Cycles of all lengths in arc-3-cyclic semicomplete digraphs

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Abstract

Let D be an arc-3-cyclic, semicomplete digraph and uv be an arc of D contained in a cycle of length r. If \( uv \notin A(D) \) then the arc \( uv \) is contained in cycles of length \( h : 3 \leq h \leq r \), or if \( \delta^+(D), \delta^-(D) \geq 3 \) then the arc \( uv \) is contained in cycles of length \( h : 6 \leq h \leq r \). Also included in this paper is a very useful crossing arc theorem.

1. Introduction

A digraph \( D \) is a tournament if for each pair of vertices there is precisely one arc between them. A digraph \( D \) is called semicomplete if for each pair of vertices there is at least one arc between them. Obviously, the semicomplete digraph is a generalization of the tournament. For years, tournaments have been studied and many results on cycles in tournaments have been published. [1,7–14,16–18]. A great number of the theorems proved for tournaments also hold for semicomplete digraphs.

One of the early results dealing with cycles in tournaments was given by Moon [11] and addresses the idea of finding cycles of every length containing each vertex of a tournament.

Definition 1. A digraph \( D \) of order \( n \) is arc-pancyclic (or vertex-pancyclic) if each arc (or vertex) of \( D \) is contained in cycles of length \( k \), \( 3 \leq k \leq n \). A digraph \( D \) is arc-k-cyclic if each arc of \( D \) is contained in a cycle of length \( k \). If \( k \) is the order of \( D \), then the property of arc-k-cyclic is also called arc-Hamilton-cyclic.

Theorem 1 (Moon [11]). Every strongly connected tournament is vertex-pancyclic.

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The arc-pancyclic property proved to be much more complicated than the property of vertex-pancyclic. The pioneer result in this area was obtained by Alspach in 1967.

**Theorem 2** (Alspach [1]). *Every regular tournament is arc-pancyclic.*

Note, a *regular* tournament is one in which all the vertices have the same out-degree. The out-degree of a vertex $v$ is the number of arcs of the form $vu$ in the tournament.

After this paper, many articles on this topic were published. In 1982, a necessary and sufficient condition for arc-pancyclicity in tournaments was obtained

**Theorem 3** (Wu et al. [18]). *A tournament $D$ is arc-pancyclic if and only if $D$ is arc-Hamilton-cyclic and arc-3-cyclic.*

The arc-3-cyclic condition is rather weak since almost all tournaments are arc-3-cyclic [14] and it can be determined whether a tournament is arc-3-cyclic in at most $O(n^3)$ operations. It was recently proved by Bang-Jensen et al. [6] that the determination of arc-Hamilton-cyclic property for tournaments is also a polynomial problem. The proof given in [18] actually implies the following result stronger than that of Theorem 3 (also see [17] for an alternative and simpler proof).

**Theorem 4** (Wu et al. [18]). *Let $D$ be arc-3-cyclic tournament and $e$ be an arc of $D$. If the arc $e$ is contained in a cycle of length $r$, then $e$ is contained in cycles of length $k : 3 \leq k \leq r$."

Later, the problem of arc-pancyclic for tournaments was completely solved by Tian et al. [16]. They showed that except for two families of arc-3-cyclic tournaments, every arc-3-cyclic tournament is arc pancyclic. Bang-Jensen [4] constructed an arc-3-cyclic digraph (see Fig. 1)\(^2\) that was not arc-pancyclic and not in the family of counter examples given by [16]. However, his example was not a tournament but a semicomplete digraph. Thus, the theorem in [16] does not hold for semicomplete digraphs.

In this paper we generalize the results in [18] to semicomplete digraphs and show that under certain conditions arc-3-cyclic semicomplete digraphs are arc-pancyclic. Theorem 6 states that if an arc $e$, of an arc-3-cyclic semicomplete digraph, is in an $r$-cycle ($r \geq 3$) and not in a 2-cycle, then $e$ is contained in a cycle of each possible length $i, 3 \leq i \leq r$. Theorem 7 adds a degree condition for the end vertices of $e$ and this is enough to insure that if $e$ is in an $r$-cycle ($r \geq 6$) then $e$ is in a cycle of each length $i, 6 \leq i \leq r$. As a corollary of Theorem 7, we have that if $D$ is arc-Hamiltonian-cyclic, arc-k-cyclic for each $k = 3, 4, 5$ and each vertex has in- and out-degree at least 3, then $D$ is arc-pancyclic.

\(^2\)The tournament induced by $v_1, \ldots, v_6$ is a transitive tournament ($v_iv_j \in A(T)$ if and only if $i < j$). The undirected edges represent arcs in both directions.
2. Definitions

A digraph $D$ consists of a pair $V(D), A(D)$, where $V(D)$ is a finite set of vertices and $A(D)$ is a set of ordered pairs $uv$ of vertices, called arcs. All digraphs considered in this paper have no loops or parallel arcs in the same direction (parallel arcs in opposite directions are allowed). A digraph $D$ is called \textit{semicomplete} if for each pair of distinct vertices $u, v \in V(D)$, either $uv, vu$ or both belong to $A(D)$. An acyclic digraph $D$ is called a $(u,v)$-\textit{acyclic subgraph} if for every vertex $w$ of $D$ there is a directed $(u,w)$-path and a directed $(w,v)$-path. A semicomplete digraph is \textit{arc-3-cyclic} if every arc of $D$ is contained in a cycle of length three. Let $I(u) = \{x \in V(D) : xu \in A(D)\}$ and $O(u) = \{x \in V(D) : ux \in A(D)\}$. For a subgraph $H$ of $D$, let $I_H(u) = I(u) \cap V(H)$ and $O_H(u) = O(u) \cap V(H)$. A midvertex of an arc $a = uv$, denoted by $\text{mid}(uv)$, in an arc-3-cyclic digraph is a vertex $w$ of $D$ such that $vw, wu \in A(D)$, that is $w \in I(u) \cap O(v)$. A directed path will be denoted $P = v_1 \cdots v_r$. A forwarding arc, of a path $P$, is an arc from $v_i$ to $v_{i+t}$ on the path, for $t \geq 2$. The pace of a forwarding arc $v_iv_{i+t}$ is the number of arcs on the path bypassed by the forwarding arc. Crossing arcs are a pair of forwarding arcs on a path $P$, $v_iv_{i+t}$ and $v_jv_{j+s}$, such that $0 \leq i < j < i+t < j+s \leq r$.

3. Theorems

The following results are presented in this paper.

\textbf{Theorem 5} (The Crossing Arcs Theorem). \textit{Let $D$ be a semicomplete digraph and $P = v_1 \cdots v_r$ be a path of length $r - 1$. If there are four integers $a, b, c$ and $d$ such that $1 \leq a < b < c < d \leq r$ and $v_av_c, v_bv_d \in A(D)$ then $D$ contains a path of length $r - 2$ from $v_1$ to $v_r$ unless $b = a + 1, c = b + 3$, and $d = c + 1$.}

\textbf{Theorem 6} (A generalization of WZZ’s Theorem [18]). \textit{Let $D$ be an arc-3-cyclic semicomplete digraph and an arc $a = uv \in A(D)$. If the arc $a = uv$ is contained in a cycle of length $r$ and $vu \notin A(D)$, then $a$ is contained in cycles of lengths $h : 3 \leq h \leq r$.}
Theorem 7. Let $D$ be an arc-3-cyclic semicomplete digraph and an arc $a = uv \in A(D)$. If the arc $a = uv$ is contained in a cycle of length $r$ and the in-degree of $u$ and the out-degree of $v$ are at least three, then $a$ is contained in cycles of lengths $6 \leq h \leq r$.

4. Proof of Theorem 5

Theorem 5 is the most important result since it is widely used in the proofs of the other theorems.

The following well-known lemma of Thomassen has been used in many papers about tournaments (for instance, see [5,16, 17]). It can be generalized to semicomplete digraphs and will be applied in the proofs of the main theorems of this paper. The following is a generalized version of the Thomassen’s Hamilton Path Lemma (with the same proof given in [15]).

Lemma 1 (Thomassen [15]). Let $D$ be a semicomplete digraph and $x$ and $y$ be two distinct vertices of $D$. If $D$ has an $(x, y)$-acyclic subgraph of order $p$, then $D$ contains a path from $x$ to $y$ of length $p - 1$.

Lemma 1 is a consequence of what is called the path merging property in [5]. (A digraph is path mergeable if whenever we have two internally disjoint $(x, y)$-paths $P$ and $Q$, we can find a new $(x, y)$-path $R$, containing exactly the vertices of $P$ and $Q$.) Digraphs with this property have been studied by Bang-Jensen.

Remark. Let $P = v_1 \cdots v_r$ be a $(v_1, v_r)$-path of length $r - 1$ in a semicomplete digraph $D$. If there is no $(v_1, v_r)$-path of length $r - 2$ in $D$, then $v_{\mu+1}v_{\mu-1} \in A(D)$, for every $\mu = 2, \ldots, r - 1$. Hence, $v_{i+3}v_i + v_{i+2}v_i$ is a path of $D$. For $1 \leq i \leq i + 3 \mu \leq r$, the union of paths $v_{i+3}v_{i+3-2i+3zi-1}v_{i+3zi}$ (for $z = 1, \ldots, \mu$) is a path of $D$ from $v_{i+3\mu}$ to $v_i$ and is denoted by $v_{i+3\mu} \cdots v_i$.

Proof of Theorem 5. Let $v_1 \cdots v_r$ be a path of $D$ of length $r - 1$ from $v_1$ to $v_r$ and there are four integers $a$, $b$, $c$ and $d$ such that $1 \leq a < b < c < d \leq r$ and $v_av_d, v bv_d \in A(D)$. We prove the theorem by contradiction. Assume that $D$ contains no path of length $r - 2$ from $v_1$ to $v_r$.

Case 1: $c = b + 1$. Let $v_av_c$ and $v bv_d$ be crossing arcs of $D$ with $c = b + 1$ such that $c - a$ and $d - b$ are as small as possible. (Note: $c - a = 2$ or $d - b = 2$ yield an obvious path of length $r - 2$, thus $c - a$ and $d - b \geq 3$.)

Subcase 1.1: $c - a = 3$ and $d - b = 3$. We have a $(v_1, v_r)$-path of length $r - 2$, $v_1 \cdots v_av_cv_{a+2}v_{b}v_d \cdots v_r$.

Subcase 1.2: $c - a \geq 4$. Since $c - a$ is smallest $v_cv_{a+2} \in A(D)$. We have a $(v_1, v_r)$-acyclic subgraph of order $r - 1$, $v_1 \cdots v_av_{a+2} \cdots v bv_d \cup v_c \cdots v_r$.

The subcase of $d - b \geq 4$ can be dealt with symmetrically.
Case 2: $c \geq b + 2$. Obviously, $c \neq b + 2$ for otherwise $[P \setminus \{v_{b+1}\}] \cup \{v_a v_c, v_b v_d\}$ is a $(v_1, v_r)$-acyclic subgraph of order $r - 1$. Thus $c > b + 2$. Now we will find a $(v_c, v_b)$-path $Q$ in $D\{v_b, \ldots, v_c\}$ of length $c - b - 2$ so that

$$[P \setminus \{v_{b+1}, \ldots, v_{c-1}\}] \cup \{v_a v_c, v_b v_d\} \cup Q$$

is a $(v_1, v_r)$-acyclic subgraph of order $r - 1$.

Subcase 2.1: $c \equiv b \equiv 2 \pmod{3}$. Then we have $Q = v_c v_{c-2} \cdots v_b$.

Subcase 2.2: $c \equiv b \equiv 1 \pmod{3}$. Note that $c \geq b + 2$ implies that $c - b \geq 4$.

(i) If $v_c v_{c-3}$ is an arc of $D$, then let $Q = v_c v_{c-3} v_{c-2} v_{c-4} \cdots v_b$. Therefore $v_{c-3} v_c$ is an arc of $D$.

(ii) If $v_{c-1} v_{c-4}$ is an arc of $D$, then let $Q = v_c v_{c-2} v_{c-1} v_{c-4} \cdots v_b$. Therefore $v_{c-4} v_{c-1}$ is an arc of $D$.

(iii) Since $v_c v_{c-3}$ and $v_{c-4} v_{c-1}$ are arcs of $D$, then $D$ has a $(v_1, v_r)$-acyclic subgraph $[P \setminus \{v_{c-2}\}] \cup \{v_{c-3} v_c, v_{c-4} v_{c-1}\}$.

Subcase 2.3: $c \equiv b \equiv 0 \pmod{3}$. Note that $c \geq b + 3$.

(i) Assume $c = b + 3$ and either $a < b - 1$ or $d > c + 1$. Then let $Q = v_c v_{c-2} v_{c-4}$ when $a < b - 1$ or $Q = v_{c+1} v_{c-1} v_{c-3}$ when $d > c + 1$ and therefore $[P \setminus \{v_{b+1}, \ldots, v_{c-1}\}] \cup \{v_a v_c, v_b v_d\} \cup Q$ is a $(v_1, v_r)$-acyclic subgraph of order $r - 1$. Since $c = b + 3$, $a = b - 1$ and $d = c + 1$ is the exceptional case of the theorem, we assume that $c \geq b + 6$.

(ii) Assume that $v_c v_{c-3}$ is an arc of $D$. We claim that $v_{c-1} v_{c-4}$ is not an arc of $D$, for otherwise, we can have a $(v_1, v_r)$-acyclic subgraph $[P \setminus \{v_{b+1}, \ldots, v_{c-1}\}] \cup \{v_a v_c, v_b v_d\} \cup Q$ with $Q = v_c v_{c-3} v_{c-2} v_{c-1} v_{c-4} v_{c-6} \cdots v_b$. We also claim that $v_{c-2} v_{c-5}$ is not an arc in $D$ for otherwise, we have a $(v_1, v_r)$-acyclic subgraph as before with $Q = v_c v_{c-3} v_{c-2} v_{c-5} v_{c-4} v_{c-6} \cdots v_b$. Hence, both $v_{c-4} v_{c-1}, v_{c-5} v_{c-2} \in A(D)$ and consequently, we obtain a $(v_1, v_r)$-acyclic subgraph $[P \setminus \{v_{c-3}\}] \cup \{v_{c-4} v_{c-1}, v_{c-5} v_{c-2}\}$. Thus $v_{c-4} v_{c-1}$ is not an arc of $D$.

(iii) If $v_{c-1} v_{c-5}$ is an arc of $D$, then we have a $(v_1, v_r)$-acyclic subgraph $[P \setminus \{v_{b+1}, \ldots, v_{c-1}\}] \cup \{v_a v_c, v_b v_d\} \cup Q$ with $Q = v_c v_{c-2} v_{c-1} v_{c-5} v_{c-4} v_{c-6} \cdots v_b$. Thus $v_{c-1} v_{c-5}$ is not an arc of $D$.

(iv) By (ii) and (iii) we have a $(v_1, v_r)$-acyclic subgraph $[P \setminus \{v_{c-2}\}] \cup \{v_{c-3} v_c, v_{c-5} v_{c-1}\}$. □

5. Lemmas

Let $D$ be an arc-3-cyclic semicomplete digraph and $P = v_1 \cdots v_r$ be a path of $D$ from $v_1$ to $v_r$ of length $r - 1$ ($r > 3$). We assume that $D$ contains no path from $v_1 = v$ to $v_r = u$ of length $r - 2$. Denote $W = V(D) \setminus V(P)$. Then we have the following lemmas.

Lemma 2. $P$ has a forwarding arc.
Proof. Suppose there is no forwarding arc on $P$. Consider $x = \text{mid}(v_{m+3}v_m)$ where $1 \leq m \leq r - 3$. If $x \in W$ then $v_1 \ldots v_m x v_{m+3} \ldots v_r$ is a path of length $r - 2$. So $x \in V(P)$. Therefore $x = v_i$ for some $1 \leq i \leq r$ and then $v_i v_{m+3}$ or $v_m v_i$ is a forwarding arc on $P$. □

Definition. Let $v_m v_{m+h}$ be a forwarding arc of $P$. If there is no other forwarding arc of $P$ with both ends in the segment $v_m \ldots v_{m+h}$, then $v_m v_{m+h}$ is called a minimal forwarding arc.

Lemma 3. The minimal forwarding arc is of pace three or four.

Proof. By the remark of Section 4, the minimal forwarding arc is of pace at least three. Let $v_m v_{m+h}$ be a minimal forwarding arc. Suppose $h \geq 5$. Since $v_m v_{m+1} v_{m+4} \in A(D)$, consider $x = \text{mid}(v_{m+4}v_{m+1})$. Then $x \in \{v_1, \ldots, v_{m-1}, v_{m+h+1}, \ldots, v_r\}$ would imply crossing arcs not allowed by Theorem 5. Also $x \in \{v_m, \ldots, v_{m+k}\}$ would imply $h$ is not minimal. Thus, $x \in W$ and this produces an obvious $(v_i, v_r)$-path of length $r - 2$, $v_1 \ldots v_{m+1} x v_{m+4} \ldots v_r$. □

Lemma 4. If $v_a v_b \in A(D)$ and $1 \leq a < b \leq r$, $b - a \geq 3$, then $v_c v_d \notin A(D)$, with $d - c \geq 3$, for $b \leq c < d \leq r$ or $1 \leq c < d \leq a$.

Proof. Suppose $v_a v_b, v_c v_d \in A(D)$, with $1 \leq a < b \leq c < d \leq r$. Here $b - a \geq 3$, and $d - c \geq 3$. Then $v_{b-2} v_{c+2} \notin A(D)$ for this would produce crossing arcs disallowed in Theorem 5. Consider $x = \text{mid}(v_{c+2}v_{b-2})$. If $x \in V(P)$ then we have either crossing arcs or a forwarding arc of pace two, so $x \in W$. But this will produce the following $(v_1, v_r)$-acyclic subgraph of order $r - 1$: $[P \setminus \{v_{b-1}, v_{c+1}\}] \cup \{v_a v_b, v_c v_d, x v_{c+2}, v_{b-2} x\}$. This contradicts the assumption that $D$ contains no path from $v_1$ to $v_r$ of length $r - 2$ and completes the proof of the lemma. □

Remark 1. By Theorem 5 and Lemmas 2–4, the path $P$ has only one or two minimal forwarding arcs:

(i) $v_m v_{m+3}$,
(ii) $v_{m-1} v_{m+3}, v_m v_{m+4}$ (one crossing),
(iii) $v_m v_{m+4}$ (no crossing).

Remark 2. Note case (ii) can always be converted to (i): Since $P' = v_1 \ldots v_{m-1} v_{m+3}$ $v_{m+1} v_{m+2} v_{m+4} \ldots v_r$ is also a path of length $r - 1$. Either $v_{m+1} v_{m+3} \in A(D)$ or $v_m v_{m+3} \in A(D)$. Then one of $P$ or $P'$ has a minimal forwarding arc of pace of three.

So we only need to consider case (i) and (iii), that is, the path $P$ has only one minimal forwarding arc with pace three or four. Case (ii) is therefore ignored and path $P$ may have a crossing pair only in case (i).
Lemma 5. Let $v_0v_b$ be a minimal forwarding arc on $P$. Then for each $i = 1, \ldots, a$,

$$O(v_i) \cap \{v_b, \ldots, v_r\} \neq \emptyset$$

and for each $j = b, \ldots, r$,

$$I(v_j) \cap \{v_1, \ldots, v_a\} \neq \emptyset.$$

**Proof.** We show that for each $i = 1, \ldots, a$,

$$O(v_i) \cap \{v_b, \ldots, v_r\} \neq \emptyset.$$

The other part of the lemma can be proved similarly. Obviously, $v_b \in O(v_a) \cap \{v_b, \ldots, v_r\} \neq \emptyset$. So we assume $O(v_i) \cap \{v_b, \ldots, v_r\} \neq \emptyset$, for some $i = 2, \ldots, a$, and we want to show that $O(v_{i-1}) \cap \{v_b, \ldots, v_r\} \neq \emptyset$. Now $v_{i+2}v_{i-1} \in A(D)$ since $v_{i-1}v_{i+2}$ would be another forwarding arc outside $v_0v_b$ or create crossing arcs contradicting Lemma 4 or Theorem 5. Then $mid(v_{i+2}v_{i-1}) \notin W$ since this would produce an obvious path of length $r - 2$. If $mid(v_{i+2}v_{i-1}) \in \{v_1, \ldots, v_{b-1}\}$, then we have crossing arcs disallowed in Theorem 5, other forwarding arcs disallowed by Lemma 4 or forwarding arcs of pace two. The only possibility would be the exceptional case in Theorem 5, when $b - a = 4$, $i = a$ and $mid(v_{i+2}v_{i-1}) = mid(v_{a+2}v_{a-1}) = v_{a+3}$. However, this means the arc $v_{a+3}v_{a+2} \in A(D)$ and we have a path of length $r - 2$, $v_1 \ldots v_{a-1}v_{a+3}v_{a+2}v_a v_{a+4} \ldots v_r$. Thus, the vertex $mid(v_{i+2}v_{i-1}) \in O(v_{i-1})$ must be in $\{v_b, \ldots, v_r\}$. This completes the proof of the lemma.  

6. Proof of Theorem 6

Suppose $D$ is an arc-3-cyclic semicomplete digraph and $v_rv_1 \in A(D)$ is contained in a cycle, $C = v_1v_2 \ldots v_rv_1$ of length $r$, and $v_1v_r \notin A(D)$. Assume that $D$ contains no cycle of length $r - 1$ containing $v_r v_1$. By Lemma 2, $P = v_1v_2 \ldots v_r$ has a forwarding arc, say $v_0v_b$. By Lemma 5, $O(v_1) \cap \{v_b, \ldots, v_r\} \neq \emptyset$ and $I(v_r) \cap \{v_1, \ldots, v_a\} \neq \emptyset$. Let $m = \max\{i : v_i \in O(v_1)\}$ and $n = \min\{i : v_i \in I(v_r)\}$. Since $v_1v_r \notin A(D)$, $m \neq r, n \neq 1$. Therefore, $v_1v_m, v_nv_r$ are crossing arcs of $P$. By Theorem 5 and our assumption that $v_r v_1$ is not contained in any cycle of length $r - 1$, we must have that $m = 5, n = 2$ and $r = 6$.

Since $v_1v_5 \in A(D)$, by Theorem 5 again, $v_6v_3 \in A(D)$. Let $x = mid(v_6v_3)$. Here $x \neq v_1$ since $v_1v_6 \notin A(D)$. It is also obvious that $x \notin W$ for otherwise $v_1v_2v_3xv_6v_1$ is a cycle of length $r - 1$. To avoid forwarding arcs of pace two, $x \neq v_3, v_4$ since $v_3x, xv_6 \in A(D)$. Thus, the only remaining case is that $x = v_2$. That is, $v_6v_3, v_2v_6 \in A(D)$. Again, we obtain a cycle of length $r - 1$: $v_1v_2v_3v_2v_6v_1$. This contradicts the assumption of non-existence of such cycle and completes the proof of the theorem.
7. Proof of Theorem 7

Let \( v_1 \cdots v_r v_1 \) (\( r \geq 7 \)) be a cycle of \( D \) of length \( r \) containing an arc \( v_r v_1 \). We assume that the arc \( v_r v_1 \) is not contained in a cycle of length \( r - 1 \). By Theorem 6, we only need to consider the case where both \( v_1 v_r v_r v_1 \in A(D) \). Here we note that \( D \) contains no forwarding arcs of pace two, since such an arc would produce an obvious \( (r-1) \)-cycle. Denote \( W = V(D) \setminus V(P) \).

We claim that either

\[
O_P(v_1) \subseteq \{v_2, v_r\} \quad \text{or} \quad I_P(v_r) \subseteq \{v_r^{-1}, v_1\}.
\]

Assume that \( |I_P(v_r)| \geq 3 \) and \( |O_P(v_1)| \geq 3 \). Let \( v_i \in I_P(v_r) \setminus \{v_1, v_r^{-1}\} \) and \( v_j \in O_P(v_1) \setminus \{v_2, v_r\} \). Consider the possibilities for placement of \( v_i \) and \( v_j \) on the path. If \( j \leq i \) then \( v_1 v_j \) and \( v_i v_r \) would be arcs that contradict Lemma 4, since there are no forwarding arcs with pace two. If \( i < j \) and since \( r > 6 \), we will have crossing arcs not allowed by Theorem 5. This proves our claim.

Without loss of generality, let \( I_P(v_r) = \{v_1, v_r^{-1}\} \). Since the in-degree of \( v_r \) is at least 3, let \( w \in I_P(v_r) \neq \emptyset \).

Let \( h \) denote the minimum pace of a forwarding arc.

Remark 3. Since \( I_P(v_r) = \{v_1, v_r^{-1}\} \) and \( r \geq 7 \), the minimal forwarding arc does not have its head \( (v_m+h) \) at \( v_r \) and therefore we have that \( m + h < r \).

Case 1: \( h = 3 \). The minimal forwarding arc is \( v_m v_{m+3} \).

Obviously \( I_P(w) \subseteq \{v_{m+1}, v_{m+2}, v_r\} \) for otherwise, let \( v_i \in I_P(w) \setminus \{v_{m+1}, v_{m+2}, v_r\} \). Then \( v_1 \cdots v_m v_{m+3} \cdots v_r \cup v_j v_r \) is a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \). Thus, \( V(P) \setminus \{v_{m+1}, v_{m+2}\} \subseteq O_P(w) \).

Subcase 1.1: \( m \neq 1 \). Since \( w v_m \in A(D) \), consider \( x = \text{mid}(w v_m) \). Then \( x \not\in W \). For suppose \( x \in W \), by Lemma 5, \( I(v_m+4) \cap \{v_1, \ldots, v_m\} \neq \emptyset \). Let \( v_i \in I(v_m+4) \cap \{v_1, \ldots, v_m\} \), then we have a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \): \( v_1 \cdots v_i v_{m+4} \cdots v_r \cup v_i \cdots v_m x w v_r \).

(Note, \( m + 4 \leq r \) by Remark 3.)

So \( x \in P \). Since \( I_P(w) \subseteq \{v_{m+1}, v_{m+2}, v_r\} \) and \( x \in I_P(w) \), we have that \( x \in \{v_{m+1}, v_{m+2}, v_r\} \). Note that \( x \neq v_r \) because \( m \neq 1 \), \( I_P(v_r) \subseteq \{v_1, v_r^{-1}\} \) and \( v_m x \in A(D) \). Now \( h = 3 \), thus \( x \neq v_{m+2} \) rather \( x = v_{m+1} \). However, this implies the existence of a \((v_1, v_r)\)-path of length \( r - 2 \), \( v_1 \cdots v_m v_{m+1} w v_{m+4} \cdots v_r \), which is a contradiction.

Subcase 1.2: \( m = 1 \). By Lemma 5 and the fact that \( r > 6 \), \( v_1 \in I(v_5) \) and \( v_1 \in I(v_6) \). We claim that \( v_2, v_3 \not\in I(w) \). For otherwise, we have a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \): \( v_1 v_5 \cdots v_r \cup v_1 v_2 w v_r \) (or \( v_1 v_5 \cdots v_r \cup v_1 v_2 v_3 w v_r \)). Thus, \( I_P(w) \subseteq \{v_r\} \). Let \( y = \text{mid}(w v_2) \). Here, \( y \in W \cup \{v_r\} \) because \( I_P(w) \subseteq \{v_r\} \). Since \( I_P(v_r) \subseteq \{v_1, v_r^{-1}\} \), \( y \neq v_r \) and thus \( y \in W \). Again by Lemma 5 and the fact that \( r > 6 \), we have \( v_1 v_6 \in A(D) \). Then this implies the existence of a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \), \( v_1 v_6 \cdots v_r \cup v_1 v_2 y w v_r \), which is a contradiction.

Case 2: \( h = 4 \). By Remark 2 in Section 2, in this case \( v_m v_{m+4} \) is the only forwarding arc with the minimum pace and \( P \) has no crossing arcs.
(i) We claim that \( w_{m+1} \in A(D) \). If \( w_{m+1} \notin A(D) \) then we have a \((v_1, v_r)\)-acyclic subgraph: \( v_1 \cdot v_m v_{m+4} \cdot v_r \cup v_{m+1} v_{m+5} w_{m+1} w_{t_r} \).

(ii) Let \( z = \text{mid}(w_{m+1}) \). We claim that \( z = v_m \).

Assume that \( z \in W \). By Lemma 5, \( v_{m+5} \) is dominated by some vertex in \( \{v_1, \ldots, v_m\} \), say \( v_r \). (Note \( m + 5 \leq r \) by Remark 3.) Then we have a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \): \( v_1 \cdot v_l v_{m+5} \cdot v_r \cup v_l \cdot v_{m+1} w_{t_r} \). So \( z \notin W \).

Note that \( O(v_{m+1}) \cap \{v_{m+3}, \ldots, v_r\} = \emptyset \), since \( h = 4 \) and \( P \) has no crossing arcs. Thus, \( z \notin \{v_{m+3}, \ldots, v_r\} \) since \( v_{m+1} w \in A(D) \).

We also have that \( v_s \notin I(w) \) for each \( s \leq m - 2 \). Otherwise, suppose \( v_s \in I(w) \) and \( s \leq m - 2 \), then we have a path: \( v_1 \cdot v_s w_{m+1} v_{s+1} \cdot v_m v_{m+4} \cdot v_r \). (Note that the arc \( v_{m+1} v_{s+1} \) exists because there is no crossing arcs in this case.)

In summary, \( \text{mid}(w_{m+1}) = z \in \{v_{m-1}, v_m, v_{m+2}\} \). However, \( v_{m+2} w \notin A(D) \). For otherwise, we can find the \((v_1, v_r)\)-acyclic subgraph: \( v_1 \cdot v_l v_{m+4} \cdot v_r \cup v_{m+4} v_{m+2} w_{t_r} \). So \( z \notin v_{m+2} \) and \( w_{m+2} \notin A(D) \). Also, \( v_{m+1} w \notin A(D) \). For otherwise, \( v_1 \cdot v_{m-1} w_{m+2} \cdot v_r \) is a path of length \( r - 2 \). So \( z \notin v_{m+1} \). Thus we have proved our claim that \( z = \text{mid}(w_{m+1}) = v_m \) and therefore \( v_m w \in A(D) \).

(iii) We have that \( v_{m+3} w \in A(D) \) for otherwise, \( v_1 \cdot v_m v_{m+3} \cdot v_r \) is a path of length \( r - 2 \).

(iv) We claim \( m + 5 = r \). By Remark 3, we have \( m + 5 \leq r \). So suppose \( m + 5 < r \). Then we can find a \((v_1, v_r)\)-acyclic subgraph

\[ v_1 \cdot v_m v_{m+4} \cdot v_r \cup v_{m+1} v_{m+5} w_{m+5} w_{t_r} \]

The arc \( v_{m+1} v_{s+1} \) exists since there are no forwarding arcs of pace two and the existence of the arc \( v_{m+3} w \) is given in (iii).

(v) Since \( r \geq 7 \) and \( r = m + 5 \), we have that \( m - 1 \geq 1 \). We claim that \( v_{m-1} v_{r-1} \notin A(D) \). If not, we obtain a \((v_1, v_r)\)-acyclic subgraph of order \( r - 1 \): \( v_1 \cdot v_l v_{m-1} v_{r-1} v_r \cup v_{r-1} v_{m+2} w_{t_r} \). Where \( m + 4 = r - 1 \) is given in (iv), the arcs \( v_{r-1} v_{m+2}, v_{m+2} v_m \) exist to avoid forwarding arcs of pace two and \( v_m w \) is given by (ii).

(vi) By Lemma 5, \( O_P(v_{m-1}) \cap \{v_{r-1}, v_r\} \neq \emptyset \). Furthermore, by (v), \( v_{m-1} v_r \in A(D) \). However, we have assumed that \( I_P(v_r) = \{v_1, v_{r-1}\} \). Thus \( m - 1 = 1 \), that is \( m = 2, r = 7 \).

(vii) Let \( x' = \text{mid}(v_7 v_1) \). We have that \( x' \notin V(P) \) since \( I_P(v_7) = \{v_1, v_6\} \) and \( v_1 v_6 \notin A(D) \), by (v). But either \( v_1 \cdot v_4 x' v_7 \) or \( v_1 x' v_4 \cdot v_7 \) is a path of length 5 since one of \( v_4 x' \) or \( x' v_4 \in A(D) \). This contradicts our assumption that \( D \) has no \((v_1 v_r)\)-path of length \( r - 2 \) and completes the proof of our theorem. \( \square \)

Remark. The following example shows the conclusion of Theorem 7 cannot be strengthened. Notice \( r = 6 \) and in-degree of \( u = v_6 \) and out-degree of \( v = v_1 \) are equal to three:

\[ V(D) = \{v_1, \ldots, v_6\} \]
and

\[ A(D) = \{v_i v_{i+1} : i = 1, 2, \ldots, 5\} \cup \{v_{i+2} v_1 : i = 1, 2, 3, 4\} \]
\[ \cup \{v_{i+3} v_1 : i = 1, 2, 3\} \cup \{v_6 v_1, v_1 v_6, v_1 v_5, v_2 v_6, v_2 v_1\}. \]

This semicomplete digraph is arc-3-cyclic and has a 6-cycle containing \(v_6 v_1\) but no 5-cycle containing that arc (see Fig. 2).

**References**