Extraconnectivity of graphs with large girth

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Abstract
Following Harary, the conditional connectivity (edge-connectivity) of a graph with respect to a given graph-theoretic property is the minimum cardinality of a set of vertices (edges), if any, whose deletion disconnects the graph and every remaining component has such a property. We study the case in which all these components are different from a tree whose order is not greater than \(n\). For instance, the recently studied superconnectivity of a maximally connected graph corresponds to this conditional connectivity for \(n = 1\). For other values of \(n\), some sufficient conditions for a graph to have the maximum possible conditional connectivity are given.

1. Introduction

The graphs considered in this paper are simple, i.e. without loops or multiple edges. Given a graph \(G\), let \(V = V(G)\) (resp. \(E = E(G)\)) denote the vertex set (resp. edge set) of \(G\). Use \(\delta = \delta(G)\) to denote the minimum degree of the vertices of \(G\). If \(x \in V\), \(\Gamma(x)\) is the set of vertices adjacent to \(x\). For any pair of vertices \(x, y \in V\), a path \(x_0x_1x_2 \ldots x_{n-1}y\) from \(x\) to \(y\), with not necessarily different vertices, is called an \(x\rightarrow y\) path. Its length \(n\) is denoted by \(|x\rightarrow y|\). The distance between two vertices \(x\) and \(y\) is denoted by \(d(x, y)\) and \(D = D(G) = \max \{d(x, y) : x, y \in V\}\) stands for the diameter of \(G\). The girth \(g = g(G)\) is the length of a shortest cycle in \(G\). If \(F \subseteq V\), the distance between \(x\) and \(F\), \(d(x, F)\), is the minimum over all the distances \(d(x, f), f \in F\). Given \(F \subseteq V\) and \(x \in V - F\), \(C(x)\) is the component of \(G - F\) to which \(x\) belongs. Of course, if \(F\) is not a disconnecting set, then \(C(x) = G - F\). The parameters \(\kappa = \kappa(G)\) and \(\lambda = \lambda(G)\) refer, respectively, to the connectivity and the edge-connectivity of \(G\). It is well known that the connectivity, the edge-connectivity and the minimum degree are related by the following inequalities (see [10]): \(\kappa(G) \leq \lambda(G) \leq \delta(G)\). For other standard graph-theoretic terms not defined in this paper, see [3].

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A connected graph with diameter \( D \) is said to be \( l \)-geodetic, for some \( 1 \leq l \leq D \), if any two vertices are joined by at most one path of length less than or equal to \( l \). If \( l = D \), \( G \) is called strongly geodetic; see [2, 7]. Note that, if \( G \) has girth \( g \), then \( G \) is \( l \)-geodetic, with \( l = \lfloor (g - 1)/2 \rfloor \). Reciprocally, if \( G \) is \( l \)-geodetic, then its girth \( g \) is at least \( 2l + 1 \). If \( G \) is a directed graph, an analogous parameter \( l \) can be defined. It has been considered by the authors in order to characterize maximally connected digraphs; see [4].

A sufficient condition for an \( l \)-geodetic graph \( G \) to have maximum connectivity (edge-connectivity) can be formulated in terms of \( l \) and the diameter \( D \); see [4, 8, 9].

**Theorem 1.1.** Let \( G \) be an \( l \)-geodetic graph with minimum degree \( \delta \), diameter \( D \) and connectivities \( \lambda \) and \( \kappa \). Then

\[
\kappa = \delta \quad \text{if} \quad D \leq 2l - 1,
\]
\[
\lambda = \delta \quad \text{if} \quad D \leq 2l.
\]

If \( G \) is a maximally connected graph, i.e. \( \kappa = \delta \), the set of vertices adjacent to a vertex \( x \) with degree \( \delta \) is a trivial minimum order disconnecting set. Following the definition given in [1], it is said that \( G \) is super-\( \kappa \) if every minimum disconnecting set is trivial. Analogously, \( G \) is said to be super-\( \lambda \) if all its minimum edge-disconnecting sets are trivial. A nontrivial set of vertices or edges refers to a vertex or edge set that does not contain a trivial disconnecting set. The following theorem is proved by the authors in [5].

**Theorem 1.2.** Let \( G = (V, E) \) be an \( l \)-geodetic graph with minimum degree \( \delta \) and diameter \( D \). Let \( F \subseteq V \) and \( A \subseteq E \), \(|F|, |A| \leq 2\delta - 3\), be nontrivial sets. Then

\[
G - F \text{ is connected} \quad \text{if} \quad D \leq 2l - 2,
\]
\[
G - A \text{ is connected} \quad \text{if} \quad D \leq 2l - 1.
\]

By the above theorem, note that \( G \) is super-\( \kappa \) if \( D \leq 2l - 2 \) and \( G \) is super-\( \lambda \) if \( D \leq 2l - 1 \).

2. The conditional connectivities \( \kappa(n) \) and \( \lambda(n) \)

Given a graph \( G \) and a graph-theoretic property \( \mathcal{P} \), let \( \kappa(G; \mathcal{P}) \) be the minimum cardinality of a set of vertices, if any, whose deletion disconnects the graph and every remaining component has property \( \mathcal{P} \). Following Harary [6], \( \kappa(G; \mathcal{P}) \) is called the conditional connectivity of \( G \) with respect to \( \mathcal{P} \). Let \( \mathcal{P}_n \) be the property that every component is not a tree with number of vertices \( k \leq n \). In this paper the conditional connectivities \( \kappa(n) = \kappa(G; \mathcal{P}_n), n \geq 0 \), are studied. In what follows it is supposed that, for the graphs considered, such a \( \kappa(n) \) exists.

Note that if \( G \) is not a complete graph, then \( \kappa(0) \) corresponds to the connectivity \( \kappa \). So, \( \kappa(0) \leq \delta \) and, by Theorem 1.1, \( D \leq 2l - 1 \) is a sufficient condition for \( G \) to be
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maximally connected, i.e. \( \kappa(0) = \delta \). Moreover, the conditional connectivity \( \kappa(1) \) is the minimum number of vertices whose deletion disconnects the graph and the remaining components are different from an isolated vertex. Thus, \( \kappa(1) \) measures the superconnectivity of the graph. Clearly, if \( G \) contains an edge with endvertices \( x, y \) of degree \( \delta \) and \( \Gamma(x) \cap \Gamma(y) = \emptyset \), then the set \( F = \Gamma(x) \cup \Gamma(y) - \{ x, y \} \) could be an example of a disconnecting set with \( 2\delta - 2 \) vertices satisfying the above conditions (since \( xy \) is a component of \( G - F \)). Thus, for such a graph \( G, \kappa(1) \leq 2\delta - 2 \) and, by the results given in Theorem 1.2, \( D \leq 2l - 2 \) is a sufficient condition for \( \kappa(1) = 2\delta - 2 \).

More generally, suppose that \( G \) contains as a subgraph a tree \( T_k \) with order \( k \). Let \( H \) be the set of vertices of \( G - T \) that are adjacent to vertices of \( T \). As \( T \) has \( k \) vertices and \( k - 1 \) edges, it is easily seen that if all the vertices of \( T \) have degree \( \delta \) (in \( G \)), then the number of vertices in \( H \) is at most \( k\delta - 2(k - 1) \). Fixing the positive integer \( n \), consider a tree \( T_{n+1} \) contained in \( G \) such that all its vertices also have degree \( \delta \). Clearly, \( T_{n+1} \) is a component of \( G - H_{n+1} \) and, thus, by the considerations given above about \( |H_k| \), the conditional connectivity \( \kappa(n) \) is, for such \( G \), at most \( (n + 1)\delta - 2n \). In this section, a sufficient condition for \( \kappa(n) \) to be maximum is derived. This condition relates the parameters \( l \) and \( D \). It is assumed that \( \delta > 2 \) and \( l > n \), i.e. \( g \geq 2n + 3 \). Moreover, a set \( F \subseteq V(G) \) will be called nontrivial if \( F \) does not contain any \( H_k \) for \( 1 \leq k \leq n \). Note that for \( n = 1 \) this definition agrees with that given in Section 1.

Lemma 2.1. Let \( G \) be \( l \)-geodetic and \( n \in \mathbb{Z}^+ \). If \( F \subseteq V, |F| < (n + 1)\delta - 2n \), is nontrivial and \( x \in V - F \), then there exists a vertex \( z \in C(x) \) such that \( d(z, F) \geq l - n \).

Proof. As \( F \) is nontrivial, \( C(x) \) is not a tree with order \( k \leq n \). Then \( C(x) \) must have at least \( n + 1 \) vertices; otherwise, \( C(x) \) would contain a cycle of length at most \( n \) and \( g(G) \leq n \), contradicting \( l > n \).

Let \( z \in C(x) \) be a vertex such that \( d(z, F) = r \) is maximum. We will prove that \( r \geq l - n \) and, thus, vertex \( z \) satisfies the lemma. As \( C(x) \) has more than \( n \) vertices, vertex \( z \) belongs to a tree \( T_{n+1} \) with order \( n + 1 \) contained in \( C(x) \). Now the idea is to obtain from \( T_{n+1} \) another tree containing \( n + 1 \) vertices which are as far away from \( F \) as possible in the sense to be precised.

Use \( U \) to denote the set of vertices of \( T_{n+1} \). For every \( t \in U \) consider a path \( t_0, t_1, \ldots, t_{s-1}, t_s, s \geq 1 \), \( t_0 = t, t_1 \notin U \), such that \( d(t_i, F) > d(t_{i-1}, F), 1 \leq i \leq s \), and \( d(h, F) \leq d(t_s, F) \) for every \( h \neq t_{s-1} \), adjacent to \( t_s \) (if such a path does not exist, let \( s = 0 \) and consider the trivial path \( t = t_0 \)). Define \( \Gamma^*(t) \) as the set of vertices adjacent to \( t_s \) that are different from \( t_{s-1} \) (if \( s = 0 \), \( \Gamma^*(t) \) contains the vertices adjacent to \( t \) that are not in \( U \)). Given two different vertices \( t, t' \in U \) let \( \neq \) be the \( t \rightarrow t' \) path in \( T_{n+1} \), and consider the \( h \rightarrow h' \) path \( h, t_0, t_1, \ldots, t_{s+1}, t'_0, t'_1, \ldots, t'_s, h' \), where \( h \in \Gamma^*(t) \) and \( h' \in \Gamma^*(t') \). Let us prove that its length is \( |h \rightarrow h'| \leq 2n + 2 \). First, suppose that the \( t \rightarrow z \) and \( z \rightarrow t' \) paths in \( T_{n+1} \) have a common subpath of length \( k, 0 \leq k \leq n - 1 \). Then, as \( T_{n+1} \) has \( n \) edges, \( n - k \geq |t \rightarrow t'| = |t \rightarrow z| + |z \rightarrow t'| - 2k \), so that \( n + k \geq |t \rightarrow z| + |z \rightarrow t'| \). Moreover, the \( z \rightarrow t \) and \( z \rightarrow t' \) paths in \( T_{n+1} \) have lengths at least \( r - d(t, F) \) and \( r - d(t', F) \), respectively. Therefore, \( n + k \geq |t \rightarrow z| + |z \rightarrow t'| \geq 2r - (d(t, F) + d(t', F)) \). On the other
hand, $|t \to t'| \leq l - d(t, F)$ and $|t' \to t''| \leq l - d(t', F) + n - k + l - d(t', F) = 2l - (d(t, F) + d(t', F)) + n - k \leq n + k + n - k = 2n$. Hence, $|h \to h'| \leq 2n + 2$ (if $h, h' \in \Gamma^*(t)$) the considered $h \to h'$ path is $h, t, h'$ with length 2). In particular, these results imply that $h \neq h'$; otherwise, $g(G) \leq 2n + 2$, contradicting $l > n$. Thus, $\Gamma^*(t) \cap \Gamma^*(t') = \emptyset$ if $t \neq t'$.

Now, consider the union $H$ of the sets $\Gamma^*(t), t \in U$. As $|U| = n + 1$, it is clear that $|H| \geq (n + 1)\delta - 2n$. For every $h \in H$ let $f_h$ be a vertex of $F$ such that $d(h, f_h) \leq \varepsilon$ is minimum. Then $f_h = f_{h'}$ for some $h \neq h'$ because $|F| < |H|$. Moreover, note that if $h \in \Gamma^*(t)$ and $h$ is adjacent to a certain $t$, then a $h \to f_h$ path with length $d(h, f_h)$ does not contain the vertex $t$. So, $G$ contains a cycle $f_h \to h \to h' \to f_h$ with length at most $\varepsilon + 2n + 2 + \varepsilon = 2(n + \varepsilon + 1)$. Hence, $2(n + \varepsilon + 1) \geq g \geq 2l + 1$, which implies $\varepsilon \geq l - n$ because $\varepsilon$ must be an integer. \qed

The next theorem gives the aforementioned sufficient condition for $\kappa(n)$ to be optimum.

**Theorem 2.2.** Let $G$ be an $l$-geodetic graph and let $F \subseteq V$, $|F| < (n + 1)\delta - 2n$, be nontrivial. If $D \leq 2l - 2n - 1$, then $G - F$ is connected. Equivalently,

$$\kappa(n) > (n + 1)\delta - 2n \quad \text{if} \quad D < 2l - 2n - 1.$$  

**Proof.** We will show that, if $2l - 2n > D$, between any pair of vertices $x, y \in V$ there is in $G$ an $x \to y$ path that contains no vertex of $F$. According to Lemma 2.1, in $G - F$ there are $x \to x'$ and $y \to y'$ paths such that $d(x', F)$ and $d(y', F)$ are at least $l - n$. Hence, a $x' \to y'$ path of length at most $D$ avoids $F$ if $D < 2l(l - n)$. \qed

Without more information about the structure of $G$, this is all we can infer from the given conditions. As mentioned above, note that if $G$ contains a tree $T_{n+1}$ with $n + 1$ vertices, all of them with degree $\delta$, the set $F$ of vertices of $G - T_{n+1}$ that are adjacent to vertices of $T_{n+1}$ could be an example of a nontrivial disconnecting set with $|F| \leq (n + 1)\delta - 2n$ vertices. Then we would have $\kappa(n) \leq (n + 1)\delta - 2n$.

Now define $\lambda(n)$ as the minimum cardinality of a set of edges, if any, whose deletion disconnects $G$ and every remaining component is not a tree with order $k \leq n$. If $G$ contains a tree $T_k$ as a subgraph (with order $k$), the set of edges incident with vertices of $T_k$ (that are not edges of $T_k$) could be a trivial disconnecting set of edges whose deletion leaves $T_k$ as a component. Given $n \geq 1$, a set $A \subseteq E(G)$ is nontrivial if $A$ does not contain a trivial disconnecting set for any $k \leq n$. The following lemma allows one to prove that the results given in Lemma 2.1 and Theorem 2.2 are also valid for edges. As given above, $l > n$ and $\delta > 2$. Moreover, define $U$ as a vertex set obtained by choosing exactly one endvertex $u$ for each $uv \in A$.

**Lemma 2.3.** Let $G$ be $l$-geodetic and let $A \subseteq E$, $|A| < (n + 1)\delta - 2n$, be nontrivial. Then, for any given $x \in V$, there exists a vertex $z \in C(x)$ such that $d(z, U) \geq 1$.  


Proof. It suffices to deal with the case \(d(x, U) = 0\). As \(A\) is nontrivial and \(l > n\), \(|C(x)| \geq n + 1\). Consider a tree \(T_{n+1}\) contained in \(C(x)\) and let \(A' = \{ t \in A: t \notin V(T_{n+1}) \}\). For each \(t \in E\) consider the set \(A_t\) whose elements are the edges of \(A\) that are incident with vertex \(u\) and define \(A'' = \bigcup_{t \in E} A_t\). Note that the sets \(A_t\) are pairwise disjoint, and the same is true for \(A'\) and \(A''\); otherwise, there would exist a cycle of length \(p < n + 3\), contradicting \(l > n\). Clearly, \(|A'\| + |E'| \leq |A'\| + |A''| \leq |A| < (n + 1)\delta - 2n\). On the other hand, \(|A'| + |E'| + |E''| \geq (n + 1)\delta - 2n\), where \(E'' = \{ t \in E: t \notin V(T_{n+1}) \}\). Then, there exists an edge \(t \in E''\) for which \(d(t, U) > 1\).

The edge version of Theorem 2.2 is the following.

Theorem 2.4. Let \(G\) be \(l\)-geodetic and \(n \in \mathbb{Z}^+\). Then
\[
\lambda(n) \geq (n + 1)\delta - 2n \quad \text{if} \quad D \leq 2l - 2n.
\]

Proof. Let \(A \subseteq E\), \(|A| < (n + 1)\delta - 2n\), be nontrivial. Let us prove that if \(G - A\) is not connected, then \(D \geq 2l - 2n\). So, suppose \(G - A\) is not connected and let \(x\) and \(y\) be vertices belonging to different components \(C(x)\) and \(C(y)\) of \(G - A\). If \(u \in A\) is an edge joining \(C(x)\) with another component of \(G - A\), let \(u\) be the endvertex of this edge that belongs to \(C(x)\) and use \(U'(x)\) to denote the set of these vertices \(u\). Now, define \(U\) as a vertex set obtained by choosing exactly one endvertex for each edge of \(A\), but in such a way that \(U \supseteq U'(x)\). The sets \(W'(y)\) and \(W\) are defined in a similar way. Since, clearly, \(|U|, |W| < (n + 1)\delta - 2n\), by Lemmas 2.1 and 2.3, there exist vertices \(x' \in C(x)\) and \(y' \in C(y)\) such that \(d(x', U) \geq l - n\) and \(d(W, y') \geq l - n\). Thus, any \(x' \rightarrow y'\) path in \(G\) has length at least \(2l - (n + 1)\delta - 2n\) and then \(D \geq d(x', y') \geq 2l - 2n\).

3. Disconnecting sets in extraconnected graphs

When Theorem 2.2 is applied for \(n = 1\), it states that \(\kappa(1) \geq 2\delta - 2\) if \(D \leq 2l - 3\). However, by Theorem 1.2, \(D \leq 2l - 2\) suffices to assure optimum superconnectivity. In a similar way, the sufficient condition given by Theorem 2.2 when \(n = 2\) can be improved by using a more involved reasoning. To use a terminology similar to that used when the graph \(G\) is super-\(\kappa\), let us say that a maximally superconnected graph, i.e. \(\kappa(1) \geq 2\delta - 2\), is extra-\(\kappa\) if every disconnecting set with at least \(2\delta - 2\) vertices is trivial, i.e. its deletion leaves either an edge or an isolated vertex as a component.

Let \(F \subseteq V\) and \(x \in V - F\). For each \(f \in F\) such that \(d(x, f) \leq l\), the vertex adjacent to \(x\) in the unique shortest \(x \rightarrow f\) path is denoted by \(v(x \rightarrow f)\). Analogously, let \(v(x \rightarrow F) = \{ v(x \rightarrow f): f \in F, d(x, f) \leq l\}\).

Lemma 3.1. Let \(G\) be \(l\)-geodetic with minimum degree \(\delta \geq 3\). For \(n = 2\), let \(F \subseteq V\), \(|F| < 3\delta - 4\), be a nontrivial set of vertices and let \(x \in V - F\). Then there exists in \(G - F\) an \(x \rightarrow x'\) path such that \(d(x', F) \geq l - 1\).
Proof. Assume that \(|F|=3\delta-5, l>2\) and \(d(x, F)<l-1\). First of all, note that if there exists a vertex \(y\in \Gamma(x)-v(x\to F)\) then \(d(y, F)=d(x, F)+1\) and the distance from \(x\) to \(F\) is increased. Thus, suppose that \(v(x\to F)=\Gamma(x)\).

Let \(N\) be the set of \(\delta-1\) vertices of \(F\) nearest from \(x\) and let \(H=F-N\). As \(F\) is nontrivial, \(d(x, H)\geqslant 2\) and there is in \(V-F\) a vertex \(y\) adjacent to \(x\) and not in \(v(x\to N)\).

Case 1: Suppose that \(d(x, H)\geqslant l\). As \(y\) is adjacent to \(x\), \(d(y, H)\geqslant d(x, H)-1\) and \(d(y, F)=d(x, F)+1\). Moreover, as \(y\) is not in \(v(x\to N)\), \(d(y, N)=d(x, N)+1\). So, if \(d(x, F)=d(x, N)\leqslant l-3\), then \(d(y, F)=d(x, F)+1\) and the distance from \(x\) to \(F\) is increased. On the other hand, if \(d(x, N)\geqslant l-2\), then \(d(y, N)=l-1\) and the theorem holds.

Case 2: Suppose that \(d(x, H)=l-1\).

Case 2.1: If \(d(x, F)=d(x, N)\leqslant l-3\), then, reasoning as above, \(d(y, H)\geqslant l-2\) and \(d(y, N)=d(x, N)+1\). Thus, again \(d(y, F)=d(x, F)+1\).

Case 2.2: \(d(x, F)=d(x, N)=l-2\). In this case let \(J\) be the set of \(\delta-2\) vertices of \(H\) nearest from \(y\) and let \(v(y\to J)\). For if \(z\notin F\), then \(F-N\) or \(\{x\}\) and, as \(F\) does not contain the trivial set \(\Gamma(y)\), \(y\) can be replaced by \(y\). Let \(K=H-J\).

Case 2.2.1: Let \(d(x, K)=l\). Hence, \(d(z, K)>d(x, K)-1\) and \(d(z, N)=d(x, N)+2=l\) because \(v(y\to f)=x, \forall f\in N\) such that \(d(x, f)\leqslant l-1\).

Case 2.2.2: Let \(d(x, K)=l-1\). Note that in this case \(d(x, f)\geqslant l-1\) for any \(f\in J\). Reasoning as in case 2.2.1, it is proved that \(d(z, K)\geqslant l-3, d(z, J)\geqslant l-1\) and \(d(z, N)=l\). Moreover, if \(l=3, d(z, K)\geqslant 1\) since \(z\) is not in \(F\). So, assume that \(l\geqslant 4\).

Suppose that \(v(x\to J)=y\). Now, let \(t\neq y\) be a vertex adjacent to \(z\) not in \(v(z\to K)\). As \(l\geqslant 3\), vertex \(t\) does not belong to \(F\). Indeed, the path \(x, y, z, t\) with length \(3\) is unique and then \(v(x\to t)=y\) and \(v(y\to t)=z\). Thus, \(t\notin N\) and \(t\notin J\) since \(y\) is not in \(v(x\to N)\) and \(z\) is not in \(v(z\to J)\). Moreover, \(t\) is not in \(v(z\to K)\) and then \(t\notin K\). Now, for any \(f\in N\) such that \(d(x, f)\leqslant l-2, d(z, f)=l\) and \(v(z\to f)=y\). Then \(d(t, f)=l\). If \(f\in N\) and \(d(x, f)=l-1\), then it can only be assured that \(d(t, f)\geqslant l-1\). Thus, \(d(t, N)\geqslant l-1\). As \(d(z, K)\geqslant l-3\) and \(t\) is not in \(v(z\to K)\), \(d(t, K)\geqslant l-2\). Finally, \(d(z, J)=l-1\) and \(v(z\to J)=y\). Thus, \(d(t, J)=l\).

On the other hand, if there exists a vertex \(f\in J\) such that \(v(x\to f)\neq y\), then \(d(y, f)=l\) and \(v(y\to f)=x\). As \(J\) is the set of \(\delta-2\) vertices of \(H\) nearest to \(y\), \(d(y, K)\geqslant l\) and thus \(d(z, K)\geqslant l-1\). Reasoning as in the case before, \(d(z, J)\geqslant l-1\) and \(d(z, N)\geqslant l\).

Case 3: \(d(x, H)\leqslant l-2\). In this case we again have that \(d(y, N)=d(x, N)+1\) and \(v(y\to N)=x\). This implies \(d(z, N)=d(y, N)+1=d(x, N)+2\). More precisely, as \(d(x, f)\leqslant l-2\) for any vertex \(f\in F, d(z, f)=d(x, f)-2\). Note that \(d(z, f)\leqslant l\). Besides, as \(d(x, H)\leqslant l-2\) we have \(d(y, H)=d(y, J)\leqslant l-1\) and then \(d(z, J)\geqslant d(y, J)+1\). More precisely, if \(d(y, f)\geqslant l\) for some \(f\in J\), then \(d(z, f)\geqslant l\). But \(d(x, f)\geqslant l-1\) if \(v(y\to f)=x\); otherwise, \(d(x, f)\geqslant l\). In any case, if \(d^*(u, v)\) is the shortest path from \(u\) to \(v\), we have that \(d^*(z, f)\geqslant d^*(x, f)\). Moreover, \(d(z, f)\geqslant d(x, f)-2\) for any \(f\in K\) (of course, as \(z\) is not in \(F\), if \(d(x, f)=2\) then \(d(y, f)\geqslant 1\) but \(d(z, f)\geqslant 1\)). It is clear that \(d^*(z, f)\geqslant d^*(x, f)+l/2\).
Now, given any vertex $x$ define its global distance to $F$ as $S(x) = \sum_{f \in F} d^*(x, f)$. Then

\[
S(z) = \sum_{f \in F} d^*(z, f) = \sum_{f \in F} d^*(z, f) + \sum_{f \in F} d^*(z, f) + \sum_{f \in F} d^*(z, f)
\]

\[
= \sum_{f \in F} d(z, f) + \sum_{f \in F} d^*(z, f) + \sum_{f \in F} d^*(z, f)
\]

\[
\geq \sum_{f \in F} (d(x, f) + 2) + \sum_{f \in F} d^*(x, f) + \sum_{f \in F} (d^*(x, f) - 2)
\]

\[
= \sum_{f \in F} (d^*(x, f) + 2) + \sum_{f \in F} d^*(x, f) + \sum_{f \in F} (d^*(x, f) - 2)
\]

\[
= S(x) + 2(\delta - 1) - 2(\delta - 2) = S(x) + 2,
\]

i.e. the global distance to $F$ has been increased.

Now, it is clear that by repeatedly applying the above reasonings, there is in $G - F$ an $x \rightarrow x'$ path such that $d(x', F') \geq l - 2$ for some $F' \subset F$, $|F'| = \delta - 2$, and $d(x', F'') \geq l - 1$, where $F'' = F - F'$. Let $H'$ be the set of vertices $h$ at distance 2 from $x'$ such that $v(x \rightarrow h)$ is not in $v(x \rightarrow F')$. Clearly $|H'| = 2(\delta - 1)$. As $|F''| = |F| - |F'| = (3\delta - 5) - (\delta - 2) = 2\delta - 3$, there is a vertex $t \in H'$ that is not in any $x' \rightarrow f$ path for $f \in F''$. For such $t$, $d(t, F') \geq l - 1$. Moreover, if $v(x \rightarrow t) = y$, let $A$ be the set of vertices $f \in F''$ such that $v(x' \rightarrow A) = y$. Then $d(y, A) \geq l - 2$ and $d(t, A) \geq l - 1$. Besides, $d(y, F'' - A) \geq l$ and $d(z, F'' - A) \geq l - 1$.

Now, reasoning as in the proof Theorem 2.2, the following result is easily proved.

**Theorem 3.2.** Let $G$ be $l$-geodetic. Then

\[
\kappa(2) \geq 3\delta - 4 \quad \text{if} \quad D \leq 2l - 3.
\]

This result implies that $G$ is maximally extraconnected if $D \leq 2l - 3$.

The corresponding result for edges is a straightforward consequence of Lemmas 2.3 and 3.1.

**Theorem 3.3.** Let $G$ be $l$-geodetic. Then

\[
\lambda(2) \geq 3\delta - 4 \quad \text{if} \quad D \leq 2l - 2.
\]

In view of the results given in Theorems 1.1, 1.2, 3.2 and 3.3, the authors conjecture that the following result holds.

**Conjecture 3.4.** Let $G$ be $l$-geodetic and let $n \in \mathbb{Z}^+$. Then

\[
\kappa(n) \geq (n + 1)\delta - 2n \quad \text{if} \quad D \leq 2l - n - 1,
\]

\[
\lambda(n) \geq (n + 1)\delta - 2n \quad \text{if} \quad D \leq 2l - n.
\]
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References