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Category $\mathcal{O}$ over a deformation of the symplectic oscillator algebra

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Abstract

We discuss the representation theory of H_f, which is a deformation of the symplectic oscillator algebra $\mathfrak{sp}(2n)\ltimes\mathfrak{h}_n$, where $\mathfrak{h}_n$ is the ($(2n+1)$-dimensional) Heisenberg algebra. We first look at a more general algebra with a triangular decomposition. Assuming the PBW theorem, and one other hypothesis, we show that the BGG category $\mathcal{O}$ is abelian, finite length, and self-dual.

We decompose $\mathcal{O}$ as a direct sum of blocks $\mathcal{O}(\lambda)$, and show that each block is a highest weight category.

In the second part, we focus on the case H_f for $n=1$, where we prove all these assumptions, as well as the PBW theorem.

MSC: Primary: 16D90; secondary: 16S30; 17B10

0. Introduction

We discuss here the BGG category over a deformation of a well-known algebra $H_0 = \mathfrak{U}(\mathfrak{sp}(2n))\ltimes A_n$. The relation $[Y_i, X_i]=1$ in A_n is deformed using the quadratic Casimir operator Δ of $\mathfrak{sp}(2n)$. We work throughout over a ground field k of characteristic zero.

In the first half, we work in a more general setup, involving an algebra with a triangular decomposition. We carry out many of the classical constructions, including standard (Verma) and co-standard modules, and introduce the BGG category $\mathcal{O}$. Next, we introduce the duality functor, which is exact, and show some homological properties

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of $\mathcal{O}$. Assuming the nonvanishing and finite length of all Verma modules, we show that $\mathcal{O}$ has many good properties (in particular, it is abelian, finite length, and self-dual).

Under additional assumptions, we decompose $\mathcal{O}$ as a direct sum of subcategories—or *blocks*—$\mathcal{O}(\lambda)$. We show that each of these blocks $\mathcal{O}(\lambda)$—and hence $\mathcal{O}$—has enough projectives. This helps us construct projective covers, injective hulls, and progenerators in each block. There is also an equivalence from $\mathcal{O}(\lambda)$ to the category of finitely generated modules over a finite-dimensional algebra. Assuming the PBW theorem, each block is a highest weight category, so that BGG reciprocity holds here.

In the second half, we introduce our algebra H_0 (and later on, H_f), and produce explicit automorphisms and an anti-involution (which is used to consider duality). We then focus on the case $n=1$. Analogous to $\mathfrak{sl}_2$-theory, we first look at standard cyclic modules via explicit calculations. We then show that a large set of Verma modules are nonzero.

Next, we show that an important constant α_{rm} is actually a polynomial. This shows the PBW Theorem. We then take a closer look at Verma modules. There is an important condition for a Verma module $Z(r)$ to have a submodule $Z(t)$: the constant $\alpha_{r,r-t+1}$ above must vanish. This helps partition k into the blocks $S(r)$.

The structure of finite dimensional simple modules is very similar to the $\mathfrak{sl}_2$-case; we state the well-known character formulae here. We completely classify all Vermas with noninteger weights, and give some results on Vermas with integer weights. Therefore, all the assumptions (and results) of the first half are shown to hold for H_f.

Part 1: General theory

In this first part, we examine in detail the structure of the category $\mathcal{O}$, and several duality and homological properties, under a general setup involving a general algebra with a triangular decomposition. (In particular, this treatment is valid for a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.) We end by showing that the category is a direct sum of blocks, each of which is a ("finite dimensional") highest weight category. The main goal of the second part, will be to prove (for the algebra H_f) the assumptions used in this part (including the PBW theorem), so that the results proved here all hold. Thus, one may read the second part independently of the first.

1. Standard cyclic modules in the Harish–Chandra (or BGG) category

Setup: We work throughout over a ground field k of characteristic zero. We define $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We work over an associative k-algebra A, having the following properties:

(1) The multiplication map: $B_- \oplus_k H \oplus_k B_+ \twoheadrightarrow A$ is surjective, where all symbols denote associative k-subalgebras of A (this is the *triangular decomposition*).
(2) There is a finite-dimensional subspace $\mathfrak{h}$ of H so that $H=\mathrm{Sym}(\mathfrak{h})$. Thus $\mathfrak{h}$ is an abelian Lie algebra (or H is abelian).

(3) There exists a *base of simple roots* Δ, i.e. a basis Δ of $\mathfrak{h}^* = \mathrm{Hom}_k(\mathfrak{h}, k)$. Define a partial ordering on $\mathfrak{h}^*$ by: $\lambda \geqslant \mu$ iff $\lambda - \mu \in \mathbb{N}_0\Delta$, i.e. $\lambda - \mu$ is a sum of finitely many elements of Δ (repetitions allowed).
(4) $A = \bigoplus_{\mu \in \mathbb{Z}\Delta} A_\mu$, where A_μ is a weight space for ad $\mathfrak{h}$. In other words, $[h, a_\mu] = ha_\mu - a_\mu h = \mu(h)a_\mu$ for all $h \in \mathfrak{h}$, $\mu \in \mathbb{Z}\Delta$, $a_\mu \in A_\mu$. Further, $B_+ \subset \bigoplus_{\mu \in \mathbb{N}_0\Delta} A_\mu$ and $H \subset A_0$.
(5) $(B_+)_0 = k$, and $\dim_k (B_+)_\mu < \infty$ for every μ.
(6) There exists an anti-involution i of A (i.e. $i^2|_A = \mathrm{id}|_A$) that takes $(B_+)_\mu$ to $(B_-)_{-\mu}$ for each μ, and acts as the identity on all of H.

Remarks. Because of the anti-involution i, similar properties are true for B_-, as are mentioned for B_+ above. We also have subalgebras (actually, ideals) $N_+ = \bigoplus_{\mu \neq 0} (B_+)_\mu$ in B_+, and similarly, N_- in B_-.

For an (A- or) H-module V, denote by $\Pi(V)$ the set of *weights* $\mu \in \mathfrak{h}^*$, so that the *weight space* $V_\mu := \{v \in V : hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$ is nonzero. Then standard arguments say that $\sum_{\mu \in \mathfrak{h}^*} V_\mu = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ is the largest $\mathfrak{h}$-semisimple submodule of V.

We now introduce the *Harish–Chandra category* $\mathscr{H}$. Its objects are A-modules with a (simultaneously) diagonalizable $\mathfrak{h}$-action, and finite-dimensional weight spaces. Clearly, $\mathscr{H}$ is a full abelian subcategory of A-mod. Inside this, we also introduce the (full) BGG subcategory $\mathcal{O}$, whose objects are finitely generated objects of $\mathscr{H}$ with a locally finite action of B_+, i.e. $\forall M \in \mathcal{O}$, $B_+ m$ is finite dimensional for each $m \in M$. Note that $\mathcal{O}$ is not extension-closed in A-mod (cf. [14]).

Definitions. A *maximal vector* in an A-module V is a weight vector for $\mathfrak{h}$ that is killed by N_+; in other words, it is an eigenvector for B_+.

A *standard cyclic module* is an A-module generated by exactly one maximal vector. Certain universal standard cyclic modules are called *Verma modules*, just as in the classical case of [4] or [12].

There exist maximal vectors (i.e. eigenvectors for B_+) in any object of $\mathcal{O}$. We now look at standard cyclic modules, namely $V = A \cdot v_\lambda$, where v_λ is maximal with weight λ. Most (if not all) of the results in [12, Section 20] now hold. We can construct standard cyclic modules $B_- v_\lambda$ and Verma modules $Z(\lambda) = A/(N_+, \{(h - \lambda(h) \cdot 1) : h \in \mathfrak{h}\})$ with unique simple quotients $V(\lambda)$, for each $\lambda \in \mathfrak{h}^*$.

Standing assumption: Until Section 9, we keep the assumption that every Verma module $Z(\lambda)$ is nonzero.

The $V(\lambda)$'s are pairwise nonisomorphic, exhaust all simple objects in $\mathcal{O}$, and are in bijective correspondence with $\mathfrak{h}^*$, as well as each of the sets of finite-dimensional simple $\mathfrak{h}$-modules, and finite-dimensional simple $(H \oplus_k B_+)$-modules. (For the last two bijective correspondences, we also need k to be algebraically closed, so that we can use Lie's theorem. For the same reason, all finite-dimensional modules are also in $\mathcal{O}$, whenever k is also algebraically closed.)

Notation: Any standard cyclic module V of highest weight λ is a quotient of $Z(\lambda)$. We denote this (or V) by $Z(\lambda) \to V \to 0$. We also denote the annihilator of $V(\lambda)$

in A by $J(\lambda)$, and the (unique) maximal submodule of $Z(\lambda)$ by $Y(\lambda)$, so that $V(\lambda) = A/J(\lambda) = Z(\lambda)/Y(\lambda)$.

Theorem 1. *Suppose $V \in \mathcal{O}$. Then the following are equivalent*:

(1) $\mathrm{Hom}_A(Z(\lambda), V) \neq 0$.
(2) *V has a maximal weight vector v_λ of weight λ.*
(3) *V has a standard cyclic submodule V' of highest weight λ.*

Now, by seeing where the maximal vector goes, we also have

Corollary 1. *If $Z(\lambda) \to V \to 0$, then* $\dim_k(\mathrm{Hom}_A(V, V(\mu))) = \delta_{\mu\lambda} \in \{0, 1\}$.

Lemma 1. *If V and V' are standard cyclic of highest weight λ, then the following are equivalent*:

(1) $V \to V' \to 0$.
(2) $\mathrm{Hom}_A(V, V') = k$.
(3) $\mathrm{Hom}_A(V, V') \neq 0$.

We now define the *formal character* (cf. [12, Sections 13, 21]) of an A-module $V = \bigoplus_\mu V_\mu \in \mathscr{H}$. This is just the formal sum $\mathrm{ch}_V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e(\mu)$, where $\mathbb{Z}[\mathfrak{h}^*] = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z} \cdot e(\lambda)$. Finally, define the *Kostant function* $p(\lambda)$ to be $p(\lambda) = \dim_k (B_+)_{-\lambda} = \dim_k (B_-)_\lambda$.

2. Duality and homological properties

As is standard, we give $M^* = \mathrm{Hom}_k(M, k)$ a *left* A-module structure (for each $M \in \mathcal{O}$), using the anti-involution i mentioned above. Now define the functor F from $\mathcal{O}$ to the opposite category $\mathcal{O}^{\mathrm{op}}$ (defined presently), by taking $F(M)$ to be the submodule of M^* generated by all $\mathfrak{h}$-weight vectors in M^*. Thus, $\mathcal{O}^{\mathrm{op}}$ has $F(M)$ for its objects (for $M \in \mathcal{O}$), and induced homomorphisms for its morphisms. More generally, we can define $F : \mathscr{H} \to \mathscr{H}$ in the same way.

Our analysis in the next few sections is in the spirit of [4,10,5].

Notation: Throughout the rest of this paper (resp. in the appendix), by the long exact sequence of Ext's, we mean the long exact sequence of $\mathrm{Ext}_{\mathcal{O}}$'s in the abelian, self-dual category $\mathcal{O}_{\mathbb{N}}$, consisting of all objects of finite length in $\mathcal{O}$ (resp. in the abelian category $\mathcal{O}$). (That $\mathcal{O}_{\mathbb{N}}$ is abelian and self-dual will be proved below.)

Proposition 1. *Let $M \in \mathscr{H}$.*

(1) $ch_{F(M)} = ch_M$.
(2) *$F(F(M))$ is canonically isomorphic to M.*
(3) $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(F(N), F(M))$ *if $M, N \in \mathscr{H}$.*

The proof is standard, given that all weight spaces are finite dimensional, and hence reflexive.

Proposition 2.

(1) *F is an exact contravariant functor in $\mathscr{H}$.*
(2) *If $M \in \mathscr{H}$ is simple, then so is $F(M)$. Further, $M = V(\lambda) \Leftrightarrow F(M) = V(\lambda)$.*
(3) *If $M \in \mathcal{O}$ has a filtration in $\mathcal{O}$ with subquotients $V_i \in \mathcal{O}$, then $F(M)$ has a filtration in $\mathcal{O}^{\text{op}}$, with subquotients $F(V_i)$ occurring in* reverse order *to that of the V_i's.*

Proof. We only show that if $M = V(\lambda)$, then $F(M) = V(\lambda)$. Now, $\dim_k(F(M)_\lambda) = \dim_k(M_\lambda)$ (from Proposition (1)) $=1$, hence say m^* spans $F(M)_\lambda$. Now, $m_\lambda \in M_\lambda$ is of maximal weight, so m^* is also maximal, and of weight λ. Therefore $Z(\lambda) \to B_- m^* \to 0$, whence $0 \neq B_- m^* \subset F(M)$ simple. Thus, $F(M) = V(\lambda)$. $\square$

Remarks. The last part is standard, once we verify that $\mathcal{O}$ is closed under quotienting. Further, if $M \in \mathcal{O}$ has finite length, then so does $F(M)$, and $l(M) = l(F(M))$.

$\mathcal{O}$ is an additive category, with finite direct sums. All morphism spaces are finite dimensional. Inside $\mathcal{O}$ we define a new subcategory $\mathcal{O}_{\mathbb{N}}$, whose objects are all $M \in \mathcal{O}$ with finite length (including the zero module). Morphisms are module maps, as always.

Theorem 2.

(1) *$\mathcal{O}$ is a full subcategory of A*-mod, *closed under taking quotients.*
(2) *In particular, every $M \in \mathcal{O}_{\mathbb{N}}$ is a finite direct sum of indecomposable objects.*
(3) *$\mathcal{O}_{\mathbb{N}}$ is abelian, self-dual (i.e. $\mathcal{O}_{\mathbb{N}} = \mathcal{O}_{\mathbb{N}}^{\text{op}}$), and a full subcategory of A*-mod.

Proof. For (1), if $M = \sum Am_i$, then $M/N = \sum A\overline{m_i}$, where $0 \subset N \subset M$ is a submodule of $M \in \mathcal{O}$. For (3), note that if $M \in \mathcal{O}_{\mathbb{N}}$ and N is as above, then $l(N) \leqslant l(M) < \infty$, so N is finitely generated, and hence in $\mathcal{O}$, thus in $\mathcal{O}_{\mathbb{N}}$ as well. Thus $\mathcal{O}_{\mathbb{N}}$ is abelian. (This argument fails for $\mathcal{O}$.)

To show that $\mathcal{O}_{\mathbb{N}}$ is self-dual, apply the previous Proposition 2, to any composition series for $M \in \mathcal{O}_{\mathbb{N}}$. $\square$

Inside $\mathcal{O}$ we have two sets of subcategories. For each $\lambda \in \mathfrak{h}^*$, we have the subcategory $\mathcal{O}^{\leqslant \lambda}$ whose objects are $M \in \mathcal{O}$ so that $\Pi(M) \leqslant \lambda$. And for each $\bar{\lambda} \in \mathfrak{h}^*/(\mathbb{Z} \cdot \Delta)$, we have the subcategory $\mathcal{O}_{\bar{\lambda}}$, whose objects are $M \in \mathcal{O}$ so that $\Pi(M) \subset \lambda + \mathbb{Z} \cdot \Delta$.

Proposition 3. *We work in the BGG category $\mathcal{O}$.*

(1) *$\mathcal{O}^{\leqslant \lambda}$ is a full subcategory of A*-mod, *closed under taking quotients.*
(2) *If $N_\lambda = 0$ for some $N \in \mathcal{O}$ and all $\lambda > \mu$, then* $\operatorname{Ext}^1_{\mathcal{O}}(Z(\mu), N) = 0$.
(3) *If $Z(\mu) \to V \to 0$, then* $\operatorname{Ext}^1_{\mathcal{O}}(V, V(\mu)) = 0$.

Proof. (1) is easy to check, and the proof of (2) is as in [11, Lemma (16)]. The proof of (3) is similar to that of (2), and we give it below.

Say $0 \to V(\mu) \to M \xrightarrow{\pi} V \to 0$ is exact. Let v_μ be the highest weight vector in V. Choose any (nonzero) $m \in \pi^{-1}(v_\mu) \subset M_\mu$. Now, v_μ is maximal, so $\pi(N_+ m)=0$, whence $N_+ m \subset V(\mu) \subset M$. But $V(\mu)$ has no weights $> \mu$, so $N_+ m=0$. Thus, $Z(\mu) \twoheadrightarrow B_- m \overset{\pi}{\twoheadrightarrow} V$.

We know $m \notin V(\mu)$ because $\pi(m) \neq 0 = \pi(V(\mu))$. Now, say $X = V(\mu) \cap B_- m$. Then X is a submodule of $V(\mu)$ with μ-weight space zero, so $X = 0$, and once more, we have $M = V(\mu) \oplus B_- m$. So $B_- m \cong V$ and we are done. $\square$

Remarks. We cannot replace $Z(\mu)$ by a general $Z(\mu) \to V \to 0$ in part (2) above, because we can have short exact sequences like $0 \to Z(v) \hookrightarrow Z(\mu) \twoheadrightarrow Z(\mu)/Z(v) \to 0$. Also, the above result says, in particular, that Verma modules and simple modules have no self-extensions.

Proposition 4.

(1) *If* $Z(\lambda) \to N \to 0$ *and* $\mathrm{Ext}^1_{\mathcal{O}}(Z(\mu), N) \neq 0$ *(e.g.* $N = Z(\lambda), V(\lambda)$, *etc.) then* $\mu < \lambda$.
(2) *If* $\mathrm{Ext}^1_{\mathcal{O}}(V(\mu), V(\lambda)) \neq 0$ *then it is finite dimensional, and* $\mu < \lambda$ *or* $\lambda < \mu$.
(3) *Thus* $\mathrm{Ext}^1_{\mathcal{O}}(M, N)$ *is finite dimensional for* $M, N \in \mathcal{O}_{\mathbb{N}}$.

Proof.
(1) This follows from the previous proposition: $\exists \omega > \mu$ so that $N_\omega \neq 0$. But since N is standard cyclic, hence $\mu < \omega \leqslant \lambda$, and we are done.

(2) That $\mu \neq \lambda$ was shown in the previous proposition (since there are no self-extensions). Now suppose $0 \to V(\lambda) \to M \xrightarrow{\pi} V(\mu) \to 0$ is a nonsplit extension. The proof here is similar in spirit to previous proofs. Say v_μ is the highest weight vector in $V(\mu)$, and m a lift to M. Then $\pi(N_+ m) = 0$, so we have two cases.

- If $N_+ m = 0$ then $B_- m \twoheadrightarrow V(\mu)$. Now, let $X = V(\lambda) \cap B_- m$, as earlier. X is nonzero since M is a nontrivial extension, and so X is a nonzero submodule of $V(\lambda)$, whence $X = V(\lambda)$. But now $V(\lambda) \hookrightarrow B_- m \twoheadrightarrow V(\mu)$, whence $\lambda < \mu$.

  Now, since $X = V(\lambda)$, hence $\exists Z \in (B_-)_{\lambda-\mu}$ so that $Zm = v_\lambda$ is the maximal vector in $V(\lambda)$. Conversely, any such relation completely determines M, because M_μ is one dimensional, and M has only two generators. Further, any such extension *has* to be of this type, so $\dim_k(\mathrm{Ext}^1_{\mathcal{O}}(V(\mu), V(\lambda))) \leqslant \dim_k((B_-)_{\lambda-\mu}) = p(\lambda - \mu) < \infty$.

- If $N_+ m \neq 0$ then $(V(\lambda))_{\mu+\alpha} \neq 0$ for some $\alpha > 0$, whence $\mu < \mu + \alpha \leqslant \lambda$. But we are in $\mathcal{O}_{\mathbb{N}}$, because M has length 2. Hence by Proposition A.2 (in the appendix), $\mathrm{Ext}^1_{\mathcal{O}}(V(\mu), V(\lambda)) \cong \mathrm{Ext}^1_{\mathcal{O}}(F(V(\lambda)), F(V(\mu))) = \mathrm{Ext}^1_{\mathcal{O}}(V(\lambda), V(\mu))$ by Proposition (2), whence by the previous case it is finite dimensional.

(3) This follows from the previous part, using the long exact sequence of $\mathrm{Ext}_{\mathcal{O}}$'s (and induction on lengths). $\square$

We now define the *co-standard modules* $A(\lambda) = F(Z(\lambda)) \in \mathcal{O}^{\mathrm{op}}$. Since $Y(\lambda)$ was the radical of $Z(\lambda)$, and $V(\lambda)$ the head, hence $V(\lambda)$ is the socle of $A(\lambda)$.

3. Filtrations and finite length modules

Note that to construct projectives in the classical case of [4], one could quotient $\mathfrak{U}\mathfrak{g}$ by $(\mathfrak{U}\mathfrak{g})\mathfrak{n}_+^l$. Over here we propose the following alternative:

Given $l \in \mathbb{N}$, look at the "minimal weights" in N_+^l. That is, define $\Sigma : (\mathbb{Z}\Delta)^l \to \mathbb{Z}\Delta$ by $(\mu_1, \ldots \mu_l) \mapsto \sum_{i=1}^l \mu_i$. Then the minimal weights in N_+^l are simply $T = \Sigma(\Delta^l) = \{\Sigma(i) : i \in \Delta^l\}$. (Here, Δ^l is the l-fold Cartesian product of Δ.) Now define $B_{+l} = \sum_{\alpha \in \mathbb{N}_0\Delta,\ \mu \in T} (B_+)_{\mu+\alpha}$.

Thus $\Pi(B_{+l})$ is closed under "adding positive weights", hence B_{+l} is a two-sided ideal in B_+.

We claim that B_+/B_{+l} is finite dimensional for all l. Indeed, Δ is finite, and any weight λ of B_+/B_{+l} has to look like $\sum_{\alpha \in \Delta} c_\alpha \alpha$, where $0 \leqslant c_\alpha\ \forall \alpha$, and $\sum_\alpha c_\alpha < l$. Thus $\dim_k(B_+/B_{+l})$ is the sum of dimensions of finitely many weight spaces of B_+, each of which is finite.

Definitions. (1) Define the A-modules $P(\lambda, l)$ and $I(\lambda, l)$ (for $\lambda \in \mathfrak{h}^*$ and $l \in \mathbb{N}$) by

$$P(\lambda, l) = A/I_0(\lambda, l) \in \mathcal{O} \quad \text{and} \quad I(\lambda, l) = F(P(\lambda, l)) \in \mathcal{O}^{\mathrm{op}}$$

where $I_0(\lambda, l)$ is the left ideal generated by B_{+l} and $\{(h - \lambda(h) \cdot 1) : h \in \mathfrak{h}\}$.
(2) Given $\lambda \in \mathfrak{h}^*$ and $l \in \mathbb{N}$, define the subcategory $\mathcal{O}(\lambda, l)$ to be the full subcategory of all $M \in \mathcal{O}$ so that $B_{+l}M_\lambda = 0$.
(3) A (finite) filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of an A-module M is a

(a) *p-filtration* (cf [4]), denoted by $M \in \mathscr{F}(\Delta)$, if for each i, $M_i \in \mathcal{O}$, and M_{i+1}/M_i is a Verma module $Z(\lambda_i)$.
(b) *q-filtration*, denoted $M \in \mathscr{F}(\nabla)$, if for each i, $M_i \in \mathcal{O}^{\mathrm{op}}$, and M_{i+1}/M_i is a module of the form $A(\lambda_i)$.
(c) *SC-filtration* if for each i, $M_i \in \mathcal{O}$, and M_{i+1}/M_i is standard cyclic.

For example, if $l = 1$, then we have $B_{+1} = N_+$, so $P(\lambda, 1) = Z(\lambda)$.

Proposition 5. *We still work in* $\mathcal{O}$.

(1) *Given* $M \in \mathcal{O}$, $M \in \mathscr{F}(\Delta)$ *iff* $F(M) \in \mathscr{F}(\nabla)$.
(2) $\mathrm{Hom}_A(P(\lambda, l), M) = M_\lambda$ *for each* $M \in \mathcal{O}(\lambda, l)$, *so* $P(\lambda, l)$ *is projective in* $\mathcal{O}(\lambda, l)$.
(3) *If* $M \to N \to 0$ *in* $\mathcal{O}$, *and* M *has an SC-filtration, then so does* N.
(4) $P(\lambda, l)$ *has an SC-filtration* $\forall \lambda, l$.

Proof.
(1) This follows from Proposition 2, where we take each V_i to be a Verma module.

We know that $B_- \otimes H \otimes B_+ \twoheadrightarrow A \twoheadrightarrow P(\lambda, l)$, and moreover, $H \otimes B_+ \twoheadrightarrow P(\lambda, l)_\lambda$. Therefore, $B_{+l}(H \otimes B_+) \twoheadrightarrow B_{+l}P(\lambda, l)_\lambda$. Because $\mathfrak{h}$ is ad-semisimple, we see that $B_{+l}(H \otimes B_+) \subset A \cdot B_{+l} \subset I_0(\lambda, l)$. Hence $B_{+l}(\lambda, l)_\lambda = 0$, and $P(\lambda, l) \in \mathcal{O}(\lambda, l)$, as required.

Next, we show the exactness of $\mathrm{Hom}_A(P(\lambda,l),-)$. Given $\varphi \in \mathrm{Hom}_A(P(\lambda,l),M)$, we get $v_\varphi = \varphi(1) \in M_\lambda$ (because $h\varphi(1) = \varphi(h \cdot 1) = \lambda(h)\varphi(1)$ for each $h \in \mathfrak{h}$). Conversely, given $m \in M_\lambda$, define $\varphi \in \mathrm{Hom}_k(k,M)$ by $\varphi(1) = m$. This extends to a map: $A \to M$ of left A-modules. Because $M \in \mathcal{O}(\lambda,l)$, hence B_{+l} is in the kernel, as is $(h - \lambda(h) \cdot 1)$. Thus φ factors through a map: $P(\lambda,l) \to M$ as desired. It is easy to see that both these operations are inverses of each other, so we are done.

(3) This is because quotients of standard cyclic modules are standard cyclic.

(4) The proof is similar to that in [4]. Moreover, the same ordering holds among the terms of the filtration: if $Z(\lambda_{j+1}) \to P_{j+1}/P_j \to 0$, and $\lambda_i \geqslant \lambda_j$, then $i \leqslant j$. □

Proposition 6. *Suppose $M \in \mathscr{F}(\Delta)$, and $S = \{\nu \in \mathfrak{h}^* : [M : Z(\nu)] \neq 0\}$.*

(1) *If λ is maximal in S, then $\exists M'' \in \mathscr{F}(\Delta)$ so that $0 \to Z(\lambda) \to M \to M'' \to 0$ is exact.*
(2) *If λ is minimal in S, then $\exists M' \in \mathscr{F}(\Delta)$ so that $0 \to M' \to M \to Z(\lambda) \to 0$ is exact.*
(3) *Suppose $M_1, M_2 \in \mathcal{O}_{\mathbb{N}}$. Then $M_1 \oplus M_2 \in \mathscr{F}(\Delta)$ iff $M_1, M_2 \in \mathscr{F}(\Delta)$.*

Proof. (1) and (3) follow from [4], and (2) is cf. [7, (A3.1)(i)]. □

The next result comes from [10], and involves $\mathfrak{h}$-diagonalizable modules M.

Proposition 7. *Suppose M is $\mathfrak{h}$-diagonalizable. Then the following are equivalent:*

(1) *$M \in \mathcal{O}$.*
(2) *M is a quotient of a direct sum of finitely many $P(\lambda,l)$'s.*
(3) *M has an SC-filtration. Further, the subquotients are standard cyclic with highest weights λ_i, and we can arrange these so that $\lambda_i \geqslant \lambda_j \Rightarrow i \leqslant j$.*

Proof. We only show, in the part (3) $\Rightarrow$ (1), that B_+ acts locally finitely on M. Since M has an SC-filtration, $\mathrm{ch}_M \leqslant \sum \mathrm{ch}_{Z(\lambda_i)}$, where we sum over a finite set. Thus, given $m \in M_\mu$, we see that $\Pi(B_+ m) \subset \bigcup_i \{\lambda : \mu \leqslant \lambda \leqslant \lambda_i\}$, and each of these sets is finite. Hence $\Pi(B_+ m)$ is finite, so $B_+ m$ is itself finite dimensional. □

Theorem 3. *Suppose every Verma module $Z(\lambda)$ has finite length.*

(1) *Then $\mathcal{O} = \mathcal{O}_{\mathbb{N}}$.*
(2) *If $\mathrm{Ext}^1_{\mathcal{O}}(Z(\mu),M)$ or $\mathrm{Ext}^1_{\mathcal{O}}(M,A(\mu))$ is nonzero for $M \in \mathcal{O}$, then M has a composition factor $V(\lambda)$ with $\mu < \lambda$.*
(3) *If $X \in \mathscr{F}(\Delta)$ and $Y \in \mathscr{F}(\nabla)$ then $\mathrm{Ext}^1_{\mathcal{O}}(X,Y) = 0$.*
(4) *If $X \in \mathscr{F}(\Delta)$ and $Y \in \mathscr{F}(\nabla)$ then*

$$\dim_k(\mathrm{Hom}_A(X,Y)) = \sum_{\nu \in \mathfrak{h}^*} [X : Z(\nu)][Y : A(\nu)],$$

where the terms on the RHS are the respective multiplicities in the various filtrations. Thus,

$$[X : Z(\mu)] = \dim_k(\mathrm{Hom}_A(X, A(\mu))), \textit{ and } [Y : A(\mu)] = \dim_k(\mathrm{Hom}_A(Z(\mu), Y)).$$

Proof.

(1) If all Verma modules have finite length, then so do all standard cyclic modules, and since every module has an SC-filtration, hence all modules have finite length.

(2) This is cf. [7, (A1.6)(ii)].

(3) The general case follows by the long exact sequence of $\mathrm{Ext}_{\mathcal{O}}$'s (and induction on lengths of filtrations) from the case $X = Z(\mu)$, $Y = A(\lambda)$. To show the latter, suppose $\mathrm{Ext}^1_{\mathcal{O}}(X, Y) \neq 0$. Applying the previous part with $\mathrm{Ext}^1_{\mathcal{O}}(Z(\mu), Y)$, we see that Y has a composition factor $V(v)$ with $\mu < v$. Since $Y = A(\lambda)$, hence we get $\mu < v \leqslant \lambda$, so $\mu < \lambda$.

By symmetry, apply the previous part with $\mathrm{Ext}^1_{\mathcal{O}}(X, A(\lambda))$, to get that X has a composition factor $V(v)$ with $\lambda < v$. Again, $v \leqslant \mu$ because $X = Z(\mu)$, so $\lambda < \mu$. Thus we have obtained: $\lambda < \mu < \lambda$, a contradiction. Hence all $\mathrm{Ext}^1_{\mathcal{O}}(Z(\mu), A(\lambda)) = 0$.

(4) For $X = Z(\mu), Y = A(\lambda)$, the result says that $\dim_k(\mathrm{Hom}_A(Z(\mu), A(\lambda))) = \delta_{\mu\lambda}$, and this is simply [7, (A1.6)]. We again build the general case up, using the long exact sequence of $\mathrm{Ext}_{\mathcal{O}}$'s and the previous part. □

4. Blocks in the BGG category $\mathcal{O}$

Note that in the classical case, we had the notion of *blocks* $\mathcal{O}(\chi)$, where $\chi \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathfrak{Z}(\mathfrak{U}(\mathfrak{g})), \mathbb{C})$. Thus, a $\mathfrak{g}$-module V is in $\mathcal{O}(\chi)$ iff for each z in the center $\mathfrak{Z}$ one can find an n so that $(z - \chi(z))^n$ kills V. Furthermore, (cf. [12, Exercise (23.9)] or [6, (7.4.8)]) every algebra map from the center to $\mathbb{C}$ is of the form χ_μ for some $\mu \in \mathfrak{h}^*$. Thus, the irreducible module $V = V(\lambda)$ is in $\mathcal{O}(\chi)$ iff $\chi_\lambda = \chi = \chi_\mu$, iff $\lambda + \delta$ and $\mu + \delta$ are W-conjugate (by Harish–Chandra's theorem).

Over here, we do not have any of this, so we make some additional assumptions. We make $\mathfrak{h}^*$ into a (directed) graph as follows: given $\lambda, \mu \in \mathfrak{h}^*$, we say that $\lambda \to \mu$ if $Z(\lambda)$ has a simple subquotient $V(\mu)$. Now make all edges nondirected, and for any $\lambda \in \mathfrak{h}^*$, define the set $S(\lambda)$=$\{\mu : \lambda$ and μ are in the same connected component of the graph $\mathfrak{h}^*\}$.

Standing assumption: $S(\lambda)$ is finite for each λ.

(Thus the $S(\lambda)$'s partition $\mathfrak{h}^*$, and $S(\lambda) \subset \lambda + \mathbb{Z}\Delta$.)

(For example, if $A = \mathfrak{U}\mathfrak{g}$, where $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$, then (it is well known that) the set $S(\lambda)$ is contained in $W \bullet \lambda$, where the $\bullet$ denotes the twisted action of the Weyl group: $w \bullet \lambda = w(\lambda + \delta) - \delta$, where δ is the half-sum of positive roots.)

Note that category $\mathcal{O}$ has the full subcategories $\mathcal{O}(\lambda)$, defined as follows: Given $\lambda \in \mathfrak{h}^*$, $\mathcal{O}(\lambda)$ contains precisely those $M \in \mathcal{O}_{\mathbb{N}}$, all of whose composition factors are of the form $V(\mu)$, for some $\mu \in S(\lambda)$.

Lemma 2. $\mathcal{O} = \mathcal{O}_{\mathbb{N}}$.

Proof. It suffices to show that every Verma module $Z(\lambda)$ has finite length. Suppose V is any subquotient of $Z(\lambda)$. Then V has a maximal vector v_μ, so we get a nonzero module map: $Av_\mu = B_- v_\mu \hookrightarrow V$. Hence $V(\mu) = Av_\mu/\mathrm{rad}(Av_\mu) \hookrightarrow V/\mathrm{rad}(Av_\mu)$, so $V(\mu)$ is a subquotient of $Z(\lambda)$, and thus $\mu \in S(\lambda)$ by definition. We then claim that

$$l(Z(\lambda)) \leqslant \sum_{\mu \in S(\lambda)} \dim_k (Z(\lambda))_\mu = \sum_{\mu \in S(\lambda)} p(\mu - \lambda) < \infty$$

because if $Z(\lambda) = V_0 \supset V_1 \supset \cdots$, then each V_i/V_{i+1} has a maximal vector of weight μ for some $\mu \in S(\lambda)$. Hence there can only be "RHS-many" submodules in a chain, as claimed. $\square$

Theorem 4.

(1) $\mathrm{Ext}^1_{\mathcal{O}}(V(\lambda'), V(\lambda)) = 0$ *if* $\lambda' \notin S(\lambda)$.
(2) *Given* $M \in \mathcal{O}$, *let* S_M *be the union of all* $S(\lambda)$*'s corresponding to all simple subquotients of* M. *Suppose* S_M *and* $S_{M'}$ *are disjoint for* $M, M' \in \mathcal{O}$. *Then* $\mathrm{Hom}_{\mathcal{O}}(M, M') = \mathrm{Ext}^1_{\mathcal{O}}(M, M') = 0$.
(3) $\mathcal{O} = \sum \mathcal{O}(\lambda) = \bigoplus \mathcal{O}(\lambda)$, *where we sum over all distinct blocks.*

Proof.

(1) Say $0 \to V(\lambda) \to M \xrightarrow{\pi} V(\lambda') \to 0$ is a nontrivial extension. Then we know from Proposition 4 that $\lambda < \lambda'$ or $\lambda' < \lambda$. Assume first that $\lambda' > \lambda$. Choose $m \in \pi^{-1}(v_{\lambda'})$. Then from the proof of Proposition 4, we see that $V(\lambda) \hookrightarrow M = B_- m \twoheadrightarrow V(\lambda')$. Hence M is standard cyclic, so $Z(\lambda')$ has a simple subquotient $V(\lambda)$, whence $\lambda' \in S(\lambda)$. On the other hand, if $\lambda' \notin S(\lambda)$ and $\lambda > \lambda'$, then by Proposition A.2 in the appendix, $\mathrm{Ext}^1_{\mathcal{O}}(V(\lambda'), V(\lambda)) \cong \mathrm{Ext}^1_{\mathcal{O}}(V(\lambda), V(\lambda')) = 0$ (since $\mathcal{O} = \mathcal{O}_{\mathbb{N}} = \mathcal{O}^{\mathrm{op}}$).

(2) This follows from (1) above, using induction on length, and the long exact sequence of $\mathrm{Ext}_{\mathcal{O}}$'s. For the $\mathrm{Hom}_{\mathcal{O}}$'s, use Corollary 1 in place of part (1) above.

(3) Given $M \in \mathcal{O}$, we claim we can write it as $M = \bigoplus M(\lambda)$, where $M(\lambda) \in \mathcal{O}(\lambda)$. We prove this by using induction on the length of M. For $l(M) = 0$ or 1, we are easily done. Suppose we have $0 \to N \to M \to V(\mu) \to 0$. We know that $N = \bigoplus N(\lambda)$ because N has lesser length.

Now $N = N' \oplus N(\mu)$, say, where N' is the direct sum of all other components of N. By Proposition A.1 (in the appendix), $M = N' \oplus M(\mu)$, where $0 \to N(\mu) \to M(\mu) \to V(\mu) \to 0$. This is because $\mathrm{Ext}^1_{\mathcal{O}}(V(\mu), N') = 0$ from the previous part.

Thus $M = \bigoplus M(\lambda)$, where $M(\lambda) = M(\mu)$ if $\lambda = \mu$, and $N(\lambda)$ otherwise. $\square$

Definition. Fix any indexing $S(\lambda) = \{\lambda_1, \ldots, \lambda_n\}$ that satisfies the following condition: If $\lambda_i \geqslant \lambda_j$, then $i \leqslant j$. Now define the *decomposition matrix* D in any block $\mathcal{O}(\lambda)$ (where $S(\lambda) = \{\lambda_i\}$, under the above reordering) to be $D_{ij} = [Z(\lambda_i) : V(\lambda_j)]$.

Proposition 8. *We work in a fixed block* $\mathcal{O}(\lambda)$.

(1) *D is unipotent.*
(2) *The Grothendieck group* $\mathrm{Grot}(\mathcal{O}(\lambda))$ *has the following* $\mathbb{Z}$*-bases*: $\{[V(\mu)] : \mu \in S(\lambda)\}$, $\{[Z(\mu)] : \mu \in S(\lambda)\}$, $\{[A(\mu)] : \mu \in S(\lambda)\}$.

Remarks. Given $M \in \mathcal{O}(\lambda)$, we now define the *multiplicities* $[M : V(\lambda)]$, $[M : Z(\lambda)] = [M : A(\lambda)]$ to be the coefficients of the respective basis elements, when writing $[M]$ as a linear combination of each of these bases. Then these actually equal the multiplicities of $Z(\lambda)$'s and $V(\lambda)$'s in various p- and SC-filtrations (whenever M does have such a filtration).

5. Projective modules in the blocks $\mathcal{O}(\lambda)$

Now fix $\lambda \in \mathfrak{h}^*$. From above, we see that $\mathcal{O}(\lambda)$ is a full subcategory of $\mathcal{O}$ that is abelian, self-dual, and finite length. We now construct projectives and progenerators in these blocks. Given $\mu \in S(\lambda)$, as above we define $\mathcal{O}(\lambda)^{\leqslant \mu}$ to be $\mathcal{O}(\lambda) \cap \mathcal{O}^{\leqslant \mu}$.

Proposition 9.

(1) *If* $V \in \mathcal{O}(\lambda)^{\leqslant \lambda}$, *then* $\mathrm{Hom}_A(Z(\lambda), V) \cong V_\lambda$.
(2) $Z(\mu)$ *is the projective cover of* $V(\mu)$ *in* $\mathcal{O}(\lambda)^{\leqslant \mu}$.

Proof.

(1) We see that $\mathcal{O}(\lambda)^{\leqslant \lambda} \subset \mathcal{O}(\lambda, 1)$, so $P(\lambda, 1) = Z(\lambda)$ is projective here.
(2) We already know $Z(\mu)$ is an indecomposable projective in $\mathcal{O}(\lambda)^{\leqslant \mu} = \mathcal{O}(\mu)^{\leqslant \mu}$, and $Y(\mu) = \mathrm{rad}(Z(\mu))$. Now use Theorem A.1 from the appendix. $\square$

Theorem 5.

(1) $\mathcal{O}(\lambda)$ *has enough projectives.*
(2) *There is a bijection between* $S(\lambda)$ *and each of the following sets*: *indecomposable projectives* (*i.e.* projective covers), *indecomposable injectives* (*i.e.* injective hulls), *Verma modules*, *co-standard modules*, *and simple modules* (*all in* $\mathcal{O}(\lambda)$).
(3) $\mathcal{O}(\lambda)$ *is equivalent to* $(\text{mod-}B_\lambda)^{fg}$, *where* B_λ *is a finite-dimensional semisimple* k*-algebra*.

Remarks. In fact, everything in Theorems (A.1) and (A.2) holds here, if we show the first part. For example, if λ_0 is maximal in $S(\lambda)$, then $P(\lambda_0) = Z(\lambda_0)$ is the projective cover of $V(\lambda_0)$, and $I(\lambda_0) = A(\lambda_0)$ is the injective hull.

Proof. We only have to show that enough projectives exist in our abelian category $\mathcal{O}(\lambda)$. We refer to [2, Section 3.2]. Following Remark (3) there, we only need to verify five things (here) about $\mathcal{O}(\lambda)$, to conclude that enough projectives exist. We do so now.

(1) $\mathscr{A} = \mathcal{O}(\lambda)$ is a finite length abelian k-category.
(2) There are only finitely many simple isomorphism classes here (because $S(\lambda)$ is finite).
(3) Endomorphisms of any simple object (in fact, of any standard cyclic object) are scalars, by Lemma 1.

The notation $\mathscr{A}_T$ refers precisely to $\mathcal{O}(\lambda)^{\leqslant\mu}$. It *is* a full subcategory. Further, $L(s)=V(s)$, $\Delta(s)=Z(s)$, and $\nabla(s)=A(s)$ here. We also have maps $\Delta(s)\to L(s)$ and $L(s)\to\nabla(s)$.

(4) As seen earlier, $Z(\mu)\to V(\mu)$ is a projective cover in $\mathcal{O}(\lambda)^{\leqslant\mu}$, and therefore $V(\mu)\to A(\mu)$ is an injective hull, by duality. Both $Z(\mu)$ and $A(\mu)$ are indecomposable, in particular.

(5) $Y(s)=\ker(\Delta(s)\to L(s))$ and $F(Y(s))=\operatorname{coker}(L(s)\to\nabla(s))$ both lie in $\mathcal{O}(\lambda)^{<s}$ for each $s\in S(\lambda)$ (meaning that they are in $\mathcal{O}(\lambda)^{\leqslant s}$ and have no subquotients $V(s)$). □

Remarks.

(1) The simple module, Verma module, co-standard module, projective cover, and injective hull (of $V(\mu)$) corresponding to $\mu\in S(\lambda)$ are denoted, respectively, by $V(\mu)$, $Z(\mu)$, $A(\mu)$, $P(\mu)$, $I(\mu)$.

(2) By duality, there are enough injectives in $\mathcal{O}(\lambda)$. Since $\mathcal{O}=\bigoplus\mathcal{O}(\lambda)$, hence $\mathcal{O}$ has enough projectives and injectives; in particular, $P(\lambda)$ is projective and $I(\lambda)$ is injective in $\mathcal{O}$ too. Every projective module $P\in\mathcal{O}$ is of the form $P=\bigoplus P(\lambda)^{\oplus n_\lambda}$, where only finitely many n_λ's are nonzero (and positive).

We conclude this section with one last result, cf. [4]. It holds because $\mathcal{O}=\bigoplus\mathcal{O}(\lambda)$.

Proposition 10. *Given* $\lambda\in\mathfrak{h}^*$ *and* $M\in\mathcal{O}$,
one has $\dim_k(\mathrm{Hom}_A(P(\lambda),M))=\dim_k(\mathrm{Hom}_A(M,I(\lambda)))=[M:V(\lambda)]$.

6. Every block $\mathcal{O}(\lambda)$ is a highest weight category

We now introduce the notion of a *highest weight category*, cf. [5; 7, (A2.1)]. Let $\mathscr{C}$ be an abelian category over a field k. Let S index a complete collection of nonisomorphic simple objects in $\mathscr{C}$, say $\{V(\lambda):\lambda\in S\}$. We assume that $\mathscr{C}$ is locally Artinian and satisfies the Grothendieck condition (these are technical, though for our purposes, finite length would suffice), and contains enough injectives.

The category $\mathscr{C}$ is then said to be a *highest weight category* if S satisfies the following conditions:

(1) S is an *interval finite* poset, i.e. there is a partial ordering $\leqslant$ on S, and for each $\mu\leqslant\lambda\in S$, the set of intermediate elements $[\mu,\lambda]=\{\nu\in S:\mu\leqslant\nu\leqslant\lambda\}$ is finite.

(2) There is a collection of objects $\{A(\lambda):\lambda\in S\}$ of $\mathscr{C}$, and for each λ, an embedding $V(\lambda)\hookrightarrow A(\lambda)$, such that all composition factors $V(\mu)$ of $A(\lambda)/V(\lambda)$ satisfy $\mu<\lambda$. For $\mu,\lambda\in S$, we have that $\dim_k\mathrm{Hom}_{\mathscr{C}}(A(\lambda),A(\mu))$ and $\sup_{M\in J}[M:V(\mu)]$ are finite. Here, J is the set of all subobjects of $I(\lambda)$ of finite length, and $[M:V(\mu)]$ denotes the multiplicity in M of the simple module $V(\mu)$.

(3) Each simple $V(\lambda)$ has an injective envelope $I(\lambda)$ in $\mathscr{C}$. Further, the $I(\lambda)$'s each have a "good filtration" which begins with $A(\lambda)$—namely, an increasing filtration

$0 = F_0(\lambda) \subset F_1(\lambda) \subset F_2(\lambda) \subset \cdots$, such that:
(a) $F_1(\lambda) \cong A(\lambda)$;
(b) for $n > 1$, $F_n(\lambda)/F_{n-1}(\lambda) \cong A(\mu)$ for some $\mu = \mu(n) > \lambda$;
(c) for a given $\mu \in S$, $\mu(n) = \mu$ for only finitely many n;
(d) $\bigcup_i F_i(\lambda) = I(\lambda)$.

Reconciling this notation to our earlier notation, we see that each block $\mathscr{C} = \mathcal{O}(\lambda)$ (is finite length, and hence) already satisfies all conditions but two, namely, that $I(\lambda)/A(\lambda) \in \mathscr{F}(\nabla)$, and each co-standard cyclic factor $A(\mu)$ of $I(\lambda)/A(\lambda)$ satisfies $\mu > \lambda$. (Here, we take S to be the finite set $S(\lambda)$.)

Standing assumption: The PBW theorem holds. In other words, $A \cong B_- \bigotimes_k H \bigotimes_k B_+$. The final result in our analysis in this first part, is

Theorem 6. *Every block $\mathcal{O}(\lambda)$ is a highest weight category.*

We need some intermediate results first.

Proposition 11.

(1) *Fix $\lambda, \lambda' \in \mathfrak{h}^*$. Then $\forall l \gg 0$, $\forall V \in \mathcal{O}(\lambda')$, we have $\mathrm{Hom}_A(P(\lambda, l), V) \cong V_\lambda$ as vector spaces.*
(2) *$P(\lambda, l) \in \mathscr{F}(\Delta)$ $\forall \lambda, l$. Moreover, $[P(\lambda, l) : Z(\lambda')] = p(\lambda - \lambda')$ if $\lambda' - \lambda \in \Pi(B_+/B_{+l})$ (otherwise it is zero). Here p is Kostant's function.*
(3) *$P(\lambda) \in \mathscr{F}(\Delta)$. If $[P(\lambda) : Z(\mu)] \neq 0$, then $\mu \geqslant \lambda$.*
(4) *$[P(\lambda) : Z(\lambda)] = 1$.*

Proof.
(1) The proof is similar to a proof in [4].
(2) Look at the analogous proof in [4]. Now that we know the PBW theorem, that proof goes through completely.
(3) Fix $l \gg 0$ so that $\mathrm{Hom}_A(P(\lambda, l), V) = V_\lambda$ for all $V \in \mathcal{O}(\lambda)$. Now suppose $P(\lambda, l) = \bigoplus_{\lambda'} N(\lambda')$. Since $\mathrm{Hom}_A(P(\lambda, l), -)$ is exact in $\mathcal{O}(\lambda)$, hence so is $\mathrm{Hom}_A(N(\lambda), -)$. Thus $N(\lambda)$ is projective in $\mathcal{O}(\lambda)$, so say $N(\lambda) = \bigoplus_{\mu \in S(\lambda)} n_\mu P(\mu)$.
Note that $\dim_k(\mathrm{Hom}_A(P(\lambda, l), V(\lambda))) = \dim_k(V(\lambda)_\lambda) = 1$, so $\dim_k(\mathrm{Hom}_A(N(\lambda), V(\lambda))) = 1$ (because $\mathcal{O} = \bigoplus \mathcal{O}(\lambda)$). Applying Proposition 10, we get $n_\lambda = 1$. Thus $P(\lambda)$ is a direct summand of $P(\lambda, l)$, and $P(\lambda, l)$ has a p-filtration, so by Proposition 6, $P(\lambda) \in \mathscr{F}(\Delta)$.
Finally, $P(\lambda)$ is a summand of $P(\lambda, l)$, hence *for all* μ we have $[P(\lambda) : Z(\mu)] \leqslant [P(\lambda, l) : Z(\mu)] \leqslant p(\lambda - \mu)$. Therefore $[P(\lambda) : Z(\mu)] \neq 0$ only if $\lambda \leqslant \mu$.
(4) Suppose $P(\lambda) \supset M_1 \supset \cdots$ is a p-filtration, with $P(\lambda)/M_1 \cong Z(\mu)$ for some $\mu \geqslant \lambda$. Then $P(\lambda) \twoheadrightarrow P(\lambda)/M_1 = Z(\mu) \twoheadrightarrow Z(\mu)/Y(\mu) = V(\mu)$ simple. Hence the composite has kernel $\mathrm{rad}(P(\lambda))$, whence $V(\mu) = V(\lambda)$, or $\mu = \lambda$. Hence $[P(\lambda) : Z(\lambda)] > 0$. Also, $[P(\lambda) : Z(\lambda)] \leqslant [P(\lambda, l) : Z(\lambda)] = p(\lambda - \lambda) = 1$, so we are done. $\square$

Proof of the theorem. Dualize the p-filtration for $P(\lambda)$ (in the last part above) to get a q-filtration for $I(\lambda)$. Clearly, $P(\lambda)/M_1 = Z(\lambda)$ means that the filtration looks like

$0 \subset A(\lambda) \subset \cdots \subset I(\lambda)$. The weights are suitably ordered, hence $\mathcal{O}(\lambda)$ is a highest weight category. $\square$

From above, we conclude that every projective module in $\mathcal{O}$ has a p-filtration, since each $P(\lambda)$ does. Also, since $\mathcal{O}(\lambda)$ is a highest weight category, we have *Brauer–Humphreys/BGG Reciprocity*, which says that $[P(\lambda):Z(\mu)]=[A(\mu):V(\lambda)]=[Z(\mu):V(\lambda)]$. Further, the cohomological dimension of $\mathcal{O}(\lambda)$ is bounded above, hence finite.

There are many more results, especially on Tilting modules and Ringel duality, which are readily found in [7], for instance, and which we do not mention here.

Part 2: The (deformed) symplectic oscillator algebra H_f

In this part, we show that all assumptions in the first part are true for the algebra H_f, which we shall define presently. We prove the PBW theorem for H_f, classify all finite-dimensional simple modules, state the well-known character formulae, and take a closer look at Verma modules. We conclude by producing a counterexample to Weyl's theorem (of complete reducibility) for a special case.

7. Introduction; automorphisms and anti-involutions

We continue to work over an arbitrary field k of characteristic zero.

Consider the Lie algebra $\mathfrak{sp}(2n)$. The Cartan subalgebra $\mathfrak{h}$ has basis $h_i = e_{ii} - e_{i+n,i+n}$ $(1 \leqslant i \leqslant n)$, though these do not correspond to the simple roots of $\mathfrak{sp}(2n)$. Now define the functionals $\eta_i \in \mathfrak{h}^*$ by $\eta_i(h_j) = \delta_{ij}$. Then the roots and root vectors are:

$u_{jk} = e_{jk} - e_{k+n,j+n} : 1 \leqslant j \neq k \leqslant n$ (root $= \eta_j - \eta_k$),
$v_{jk} = e_{j,k+n} + e_{k,j+n} : 1 \leqslant j < k \leqslant n$ (root $= \eta_j + \eta_k$),
$w_{jk} = e_{j+n,k} + e_{k+n,j} : 1 \leqslant j < k \leqslant n$ (root $= -\eta_j - \eta_k$),
$e_j = e_{j,j+n} : 1 \leqslant j \leqslant n$ (root $= 2\eta_j$),
$f_j = e_{j+n,j} : 1 \leqslant j \leqslant n$ (root $= -2\eta_j$).

The simple roots are given by $\{\eta_i - \eta_{i+1} : 0 < i < n\}$ and $2\eta_n$.

Remark. It is easier for calculations to use $e_j = 2e_{j,j+n}$ and $f_j = 2e_{j+n,j}$, because then $h_j = u_{jj}, e_j = v_{jj}, f_j = w_{jj}$.

Let $B = k[X_1, \ldots, X_n]$, and consider a $2n$-dimensional k-vector space $V \subset \mathrm{End}(B)$, with basis given by $\{X_i = \text{multiplication by } X_i : 1 \leqslant i \leqslant n,\ Y_i = (\partial/\partial X_i) : 1 \leqslant i \leqslant n\}$. Then the subalgebra generated by V in $\mathrm{End}(B)$ is called the *Weyl algebra* $= A_n$. We now construct the *Weil representation* of $\mathfrak{sp}(2n)$ on B. More precisely, define the map $\varphi : \mathfrak{U}(\mathfrak{sp}(2n)) \to A_n \subset \mathfrak{gl}(B)$ as follows:

$$h_i \mapsto X_iY_i + 1/2, \quad u_{jk} \mapsto X_jY_k, \quad v_{jk} \mapsto -X_jX_k, \quad w_{jk} \mapsto Y_jY_k,$$

$$e_j \mapsto -X_j^2/2, \quad f_j \mapsto Y_j^2/2.$$

Thus we obtain a representation $\varphi_0 : H_0 \to A_n$, where $H_0 = \mathfrak{U}(\mathfrak{sp}(2n)) \ltimes A_n$, and $\varphi_0 = \varphi \ltimes \mathrm{id}$. (It is a *faithful* map of Lie algebras: $\mathfrak{sp}(2n) \to A_n$.) Here H_0 is defined by $Za - aZ = Z(a)$ $(=[\varphi(Z), a])$, where $Z \in \mathfrak{sp}(2n)$, $a \in V$, and $Z(a)$ is the action of Z on a. Thanks to our choice of φ, this also agrees with the natural action of $\mathfrak{sp}(2n)$ on V (i.e. as $2n \times 2n$ matrices, acting on vectors in V).

Note that H_0 arises from the *symplectic oscillator algebra* $\mathfrak{sp}(2n) \ltimes \mathfrak{h}_n$ (relations as above) by: $H_0 = \mathfrak{U}(\mathfrak{sp}(2n) \ltimes \mathfrak{h}_n)/(I-1)$, where I is the central element in (the $(2n+1)$ dimensional *Heisenberg* algebra) $\mathfrak{h}_n$.

We now consider a deformation over $k[T]$ of H_0. For $f \in k[T]$, define $H_f = T(V_0)/\langle R_f \rangle$, where $V_0 = \mathfrak{sp}(2n) \bigoplus V$ and R_f is generated by $Za - aZ = Z(a)$, the usual $\mathfrak{sp}(2n)$ relations, $[X_i, X_j]$, $[Y_i, Y_j]$, and the deformed relations $[Y_i, X_j] - \delta_{ij}(1 + f(\Delta))$. Here, Δ is the quadratic Casimir element in $\mathfrak{sp}(2n)$, acting on A_n via the above map φ, as the scalar $c_\varphi = -(2n^2+n)/16(n+1) \in \mathbb{Q} \subset k$.

Remarks. We can show that $\mathfrak{sp}(2n)$ commutes with all of $[V, V]$, so that the deformation must lie in $\mathfrak{Z}(\mathfrak{U}(\mathfrak{sp}(2n)))$, and for $n = 1$, this is precisely $\mathbb{C}[\Delta]$. This explains the choice of deformed relations. (However, Δ does not commute with all of V.)

We now explicitly describe some automorphisms and an anti-involution of H_f.

Anti-involution: Define $i : V_0 \to V_0$ by sending $X_j \mapsto Y_j, Y_j \mapsto X_j, u_{jk} \mapsto u_{kj}, v_{jk} \mapsto -w_{jk}, w_{jk} \mapsto -v_{jk}$ $\forall j,k$ (as in the remark following the lists of roots and root vectors, a few paragraphs above). This extends to an anti-involution: $T(V_0) \twoheadrightarrow H_f$, defined on monomials by reversing the order, and this map *does* vanish on R_f, as desired. In addition, it takes $\mathfrak{U}(N_+)_\mu$ to $\mathfrak{U}(N_-)_{-\mu}$ for every μ, and acts on $\mathfrak{h}$ as the identity.

Automorphisms/lifts of the Weyl group: Let us now lift the Weyl group to automorphisms of H_f. Let $S = \{u_{jk}, v_{jk}, w_{jk}, X_j, Y_j\}$. Then $\forall a_\alpha \in S \cap (\mathfrak{sp}(2n))_\alpha$, we see that $\tau_{a_\alpha}(b) := \exp(\mathrm{ad}\ a_\alpha)(b)$ is a finite series $\forall b \in S$, if $\alpha \neq 0$. Further, $\tau_\alpha := \tau_{a_\alpha}\tau_{-a_{-\alpha}}\tau_{a_\alpha}$ takes $(V_0)_\mu$ to $(V_0)_{\sigma_\alpha(\mu)}$ for all (simple) roots α. In addition, it also permutes the Cartan subalgebra $\mathfrak{h}$ "appropriately". Thus each τ_α is an algebra automorphism, preserving V_0 and taking $(H_f)_\mu$ to $(H_f)_\nu$, where $\nu = \sigma_\alpha(\mu)$.

Now, we know (cf. [12, Exercise (13.5)]), that the Weyl group $W = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ of $\mathfrak{sp}(2n)$ contains -1. So we can construct an automorphism τ of H_f that restricts to -1 on $\mathfrak{h}$, preserves V_0, and takes each weight space to the corresponding negative weight space.

8. Standard cyclic H_f-modules in the BGG category

Let Φ (resp. Φ_f) be the root system of $\mathfrak{sp}(2n)$ (resp. H_f). Then $\Phi_f = \Phi \coprod \{\eta_i, -\eta_i : 1 \leqslant i \leqslant n\}$, and $\Delta_0 = 1 + f(\Delta)$. We write positive and negative roots as $\Phi_f^+ = \Phi^+ \coprod \{\eta_j\}$ and $\Phi_f^- = -\Phi_f^+$. Similar to [12], we introduce an ordering among the roots as follows: $\lambda \succ_f \mu$ if $\lambda - \mu$ is of the form $(m\eta_n + \sum_{i<n} k_i\alpha_i)$, where $m, k_i \in \mathbb{N}_0$, and $\alpha_i = \eta_i - \eta_{i+1}$ are the first $n-1$ simple roots (as above).

Now define Lie subalgebras $N_+ = [B_+, B_+] \subset B_+ \subset H_f$ as follows: $B_+ = \mathfrak{h} \oplus N_+$ is a Borel subalgebra, and $N_+ = \bigoplus_{i=1}^n kX_i \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{sp}(2n))_\alpha$ is nilpotent. Similarly, we have B_- and N_-. (Note that these are *not* the $B_\pm, N_\pm$ of Section (1) above; rather, those are given here by $\mathfrak{U}(B_\pm), \mathfrak{U}(N_\pm)$.)

We now observe that the "Setup" for the analysis in the first part of this paper is partially valid here. The assumptions in Section 1 are all satisfied. Thus Theorem 1 holds here. Assuming the PBW theorem, we introduce another equivalent condition:

Corollary 2. *Suppose $H_f \cong \mathfrak{U}(N_-) \oplus_k \mathfrak{U}(\mathfrak{h}) \oplus_k \mathfrak{U}(N_+)$. Then all nonzero maps from $Z(\mu)$ to $Z(\lambda)$ are injections.*

The proof uses the fact that $\mathfrak{U}\mathfrak{g}$ is an integral domain for *any* Lie algebra $\mathfrak{g}$ (cf. [6, (2.3.9)]).

Now suppose $V(\lambda)$ is finite dimensional. Since any H_f-module is also a $\mathfrak{sp}(2n)$-module, hence Weyl's theorem applies (cf. [16, Section 7.8]), and $V(\lambda)$ is a direct sum of finitely many $V_C(\mu)$'s, where $V_C(\mu)$ is the irreducible $\mathfrak{sp}(2n)$-module of highest weight μ (which is dominant integral because V has finite dimension). Thus if $V(\lambda)$ is finite dimensional, then $\lambda \in \Lambda^+$. Further, $\Pi(V)$ is saturated (under the action of the Weyl group W of $\mathfrak{sp}(2n)$).

We now come to character theory. W acts naturally on $\mathbb{Z}[\Lambda]$ by $\sigma e(\lambda) = e(\sigma\lambda)$. If $\dim_k(V) < \infty$, then $\dim(V_\mu) = \dim(V_{\sigma(\mu)})$, i.e. $\mathrm{ch}_V \in \mathbb{Z}[\Lambda]^W$. Let us define $\tau_\alpha \in \mathrm{Aut}(V)$ for any finite-dimensional module V. Since all nonzero root vectors in $\mathfrak{sp}(2n)$ act nilpotently on V, we can define τ_α as above. Then $\tau_\alpha \in \mathrm{Aut}(V)$ and $\tau_\alpha : V_\mu \to V_{\sigma_\alpha(\mu)}$ by $\mathfrak{sp}(2n)$-theory. In particular, we again get $\mathrm{ch}_V \in \mathbb{Z}[\Lambda]^W$.

In order to handle infinite-dimensional modules, we redefine the formal character as a function: $\Lambda \to \mathbb{Z}$. Then multiplication becomes convolution. The $e(\mu)$ becomes $\varepsilon_\mu : v \mapsto \delta_{\mu v}$, so $\sigma(\varepsilon_\mu) = \varepsilon_{\sigma\mu}$. The usual definition of the Kostant function now coincides with our previous definition (setting $B_- = \mathfrak{U}(N_-)$). The *Weyl function* q is just $\prod_{\alpha \in \Phi_f^+}(e(\alpha/2) - e(-\alpha/2))$, and we set $\delta = \frac{1}{2}\sum_{\alpha \in \Phi_f^+} \alpha$.

Lemma 3. *Assume the PBW theorem holds. Then*

(1) $p = \mathrm{ch}_{Z(0)}$
(2) $\mathrm{ch}_{Z(\lambda)} = p * \varepsilon_\lambda$
(3) $q * \mathrm{ch}_{Z(\lambda)} = q * (p * \varepsilon_\lambda) = \varepsilon_{\lambda+\delta}$.

The proof is a matter of easy calculation.

9. H_f-modules for $n = 1$

Throughout the rest of this paper, we take $n = 1$. Thus our Lie algebra is $C_1 = \mathfrak{sl}_2 = \mathfrak{sp}(2)$. We denote the generators of H_f by E, F, H, X, Y. The "root system" is $\Phi_f = \{\pm\eta, \ \pm 2\eta\}$, and the Weyl group W is simply S_2. We may also prefer to work with a related group $W' = S_2 \times S_2$, whose action on the weights will be seen later, in Section 16 below.

We write down the generators and relations explicitly here. H_f is generated by X, Y, E, F, H, with E, F, H spanning $\mathfrak{sl}_2$. The other relations are: $[E, X] = [F, Y] = 0$,

$[E,Y]=X$, $[F,X]=Y$. Further, X and Y are weight vectors for $H:[H,X]=X$, $[H,Y]=-Y$. Finally, the deformed relation is $[Y,X]=\Delta_0=1+f(\Delta)$, where Δ is the quadratic Casimir element $\frac{1}{4}(EF+FE+H^2/2)$.

Note that the original symplectic oscillator algebra contains the *oscillator algebra* $\mathscr{A}_0$ (cf. [13]), where $E_+=X$, $E_-=Y$, $H=H$, $\mathscr{E}=I=1$ (where I is the central element in $\mathfrak{h}_1$).

Our main motivation is to prove the PBW theorem, and the remaining "standing assumption" mentioned in Section 4 above (note that all Verma modules are automatically nonzero if PBW holds). However, we will also consider other things—for example, the structure of finite-dimensional modules and Verma modules.

First of all, notice (cf. [12]) that on any standard cyclic $\mathfrak{U}(\mathfrak{sl}_2)$-module, Δ acts by a scalar. Therefore Δ_0 also acts by a scalar, and let us denote this by c_{0r} if the module is of highest weight $r\in k$. Clearly, c_{0r} depends on the polynomial f as well.

We now come to calculations. First of all, observe that $\mathfrak{U}(N_-)=k[Y,F]$ because $YF=FY$. Thus we see that in $Z(r)$, a spanning set for the $(r-m)$-weight space is $Y^m, Y^{m-2}F, \ldots$. Define the constants

$$\alpha_{rm}=\sum_{i=0}^{m-2}(r+1-i)c_{0,r-i} \quad \text{and} \quad d_{r-m}=\alpha_{rm}/(r-m+2)(r-m+3) \tag{1}$$

Of course, to define d_{r-m} we should not have $r=m+2,\ m+3$. Also, we clearly have $m\in\mathbb{N}$ (for $m=1$ we can take the empty sum $=0$).

For the time being, we work only with standard cyclic modules. Consider any $Z(r)\to V=H_f v_r\to 0$, for $r\in k$. We have

Theorem 7. *Let $V=H_f v_r$. Then*

(1) *v_r and $v_{r-1}=Yv_r$ are $\mathfrak{sl}_2$-maximal vectors (i.e. $Ev_r=Ev_{r-1}=0$).*
Now say $t\in r-2-\mathbb{N}_0$. Wherever d_t can be defined, we have R_t and define S_t inductively:

$$Xv_{t+1}=EYv_{t+1}=-\frac{\alpha_{r,r-t}}{t+3}\,v_{t+2}, \tag{$\mathrm{R_t}$}$$

$$v_t\stackrel{\text{def}}{=}Yv_{t+1}+d_tFv_{t+2}. \tag{$\mathrm{S_t}$}$$

For the same values of t, we also have the following:

(2) *$v_t=p_{r-t}(Y,F)v_r$ for some polynomial $p_{r-t}(Y,F)=Y^{r-t}+c_1FY^{r-t-2}+\cdots\in k[Y,F]$ (monic in Y).*

(3) *Say $v\in V_t$. Then $Ev=0$ iff $v\in k\cdot v_t$.*

Remarks.

(1) Thus, if $r\in\mathbb{N}_0$, then the equations are valid until we reach $t=-1$. We can define v_{-1} and calculate Xv_{-1}, but cannot go beyond that. Of course, if $r\notin\mathbb{N}_0$ then we can go on indefinitely.

(2) Suppose $t>-2$ or $t\notin\mathbb{N}_0$. Then we can rewrite (R_t) as

$$Xv_{t+1}=EYv_{t+1}=-(t+2)d_tv_{t+2}. \tag{$\mathrm{R'_t}$}$$

(3) Henceforth, the phrase "where(ver) d_t can be defined" means "where(ver) $t>-2$ if $r\in\mathbb{N}_0$".

Proof of the theorem. This is just inductive calculations. □

Corollary 3. *Suppose* $v_t, v_{t+1} \neq 0$ *for some* t $(t > -2$ *if* $r \in \mathbb{N}_0)$. *Then* v_t *is maximal iff* $\alpha_{r,r-t+1} = 0$.

We will see further below that one implication holds for any $r \in k$, namely, that if v_t is maximal in $Z(r)$, then $\alpha_{r,r-t+1} = 0$.

Corollary 4. *Suppose* $v_t = 0$. *If* v_{t-n} *can be defined for* $n \in \mathbb{N}_0$, *then* $v_{t-n} = 0$.

Corollary 5. *Suppose* V (*as above*) *has another maximal vector* v_t *for some* $t \in r - \mathbb{N}$. *Then a weight vector* v_T *in* $V' = H_f v_t$ (*defined in* V' *by the relation* (S_T) *for some* T, *so that* d_{T-1} *is defined*) *is maximal in* V' *iff it is maximal in* V.

Proof. The proof is, of course, that a maximal vector generates a submodule, and a submodule of a submodule is still a submodule. However, there is a related phenomenon occurring among the α_{rm}'s. The point is that if $H_f v_T \subset H_f v_t \subset H_f v_r = V$ are all submodules of V, then these v's are maximal vectors, and Corollary 3 says that there is a relation among the various α_{rm}'s. In fact, it is easy to show (from definitions) that

$$\alpha_{r,r-T+1} = \alpha_{r,r-t+1} + \alpha_{t,t-T+1} \qquad \square \tag{2}$$

Corollary 6. *Say* $V = V(r)$ *is simple, and* d_t *can be defined for* $t \in r - 2 - \mathbb{N}_0$. *Then* $d_{t-1} = 0$ *iff* $v_t = 0$.

10. General philosophy behind the structure theory

As we shall see, many standard cyclic (resp. Verma, simple) H_f-modules $Z(r) \to V \to 0$, are a direct sum of a progression of standard cyclic (resp. Verma, simple) $\mathfrak{U}(\mathfrak{sl}_2)$-modules $V_{C,t}$ of highest weight $t \in r - \mathbb{N}_0$. (Each module $V_{C,t}$ has multiplicity one as well.)

If this progression terminates, say at $Z_C(t) \to V_{C,t} \to 0$ for some $t = r - n$, then (we show later that) $\alpha_{r,n+1} = 0$. The converse is true, for instance, when $r \notin \mathbb{N}_0$ (as the results and remarks in the previous section suggest), or if V is finite dimensional simple (as we shall see in a later section). But there are counterexamples to a general claim of this kind, which we shall provide below.

The specific equations governing such a direct sum $V = \bigoplus_i V_{C,r-i}$ are the subject of the previous subsection. Very briefly, though, if v_t is the highest weight vector (for $\mathfrak{U}(\mathfrak{sl}_2)$) in $V_{C,t}$, then we see that $E(Xv_t) = X(Ev_t) = 0$, so that Xv_t must be a highest weight vector in $V_{C,t+1}$. Since the highest weight space in each $V_{C,t}$ is one dimensional, there is some scalar a_t so that $Xv_t = a_t v_{t+1}$. And if *this* scalar vanishes, then v_t is H_f-maximal in V.

This is the scalar $\alpha_{r,n}$ (upto a constant).

11. Certain Verma modules are nonzero

We now show that $Z(r)$ is nonzero if $r \notin \mathbb{N}_0$. In fact, we show it to be isomorphic to $\mathfrak{U}(N_-)$, by constructing a standard cyclic module of highest weight r, whose character is $\mathrm{ch}_{\mathfrak{U}(N_-)} * \varepsilon_r$.

Lemma 4. *We work in H_f.*

(1) $[X, F^j Y^i] = -F^j \sum_{l=0}^{i-1} Y^{i-l-1} \Delta_0 Y^l - jF^{j-1} Y^{i+1}$,
(2) $[E, F^j Y^i] = -F^j \sum_{m=0}^{i-2} (i-1-m) Y^{i-2-m} \Delta_0 Y^m + j(r-i-j+1) F^{j-1} Y^i$.

Proof. We show by induction that $[F^j, X] = jF^{j-1}Y$. Now the proof is just small calculations. □

Now fix $r \notin \mathbb{N}_0$. Define a module V with k-basis $\{v_{ij} : i, j \in \mathbb{N}_0\}$. We now define the module structure by: $Yv_{ij} = v_{i+1,j}$, $Fv_{ij} = v_{i,j+1}$, $Hv_{ij} = (r-i-2j)v_{ij}$. For the E- and X-actions, we use the preceding lemma as follows:

We first set $Xv_{00} = Ev_{00} = 0$. From above, $Y^k F^l v_{ij} = v_{i+k,j+l}$, so $YF = FY$ (on all of V). Now we multiply both sides of the equations in the lemma above, by v_{00} on the *right*. The left-hand sides give us Xv_{ij} and Ev_{ij}, respectively. The right-hand sides are calculated inductively, starting from the fact that we set $Xv_{00} = Ev_{00} = 0$. We see that we can define Δv_{ij} inductively, using the above lemma; hence we can also define $\Delta_0 v_{ij}$ using induction on (i, j).

This is how we define Xv_{ij} and Ev_{ij} inductively. Now we need to verify that the module structure is consistent with the relations in H_f. (To start with, it is easy to compute that $Ev_{10} = EYv_{00} = 0$. Similarly, $\Delta_0 v_{00} = c_{0r} v_{00}$ and $\Delta_0 v_{10} = c_{0,r-1} v_{10}$.)

First of all, one sees from above that the E, X, Y, F, H-actions take weight vectors into appropriate weight spaces, so all relations of the form $[H, a_\mu] = \mu(H) a_\mu$ automatically hold. As seen above, $YF = FY$. We now verify the following:

$$[E, Y] = X, \quad [F, X] = Y, \quad [E, F] = H, \quad [Y, X] = \Delta_0.$$

Let us show that $EY - YE = X$; the others are similar (and easy). Note that in the calculations below, the right-hand side quantities are to be (right) multiplied by v_{00}.

$$EYv_{ij} = -F^j \sum_{m=0}^{i-1} (i-m) Y^{i-1-m} \Delta_0 Y^m + j(r-i-j) F^{j-1} Y^{i+1},$$

$$YEv_{ij} = -F^j \sum_{m=0}^{i-2} (i-1-m) Y^{i-1-m} \Delta_0 Y^m + j(r-i-j+1) F^{j-1} Y^{i+1},$$

$$Xv_{ij} = -F^j \sum_{m=0}^{i-1} Y^{i-1-m} \Delta_0 Y^m - jF^{j-1} Y^{i+1}.$$

To verify the last relation, namely $EX = XE$, we now introduce another basis of V.

Lemma 5. *The set* $\{F^j v_{r-n} : j, n \in \mathbb{N}_0\}$ *is a basis for* V, *where* v_{r-n} *is defined in* (S_t).

Proof. Equations (R_t), (S_t) hold for all $t = r - n$ (since $r \notin \mathbb{N}_0$), so define (for all n) $v_t = v_{r-n} = p_n(Y, F)v_r$, where all p_n's are *monic*. This makes a change of basis easy to carry out. □

Remarks. Until now, we have *never* used the relation $EX = XE$. We now define some module relations using the $F^j v_{r-n}$'s. That they hold can be checked from relations (R_t) and (S_t), once again without using $[E, X] = 0$.

$H \cdot F^j v_{r-n} = (r - n - 2j)F^j v_{r-n}$,

$E \cdot F^j v_{r-n} = j(r - n - j + 1)F^{j-1} v_{r-n}$,

$X \cdot F^j v_{r-n} = -jYF^{j-1} v_{r-n} - (r - n + 1)d_{r-n-1}F^j v_{r-n+1}$. (Here, $d_{r-1} = 0$ as above.)

We now verify the remaining relation, namely, $EX = XE$. Note that we are free to use the other relations now, since we showed above that they hold on all of V. We compute $EX(F^j v_{r-n}) = -j(r - n - j + 1)[(j - 1)YF^{j-2} v_{r-n} + (r - n + 1)d_{r-n-1}F^{j-1} v_{r-n+1}] = XE(F^j v_{r-1})$.

We have thus checked all relations, and hence shown that there exists a nonzero standard cyclic module $Z(r) \to V \to 0$ of highest weight $r \notin \mathbb{N}_0$. In fact,

Theorem 8. $0 \neq Z(r) \cong k[Y, F] \ \forall r \notin \mathbb{N}_0$.

12. α_{rm} is a polynomial

We now show that α_{rm} is a polynomial in two variables. Actually we show a more general result, that can be applied to various "polynomials" in our setting. Throughout, by $\deg(f)$ we mean the degree of $1 + f(T)$, because that is what we use in handling Δ_0.

Proposition 12. *Given* $d \in \mathbb{N}_0$, *there exists a polynomial* $g_d \in \mathbb{Q}[T] \subset k[T]$, *of degree* $d + 1$, *so that* $g_d(0) = 0$, *and* $g_d(T) - g_d(T - 1) = T^d$.

Proof. We inductively define

$$g_d(T) = \frac{1}{d+1}\left[(T+1)^{d+1} - 1 - \sum_{i=0}^{d-1}\binom{d+1}{i} g_i(T)\right].$$

The base case is $g_0(T) = T$. Then one checks that g_d is as desired, by induction on d. (In particular, for all $m \in \mathbb{N}_0$, we have $g_d(m) = \sum_{n=1}^{m} n^d$, e.g. $g_1(T) = T(T+1)/2$.) □

Corollary 7. α_{rm} *is a polynomial in* r, m, *of degree* $2\deg(f) + 2$ *in* m, *and degree* $2\deg(f) + 1$ *in* r.

Proof. First of all we find out what c_{0r} actually is—or more precisely, what Δ acts on $\mathfrak{U}(\mathfrak{sl}_2)v_t$ by. So suppose we have $Ev_t = 0$. Then $\Delta = (EF + FE + H^2/2)/4$ acts on v_t by: $(EFv_t + F.0 + H^2 v_t/2)/4 = (tv_t + 0 + t^2 v_t/2)/4 = [(t^2 + 2t)/8]v_t$.

Thus Δ acts on v_t by the scalar $c_t = (t^2 + 2t)/8$. Remember, of course, that t is of the form $r - m$ for some $m \in \mathbb{N}_0$. Now, we see that Δ_0 acts on v_t by $c_{0,r-m} = 1 + f(c_{r-m})$. This is clearly a polynomial in r and m, if we expand out $f(c_{r-m})$ formally.

Now Eq. (1), combined with Proposition 12, says that $\alpha_{r,m}$ is a polynomial in two variables, as required. Also, $1 + f(c_t)$ is of degree $2\deg(f)$ in each of r and t, so Eq. (1) and Proposition 12 tell us that $\deg(\alpha) = 2\deg(f) + 2$ in m, and $2\deg(f) + 1$ in r. $\square$

13. The Poincare–Birkhoff–Witt theorem for H_f

The proof of the PBW theorem below, builds on Section (11) above. We first remark, though, that the PBW theorem (and hence the analysis in Section (11)) can all be proved using the Diamond Lemma (cf. [3]). This (was suggested by Gan to the author, and) is done in detail in future work, with Gan and Guay, in [9], for a similar associative algebra—namely, the q-analog of H_f.

We now show the PBW theorem for H_f. If $\Delta_0 = 0$, then H_f is the universal enveloping algebra of a five-dimensional Lie algebra, so the PBW theorem holds. If not, then to show the PBW theorem, we need the following key lemma.

Lemma 6. *Given $s \in \mathbb{N}_0$, there is a finite subset $T \subset k$ so that if $r \notin T \cup \mathbb{N}_0$, then $X^s v_{r-s} = X^s p_s(Y,F) \neq 0$ in $Z(r)$.*

(Note that since char $k = 0$, hence $\mathbb{Z} \hookrightarrow k$, and therefore $\mathbb{N}_0 \cup \{\text{a finite set}\} \neq k$.)

Proof. If $r \notin \mathbb{N}_0$, then repeatedly applying (R_t) yields

$$\begin{aligned} X^{s-1} v_{r-s} &= X^{s-1} p_s(Y,F) \\ &= [(r-s+2)(r-s+3)\ldots r]^{-1}(-1)^{s-1}[\alpha_{r,s-1}\alpha_{r,s-2}\ldots\alpha_{r,3}]\ v_{r-1}. \end{aligned}$$

The first product of terms is nonzero if we take $r \notin \mathbb{N}_0$, so denote it by $d_0 \neq 0$. Also, $Xv_{r-1} = XYv_r = -\Delta_0 v_r = -c_{0r}v_r$. Therefore,

$$X^s v_{r-s} = X^s p_s(Y,F) = (-1)^s \left(d_0 c_{0r} \prod_{j=3}^{s-1} \alpha_{r,j} \right) v_r.$$

Clearly each term in the product is a polynomial—but this time in r (by Corollary 7), as is $c_{0,r}$ (by definition). Therefore, let us take T to be the set of roots of *all* these polynomials in k. Clearly, if $r \notin \mathbb{N}_0 \cup T$, then the right-hand side does not vanish in $Z(r) \neq 0$, and hence we are done. $\square$

We prove two claims, and then the PBW theorem. As above, we take (N_+) to be the left ideal generated by $N_+ = kX \oplus kE$. But first, we observe that $B_- = \mathfrak{h} \oplus N_- = kH \oplus kY \oplus kF$ is a Lie algebra, so we know the PBW theorem for it. Consequently, the multiplication map: $k[Y,F] \bigotimes_k k[H] \to \mathfrak{U}(B_-)$ is an isomorphism.

Proposition 13. $k[Y,F]k[H] \bigcap (N_+) = 0$.

Proof. Suppose $\exists 0 \neq b \in k[Y,F]k[H] \cap (N_+)$. Now, $b=0$ in every Verma module $Z(r)$, so b_+b is also zero, for every $b_+ \in \mathfrak{U}(N_+) = k[X,E]$.

But we will now produce b_+ and r so that $0 \neq b_+b \in k^\times \cdot \bar{1}$ in $Z(r)$, thus producing a contradiction. Suppose b_- is of the form $\sum_{i,j} Y^i F^j b_{ij}(H) \in k[Y,F]k[H]$. Firstly, we may assume w.l.o.g. that b_- is a weight vector for H, because if not, then we take the lowest weight component to $k^\times \cdot 1$, and then the other components automatically are killed.

So suppose $b_- = \sum_{j=0}^{l} F^j Y^{n-2j} b_j(H)$. Let l' be the largest number so that $b_{l'}$ is nonzero. W.l.o.g. $b_l \neq 0$ (i.e. $l' = l$), so b_l has a finite set of roots S. Also, given l, the above lemma says there exists a finite set T so that if $r \notin \mathbb{N}_0 \cup T$, then $X^{n-2l} v_{r-(n-2l)} \in k^\times v_r = k^\times \cdot \bar{1}$.

So fix $r \notin \mathbb{N}_0 \cup T \cup S$. Then $b_- = \sum_{j=0}^{l} Y^{n-2j} F^j b_j(r)$, and $b_l(r) \neq 0$. We now write b_- as a linear combination

$$b_- = a_0 v_{r-n} + a_2 F v_{r-n+2} + \cdots + a_{2l} F^l v_{r-n+2l},$$

where $a_{2l} = b_l(r) \neq 0$, because $v_{r-n} = p_n(Y,F) v_r$, and the p_n's are monic in Y.

Since the v_t's are $\mathfrak{sl}_2$-maximal, hence by $\mathfrak{sl}_2$-theory, E^l kills all summands but the last one. And since $r \notin \mathbb{N}_0 \cup T \cup S$, hence again by $\mathfrak{sl}_2$-theory (cf. [12, Section 7]), $E^l b_- = E^l(a_{2l} F^l v_{r-n+2l}) = c_0 v_{r-n+2l}$ for some nonzero scalar c_0. But then $X^{n-2l}(E^l b_-) = c_0 X^{n-2l} v_{r-(n-2l)}$, and this is nonzero by the above lemma. Hence we have produced b_+ so that $b_+ b \neq 0$ in $Z(r)$. This is a contradiction to the first paragraph in this proof, and hence we are done. □

Corollary 8. $Z(r) \cong k[Y,F]\ \forall r \in k$.

Proof. Suppose not. Then there is a relation, say of the form $b_- \in k[Y,F] \cap (N_+, (H - r \cdot 1))$. Since the multiplication map: $k[Y,F] \oplus_k k[H] \oplus_k k[X,E] \to H_f$ is onto, hence say $b_- = n_+ + p$, where $n_+ \in (N_+)$, and $p \in k[Y,F]k[H] \setminus k[Y,F]$. Clearly, then, $n_+ = b_- - p \in k[Y,F]k[H] \cap (N_+) = 0$.

Further, p is of the form $p = \sum_i b_{-i} p_i(H - r \cdot 1)$, where each p_i is a polynomial with no constant term, and the b_{-i}'s are linearly independent in $k[Y,F]$. Since we know the PBW theorem for the *Lie algebra* B_-, hence $k[Y,F] \oplus_k k[H] \cong k[Y,F]k[H]$. Thus $p_i = 0\ \forall i$, so $p = 0$, whence $b_- = 0$ as required. □

Finally, we have

Theorem 9. *The PBW theorem holds, i.e.* $\{F^a Y^b H^c X^d E^e : a,b,c,d,e \geqslant 0\}$ *is a k-basis for* H_f.

Proof. Suppose not. Then there is a relation of the form $a = \sum_{i=1}^{l} b_i X^{d_i} E^{e_i} = 0$, where $b_i \in k[Y,F]k[H]$ for each i.

We first find $b_- \in k[Y,F]$ on which exactly *one* of the $X^{d_i}E^{e_i}$'s acts nontrivially. Choose the least e, and among all d_i's, choose the least d, for which $X^d E^e$ has nonzero coefficient. By the above lemma, there exists a finite set T so that $X^d v_{r-d} \neq 0$ in $Z(r)$ if $r \notin \mathbb{N}_0 \cup T$.

Let us now look at $v=F^e v_{r-d}\in k[Y,F]$. Clearly, for $(d',e')\neq(d,e)$, either $e'>e$ (in which case $(X^{d'}E^{e'})(F^e v_{r-d})=c_0(X^{d'}E^{e'-e-1})Ev_{r-d}=0$), or $e'=e$ and $d'>d$ (in which case $(X^{d'}E^e)(F^e v_{r-d})=c_0X^{d'}v_{r-d}=c_0'X^{d'-d-1}Xv_r=0$), for some nonzero $c_0,c_0'\in k$. Thus we see that only X^dE^e acts nontrivially on $v\in Z(r)$, because $(X^dE^e)(F^e v_{r-d})=c_0X^d v_{r-d}=c_0'v_r$ for $c_0,c_0'\in k^\times$, from above. Thus we have found such a $b_-\in k[Y,F]$.

Returning to the PBW theorem, recall that we had a linear combination that was zero: $a=\sum_{i=1}^{l} b_iX^{d_i}E^{e_i}=0$, and w.l.o.g. we assume the special (d_i,e_i) (as above) corresponds to $i=l$. Now suppose that $b_l=\sum_j b_{-j}p_j(H)$, where b_{-j} are linearly independent in $k[Y,F]$, and p_j are nonzero polynomials. Then $\Pi p_j=p\neq 0$, and $k\setminus(\mathbb{N}_0\cup T)$ is infinite, so choose any $r\notin(\mathbb{N}_0\cup T)$, such that $p(r)\neq 0$. Therefore $p_j(r)\neq 0\ \forall j$.

Finally, we have $a=0$, so $0=a\cdot b_-$ (where r is chosen above) $=c_rb_l$ for some nonzero scalar c_r (note that we are working in $Z(r)$ here). Therefore b_l is zero in $Z(r)$, whence $\sum_j p_j(r)b_{-j}=0$. But the b_{-j}'s are linearly independent in $Z(r)\cong k[Y,F]$ (from above), and $p_j(r)\neq 0\ \forall j$ (by choice of r). This is a contradiction, hence such a relation $a=0$ cannot occur in the first place. □

14. Necessary condition for $Z(t)\hookrightarrow Z(r)$

The main result is

Theorem 10.

(1) *If $Z(r)$ has a maximal vector of weight $r-n=t$, then (it is unique upto scalars, and) $\alpha_{r,r-t+1}=0$.*
(2) (Verma's Theorem, *cf.* [15; 6, (7.6.6)]) $\mathrm{Hom}_{H_f}(Z(r'),Z(r))=0$ *or k for general $r,r'\in k$. All nonzero homomorphisms are injective.*

The first part of Verma's theorem is easy to show given the previous part, and the second part follows from Corollary 2. For the first part of the theorem, we need some preliminaries.

Definition. Given $T\in H_f$, denote by $W(r,n,T)$ the set of solutions to $Tv=0$ in $Z(r)_{r-n}$.

Proposition 14. *For all $n\in\mathbb{N}_0$ and $r\in k$, we have*

(1) $\dim_k(W(r,n,X))\leqslant 1$; *it equals* 1 *if n is even.*
(2) $1\leqslant\dim_k(W(r,n,E))\leqslant 2$ *if $r+1\in\mathbb{N}_0$ and $r+1\leqslant n\leqslant 2r+2$; it equals* 1 *otherwise.*

Proof. Both the proofs are similar, so we show (1) now. We know $Z(r)_{r-n}$ is spanned by $Y^n,FY^{n-2},\ldots$. Now, we claim that if $Xv=0$ for nonzero $v\in Z(r)_{r-n}$, then the contribution of Y^n to v is nonzero (i.e. $v=a_0Y^n+a_1FY^{n-2}+\cdots$, where $a_0\neq 0$).

Well, suppose $v=\sum_{i\geqslant s} a_iF^iY^{n-2i}$ for some $s\geqslant 0$, where $a_s\neq 0$. From Lemma 4, we see that $Xv=-sa_sF^{s-1}Y^{n-2s+1}+$ terms of lower degree in Y. Since $a_s\neq 0$, hence $s=0$ as required.

Thus, every $0\neq v\in W(r,n,X)$ is of the form $v=cY^n+$ lower order terms. Now suppose we have two such $0\neq v_i=c_iY^n+l.o.t.\in W(r,n,X)$ (i.e. for $i=1,2$). Then $c_2v_1-c_1v_2$ is also in $W(r,n,X)$, but without any Y^n term. Hence it is zero from above, so that $v_2\in k\cdot v_1$, as required.

Finally, we need to show that if n is even, then such a v exists. Recall the Kostant function p. Now observe that $p(-2n)=p(-2n+1)+1\ \forall n$ (because we have the sets $\{F^0Y^{2n},\dots,F^nY^0\}$ and $\{F^0Y^{2n-1},\dots,F^{n-1}Y\}$). Thus, $X:Z(r)_{r-2n}\to Z(r)_{r-2n+1}$ is a map from one space to another of lesser dimension. Hence it has nontrivial kernel, as required. □

Remarks.

(1) This makes the relation $Xv_t\in kv_{t+1}$ easier to understand: $E(Xv_t)=X(Ev_t)=0$, so Xv_t is in $W(r,r-t-1,E)$.
(2) The above result holds for any $Z(r)\to V\to 0$. In any such V, any maximal vector of a given weight r' (if it exists) is unique upto scalars.

Proposition 15. *We work again in the Verma module $Z(r)$ for any $r\in k$.*

(1) *Δ_0 acts on F^mY^n by $\Delta_0F^mY^n=F^m(c_{0,r-n}Y^n+l.o.t.)\in Z(r)_{r-n-2m}$.*
(2) *If $v\in Z(r)_{r-n}$ satisfies $Xv=0$, then upto scalars we have*

$$v=Y^n-FY^{n-2}\sum_{l=0}^{n-1}c_{0,r-l}+l.o.t.$$

(3) *If $v\in Z(r)_{r-n}$ satisfies $Ev=0$, then upto scalars, v is one of the following*:
  (a) *$v=F^{j+1}v_j$, where $-1\leqslant j\leqslant r,\ r+1\in\mathbb{N}_0$, and $r+1\leqslant n\leqslant 2(r+1)$*
  OR
  (b) *$v=(r+2-n)Y^n+FY^{n-2}\sum_{m=0}^{n-2}(n-1-m)c_{0,r-m}+l.o.t.$*

Remarks.

(1) Here, *l.o.t.* denotes monomials of lower order in Y.
(2) Thus, a necessary condition for $Z(r)$ not to be simple (for general $r\notin k$) is that $\alpha_{r,r-t+1}=0$ for some $t\in r-\mathbb{N}$. Further, if $r\notin\mathbb{N}_0$, then Corollary 3 says that this condition is also sufficient, i.e. the converse to (4) holds as well, if the maximal vector v_t is nonzero.

Proof.

(1) W.l.o.g. $m=0$, because Δ (and hence Δ_0) commutes with F. We now proceed by induction on n. For $n=0$, v_r is maximal, hence (e.g. cf. Corollary 7) $\Delta_0v_r=c_{0r}v_r$. Further, $\Delta_0=1+f(\Delta)$ and hence $\Delta_0\in\mathrm{End}_k(Z(r)_t)$ for any $t\in r-\mathbb{N}_0$.

Thus, $\Delta_0 Y^n$ is a linear combination of Y^n, FY^{n-2}, and lower order terms in Y. Now, $4\Delta = 2FE + [(H^2+2H)/2]$, so $4\Delta Y^n = 2FEY^n + [(H^2+2H)/2]Y^n$. Of course, EY^n is a linear combination of $Y^{n-2-i}\Delta_0 Y^i$ from above, and $\Delta_0 Y^i$ is a linear combination of lower order terms, by induction. So EY^n and hence $2FEY^n$ are l.o.t. in Y.

Thus, $\Delta Y^n = [(r-n)^2 + 2(r-n)]Y^n/8 + l.o.t. = c_{r-n}Y^n + l.o.t.$ (because H acts on $Z(r)_{r-n}$ by $r-n$). Also, we have $\Delta(l.o.t.) = l.o.t.$ by the induction hypothesis, so $\Delta^2 Y^n = c_{r-n}^2 Y^n + l.o.t.$, and so on. Hence $\Delta_0 Y^n = (1+f(\Delta))Y^n = (1+f(c_{r-n}))Y^n + l.o.t. = c_{0,r-n}Y^n + l.o.t.$ as required.

(2) From Lemma 4, $XY^n = -\sum_{l=0}^{n-1} Y^{n-1-l}\Delta_0 Y^l = -Y^{n-1}\sum_{l=0}^{n-1} c_{0,r-l} + l.o.t.$ by what we just proved. Similarly, $XFY^{n-2} = -Y^{n-1} + l.o.t.$, and hence if $Xv = 0$, then v is monic in Y, and it *must* look like $v = Y^n - FY^{n-2}\sum_{l=0}^{n-1} c_{0,r-l} + l.o.t.$, in order that the two highest degree (in Y) terms vanish.

(3) The argument is the same as the one just above; the coefficients are slightly different. □

Proof of Theorem 10. If $v = Y^n + l.o.t. \in Z(r)_{r-n}$ is maximal, then so is $(r-n+2)v$, and then both conditions (the ones in (2) and (3)(b) above) must be satisfied, whence the coefficient of FY^{n-2} is the same in both the forms. Therefore, we have

$$-(r-n+2)\sum_{l=0}^{n-1} c_{0,r-l} = \sum_{m=0}^{n-2}(n-1-m)c_{0,r-m} = \sum_{l=0}^{n-1}(n-1-l)c_{0,r-l}$$

because for $l = n-1$ the summand on the RHS vanishes. Simplifying this, we get $\sum_{l=0}^{n-1}[(r-n+2)+(n-1-l)]c_{0,r-l} = 0$, which by definition means $\alpha_{r,n+1} = \alpha_{r,r-t+1} = 0$ as required. □

Suppose $\Delta_0 \neq 0$. Given $r \in k$, let r_0 be the maximal $t \in r + \mathbb{N}_0$, such that $t = r$ is a root of α_{r_0,r_0-t+1} (this exists because α_{rm} is a polynomial, as in Corollary 7). Define the set $S(r)$ to be the set of roots t of α_{r_0,r_0-t+1}, that are in $r_0 - \mathbb{N}_0$.

We claim that if $\alpha_{t,t-t'+1} = 0$, then $t \in S(r)$ iff $t' \in S(r)$. (Thus, $S(r)$ is the transitive (and symmetric) closure of $\{r\}$, under the relation of "being a root of $\alpha_{t,m}$".) This follows from Eq. (2) (mentioned in the proof of Corollary 5).

Lemma 7. Suppose $\Delta_0 \neq 0$.

(1) *For any* $r \in k$, *the set* $S(r)$ *is finite*, *of size at most* $2\deg(f) + 2$.
(2) *The sets* $S(r)$ *partition* k.

Proof. The first part follows from Corollary 7, and the second part from Eq. (2). □

Warning: The set $S(r)$ need *not* serve the role of the $S(\lambda)$'s of the first part (of this paper), but might split into a disjoint union of sets $S(\lambda)$. As we shall see later, in most cases the $S(r)$'s do serve as $S(\lambda)$'s, though.

15. Finite dimensional simple H_f-modules

Suppose $V = V(r)$ is finite dimensional and simple. Then $r \in \mathbb{N}_0$, and $V = \bigoplus V_C(n)$, as mentioned earlier (or cf. [16, Section 7.8]). (Here, $0 \leqslant n \leqslant r$ for each summand.) Thus any nonzero $\mathfrak{sl}_2$-maximal weight vector in $V(r)$ has non-negative weight. In particular, $v_{-1} = 0$ in $V(r)$.

The highest weight space has $\dim_k(V_r) = 1$, so $[V(r) : V_C(r)] = 1$. Let us use (R_t), (S_t) now. We know $v_{-1} = 0$ in $V(r)$, so let s be the largest integer in $\mathbb{N}_0$ such that $v_{s-1} = 0$ but v_s is nonzero in $V(r)$. Thus, $v_t \neq 0$ if $s \leqslant t \leqslant r$ by Corollary 4. Also, by Corollary 3, we have $\alpha_{r,r-s+2}$ (and hence d_{s-2} if $s > 0$) $= 0$ (but $d_t \neq 0$ $\forall t \in s - 1 + \mathbb{N}_0$). Thus $Yv_s = -d_{s-1}Fv_{s+1}$ etc. Now, Equations (R_t) and (S_t) show us that the subspace $\bigoplus_{i=s}^{r} V_C(i)$ is an H_f-submodule of $V(r)$. Since $V(r)$ is simple, they are equal, and we have just proved.

Theorem 11. *If $V = V(r)$ is finite dimensional, then $r \in \mathbb{N}_0$ and $\exists s \leqslant r \in \mathbb{N}_0$ so that $V = \bigoplus_{i=s}^{r} V_C(i)$. Also, $\alpha_{r,r-s+2} = v_{s-1} = 0$ and $\Pi(V) = \{\pm r, \pm(r-1), \ldots, \pm s\}$ is W-stable. Conversely, if $\exists 0 \leqslant s \leqslant r$ so that $\alpha_{r,r-s+2} = 0$, but $d_t \neq 0$ $\forall s - 2 < t < r - 1$, then $V = \bigoplus_{i=s}^{r} V_C(i)$, where $V_C(i)$ is a simple $\mathfrak{sl}_2$-module with $\mathfrak{sl}_2$-maximal vector v_i.*

Remark. The module structure is completely determined by relations (R_t), (S_t), and $\mathfrak{sl}_2$-theory.

(The Weyl group W acts on $\Pi(V)$ (and V) by permuting $\{\mu, -\mu\}$ (and $\{V_\mu, V_{-\mu}\}$), as seen in the next section.) We say an ideal I of H_f is *primitive* if H_f/I is a simple H_f-module. Define $J(r)$ to be the annihilator of $V(r) = V(r, s)$ in H_f (we still have $r \in \mathbb{N}_0$, of course), and let $Y(r) = \operatorname{rad}(Z(r))$.

Proposition 16. *$J(r)$ is generated by $\{F^{j+1} p_{r-j}(Y, F) = F^{j+1} v_j : s \leqslant j \leqslant r\}$ along with $p_{r-s+1}(Y, F) = v_{s-1}, N_+$, and $(H - r \cdot 1)$. Further, if $j \in \mathbb{N}_0$ then we have $XF^{j+1} v_j = -(j+1)F^j v_{j-1}$.*

Proof. Observe that $J(r)$ definitely contains all these terms because these relations vanish in $V(r, s)$ (where $1 = v_r$). So let these relations generate the (left) ideal I. (Thus $H_f/I \twoheadrightarrow V(r, s) = H_f/J(r) = Z(r)/Y(r)$.) Since $p_{r-s+1} \in I$, we see that every element in H_f/I is of the form $Y^j F^k$ where $j \leqslant r - s$. Then the other relations tell us that $0 \leqslant k \leqslant r - j$. Thus $\dim_k(H_f/I) \leqslant (r+1) + r + \cdots + (s+1)$, and we can easily verify (using Theorem 11) that this *is* $\dim_k(V(r, s))$. Hence we are done.

For the second part, we calculate: $[F^n, X] = nF^{n-1}Y$. Then the rest is (also) calculation. $\square$

16. Characters, and an automorphism

Recall that we have already defined the group ring $\mathbb{Z}[\mathbb{Z}]$, Kostant and Weyl's functions, and the formal character earlier. Since we know that $Z(r) \cong k[Y, F]$ as

$\mathfrak{U}(N_-)$-modules, hence $p(-n) = 1 + \lfloor n/2 \rfloor$ for $n \in \mathbb{N}_0$, if we identify η in Φ_f^+ with 1, and hence 2η with 2. The *Weyl function* q is just $(e(1) - e(-1))(e(1/2) - e(-1/2))$. Also define

$$\omega(r+\delta, s+\delta') = \left[\sum_{\sigma \in W} \mathrm{sn}(\sigma) e\left(\sigma\left(\frac{r+s+2}{2}\right)\right)\right]\left[\sum_{\sigma \in W} \mathrm{sn}(\sigma) e\left(\sigma\left(\frac{r-s+1}{2}\right)\right)\right],$$

where $\delta = 3/2 = 3\eta/2 = \frac{1}{2}\sum_{\alpha \in \Phi_f^+} \alpha$, and $\delta' = \eta/2$. Thus we have $\omega(r+\delta,\ s+\delta') = e(r+\delta) - e(s+\delta') - e(-s-\delta') + e(-r-\delta)$.

Lemma 8.

(1) $\mathrm{ch}_{Z(\lambda)} = e(\lambda)(1+t)(1+2t^2+3t^4+\cdots) = p * \varepsilon_\lambda$, *where* $t = e(-1)$.
(2) $q = \omega(\delta,\ \delta')$.

The proof is a matter of easy calculation.

Digression on W': We now discuss the action of a different group $W' = (\mathbb{Z}/2\mathbb{Z})^2$ on the roots. We work here with Z_2, the ring of *dyadic fractions* $\{a/2^b : a, b \in \mathbb{Z}\}$. First of all, $W' = \{1,\ \sigma_1,\ \sigma_2,\ \sigma_1\sigma_2 = \sigma_2\sigma_1 = -1\}$. Further, it acts on $M = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ by invertible linear maps, i.e. $W' \subset GL_2(\frac{1}{2}\mathbb{Z})$.

To compute the explicit action, let $e_1 = (1,0),\ e_2 = (0,1)$ be a $\frac{1}{2}\mathbb{Z}$-basis for the free $\frac{1}{2}\mathbb{Z}$-module M of rank 2. Then $\sigma_i(e_j) = (-1)^{\delta_{ij}} e_j$ where $i, j \in \{1,2\}$. Further, there is a sign homomorphism sn : $W' \to \{\pm 1\}$, given by $\mathrm{sn}(\sigma) = \det(\sigma) = (-1)^{l(\sigma)}$, l being the *length*. Thus the σ_i's are transpositions, or more accurately, reflections, and $\mathrm{sn}(\sigma_1\sigma_2) = 1$, because $\sigma_1\sigma_2 = (-1) \cdot \mathrm{id}$ on all of M.

We now define a map $\varphi : \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \to \frac{1}{2}\mathbb{Z}$, given by $\varphi(m,n) = m + 2n$. This corresponds to identifying the first coordinate with the coefficient of $\eta =$ root of X, and the second with the coefficient of $2\eta =$ root of E. Thus, the half sum of the roots would be $\delta = \frac{1}{2}\varphi(1,1) = \varphi(\frac{1}{2}, \frac{1}{2})$. Similarly, $\varphi(-1,1) = 2\delta'$.

Given $(m,n) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we draw a "square" of its orbit under W'. In what follows below, we "cut" off a side of the square and expand out the sides in one line. For instance, we have the following map (essentially, we *want* this to hold, in order to write formulae for $V(r,s)$ analogous to the $\mathfrak{sl}_2$-case):

$$\varphi\left(W'\left(\frac{1}{2}, \frac{1}{2}\right)\right) : \left[\delta \xleftrightarrow{\sigma_1} \delta' \xleftrightarrow{\sigma_2} -\delta \xleftrightarrow{\sigma_1} -\delta' \xleftrightarrow{\sigma_2} \delta\right].$$

Identifying δ, η, δ' etc. with numbers in $\mathbb{Z}$, we can write

$$\varphi\left(W'\left(\frac{1}{2}, \frac{1}{2}\right)\right) : \left[3/2 \xleftrightarrow{\sigma_1} 1/2 \xleftrightarrow{\sigma_2} -3/2 \xleftrightarrow{\sigma_1} -1/2 \xleftrightarrow{\sigma_2} 3/2\right]. \tag{3}$$

The orbit of W'—or more precisely, $\varphi \circ W'$—on the roots $\eta = e_1$ and $2\eta = e_2$, is given by

$$\varphi(W'(1,0)) : \left[1 \overset{\sigma_1}{\longleftrightarrow} -1 \overset{\sigma_2}{\longleftrightarrow} -1 \overset{\sigma_1}{\longleftrightarrow} 1 \overset{\sigma_2}{\longleftrightarrow} 1\right], \tag{4}$$

$$\varphi(W'(0,1)) : \left[2\eta \overset{\sigma_1}{\longleftrightarrow} 2\eta \overset{\sigma_2}{\longleftrightarrow} -2\eta \overset{\sigma_1}{\longleftrightarrow} -2\eta \overset{\sigma_2}{\longleftrightarrow} 2\eta\right]. \tag{5}$$

And then we see that $(4)+(5)=(3)+(3)$, which *should* hold, because we defined δ as the half sum of positive roots—and which *does* hold, because the actions of φ and $\sigma \in W'$ are all linear.

Finally, if $V = V(r,s)$ is a simple H_f-module, then we also have

$$\varphi\left(W'\left(\frac{r-s+1}{2}, \frac{r+s+2}{4}\right)\right)$$
$$: \left[r+\delta \overset{\sigma_1}{\longleftrightarrow} s+\delta' \overset{\sigma_2}{\longleftrightarrow} -r-\delta \overset{\sigma_1}{\longleftrightarrow} -s-\delta' \overset{\sigma_2}{\longleftrightarrow} r+\delta\right]. \tag{6}$$

Note that Eq. (3) is a special case of this last Eq. (6), if we take $r = s = 0$. Now denote by ψ the endomorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$, sending (r,s) to $((r-s+1)/2, (r+s+2)/4)$. We can now use this to write the standard character formulae.

Back to characters. We see now that $\omega(r+\delta, s+\delta') = \sum_{\sigma\in W'} \mathrm{sn}(\sigma)e(\varphi\sigma\psi(r,s))$ and $\omega(\delta,\delta') = \sum_{\sigma\in W'} \mathrm{sn}(\sigma)e(\varphi\sigma\psi(0,0))$.

Let us now look at $\mathrm{ch}_{V(r,s)} = \mathrm{ch}_{r,s}$, say, where $V(r,s)$ is *simple*. Theorem 11 says that $\mathrm{ch}_{r,s} = \sum_{i=s}^{r} \mathrm{ch}(V_C(i))$, and so we have (exactly as in $\mathfrak{sl}_2$-theory).

Theorem 12. *Say $V = V(r,s)$ is a simple H_f-module. Then we have*

(1) (*Weyl's character formula*)

$$\omega(\delta,\delta') * \mathrm{ch}_{r,s} = \omega(r+\delta, s+\delta'), \ \textit{or}\ \mathrm{ch}_{r,s} = \frac{\sum_{\sigma\in W'} \mathrm{sn}(\sigma)e(\varphi\sigma\psi(r,s))}{\sum_{\sigma\in W'} \mathrm{sn}(\sigma)e(\varphi\sigma\psi(0,0))}.$$

(2) (*Alternate version of the Weyl character formula*)

$$e(\delta)\mathrm{ch}_{r,s} = \omega(r+\delta, s+\delta') * \mathrm{ch}_{Z(0)} = \sum_{\sigma\in W'} \mathrm{sn}(\sigma)\mathrm{ch}_{Z(\varphi\sigma\psi(r,s))}$$

(3) (*Kostant's multiplicity formula*) *Say $m_r(t) = \dim(V(r,s)_t)$. Then*

$$m_r(t) = (p * \varepsilon_{-\delta} * \omega(r+\delta, s+\delta'))(t) = \sum_{\sigma\in W'} \mathrm{sn}(\sigma)\, p(t+\delta-\varphi\sigma\psi(r,s))$$

(4) (*Weyl's dimension formula*)

$$\deg(r,s)\ (\overset{\text{def}}{=} \dim V(r,s))$$
$$= \lim_{e(1)\to 1} \mathrm{ch}_{r,s} = \frac{(r+s+2)(r-s+1)}{2} = \frac{\psi_1(r,s)\psi_2(r,s)}{\psi_1(0,0)\psi_2(0,0)},$$

where $\psi = (\psi_1, \psi_2)$.

17. Standard cyclic modules for $r \notin \mathbb{N}_0$

Standing Assumption. For the rest of this paper, we assume that $\Delta_0 \neq 0$.

We now examine the structure of standard cyclic modules $Z(r) \to V \to 0$, for various $r \in k$. The easier choice is $r \notin \mathbb{N}_0$. Theorem 7 says that Equations (R_t), (S_t) are valid for all $t \in r-2-\mathbb{N}_0$, so we can define the $\mathfrak{sl}_2$-maximal vectors v_t for all t. Theorem 7 tells us that these span all the $\mathfrak{sl}_2$-maximal vectors.

Hence the only maximal vectors in V are those v_t's for which $\alpha_{r,r-t+1} = d_{t-1} = 0$. (Thus there are finitely many maximal vectors.) Now say W is a submodule of highest weight t for some such t. We claim that $W = Z(t)$. Suppose not, i.e. say W contains a vector of the form $a_1 F^{i_1} v_r + \cdots + a_m F^{i_m} v_{t+1}$ (in addition to $Z(t)$). Repeatedly applying E, we conclude that W contains a vector of weight higher than t, a contradiction. (We use similar arguments in Section 18 below.) Thus there are finitely many submodules, and V has a finite composition series, given by the *distinct* roots of $\alpha_{r,m}$ that are in $r-\mathbb{N}_0$.

Theorem 13. *Suppose $r \notin \mathbb{N}_0$, and $Z(r) \to V \to 0$.*

(1) *The only submodules of V are $H_f v_t = \mathfrak{U}(N_-) v_t$, where $t = r-m+1$ is a root of α_{rm}, i.e. $\alpha_{r,r-t+1} = d_{t-1} = 0$. These are only finitely many.*
(2) *V has a unique composition series with length at most $\deg \alpha_{rt} = 2(\deg(f)+1)$.*
(3) *The composition factors are isomorphic to $Z(t_i)/Y(t_i) = V(t_i)$, one for each root $t_i \in r-\mathbb{N}_0$ and nonzero maximal vector v_{t_i}.*
(4) *Given $r' \in k$, $\operatorname{Hom}_{H_f}(Z(r'), Z(r)) \neq 0$ iff $r' = t_i$ for some i.*
(5) *The primitive ideal here is generated by $v_{t_1} = p_{r-t_1}(Y,F)$ (for the "largest" such t_1).*

18. Standard cyclic modules for $r \in \mathbb{N}_0$

We now consider the case when $r \in \mathbb{N}_0$. Let $r = t_0 > t_1 > \cdots > t_k \geqslant -1$ be all the distinct integers so that v_{t_j} is a maximal vector in $Z(r)$ (i.e. all the distinct roots $(\geqslant -1)$ of $\alpha_{r,r-t+1}$). We define the H_f-submodule $Y(t_i,t_j)$ to be the $\mathfrak{U}(\mathfrak{sl}_2)$-submodule generated by $\{F^{m+1} v_m : t_i \leqslant m \leqslant t_j\}$, and $Z(t_i)$. Clearly, we have $t_i \leqslant t_j$, or $i \geqslant j$, and we also have the obvious inclusions $Y(t_i,t_j) \subset Y(t_{i'},t_{j'})$ iff $t_i \leqslant t_{i'}$ and $t_j \leqslant t_{j'}$.

Now if $V(r) = V(r,s)$ is simple, then $r = t_0$, $s = t_1$. Also, we clearly have $Z(t_i) = Y(t_i,t_i)$ and $Y(t_i) = Y(t_{i+1},t_i)$ is the maximal submodule of $Z(t_i)$. We now classify some submodules of $Z(r) = Z(t_0) = Y(t_0,t_0)$, and show that $Z(r)$ has finite length.

Proposition 17. *$Y(r) = Y(t_1,t_0)$, and every submodule of $Z(r)$ is either of the form $Y(t_l,t_s)$ (for some $k \geqslant l \geqslant s \geqslant 0$), or all its weights are (strictly) below t_k.*

Proof. (a) Suppose V is a submodule. We first show that if $F^{j+1} v_j \in V$ (for some $j \geqslant -1$), then V is of the form $Y(t_l,t_s)$ for some $k \geqslant l \geqslant s \geqslant 0$.

Suppose $F^{j+1} v_j \in V$. Then V also contains $XF^{j+1} v_j = -(j+1) F^j v_{j-1}$ (by Proposition 16), and repeatedly applying X, we conclude that $v_{-1} \in V$. Keep on applying X, to get that v_0, v_1, and so on are in V, until $v_{t_k} \in V$, because this is the first point where we cannot get further ahead (because $d_{t_k-1} = \alpha_{r,r-t_k+1} = 0$). Thus, if v' is a weight vector of highest possible weight x in V, then $x \geqslant t_k \geqslant -1$. Also, $Ev' = 0$, meaning that $v' = v_x$ upto scalar, from part (4) of Theorem 7. Next, $Xv' = Xv_x = 0$, so $d_x = 0$, meaning that $x = t_l$ for some l (by Corollary 3).

Thus, if $F^{j+1}v_j \in V$ for some j, then V contains $Z(t_l)$ as well as the $\mathfrak{U}(\mathfrak{sl}_2)$-span of $F^{j+1}v_j$'s, say for $0 \leqslant j \leqslant m (\leqslant r)$ (m maximal). Again, if $F^{m+1}v_m \in V$, then $XF^{m+1}v_m = -(m+1)F^m v_{m-1} \in V$, and as above, $YF^{m+1}v_m = F^{m+1}(v_{m-1} - d_{m-1}Fv_{m+1}) \in V$. But now, $d_{m-1} = 0$ iff v_m is maximal (by Corollary 3). Thus if v_m is not maximal then $F^{m+2}v_{m+1} \in V$ as well. But m was chosen to be maximal; hence v_m has to be maximal, and $m = t_s$ for some s. Thus we conclude that $Y(t_l, t_s) \subset V$.

If this inclusion is proper, then V contains a linear combination of terms of the form $F^{j+1+m}v_j$ ($m \geqslant 0, j > t_s$) and $F^m v_i$ ($0 \leqslant m \leqslant i, i > t_l$). Since all $F^{j+1}v_j$'s and v_i's are $\mathfrak{sl}_2$-maximal, hence repeatedly applying E gives that a linear combination of $F^{j+1}v_j$'s and v_i's is in V. We now use the H-action to separate all these terms, and we conclude that V contains a term of the form $F^{j+1}v_j$ for $j > t_s$, or v_i for $i > t_l$. This contradicts the maximality of t_s, t_l, hence $V = Y(t_l, t_s)$ as claimed.

(b) Now, if V contains no vector of the form $F^{j+1}v_j$ (for $-1 \leqslant j \leqslant r$), then we claim that V has weight vectors with weights only below t_k. For if not, then V contains a vector in the $\mathfrak{U}(\mathfrak{sl}_2)$-span of higher weight vectors $v_t (t_k \leqslant t \leqslant r)$, which would mean it would contain $F^i v_j$ for some i, j (by similar application of E, H as above), and multiplying by a suitable power of F gives us that $F^{j+1}v_j \in V$ for some j. This is false. $\square$

In general, we know that either $Z(t_k)$ is simple, or $Y(t_k)$ has a maximal vector of highest possible weight t, say, which is $\leqslant -2$. We now find all submodules of $Z(t_k)$, or equivalently, of $Y(t_k)$. (Of course, if $t_k = -1$ then we are already done, because $Z(-1)$ is already known by Theorem 13.) So now $t_k > -1$, and v_t is maximal of highest weight in $Y(t_k)$. Then we have

Proposition 18. $Y(t_k) = Z(t)$ (*and* $t \notin \mathbb{N}_0$).

Proof. The same sort of reasoning, using linear combinations of $F^i Y^j$, is used here. We are looking at $V \subset Y(t_k) \subset Z(t_k)$. So let us assume that $v_x = p(Y, F)v_{t_k} \in V$. Thanks to the H-action, we may assume that v_x is in a single weight space. Again, we know $v_t = p_{t_k - t}(Y, F)v_{t_k}$, so we may say w.l.o.g. that $v_x = p'(Y, F)v_t + F^l q(Y, F)v_{t_k} \in V$, by the Euclidean algorithm (considering all these as polynomials in Y). Here, we can choose q to be monic in Y, and we of course have $l > 0$ and $\deg(q) < t_k - t$ (thereby splitting v_x into the "higher degree" and "$Z(t)$" components).

The key fact to be shown is that $q = 0$. Suppose not, and let v_x be a vector in V of highest weight x for which $q \neq 0$. Now, we see that $Ev_x = Ep' \cdot v_t + EF^l q \cdot v_{t_k} \in V$, and the second term equals $(([E,F]F^{l-1} + \cdots + F^{l-1}[E,F])q + F^l Eq) \cdot v_{t_k} = F^{l-1}(\lambda + FE)q \cdot v_{t_k}$ for some scalar λ. Clearly, this is in the $\mathfrak{U}(\mathfrak{sl}_2)$-span of the vectors $1, Y, \ldots, Y^{t_k - t - 1}$ (inside $Z(t_k)$), by Lemma 4, since q is monic. Hence by maximality of weight of v_x, this second term is zero, because the other term $Ep' \cdot v_t$ is in $H_f v_t$ (as v_t is maximal).

Thus, $EF^l q \cdot v_{t_k} = 0$. But here, $l > 0$, so by Proposition 15 we know that $F^l q \cdot v_{t_k} = F^{j+1} v_j$ for some j. Now look at $X^{-x-1} v_x \in V$. Since $t < -1$, hence the first term of v_x goes to $X^{-x-1} p' \cdot v_t \in Z(t)_{-1} = 0$. Thus $X^{-x-1} v_x = X^{-x-1} F^{j+1} v_j$ and this has weight -1. Thus $X^{-x-1} v_x = c_0 v_{-1}$ for some nonzero scalar c_0, so that $v_{-1} \in V \subset Y(t_k)$. This is impossible, and hence $q = 0$ to start with. $\square$

Let us look at composition series now. We can directly see that $Y(r)/Z(s-1) = Y(t_0)/Z(t_1) = Y(t_1, t_0)/Y(t_1, t_1)$ is simple (by Proposition (17) above), and has highest weight vector $F^{s+1} v_s$. Again, $YF^{j+1} v_j = F^{j+1}(v_{j-1} - d_{j-1} F v_{j+1})$, so we claim inductively that $F^{j+1} v_j$ lies in $\mathfrak{U}(N_-)(F^{s+1} v_s)$. This holds in the base case because $v_{s-1} = 0$ in the simple quotient $V(r, s)$.

Therefore, $Y(r)/Z(s-1)$ is a simple standard cyclic module with highest weight vector $F^{s+1} v_s$, hence of highest weight $-s-2$. So it is isomorphic to $V(-s-2)$. We can now go to "lower" t_i's, and easily calculate the composition factors.

Thus $Z(r)$ has a finite composition series. The set of composition factors is $V(t_0)$, $V(-t_1-3), V(t_1), \ldots, V(-t_k-3), V(t_k)$, and the set of composition factors of $Y(t_k)$ (which is 0 or $Z(t)$ from above). If $Y(t_k) = Z(t)$ or $t_k = -1$ then we know everything about the composition series of $Z(t_k)$, from Theorem 13. Thus, in either case we know the composition factors of $Z(r)$ completely, modulo the following remark.

Remark. The only question that needs answering is: Given r, t_k as above, when is $Z(t_k)$ simple ?

If $r \notin \mathbb{N}_0$ then there is only one Jordan–Holder series, and we know *all* submodules of $Z(r)$. If $r \in \mathbb{N}_0$, then there may be more than one series; one example is

$$\begin{aligned} Z(r) &= Y(t_0, t_0) \supset Y(t_1, t_0) \supset Y(t_1, t_1) \supset \cdots \supset Y(t_k, t_k) \\ &= Z(t_k) \supset Y(t_k) = Z(t')(\supset \cdots), \end{aligned}$$

where $Y(t_k) = Z(t)$ or 0. We have thus shown the analogue of Theorem 13, namely

Theorem 14. *Suppose $r \in \mathbb{N}_0$, and $r = t_0 > t_1 > \cdots > t_k \geqslant -1$ are the various roots (in $\mathbb{Z}$) of $\alpha_{r,r-t+1}$.*

(1) *The submodules of $Z(r)$ with highest weight vector of weight $\geqslant -1$ are of the form $Y(t_i, t_j)$.*
(2) *If $t_k > -1$, then either $Z(t_k)$ is simple, or $Y(t_k)$ has a maximal vector of (highest) weight $t < -1$, whence $Y(t_k) = Z(t)$. In this case, or if $t_k = -1$, we know the rest of the submodules from Theorem* 13.
(3) *$Z(r)$ has a finite composition series, of length at most* $4(\deg(f) + 1)$.
(4) *The composition factors are simple modules $V(\lambda)$ with highest weights $\{t_i, -t_{i+1} - 3 : 0 \leqslant i \leqslant k-1\}$ and t_k if $Z(t_k)$ is simple. If $Y(t_k) = Z(t)$, then we add the composition factors of $Z(t)$ to this. Each simple module occurs with multiplicity* 1 *or* 2.

Thus, we can find all simple modules and primitive ideals in this case. We can make similar claims for any $Z(r) \to V \to 0$ (where $r \in \mathbb{N}_0$). Some of the multiplicities may be 2, as we shall see below.

19. The (finite) sets $S(r)$ satisfy all the assumptions

We are now ready to show that all the assumptions (and hence the analysis) in the first part of the paper, hold in the case of H_f.

Lemma 9. *Every Verma module $Z(r)$ has finite length, so $\mathcal{O} = \mathcal{O}_{\mathbb{N}}$.*

Proof. This follows from Lemmas 2 and 7. □

Thus, the assumptions and results of Theorem 2 hold in this case. Therefore every module in $\mathcal{O}$ has an SC-filtration, is of finite length, and $\mathcal{O}$ is an abelian category that is self-dual as well.

Theorem 15. *If $Z(r)$ has a simple subquotient $V(t)$, then $S(r) = S(t)$.*

Proof. This follows from Theorems 13 and 14, since we now explicitly know what composition factors any given Verma module can have. □

Remarks. Thus the $S(r)$'s decompose into a disjoint union of subsets, each of which is finite, and plays the role of the $S(\lambda)$'s of the first part of this paper. (We shall see below that in most cases the $S(r)$'s are irreducible—and hence of the form $S(\lambda)$.)

Over here, just as in the first part, we do not have the classical notion of blocks. However, we can construct blocks as in the first part (using the connected components of the $S(r)$'s), because all the assumptions now hold. We define the block $\mathcal{O}(r)$ to consist of all $M \in \mathcal{O}$, all of whose simple subquotients are of the form $V(t)$ for some $t \in S(r)$.

Now all the results mentioned above hold, and we have enough projectives, progenerators, and BGG reciprocity in the highest weight category $\mathcal{O}(r)$. We also have $\mathcal{O} = \bigoplus \mathcal{O}(r)$.

20. More on the roots of α_{rt}

We actually know more about the roots of α_{rt}, from the following proposition.

Proposition 19.

(1) *For all $r \in k$, $c_r = c_{-r-2}$, and hence $c_{0r} = c_{0,-r-2}$.*
(2) $\alpha_{r,2r+4} = 0$ *if $r + 1 \in \mathbb{N}_0$.*

(3) *Suppose* $r+1 \in \mathbb{N}_0$. *Then* $Z(r)_{-2}$ *has a maximal vector iff* $\alpha_{r,r+2} = 0$, *iff* $Z(r)_{-1}$ *has a maximal vector*.

(4) *If* $r \notin \mathbb{N}_0$ *then the roots of* α_{rt} *in* $r - \mathbb{N}_0$ *are finitely many, as seen above. If* $r \in \mathbb{N}_0$, *then let* r_0 *be maximal in* $S(r)$. *Suppose* $r_0 = t_0 > \cdots > t_k \geqslant -1$ *are all roots of* α_{r_0,r_0-t+1} *in* $r_0 - \mathbb{N}_0 \cap \mathbb{N}_0 - 1$. *Then the roots of* $\alpha_{r,r-t+1}$ *in* $r - \mathbb{N}_0$ *are all* t_j*'s less than* r, *and* $\{-t_j - 3 : 0 \leqslant j \leqslant k\}$.

(5) *The length of any Verma module* $Z(r)$ *is at most* $3\deg(f) + 4$.

Remarks. If $Z(r)_{-1}$ has a maximal vector ($r \in \mathbb{N}_0$) then $\alpha_{r,r+2} = 0$ and from part (3) above we see that $Z(r)_{-2}$ also has a maximal vector. In this case, Corollary 5 seems to, but *does not* imply, that $\mathfrak{U}(N_-)v_{-2} \hookrightarrow \mathfrak{U}(N_-)v_{-1} \hookrightarrow Z(r)$. It may happen, actually, that $\mathfrak{U}(N_-)v_{-2} \subset Z(r) \supset \mathfrak{U}(N_-)v_{-1}$, but $\mathfrak{U}(N_-)v_{-2} \not\subset \mathfrak{U}(N_-)v_{-1}$. The reason this does not go through, is that d_{-3} is not defined.

Also note that not all multiplicities are zero; in particular, if r_0 is maximal in $S(r)$, then every single $V(t)$ (for $t \in S(r)$), except at most for $V(-r_0-3)$, is a subquotient of $Z(r_0)$. Further, part (5) holds for any $Z(r) \to V \to 0$, and is a better estimate than above.

Next, we observe that if a block $S(r) \subset \mathbb{Z}$ has size 2, then it may not be irreducible, as in the original definition of $S(\lambda)$ (in the general case) ! In this case, we work with each element as a block by itself. But in all other cases, each set $S(r)$ is a block by itself (i.e. “irreducible”, as in the first part). This follows from the remarks above, and Theorems 13 and 14.

Finally, observe that if D is the unipotent decomposition matrix, then each entry of D is 0, 1 or 2, as we saw in the separate cases $r \in \mathbb{N}_0$ and $r \notin \mathbb{N}_0$ above.

Proof. (1), (2) and (4) are calculations. As for (3), one way is clear, by Theorem 10. Conversely, suppose $\alpha_{r,r+2} = 0$. Then we can verify that $v_{-2} = Yv_{-1} - c_{0,-1}Fv_0$ is indeed a maximal vector.

(5) For $r \notin \mathbb{N}_0$ this is clear from Theorem 13. For $r \in \mathbb{N}_0$ we recall the structure of $Z(r)$. We know from the previous part, that $n_+ \geqslant k$. Here, we define n_+ to be the number of roots of α_{rt} (out of a total of $2k+2$ roots, as given), that are in $\mathbb{N}_0$.

Thus the number of negative integer roots n_- is at most $k+2$. There are at most two simple subquotients (in $Y(-3-t)$ and then in $Z(t)$, as earlier) for each of these, and one simple subquotient for each positive root.

Hence the total number of terms in a composition series is at most $2n_- + n_+ = (n_- + n_+) + n_- \leqslant (2k+2) + (k+2) = 3k+4$. But $2k+2 \leqslant 2\deg(f)+2$ by Corollary (7), so $k \leqslant \deg(f)$, whence the length of a composition series is $\leqslant 3k+4 \leqslant 3\deg(f)+4$, as claimed. $\square$

Remarks. It remains to find out the composition series of a Verma module for the case $r \in \mathbb{N}_0$, or equivalently, the composition series for $Z(t_k)$ in this case. This would lead to a complete knowledge of all multiplicities $[Z(\lambda) : V(\mu)]$. However, we do not know the answer to this question.

One guess would be that $Z(t) \hookrightarrow Z(r)$ iff $\alpha_{r,r-t+1} = 0$, since one implication holds in general, and the other holds as well, if $r \notin \mathbb{N}_0$. However, this converse implication is *false* for $r \in \mathbb{N}_0$. For example, setting $g(T)=1+f(T)$, direct calculations yield that when $t_k=-1$, $Z(-2) \hookrightarrow Z(-1)$ iff $c_{0,-1}=g(-1/8)=0$. Similarly, when $t_k=0$, $Z(-3) \hookrightarrow Z(0)$ iff $g(0)(g'(0)/2+g(-1/8))=0$, and this is not true for general g (e.g. $g=1$, or $f=0$).

21. Weyl's theorem fails, multiplicities may be 2, and more

We now look at a specific module $Z(0)$. Suppose f has the property that $c_{00}=c_{0,-1}=0$. Then $Z(0)$ has maximal vectors $v_0, v_{-1}, v_{-2}, v_{-3}$, and $v_i = Y^i v_0$ for each of these.

Observe that in general, we cannot obtain a resolution for $V=V(r,s)$ in terms of the $Z(\lambda)$'s. In any such resolution, the first term would be $Z(r) \twoheadrightarrow V(r,s)$. We then need some μ so that $Z(\mu) \twoheadrightarrow Y(r)$. But this is not true in general: look at the above example $V=Z(0)$. Clearly, $Z(0) \twoheadrightarrow V(0,0)$ has kernel $Y(0)=(Y,F)$. Clearly, if $\varphi : Z(\mu) \twoheadrightarrow Y(0)$, then $v_\mu \mapsto Y$ (for if it maps to zero, then $\varphi=0$). But then we see that $F \notin \mathrm{im}(\varphi)$.

Also, observe that the multiplicities $[Z(r):V(r')]$ are not 0 or 1 in general: in the above example, we see that $[Z(0):V(-2)]=2$. This is because we have the series $Z(0) \supset Y(0)=(F,Y) \supset Z(-1)=(Y) \supset Y(-1)=Z(-2)=(Y^2) \supset Y(-2)=Z(-3)=(Y^3) \supset Y(-3) \supset \cdots$, and the subquotients are $V(0), V(-2), V(-1), V(-2), V(-3), \ldots$.

Finally, we provide a counterexample to Weyl's theorem—namely, a (finite dimensional) H_f-module M and a submodule N in it that has no complement. Take $M=V(1,0) \supset V(0,0)=N$, i.e. $M=H_f/I$, where the left ideal I is generated by $(H-1)$, E, X, Y^2, FY, F^2. In other words, $M=kw_1 \oplus kw_0 \oplus kw_{-1}$, and $N=kw_0$, with module relations as follows:

$$Ew_1 = Xw_1 = 0; \quad Fw_{-1} = Yw_{-1} = 0;$$

$$Fw_1 = w_{-1}, \; Ew_{-1} = w_1; \quad Yw_1 = w_0, \; Xw_{-1} = -w_0$$

and $Xw_0 = Yw_0 = Hw_0 = Ew_0 = Fw_0 = 0$ (i.e. w_0 is killed by X, Y, E, F, H).

It can be checked that this is a valid H_f-module structure on M, if we have $c_{00}=c_{01}=0$. However, it is obvious that kw_0 is a submodule (with a trivial module structure). Any complement must contain w_1+lower weight vectors, but when we apply Y to this, we get w_0. Thus w_0 lies in the submodule and in its complement; a contradiction. Hence there does not exist a complement to kw_0 in M, and Weyl's theorem fails for this case.

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Appendix A. Algebraic preliminaries

Throughout, R denotes a ring, and $\mathcal{O}$ denotes an abelian subcategory of R-mod.

Proposition A.1. *If* $0 \to A \oplus B' \to C \to B'' \to 0$ *in* $\mathcal{O}$, *and* $\operatorname{Ext}^1_{\mathcal{O}}(B'', A) = 0$, *then* $C = A \oplus B$, *where* $0 \to B' \to B \to B'' \to 0$ *in* $\mathcal{O}$.

Proof. Apply $\operatorname{Hom}_{\mathcal{O}}(B'', -)$ to the s.e.s. $0 \to B' \to B' \oplus A \to A \to 0$. Then our result follows by considering the long exact sequence of $\operatorname{Ext}_{\mathcal{O}}$'s. □

Proposition A.2. *Suppose* R *is a* k*-algebra, where* k *is a field, and say we have an exact contravariant duality functor* $F : \mathcal{O} \to \mathcal{O}$ (*i.e.* $F(M) \subset \operatorname{Hom}_k(M, k)$, $F(F(M)) = M$). *Then* $F : \operatorname{Ext}^1_{\mathcal{O}}(M'', M') \to \operatorname{Ext}^1_{\mathcal{O}}(F(M'), F(M''))$ *is an isomorphism of* k*-vector spaces.*

The proof more or less follows from the way we define the vector space operations; they use pullbacks, push-forwards, and element chasing in commutative diagrams, e.g. cf. [8].

Setup: Now suppose also that $\mathcal{O}$ is finite length, and a full subcategory of R-mod. Let $\mathscr{P}$ denote all indecomposable projective objects in $\mathcal{O}$, and let $\mathscr{S}$ denote all simple objects. (Thus Fitting's Lemma holds.)

Theorem A.1.

(1) *Every object* P *in* $\mathscr{P}$ *has a unique maximal sub-object* ($\operatorname{rad}(P)$). P *is the projective cover of* $P/\operatorname{rad}(P) \in \mathscr{S}$.
(2) *The map* $F : \mathscr{P} \to \mathscr{S}$ *given by* $F(P) = P/\operatorname{rad}(P)$ *is one-one. If enough projectives exist in* $\mathcal{O}$, *then* F *is a bijection.*

Theorem A.2. *Suppose now that enough projectives exist in* $\mathcal{O}$, *and* $\mathscr{P}$ *is finite.*

(1) $Q = \bigoplus_{P \in \mathscr{P}} n_P P$ *is a progenerator for* $\mathcal{O}$, *as long as all* $n_P \in \mathbb{N}$.
(2) *Set* $B = \operatorname{Hom}_{\mathcal{O}}(Q, Q)$. *Then* B *is unique upto Morita equivalence, and the functor* $D = \operatorname{Hom}_{\mathcal{O}}(Q, -)$ *is an equivalence between* $\mathcal{O}$ *and* $(\text{mod-}B)^{fg}$ (*i.e. finitely generated right* B*-modules*).
(3) D *and* $E = Q \oplus_B$*—are inverse equivalences between* $\mathcal{O}$ *and* $(\text{mod-}B)^{fg}$.

(Part (2) of Theorem (A.2) is from [1, p. 55].)

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