the definition of the K-L expansion, we can think of the W-I chaos expansion of a stochastic function as a special case of the K-L expansion of it in terms of their random variables.

In the paper, we focus on our problems with the W-I chaos expansion consisting of known random variables and stochastic Hermite polynomials. Note that although we only study our stochastic control problem with SPDE constraint equations based on the W-I chaos expansion, problems with the K-L expansions can be treated similarly and analysis of that can be obtained directly from our results.

5. The high-dimensional elliptic PDEs

In this chapter we make some assumptions on our stochastic problem and then we change our SPDE into deterministic high-dimensional PDE so that we can use traditional finite element method to solve stochastic PDEs and stochastic control problems eventually.

5.1. Finite dimensional information and notation

In this section, we talk about (2.1) and also talk about the following problem

$$-\operatorname{div}\big[a(x,\omega)\nabla u(x,\omega)\big] = g(x,\omega) \quad \text{in } D,$$
$$u(x,\omega) = 0 \quad \text{on } \partial D, \tag{5.1}$$

where $g : D \times \Omega \to \mathbb{R}$ is a stochastic function. Note that we have a unique weak solution for (5.1) by the Lax–Milgram theorem and that the error estimates for the solutions for (2.1) and (5.1) will be similar and both will be used later for error estimates of the solution of the stochastic optimal control problem.

In realistic models, the source of randomness can be expressed by a finite number of random variables that are mutually independent. For that reason, in the paper, we assume that

$$a(x,\omega) = \sum_{n=1}^{N} c_n(x) H_n(\omega) \tag{5.2}$$

and

$$g(x,\omega) = g\big(x, H_1(\omega), H_2(\omega), \ldots, H_N(\omega)\big). \tag{5.3}$$

To have the existence of the solution for stochastic elliptic PDEs with finite dimensional information, it is necessary that $\sum_{n=1}^{N} c_n(x) H_n(\omega)$ satisfy a similar condition to (2.2); i.e., there exist $m, M > 0$ such that

$$m \leqslant \sum_{n=1}^{N} c_n(x) H_n(\omega) \leqslant M \quad \text{a.e. } (x,\omega) \in D \times \Omega. \tag{5.4}$$

Remark 2. Some comments are in order concerning the finite dimensional assumption (5.2) and the ellipticity assumption (5.4). As was discussed in [4], assumption (5.2) may be valid in its own right in practical applications; also, similar to the convergence analysis of [4] for the truncated problem, we may prove the convergence of the truncated control problem based on Mercer's theorem. We refer the readers to [12] for a discussion of the ellipticity condition for $a(x,\omega)$, where $\log a(x,\omega)$ is a Gaussian field.

We also assume that each $H_n(\Omega) \equiv \Gamma_n \subset \mathbb{R}$ is a bounded interval for $n = 1, 2, \ldots, N$ and that each H_n has a density function $\rho_n : \Gamma_n \to \mathbb{R}^+$. We use the joint density $\rho(y)$ for any $y \in \Gamma \equiv \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$ of $(H_1, H_2, \ldots, H_N)$. Under these assumptions, the solution of (2.4) can be expressed by the finite number of random variables; i.e., $u(x,\omega) = u(x, H_1(\omega), H_2(\omega), \ldots, H_N(\omega))$; see e.g., [9,2,4].

Remark 3. Notice that $H := (H_1, H_2, \ldots, H_N)$ is a Γ-valued random variable. Thus, for $u \in \mathcal{H}_0^1(D)$, we have

$$E \int_D u\,dx = \int_\Omega \int_D u(x,\omega)\,dx\,dP(\omega) = \int_\Gamma \rho(y) \int_D u(x,y)\,dx\,dy.$$

Under the above assumptions, we also have the following high-dimensional deterministic equivalent variational formulation of (2.4) with the finite dimensional information:

$$\int_\Gamma \rho(y) \int_D a(x,y)\nabla u(x,y) \cdot \nabla v(x,y)\,dx\,dy = \int_\Gamma \rho(y) \int_D f(x) v(x,y)\,dx\,dy. \tag{5.5}$$

The strong formulation of this is

$$-\operatorname{div}\big[a(x,y)\nabla u(x,y)\big] = f(x) \quad \forall (x,y) \in D \times \Gamma,$$
$$u(x,y) = 0 \quad \forall (x,y) \in \partial D \times \Gamma. \tag{5.6}$$

Remark 4. First, from (2.2), we know that a is bounded and hence, we have well-posedness of (5.6). Second, the solution of a SPDE can be found by solving a deterministic PDE. Third, the finite element method can be used for stochastic problems. Finally, with $g(x,\omega)$ and $g(x,y)$ instead of $f(x)$, we obtain the same result.

For the high-dimensional elliptic PDE, we recall Sobolev spaces as follows:

$$L^2\big(\Gamma; H^r(D)\big) = \big\{ v : D \times \Gamma \to \mathbb{R} \;\big|\; \|v\|_{L^2(\Gamma; H^r(D))} < \infty \big\},$$

where

$$\|v\|^2_{L^2(\Gamma;H^r(D))} = \int_\Gamma \rho \|v\|^2_{H^r(D)}\,dy = E\|v\|^2_{H^r(D)}.$$

We now give the Banach spaces that will be used as solution spaces for the solution of the optimality system of stochastic equations.

For $r = -1, 0, 1,$ define

$$S^{p,r}(D) = C^p\big(\Gamma; H^r(D)\big),$$

with the norm $\|u\|_{S^{p,r}(D)} = \|u\|_{S^{0,r}(D)} + \sum_{j=1}^N \sum_{k=1}^{p_j} \|\partial^k_{y_j} u\|_{S^{0,r}(D)}$, where

$$\|u\|_{S^{0,r}(D)} = \sup_{y\in\Gamma} \big\|u(\cdot, y)\big\|_{H^r(D)}.$$

Also define

$$S_0^{p,1}(D) = C^p\big(\Gamma; H_0^1(D)\big).$$

Let us recall some notation:

$$b[u,v] = \int_\Gamma \rho \int_D a\nabla u\cdot\nabla v\,dx\,dy \quad\text{and}\quad [u,v] = \int_\Gamma \rho \int_D uv\,dx\,dy.$$

5.2. Finite element spaces

Let us first consider finite element spaces on $D \subset \mathbb{R}^d$. Let X^h and G^h be families of finite element approximation subspaces of $H_0^1(D)$ and $L^2(D)$ that consist of piecewise linear continuous functions defined over a family of regular triangulations of D with a maximum grid size parameter $h > 0$. We assume that X^h and G^h satisfy the following approximation properties:

(i) for all $\phi \in H^{\alpha+1}(D) \cap H_0^1(D)$, there exists $C > 0$ and an integer l such that

$$\inf_{\phi^h\in X^h} \big\|\phi - \phi^h\big\|_{H_0^1(D)} \leqslant Ch^\alpha \|\phi\|_{H^{\alpha+1}(D)}, \quad 0 \leqslant \alpha \leqslant l, \tag{5.7}$$

where $l \geqslant 1$ is usually determined by the order of the piecewise polynomials used to define X^h;

(ii) for all $\phi \in H_0^1(D)$, there exists $C > 0$ such that

$$\inf_{\phi^h\in G^h} \big\|\phi - \phi^h\big\|_{L^2(D)} \leqslant Ch\|\phi\|_{H_0^1(D)}. \tag{5.8}$$

Next, we consider finite element spaces on $\Gamma \subset \mathbb{R}^N$. We partition Γ into a finite number of disjoint $\mathbb{R}^N$ boxes B_i^N, that is, for a finite index set I, we have

$$\Gamma = \bigcup_{i\in I} B_i^N = \bigcup_{i\in I} \prod_{j=1}^N \big(a_i^j, b_i^j\big),$$

where $B_k^N \cap B_l^N = \emptyset$ for $k \neq l \in I$ and $(a_i^j, b_i^j) \subset \Gamma_j$.

A maximum grid size parameter $0 < \delta < 1$ is denoted by

$$\delta = \max\big\{\big|b_i^j - a_i^j\big|/2\colon 1 \leqslant j \leqslant N \text{ and } i \in I\big\}.$$

Let $Y^\delta \subset L^2(\Gamma)$ be the finite element approximation space of piecewise polynomials with degree at most p_j on each direction y_j. Thus if $\psi^\delta \in Y^\delta$, then $\psi^\delta|_{B_i^N} \in \operatorname{span}(\prod_{j=1}^N y_j^{n_j}\colon n_j \in \mathbb{N}$ and $n_j \leqslant p_j)$. Letting $p = (p_1, p_2, \ldots, p_N)$, we have (cf. see [6]) the following property: for all $\psi \in C_y^{p+1}(\Gamma)$,

$$\inf_{\psi^\delta\in Y^\delta} \big\|\psi - \psi^\delta\big\|_{C^0(\Gamma)} \leqslant \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1}\psi\|_{C^0(\Gamma)}}{(p_j+1)!}, \tag{5.9}$$

where $\gamma = \min_{1\leqslant j\leqslant N}\{p_j + 1\}$.

We now think of finite element spaces on $D \times \Gamma$, say $V^{h\delta}$. Here, if $v^{h\delta} \in V^{h\delta}$, $v^{h\delta} \in \operatorname{span}(\phi^h\psi^\delta\colon \phi^h(x) \in X^h$ and $\psi^\delta(y) \in Y^\delta)$.

We denote by R^h the $H^1(D)$-projection from $H_0^1(D)$ onto X^h and P^δ the $L^2(\Gamma)$-projection from $L^2(\Gamma)$ onto Y^δ. Namely for each $\phi \in H_0^1(D)$,

$$\left(R^h\phi, \phi^h\right)_{H_0^1(D)} = \left(\phi, \phi^h\right)_{H_0^1(D)} \quad \forall \phi^h \in X^h;$$

for each $\psi \in L^2(\Gamma)$,

$$\left(P^\delta\psi, \psi^\delta\right)_{L^2(\Gamma)} = \left(\psi, \psi^\delta\right)_{L^2(\Gamma)} \quad \forall \psi^\delta \in Y^\delta.$$

It follows from (5.7) that for all $\phi \in H_0^1(D) \cap H^{\alpha+1}$ and for some $C > 0$, we have

$$\left\|\phi - R^h\phi\right\|_{H_0^1(D)} \leqslant Ch^\alpha \|\phi\|_{H^{\alpha+1}(D)}, \quad 0 \leqslant \alpha \leqslant l \tag{5.10}$$

and from (5.9), for all $\psi \in C_y^{p+1}(\Gamma)$, we obtain

$$\left\|\psi - P^\delta\psi\right\|_{C^0(\Gamma)} \leqslant \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1}\psi\|_{C^0(\Gamma)}}{(p_j+1)!}. \tag{5.11}$$

By using the last two inequalities, we have the following property; see [4]: for all $u \in C^{p+1}(\Gamma; H^{\alpha+1}(D) \cap H_0^1(D))$, $0 \leqslant \alpha \leqslant l$, there exists $C > 0$, which is independent of h, δ, N, and p, such that

$$\inf_{u^{h\delta} \in V^{h\delta}} \left\|u - u^{h\delta}\right\|_{S_0^{0,1}(D)} \leqslant C\left(h^\alpha \|u\|_{S^{0,\alpha+1}(D)} + \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1} u\|_{S_0^{0,1}(D)}}{(p_j+1)!}\right). \tag{5.12}$$

Note that from the definition of the space $S^{p,r}(D)$ and its norm in Section 5.1, we may assume that $0 < \|\partial_{y_j}^k\|_{op} < 1$ for all $1 \leqslant j \leqslant N$ and $1 \leqslant k \leqslant p_j + 1$. Also we know that for the solution u of (5.6), each $\partial_{y_j}^k u$ is continuous on the bounded domain Γ_j and, hence, $\partial_{y_j}^k u$ is bounded. We now assume further that for each $\partial_{y_j}^k$ there is a constant $\tilde{c} > 0$ such that $0 < \|\partial_{y_j}^k\|_{op} \leqslant \tilde{c} < 1$. With the help of (5.12), we state the approximation property for the solution:

$$\inf_{u^{h\delta} \in V^{h\delta}} \left\|u - u^{h\delta}\right\|_{S_0^{p+1,1}(D)} \leqslant C\left(h^\alpha \|u\|_{S^{0,\alpha+1}(D)} + \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1} u\|_{S_0^{0,1}(D)}}{(p_j+1)!}\right). \tag{5.13}$$

Remark 5. Because $a(x,y) = \sum_{j=1}^N c_j(x) y_j \in C^{p+1}(\overline{D \times \Gamma})$, it is well known that the solution u of (5.6) satisfies $u \in C^{p+1}(\Gamma; H^2(D) \cap H_0^1(D))$; see e.g., Lemma 4.1 in [21] and Remark 5.1 in [5]. Also if we assume that $g(x,y) \in S^{p+1,0}(D)$ to carry out the analysis of the finite element method, then the solution u of the following problem also satisfies $u \in C^{p+1}(\Gamma; H^2(D) \cap H_0^1(D))$:

$$\begin{aligned} &-\operatorname{div}\left[a(x,y)\nabla u(x,y)\right] = g(x,y) \quad \forall (x,y) \in D \times \Gamma, \\ &u(x,y) = 0 \quad \forall (x,y) \in \partial D \times \Gamma. \end{aligned} \tag{5.14}$$

Hence, under our assumptions, (5.13) makes sense for some α.

5.3. Error estimates for high-dimensional elliptic PDEs

Recall that our goal is to solve (5.6). The stochastic weak formulation of (5.6) is as follows: seek $u \in S_0^{p+1,1}(D)$ such that for all $v \in S_0^{p+1,1}(D)$,

$$b[u, v] = [f, v]. \tag{5.15}$$

Then we have the finite element weak formulation: find $u^{h\delta} \in V^{h\delta}$ such that for all $v^{h\delta} \in V^{h\delta}$,

$$b\left[u^{h\delta}, v^{h\delta}\right] = \left[f, v^{h\delta}\right]. \tag{5.16}$$

We want error estimate of solutions for (5.15) and (5.16) in $S_0^{p+1,1}(D)$. Also we do the same thing with a finite data $g(x,y)$ instead of $f(x)$. For these, using Remark 5, we have the following lemmas.

Lemma 5. *Let $0 \leqslant \varepsilon \leqslant 1$ and $f(x) \in H^{-1+\varepsilon}(D)$. Then for any $y \in \Gamma$, $u(\cdot, y) \in H^{1+\varepsilon}(D)$ and there exists $C > 0$ such that*

$$\left\|u(\cdot, y)\right\|_{H^{1+\varepsilon}(D)} \leqslant C\|f\|_{H^{-1+\varepsilon}(D)}.$$

Proof. This follows from the interpolation theorem (see [22]). □

Remark 6. For problems with $g(\cdot, y) \in H^{-1+\varepsilon}(D)$, we have

$$\|u(\cdot, y)\|_{H^{1+\varepsilon}(D)} \leqslant C\|g(\cdot, y)\|_{H^{-1+\varepsilon}(D)}.$$

Lemma 6. *Let $0 \leqslant \varepsilon \leqslant 1$, $f(x) \in H^{-1+\varepsilon}(D)$, $u \in S_0^{p+1,1}(D)$, and $c_j(x) \in L^\infty(D)$. Then for all $j = 1, 2, \ldots, N$ and for any $y \in \Gamma$, there exists $C > 0$ such that*

$$\frac{\|\partial_{y_j}^{p_j+1} u(\cdot, y)\|_{H_0^1(D)}}{(p_j+1)!} \leqslant C\|c_j\|_{L^\infty(D)}^{p_j+1}\|f\|_{H^{-1+\varepsilon}(D)}.$$

Proof. This can be proved using the mathematical induction. □

Remark 7. For problems with $g(x, y) \in C^{p+1}(\Gamma; H^{-1+\varepsilon}(D))$, we have

$$\frac{\|\partial_{y_j}^{p_j+1} u(\cdot, y)\|_{H_0^1(D)}}{(p_j+1)!} \leqslant C \sum_{k=0}^{p_j+1} \frac{1}{k!}\|c_j\|_{L^\infty(D)}^{p_j+1-k} \left\|\partial_{y_j}^k g(\cdot, y)\right\|_{H^{-1+\varepsilon}(D)}.$$

As a consequence of (5.13), Lemma 5, and Lemma 6, we have the following theorem.

Theorem 7. *Let $0 \leqslant \varepsilon \leqslant 1$, $f(x) \in H^{-1+\varepsilon}(D)$, u be the solution of (5.15), and $u^{h\delta}$ be the finite element solution of (5.16). Then there exists $C > 0$ such that*

$$\|u - u^{h\delta}\|_{S_0^{p+1,1}(D)} \leqslant C\left(h^\varepsilon + \delta^\gamma\right) \sum_{j=1}^N \max\left\{1, \|c_j\|_{L^\infty(D)}^{p_j+1}\right\}\|f\|_{H^{-1+\varepsilon}(D)}.$$

Similarly, (5.13), Remark 6, and Remark 7 give the following remark.

Remark 8. For problems with $g(x, y) \in S^{p+1,-1+\varepsilon}(D)$, we have

$$\|u - u^{h\delta}\|_{S_0^{p+1,1}(D)} \leqslant C\left(h^\varepsilon + \delta^\gamma\right) \sum_{j=1}^N \max_{0 \leqslant k \leqslant p_j+1}\left\{1, \frac{1}{k!}\|c_j\|_{L^\infty(D)}^{p_j+1-k}\right\}\left(\sum_{k=0}^{p_j+1}\left\|\partial_{y_j}^k g\right\|_{S^{0,-1+\varepsilon}(D)}\right).$$

6. Discrete approximation of the optimality system

In this chapter, we solve stochastic optimal control problems using results from previous chapter and the Brezzi–Rappaz–Raviart theory. For this, we first introduce the theory.

6.1. Description of the Brezzi–Rappaz–Raviart theory

The B-R-R theory implies that the error of approximation of solutions of certain nonlinear problems under certain hypotheses is basically the same as the error of approximation of solutions of related linear problems; see [7,8,14]. Here for the sake of completeness, we will state the relevant results, specialized to our needs.

Consider the following type of nonlinear problems: seek $\psi \in \mathcal{X}$ such that

$$\psi + \mathcal{T}\mathcal{G}(\psi) = 0, \tag{6.1}$$

where $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, $\mathcal{G}$ is a C^2 mapping from $\mathcal{X}$ into $\mathcal{Y}$, and $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. We say that ψ is a regular solution of (6.1) if (6.1) holds and $\psi + \mathcal{T}\mathcal{G}_\psi(\psi)$ is an isomorphism from $\mathcal{X}$ into $\mathcal{X}$. Here $\mathcal{G}_\psi$ denotes the Fréchet derivative of $\mathcal{G}$ with respect to ψ. We assume that there exists another Banach space $\mathcal{Z}$, contained in $\mathcal{Y}$, with continuous imbedding, such that

$$\mathcal{G}_\psi(\psi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z}) \quad \forall \psi \in \mathcal{X}. \tag{6.2}$$

Approximations are defined by introducing a subspace $\mathcal{X}^h \subset \mathcal{X}$ and an approximating operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y}; \mathcal{X}^h)$. We seek $\psi^h \in \mathcal{X}^h$ such that

$$\psi^h + \mathcal{T}^h\mathcal{G}\left(\psi^h\right) = 0. \tag{6.3}$$

Concerning the operator $\mathcal{T}^h$, we assume the approximation properties

$$\lim_{h\to 0}\left\|\left(\mathcal{T}^h-\mathcal{T}\right)\omega\right\|_{\mathcal{X}}=0 \quad \forall \omega\in\mathcal{Y} \tag{6.4}$$

and

$$\lim_{h\to 0}\left\|\mathcal{T}^h-\mathcal{T}\right\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})}=0. \tag{6.5}$$

Note that whenever the imbedding $\mathcal{Z}\subset\mathcal{Y}$ is compact, (6.5) follows from (6.4) and, moreover, (6.2) implies that the operator $\mathcal{T}\mathcal{G}_\psi(\psi)\in\mathcal{L}(\mathcal{X};\mathcal{X})$ is compact.

We now state the result of [7] that will be used in the sequel. In the statement of the theorem, $D^2\mathcal{G}$ represents any and all second Fréchet derivatives of $\mathcal{G}$.

Theorem 8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Assume that $\mathcal{G}$ is a C^2 mapping from $\mathcal{X}$ to $\mathcal{Y}$ and that $D^2\mathcal{G}$ is bounded on all bounded sets of $\mathcal{X}$. Assume that* (6.2), (6.4), *and* (6.5) *hold and that ψ is a regular solution of* (6.1)*. Then there exists a neighborhood $\mathcal{O}$ of the origin in $\mathcal{X}$ and, for $h\leqslant h_0$ small enough, a unique $\psi^h\in\mathcal{X}^h$ such that ψ^h is a regular solution of* (6.3)*. Moreover, there exists a constant $C>0$, independent of h, such that*

$$\left\|\psi^h-\psi\right\|_{\mathcal{X}}\leqslant C\left\|\left(\mathcal{T}^h-\mathcal{T}\right)\mathcal{G}(\psi)\right\|_{\mathcal{X}}. \tag{6.6}$$

6.2. Recasting the optimality system and its discrete approximation into the B-R-R framework

We first fit our optimality system and its discrete approximation into the B-R-R framework to derive error estimates for the discrete approximation of the optimality system. Then we obtain the desired error estimates by verifying assumptions in the B-R-R theory.

We set $\mathcal{X}=S_0^{p+1,1}(D)\times L^2(D)\times S_0^{p+1,1}(D)$ and $\mathcal{Y}=H^{-1}(D)\times S^{p+1,-1}(D)$. We define the linear operator $\mathcal{T}\in\mathcal{L}(\mathcal{Y};\mathcal{X})$ as follows:

$$(\tilde{u},\tilde{f},\tilde{\xi})=\mathcal{T}(\tilde{r},\tilde{\tau})$$

if and only if

$$b[\tilde{u},v]=[\tilde{r},v] \quad \forall v\in S_0^{p+1,1}(D), \tag{6.7}$$

$$b[\tilde{\xi},\zeta]=[\tilde{\tau},\zeta] \quad \forall \zeta\in S_0^{p+1,1}(D), \tag{6.8}$$

and

$$[\beta\tilde{f}+\tilde{\xi},z]=0 \quad \forall z\in L^2(D). \tag{6.9}$$

We define $\mathcal{G}:\mathcal{X}\to\mathcal{Y}$ by

$$\mathcal{G}(\tilde{u},\tilde{f},\tilde{\xi})=(-\tilde{f},-\tilde{u}+U).$$

Then it is clear that the optimality system (2.4), (3.12), and (3.16) can be written as

$$(u,f,\xi)+\mathcal{T}\big(\mathcal{G}(u,f,\xi)\big)=0. \tag{6.10}$$

Hence, the optimality system is recast into the form of (6.1).

We now set $\mathcal{X}^{h\delta}=V^{h\delta}\times G^h\times V^{h\delta}$, where $V^{h\delta}$ and G^h are from Section 5.2.

We define the discrete operator $\mathcal{T}^{h\delta}\in\mathcal{L}(\mathcal{Y};\mathcal{X}^{h\delta})$ as follows:

$$\left(\tilde{u}^{h\delta},\tilde{f}^h,\tilde{\xi}^{h\delta}\right)=\mathcal{T}^{h\delta}(\tilde{r},\tilde{\tau})$$

if and only if

$$b\left[\tilde{u}^{h\delta},v^{h\delta}\right]=\left[\tilde{r},v^{h\delta}\right] \quad \forall v^{h\delta}\in V^{h\delta}, \tag{6.11}$$

$$b\left[\tilde{\xi}^{h\delta},\zeta^{h\delta}\right]=\left[\tilde{\tau},\zeta^{h\delta}\right] \quad \forall \zeta^{h\delta}\in V^{h\delta}, \tag{6.12}$$

and

$$\left[\beta\tilde{f}^h+\tilde{\xi}^{h\delta},z^h\right]=0 \quad \forall z^h\in G^h. \tag{6.13}$$

Then it is clear that the discrete optimality system,

$$b\left[u^{h\delta},v^{h\delta}\right]=\left[f^h,v^{h\delta}\right] \quad \forall v^{h\delta}\in V^{h\delta}, \tag{6.14}$$

$$b\left[\xi^{h\delta},\zeta^{h\delta}\right]=\left[u^{h\delta}-U,\zeta^{h\delta}\right] \quad \forall \zeta^{h\delta}\in V^{h\delta}, \tag{6.15}$$

and

$$\big[\beta f^h + \xi^{h\delta}, z^h\big] = 0 \quad \forall z^h \in G^h, \tag{6.16}$$

can be written as

$$\big(u^{h\delta}, f^h, \xi^{h\delta}\big) + \mathcal{T}^{h\delta}\big(\mathcal{G}\big(u^{h\delta}, f, \xi^{h\delta}\big)\big) = 0.$$

Hence, the discrete optimality system is recast into the form of (6.3).

6.3. Error estimates for the approximation of solutions of the optimality system

In this section, we proceed to verify all assumptions in Theorem 8. We define first a space $\mathcal{Z} = L^2(D) \times S^{p+1,0}(D)$. Then clearly this space is continuously embedded into $\mathcal{Y} = H^{-1}(D) \times S^{p+1,-1}(D)$.

Denote the Fréchet derivative of $\mathcal{G}(u, f, \xi)$ with respect to (u, f, ξ) by $D\mathcal{G}(u, f, \xi)$ or $\mathcal{G}_{(u,f,\xi)}(u, f, \xi)$. Then from $\mathcal{G}(u, f, \xi)$, we obtain for $(u, f, \xi) \in \mathcal{X}$,

$$D\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi}) = (-\tilde{f}, -\tilde{u}) \quad \forall(\tilde{u}, \tilde{f}, \tilde{\xi}) \in \mathcal{X}.$$

Proposition 1. *$D\mathcal{G}(u, f, \xi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z})$ for all $(u, f, \xi) \in \mathcal{X}$.*

Proof. The result is followed from

$$\big\|D\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi})\big\|_{\mathcal{Z}} = \|\tilde{f}\|_{L^2(D)} + \|\tilde{u}\|_{S^{p+1,0}(D)} < \infty. \qquad \square$$

Proposition 2. *$\mathcal{G}$ is twice continuously differentiable and $D^2\mathcal{G}$ is bounded on all bounded sets of $\mathcal{X}$.*

Proof. For any $(u, f, \xi) \in \mathcal{X}$,

$$D^2\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi}) = (0, 0) \quad \forall(\tilde{u}, \tilde{f}, \tilde{\xi}) \in \mathcal{X}.$$

Thus, it is easy to show that $D^2\mathcal{G}$ is well defined, continuous, and bounded on all bounded sets of $\mathcal{X}$. $\square$

Before we show (6.4) and (6.5), we consider the following lemmas.

Lemma 9. *Let $\tilde{f} \in L^2(D)$ and $\tilde{\xi} \in S_0^{p+1,1}(D)$ in (6.9). Let $\tilde{f}^h \in G^h$ and $\tilde{\xi} \in V^{h\delta}$ in (6.13). Then there exists $C > 0$ such that*

$$\|\tilde{f} - \tilde{f}^h\|^2_{\mathcal{L}^2(D)} \leqslant C\big(\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \|\tilde{f} - g^h\|^2_{\mathcal{L}^2(D)}\big) \tag{6.17}$$

for all $g^h \in G^h$.

Proof. From (6.9) and from (6.13), we see that

$$\big[\beta\tilde{f}, z^h\big] = -\big[\tilde{\xi}, z^h\big] \quad \forall z^h \in G^h \tag{6.18}$$

and

$$\big[\beta\tilde{f}^h, z^h\big] = -\big[\tilde{\xi}^{h\delta}, z^h\big] \quad \forall z^h \in G^h. \tag{6.19}$$

Subtracting (6.19) from (6.18) leads us to

$$\big[\beta\big(\tilde{f} - \tilde{f}^h\big), z^h\big] = -\big[\tilde{\xi} - \tilde{\xi}^{h\delta}, z^h\big] \quad \forall z^h \in G^h. \tag{6.20}$$

Thus, for any $g^h \in G^h$, we find

$$\begin{aligned}\big[\tilde{f} - \tilde{f}^h, \tilde{f} - \tilde{f}^h\big] &= \big[\tilde{f} - \tilde{f}^h, \tilde{f} - g^h\big] + \big[\tilde{f} - \tilde{f}^h, g^h - \tilde{f}^h\big] \\ &= \big[\tilde{f} - \tilde{f}^h, \tilde{f} - g^h\big] + \frac{1}{\beta}\big[\tilde{\xi} - \tilde{\xi}^{h\delta}, \tilde{f}^h - g^h\big].\end{aligned} \tag{6.21}$$

The Hölder inequality implies

$$\begin{aligned}\|\tilde{f} - \tilde{f}^h\|^2_{\mathcal{L}^2(D)} \leqslant{}& \|\tilde{f} - \tilde{f}^h\|_{\mathcal{L}^2(D)}\|\tilde{f} - g^h\|_{\mathcal{L}^2(D)} \\ &+ \frac{1}{\beta}\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|_{\mathcal{L}^2(D)}\|\tilde{f}^h - \tilde{f}\|_{\mathcal{L}^2(D)} + \frac{1}{\beta}\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|_{\mathcal{L}^2(D)}\|\tilde{f} - g^h\|_{\mathcal{L}^2(D)}.\end{aligned} \tag{6.22}$$

By Cauchy's inequality with $\epsilon > 0$, for some C_ϵ that depends on ϵ, we have

$$\begin{aligned}\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} \leqslant{}& \epsilon\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} + C_\epsilon\|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)} \\ &+ \frac{C_\epsilon}{\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \epsilon\|\tilde{f}^h-\tilde{f}\|^2_{\mathcal{L}^2(D)} \\ &+ \frac{1}{2\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \frac{1}{2}\|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)}.\end{aligned} \tag{6.23}$$

Now choose $\epsilon = \frac{1}{4}$. Then there exists $C > 0$ such that

$$\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} \leqslant C\left(\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)}\right) \tag{6.24}$$

for any $g^h \in G^h$. $\square$

Remark 9. $\|\tilde{f}-\tilde{f}^h\|_{L^2(D)} = \|\tilde{f}-\tilde{f}^h\|_{\mathcal{L}^2(D)}$ by the property of the density function ρ.

Lemma 10. *Let $\tilde{r} \in H^{-1}(D)$, $\tilde{u}$ be the solution of*

$$b[\tilde{u}, v] = [\tilde{r}, v] \quad \forall v \in S_0^{p+1,1}(D), \tag{6.25}$$

and $\tilde{u}^{h\delta}$ be the solution of

$$b\left[\tilde{u}^{h\delta}, v^{h\delta}\right] = \left[\tilde{r}, v^{h\delta}\right] \quad \forall v^{h\delta} \in V^{h\delta}. \tag{6.26}$$

Then we have

$$\|\tilde{u}-\tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} \to 0 \quad \text{as } h, \delta \to 0.$$

Proof. Let $\epsilon > 0$ be given. Let $\tilde{r} \in H^{-1}(D)$. Then there is a sequence of C^∞-functions $\{\tilde{r}_k\} \subset L^2(D)$ such that $\tilde{r}_k \to \tilde{r}$ in $H^{-1}(D)$; i.e., there exists k_0 such that

$$\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)} < \epsilon. \tag{6.27}$$

Now consider the following problems:

$$b[\tilde{u}_{k_0}, v] = [\tilde{r}_{k_0}, v] \quad \forall v \in S_0^{p+1,1}(D) \tag{6.28}$$

and

$$b\left[\tilde{u}^{h\delta}_{k_0}, v^{h\delta}\right] = \left[\tilde{r}_{k_0}, v^{h\delta}\right] \quad \forall v^{h\delta} \in V^{h\delta}. \tag{6.29}$$

Then from (6.25) and (6.28) and from (6.26) and (6.29), there exists $C > 0$ such that

$$\|\tilde{u}-\tilde{u}_{k_0}\|_{S_0^{p+1,1}(D)} \leqslant C\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)} \tag{6.30}$$

and

$$\|\tilde{u}^{h\delta}-\tilde{u}^{h\delta}_{k_0}\|_{S_0^{p+1,1}(D)} \leqslant C\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)}, \tag{6.31}$$

respectively.

Hence, obviously,

$$\begin{aligned}\|\tilde{u}-\tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} &\leqslant \|\tilde{u}-\tilde{u}_{k_0}\|_{S_0^{p+1,1}(D)} + \|\tilde{u}_{k_0}-\tilde{u}^{h\delta}_{k_0}\|_{S_0^{p+1,1}(D)} + \|\tilde{u}^{h\delta}_{k_0}-\tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} \\ &\leqslant C\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)} + \|\tilde{u}_{k_0}-\tilde{u}^{h\delta}_{k_0}\|_{S_0^{p+1,1}(D)} + C\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)}.\end{aligned} \tag{6.32}$$

On the other hand, because $\tilde{r}_{k_0} \in L^2(D)$, Theorem 7 yields

$$\|\tilde{u}_{k_0}-\tilde{u}^{h\delta}_{k_0}\|_{S_0^{p+1,1}(D)} \leqslant C\left(h+\delta^\gamma\right)\sum_{j=1}^{N}\max\left\{1, \|c_j\|^{p_j+1}_{L^\infty(D)}\right\}\|\tilde{r}_{k_0}\|_{L^2(D)}. \tag{6.33}$$

Thus, from the last inequality, by letting $h, \delta \to 0$, we obtain

$$\|\tilde{u}_{k_0}-\tilde{u}^{h\delta}_{k_0}\|_{S_0^{p+1,1}(D)} < \epsilon. \tag{6.34}$$

Combining (6.27), (6.32) and (6.34),

$$\left\|\tilde{u}-\tilde{u}^{h\delta}\right\|_{S_0^{p+1,1}(D)}<(2C+1)\epsilon. \tag{6.35}$$

Because ϵ is arbitrary, this complete the proof of Lemma 10. □

Remark 10. Likewise, $\tilde{\xi}$ in (6.8), $\tilde{\xi}^{h\delta}$ in (6.12), and for $\tilde{\tau}\in S^{p+1,-1}(D)$, we have

$$\left\|\tilde{\xi}-\tilde{\xi}^{h\delta}\right\|_{S_0^{p+1,1}(D)}\to 0 \quad \text{as } h,\delta\to 0.$$

Lemma 11. *Let $\tilde{f}\in L^2(D)$, $g^h\in G^h$, and $\tilde{\tau}\in S^{p+1,-1}(D)$ in (6.8). Then there exists $C>0$ such that*

$$\left\|\tilde{f}-g^h\right\|_{L^2(D)}\leqslant Ch\|\tilde{\tau}\|_{\mathcal{H}^{-1}(D)}. \tag{6.36}$$

Proof. From (6.9), we see that

$$\int_D \beta\tilde{f}z\,dx=-\int_D (E\tilde{\xi})z\,dx.$$

The Hölder inequality implies that

$$\begin{aligned}\int_D |\nabla\tilde{f}|^2\,dx&=\frac{1}{\beta^2}\int_D (E\nabla\tilde{\xi})^2\,dx\\&\leqslant\frac{1}{\beta^2}\int_D E|\nabla\tilde{\xi}|^2\,dx=\frac{1}{\beta^2}E\int_D|\nabla\tilde{\xi}|^2\,dx<\infty.\end{aligned} \tag{6.37}$$

On the other hand, choose $g^h=\tilde{P}^h\tilde{f}$, where $\tilde{P}^h$ is an $L^2(D)$-projection from $L^2(D)$ onto G^h. Because $\tilde{f}\in H_0^1(D)$, by the approximation property (5.8), there exists $C>0$ such that

$$\left\|\tilde{f}-g^h\right\|_{L^2(D)}=\left\|\tilde{f}-\tilde{P}^h\tilde{f}\right\|_{L^2(D)}\leqslant Ch\|\tilde{f}\|_{H_0^1(D)}. \tag{6.38}$$

Thus, (6.36) follows by combining the last two inequalities and (6.8) because $\tilde{\tau}\in S^{p+1,-1}(D)\subset\mathcal{H}^{-1}(D)$. □

Proposition 3. *For any $(\tilde{r},\tilde{\tau})\in\mathcal{Y}$, $\|(\mathcal{T}-\mathcal{T}^{h\delta})(\tilde{r},\tilde{\tau})\|_{\mathcal{X}}\to 0$ as $h,\delta\to 0$.*

Proof. By Lemma 9, we see that for any $g^h\in G^h$, there exists $C>0$ such that

$$\begin{aligned}\left\|\left(\mathcal{T}-\mathcal{T}^{h\delta}\right)(\tilde{r},\tilde{\tau})\right\|_{\mathcal{X}}&=\left\|\left(\tilde{u}-\tilde{u}^{h\delta},\tilde{f}-\tilde{f}^h,\tilde{\xi}-\tilde{\xi}^{h\delta}\right)\right\|_{\mathcal{X}}\\&=\left\|\tilde{u}-\tilde{u}^{h\delta}\right\|_{S_0^{p+1,1}(D)}+\left\|\tilde{f}-\tilde{f}^h\right\|_{L^2(D)}+\left\|\tilde{\xi}-\tilde{\xi}^{h\delta}\right\|_{S_0^{p+1,1}(D)}\\&\leqslant\left\|\tilde{u}-\tilde{u}^{h\delta}\right\|_{S_0^{p+1,1}(D)}+C\left(\left\|\tilde{\xi}-\tilde{\xi}^{h\delta}\right\|_{S_0^{p+1,1}(D)}+\left\|\tilde{f}-g^h\right\|_{L^2(D)}\right).\end{aligned} \tag{6.39}$$

Thus, by Lemma 10, Remark 10, and Lemma 11, we have

$$\left\|\left(\mathcal{T}-\mathcal{T}^{h\delta}\right)(\tilde{r},\tilde{\tau})\right\|_{\mathcal{X}}\to 0\quad\text{as } h,\delta\to 0. \qquad \square$$

Proposition 4. $\|\mathcal{T}-\mathcal{T}^{h\delta}\|_{\mathcal{L}(\mathcal{Z},\mathcal{X})}\to 0$ *as $h,\delta\to 0$.*

Proof. Let $\tilde{\tau}\in S^{p+1,0}(D)$. From Remark 8 there exists $C>0$ such that

$$\left\|\tilde{\xi}-\tilde{\xi}^{h\delta}\right\|^2_{S_0^{p+1,1}(D)}\leqslant C\left(h^2+\delta^{2\gamma}\right)K\|\tilde{\tau}\|^2_{S^{p+1,0}(D)}, \tag{6.40}$$

where $K=\max\{1,\frac{1}{(k!)^2}\|c_j\|^{2(p_j+1-k)}_{L^\infty(D)}\colon\ 1\leqslant j\leqslant N,\ 0\leqslant k\leqslant p_j+1\}$.

Theorem 7, Lemma 9, Lemma 11, and (6.40) yield

$$\begin{aligned}\left\|\left(\mathcal{T}-\mathcal{T}^{h\delta}\right)(\tilde{r},\tilde{\tau})\right\|^2_{\mathcal{X}}&\leqslant C\left(h^2+\delta^{2\gamma}\right)K\left(\|\tilde{r}\|^2_{L^2(D)}+\|\tilde{\tau}\|^2_{S^{p+1,0}(D)}\right)\\&\leqslant C\left(h^2+\delta^{2\gamma}\right)K\left\|(\tilde{r},\tilde{\tau})\right\|^2_{\mathcal{Z}}\end{aligned} \tag{6.41}$$

for some $C>0$.

Hence, we see that

$$\begin{aligned} \|\mathcal{T}-\mathcal{T}^{h\delta}\|^2_{\mathcal{L}(\mathcal{Z},\mathcal{X})} &= \sup_{\|(\tilde{r},\tilde{\tau})\|_{\mathcal{Z}}\neq 0} \frac{\|(\mathcal{T}-\mathcal{T}^{h\delta})(\tilde{r},\tilde{\tau})\|^2_{\mathcal{X}}}{\|(\tilde{r},\tilde{\tau})\|^2_{\mathcal{Z}}} \\ &\leqslant \sup_{\|(\tilde{r},\tilde{\tau})\|_{\mathcal{Z}}\neq 0} C\big(h^2+\delta^{2\gamma}\big)K \to 0 \quad \text{as } h,\delta\to 0. \qquad \square \end{aligned} \tag{6.42}$$

Proposition 5. *A solution of* (6.10) *is regular.*

Proof. A proof follows from the linearity and well-posedness of (6.11), (6.12), and (6.13). □

Through Propositions 1–5 we have verified all of the assumptions of Theorem 8. Thus, by that theorem, we obtain the following results.

Theorem 12. *Assume that $U \in S_0^{p+1,1}(D)$. Let $(u, f, \xi) \in S_0^{p+1,1}(D) \times L^2(D) \times S_0^{p+1,1}(D)$ be the solution of the optimality system* (2.4)*,* (3.12)*, and* (3.16)*. Let $(u^{h\delta}, f^h, \xi^{h\delta}) \in V^{h\delta} \times G^h \times V^{h\delta}$ be the solution of the discrete optimality system* (6.14)*,* (6.15)*, and* (6.16)*. Then we have*

$$\|u-u^{h\delta}\|_{S_0^{p+1,1}(D)} + \|f-f^h\|_{L^2(D)} + \|\xi-\xi^{h\delta}\|_{S_0^{p+1,1}(D)} \to 0 \quad \text{as } h,\delta\to 0.$$

Moreover, there exists $C > 0$ such that

$$\|u-u^{h\delta}\|^2_{S_0^{p+1,1}(D)} + \|f-f^h\|^2_{L^2(D)} + \|\xi-\xi^{h\delta}\|^2_{S_0^{p+1,1}(D)} \tag{6.43}$$

$$\leqslant C\big(h^2+\delta^{2\gamma}\big)K\big(\|f\|^2_{L^2(D)} + \|u-U\|^2_{S^{p+1,0}(D)}\big), \tag{6.44}$$

where $K = \max\{1, \frac{1}{(k!)^2}\|c_j\|^{2(p_j+1-k)}_{L^\infty(D)}\colon 1 \leqslant j \leqslant N,\ 0 \leqslant k \leqslant p_j+1\}$.

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References

[1] R. Adams, Sobolev Spaces, Academic, New York, 1975.
[2] I. Babuska, P. Chatzipantelidis, On solving elliptic stochastic partial differential equations, Comput. Methods Appl. Mech. Engrg. 191 (2002) 4093–4122.
[3] I. Babuska, K. Liu, R. Tempone, Solving stochastic partial differential equations based on the experimental data, Math. Models Methods Appl. Sci. 13 (3) (2003) 415–444.
[4] I. Babuska, R. Tempone, G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, SIAM J. Numer. Anal. 42 (2) (2004) 800–825.
[5] I. Babuska, R. Tempone, G.E. Zouraris, Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation, Comput. Methods Appl. Mech. Engrg. 194 (2005) 1251–1294.
[6] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, second ed., Springer, 2002.
[7] F. Brezzi, J. Rappaz, P. Raviart, Finite-dimensional approximation of nonlinear problems. Part I: Branches of nonsingular solutions, Numer. Math. 36 (1980) 1–25.
[8] M. Crouzeix, J. Rappaz, On Numerical Approximation in Bifurcation Theory, Masson, Paris, 1990.
[9] M.K. Deb, I. Babuska, J.T. Oden, Solution of stochastic partial differential equations using Galerkin finite element techniques, Comput. Methods Appl. Mech. Engrg. 190 (2001) 6359–6372.
[10] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., 1998.
[11] P. Frauenfelder, C. Schwab, R.A. Todor, Finite elements for elliptic problems with stochastic coefficients, Comput. Methods Appl. Mech. Engrg. 194 (2005) 205–228.
[12] J. Galvis, M. Sarkis, Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity, SIAM J. Numer. Anal. 47 (5) (2009) 3624–3651.
[13] R.G. Ghanem, P.D. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer-Verlag, 1991.
[14] V. Girault, P. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer, Berlin, 1986.
[15] M.D. Gunzburger, L.S. Hou, T. Svobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier–Stokes equations with distributed and Neumann controls, Math. Comp. 57 (1991) 123–151.
[16] M.D. Gunzburger, L.S. Hou, T. Svobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier–Stokes equations with Cirichlet controls, RAIRO Model. Math. Anal. Numer. 25 (1991) 711–748.
[17] M.D. Gunzburger, L.S. Hou, Finite-dimensional approximation of a class of constrained nonlinear optimal control problems, SIAM J. Control Optim. 34 (3) (May 1996) 1001–1043.
[18] H. Holden, B. Øksendal, J. Ubøe, T. Zhang, Stochastic Partial Differential Equations; A Modeling, White Noise Functional Approach, Birkhäuser, 1996.
[19] L.S. Hou, S.S. Lavindran, A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier–Stokes equations, SIAM J. Control Optim. 36 (5) (September 1998) 1795–1814.

[20] L.S. Hou, J. Lee, A Robin–Robin non-overlapping domain decomposition method for an elliptic boundary control problem, Int. J. Numer. Anal. Model., in press.
[21] J.E. Lagness, General boundary value problems for differential equations of Sobolev type, SIAM J. Math. Anal. 3 (1972) 105–119.
[22] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. 1, Springer, New York, 1972.
[23] W. Luo, Wiener chaos expansion and numerical solutions of stochastic partial differential equations, PhD thesis, California Institute of Technology, Pasadena, California, 2006.
[24] C. Schwab, R.A. Todor, Sparse finite elements for elliptic problems with stochastic loading, Numer. Math. (2003) 707–734.