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On Euler characteristic of equivariant sheaves

Alexander Braverman

Department of Mathematics, Harvard University, 1 Oxford St., Cambridge, MA 02138, USA

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Abstract

Let k be an algebraically closed field of characteristic $p>0$ and let ℓ be another prime number. Gabber and Looser proved that for any algebraic torus T over k and any perverse ℓ-adic sheaf $\mathcal{F}$ on T the Euler characteristic $\chi(\mathcal{F})$ is non-negative.

We conjecture that the same result holds for any perverse sheaf $\mathcal{F}$ on a reductive group G over k which is equivariant with respect to the adjoint action. We prove the conjecture when $\mathcal{F}$ is obtained by Goresky–MacPherson extension from the set of regular semi-simple elements in G. From this we deduce that the conjecture holds for G of semi-simple rank 1.

1. The main conjecture

1.1. Notations

In what follows, k denotes an algebraically closed field of characteristic $p>0$. Let ℓ be a prime number different from p. For an algebraic variety X over k we denote by $\mathcal{D}(X)$ the bounded derived category of ℓ-adic sheaves on X. Also we denote by $\mathrm{Perv}(X)\subset\mathcal{D}(X)$ the subcategory of perverse sheaves. For any $\mathcal{F}\in\mathcal{D}(X)$ we denote by $\chi(\mathcal{F})$ the Euler characteristic of $\mathcal{F}$, i.e.

$$\chi(\mathcal{F}) = \sum (-1)^i \dim H^i_c(X,\mathcal{F}) = \sum (-1)^i \dim H^i(X,\mathcal{F}) \tag{1.1}$$

(cf. [5] for the proof of the equality).

E-mail address: braval@math.harvard.edu.

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Let T be an algebraic torus over k. The following theorem is proved in [3] (cf. also [2] for a partial analogue in characteristic zero):

Theorem 1.2. *Let* $\mathcal{F} \in \mathrm{Perv}(T)$. *Then* $\chi(\mathcal{F}) \geqslant 0$.

Let G be a connected reductive algebraic group over k. We shall denote by $G_{\mathrm{rs}} \subset G$ the open subspace of regular semi-simple elements. We denote by $\mathrm{Perv}_G(G)$ the subcategory of $\mathrm{Perv}(G)$ consisting of perverse sheaves which are equivariant with respect to the adjoint action. We propose the following generalization of Theorem 1.2.

Conjecture 1.3. *Let* $\mathcal{F} \in \mathrm{Perv}_G(G)$. *Then* $\chi(\mathcal{F}) \geqslant 0$.

Theorem 1.4. *Assume that* $\mathcal{F} \in \mathrm{Perv}_G(G)$ *is equal to the Goresky–MacPherson extension of its restriction to* G_{rs}. *Then* $\chi(\mathcal{F}) \geqslant 0$.

Corollary 1.5. *Conjecture* 1.3 *holds for G of semi-simple rank* 1.

Proof. Let $\mathcal{N}$ be the cone of unipotent elements in G and let Z be the center of G. Clearly it is enough to prove Conjecture 1.3 only for irreducible sheaves. However, if $\mathcal{F} \in \mathrm{Perv}_G(G)$ is irreducible then either $\mathcal{F}$ is equal to the Goresky–MacPherson extension of its restriction to G_{rs} or it is supported on $Z\mathcal{N}$. The former case is covered by Theorem 1.4; hence it is enough to deal with the latter.

First of all there exists a connected component Z' of Z such that $\mathcal{F}$ is supported on $Z'\mathcal{N}$. Thus one of the following is true:

(1) $\mathcal{F}$ is supported on Z'.

(2) $\mathcal{F}$ is equal to $F' \boxtimes (\bar{\mathbb{Q}}_l)_{\mathcal{N}}$ [2] where $\mathcal{F}'$ is an irreducible perverse sheaf on Z' and $(\bar{\mathbb{Q}}_l)_{\mathcal{N}}$ is the constant sheaf on $\mathcal{N}$.

(3) $\mathcal{F} = \mathcal{F}' \boxtimes \mathcal{E}$ where $\mathcal{F}'$ is an irreducible perverse sheaf on Z' and $\mathcal{E}$ is an irreducible perverse sheaf on $\mathcal{N}$ whose restriction to the open orbit is isomorphic to (the only) non-trivial equivariant irreducible local system on this orbit (this local system is of rank 1 and its square is isomorphic to the constant sheaf).

In cases (1) and (2) our result follows immediately from Theorem 1.2 (note that Z' is a torus and that in case (2) we have $\chi(\mathcal{F}) = \chi(\mathcal{F}')$). In case (3) it is known that $H^*(\mathcal{N}, \mathcal{E}) = 0$, hence $H^*(G, \mathcal{F}) = 0$, hence $\chi(\mathcal{F}) = 0$. $\square$

Remark. Theorem 1.2 has an "ideological" explanation: namely in [3] Gabber and Loeser construct a "Mellin transform" functor $\mathcal{M} : \mathcal{D}(T) \to \mathcal{D}^b_{\mathrm{coh}}(\mathrm{Loc}_T)$ where Loc_T denotes the moduli space of tame rank one local systems on T and $\mathcal{D}^b_{\mathrm{coh}}(\mathrm{Loc}_T)$ is the bounded derived category of quasi-coherent sheaves on Loc_T. Moreover, for every $\mathcal{F} \in \mathrm{Perv}(T)$ the complex $\mathcal{M}(F)$ is actually a sheaf and $\chi(\mathcal{F})$ is equal to the generic rank of $\mathcal{M}(\mathcal{F})$ (hence $\chi(\mathcal{F}) \geqslant 0$). We do not know whether a similar explanation of Conjecture 1.3 is possible.

We expect that Conjecture 1.3 holds also when k has characteristic 0 and perverse sheaves are replaced by holonomic D-modules (the analogues of Theorem 1.4 and Corollary 1.5 do hold in this case—the proofs discussed in this paper apply without change to holonomic D-modules instead of l-adic perverse sheaves). When G is a torus and the D-modules in question have regular singularities a beautiful geometric proof of Theorem 1.2 was given by Franecki and Kapranov [2] where he explains how to compute the corresponding Euler characteristics using certain non-compact generalization of Dubson–Kashiwara index theorem. This technique has been very recently generalized by Kiritchenko [4] to the case of ad-equivariant perverse sheaves on a reductive group G over $\mathbb{C}$ (in particular she proves the analogue of Theorem 1.4 for perverse sheaves over $\mathbb{C}$). It would be interesting to generalize the arguments of [2,4] to the case of arbitrary holonomic D-modules (unfortunately, it is probably impossible to obtain a similar proof for l-adic sheaves when the characteristic of k is $p>0$).

2. Proof of Theorem 1.4

2.1. The space $\tilde{G}$. Let $\mathscr{B}$ denote the flag variety of G, i.e. the variety of all Borel subgroups of G. Let $\tilde{G}$ denote the variety of pairs $(B\in\mathscr{B},\ g\in B)$. We have the natural maps $\pi:\tilde{G}\to G$, $\alpha:\tilde{G}\to T$ and $\beta:\tilde{G}\to\mathscr{B}$. It is easy to see that α is smooth and π is proper. Let $d=\dim G-\dim T=2\dim\mathscr{B}$. Set $\tilde{G}_{\mathrm{rs}}=\pi^{-1}(G_{\mathrm{rs}})=\alpha^{-1}(T_{\mathrm{rs}})$.

Lemma 2.2. (1) *Let $\mathscr{G}\in\mathrm{Perv}(T)$. Assume that $\mathscr{G}$ is equal to the Goresky–MacPherson extension of its restriction on T_{rs}. Then $\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2})$ is a perverse sheaf which is equal to the Goresky–MacPherson extension of its restriction to G_{rs}.*

(2) *Every W-equivariant structure on a sheaf $\mathscr{G}\in\mathrm{Perv}(T)$ as above gives rise to a W-action on $\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2})$.*

(3) *Let $\mathscr{F}\in\mathrm{Perv}_G(G)$. Assume that $\mathscr{F}$ is equal to the Goresky–MacPherson extension of its restriction to G_{rs}. Then there exists a W-equivariant sheaf $\mathscr{G}\in\mathrm{Perv}(T)$ which is equal to the Goresky–MacPherson extension of its restriction to T_{rs} such that*

$$\mathscr{F}=\left(\pi_!\alpha^*\mathscr{G}[d]\left(\frac{d}{2}\right)\right)^W. \tag{2.1}$$

Proof. The first statement of Lemma 2.2 is well-known (it follows from the smallness property of π—cf. [1] and references therein).

It follows from 1 that in order to prove 2 it is enough to construct an action of W on the restriction of $\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2})$ to G_{rs}. Note that W acts freely on $\tilde{G}_{\mathrm{rs}}$ in the natural way and

$$G_{\mathrm{rs}}=\tilde{G}_{\mathrm{rs}}/W. \tag{2.2}$$

Moreover, the restriction of α to $\tilde{G}_{\rm rs}$ is W-equivariant. Hence $\alpha^*\mathscr{G}[d](\frac{d}{2})|_{\tilde{G}_{\rm rs}}$ is W-equivariant and thus (2.2) implies that $\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2})|_{G_{\rm rs}}$ has a natural action of W.

Let us prove (3). Choose an embedding $i: T\to G$. Let $i_{\rm rs}: T_{\rm rs}\to G$ be its restriction to $T_{\rm rs}$. Let $\mathscr{G}_{\rm rs} = i^*_{\rm rs}\mathscr{F}[-d](-\frac{d}{2})$. It follows from the G-equivariance of $\mathscr{F}$ that $\mathscr{G}_{\rm rs}$ is perverse. We let $\mathscr{G}$ be its Goresky–MacPherson extension to T.

Let us prove that $\mathscr{G}$ satisfies (2.1). Since both $\mathscr{F}$ and $(\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2}))^W$ are equal to the Goresky–MacPherson extensions of their restrictions to $G_{\rm rs}$ it is enough to construct an isomorphism between $\mathscr{F}|_{G_{\rm rs}}$ and $((\pi_!\alpha^*\mathscr{G}[d](\frac{d}{2}))^W)|_{G_{\rm rs}}$. Since both sheaves are equivariant with respect to the adjoint action of G it is enough to construct a W-equivariant isomorphism of their restrictions to $T_{\rm rs}$ where both sheaves are canonically isomorphic to $\mathscr{G}_{\rm rs}$. □

Proposition 2.3. *Let $\mathscr{F}$ be as in Theorem* 1.4 *and let $\mathscr{G}$ be as in Lemma* 2.2(3). *Then* $\chi(\mathscr{F}) = \chi(\mathscr{G})$.

Clearly Proposition 2.3 together with Theorem 1.2 imply Theorem 1.4.

2.4. Proof of Proposition 2.3. Set

$$\mathscr{H} = \alpha^*\mathscr{G}[d]\left(\frac{d}{2}\right)\in \mathrm{Perv}(\tilde{G}).$$

Clearly, $H^*_c(G,\mathscr{F}) = H^*_c(\tilde{G},\mathscr{H})^W$.

Consider $\alpha_!(\mathscr{H})$. By the projection formula it is isomorphic to $\mathscr{G}\otimes\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}[d](\frac{d}{2})$. It is easy to see that the complex $\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}[d](\frac{d}{2})$ is constant with fibers isomorphic to $H^*(\mathscr{B},\bar{\mathbb{Q}}_l)$. Indeed, let $\delta:\mathscr{B}\times T\to T$ denote the projection to the first multiple. Then $\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}} = \delta_!(\beta\times\alpha)_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}$. However, the map $(\beta\times\alpha):\tilde{G}\to\mathscr{B}\times T$ is a locally trivial fibration (in Zariski topology) with fiber isomorphic to $\mathbb{A}^d$. Thus $(\beta\times\alpha)_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}\simeq(\bar{\mathbb{Q}}_l)_{\mathscr{B}\times T}[-d](-\frac{d}{2})$ and hence

$$\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}\simeq(\bar{\mathbb{Q}}_l)_T\otimes H^*(\mathscr{B},\bar{\mathbb{Q}}_l)[-d]\left(-\frac{d}{2}\right). \tag{2.3}$$

This clearly gives rise to the isomorphism

$$H^*_c(\tilde{G},\mathscr{H})\simeq H^*_c(T,\mathscr{G})\otimes H^*(\mathscr{B},\bar{\mathbb{Q}}_l). \tag{2.4}$$

We can also characterize isomorphism (2.3) in the following way. To determine it uniquely it is enough to construct it on $T_{\rm rs}$. Choose as before an embedding $i: T\to G$. There is a canonical Borel subgroup B containing $i(T)$ (recall that the abstract Cartan group T comes with a root system with a preferred set of positive roots). Then for every $t\in T_{\rm rs}$ we have canonical isomorphism

$$\alpha^{-1}(t)\simeq G/T \tag{2.5}$$

(which depends however on the above choice). Clearly

$$H_c^*(G/T, \bar{\mathbb{Q}}_l)[d]\left(\frac{d}{2}\right) = H^*(\mathscr{B}, \bar{\mathbb{Q}}_l).$$

Hence, we have constructed an isomorphism between $\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}[d](\frac{d}{2})|_{T_{\mathrm{rs}}}$ and the constant complex with fiber $H^*(\mathscr{B}, \bar{\mathbb{Q}}_l)$. It is easy to see that this isomorphism does not depend on the choice of the embedding $T \to G$ and coincides with (2.3).

Let W act on the left-hand side of (2.4) by means of the identification $H_c^*(\tilde{G}, \mathscr{H}) = H_c^*(G, \pi_!\mathscr{H})$ (note that by Lemma 2.2 the group W acts on $\pi_!\mathscr{H}$). Let W also act on the right-hand side by means of the tensor product of the W-action on $H_c^*(T, \mathscr{G})$ coming from the W-equivariant structure on $\mathscr{G}$ and the natural W-action on $H^*(\mathscr{B}, \bar{\mathbb{Q}}_l)$.

Lemma 2.5. *Isomorphism* (2.4) *is also an isomorphism of W-modules with respect to the above actions.*

Proof. Let $Z = G \times_{T/W} T$ be the image of $\tilde{G}$ in $G \times T$ under the map $\gamma = \pi \times \alpha$. This is a closed subvariety of $G \times T$. It is invariant with respect to the W action on the second multiple. Let Z_{rs} denote the set of "regular semi-simple" elements of Z (i.e. the set of all elements of Z whose projection to G is regular semi-simple).

Let $\mathscr{K} = \gamma_!(\mathscr{H})$. Then $\mathscr{K}$ is equal to the Goresky–MacPherson extension of its restriction to Z_{rs}. Since γ induces an isomorphism $\tilde{G}_{\mathrm{rs}} \simeq Z_{\mathrm{rs}}$ and since the restriction of $\mathscr{H}$ to $\tilde{G}_{\mathrm{rs}}$ is W-equivariant it follows that $\mathscr{K}$ also has a natural W-equivariant structure as a perverse sheaf on $G \times T$ (where the W-action is as before on the second multiple). Thus W acts naturally on $H_c^*(G \times T, \mathscr{K})$. We have the natural isomorphism $H_c^*(\tilde{G}, \mathscr{H}) \simeq H_c^*(G \times T, \mathscr{K})$ and by the definition the action of W on $H_c^*(\tilde{G}, \mathscr{H})$ introduced before Lemma 2.5 corresponds to the action of W on $H_c^*(G \times T, \mathscr{K})$ introduced above.

Consider, on the other hand, the complex $p_!(\mathscr{K})$ where $p : G \times T \to T$ denotes the natural projection. Clearly, we have $p_!(\mathscr{K}) = \alpha_!(\mathscr{H}) \simeq \mathscr{G} \otimes H^*(\mathscr{B}, \bar{\mathbb{Q}}_l)$. On the other hand, since p commutes with W it follows that $p_!(\mathscr{K})$ admits a natural W-equivariant structure. To prove Lemma 2.5 it is enough to show that under the identification $p_!(\mathscr{K}) \simeq \mathscr{G} \otimes H^*(\mathscr{B}, \bar{\mathbb{Q}}_l)[d](\frac{d}{2})$ this structure is equal to the tensor product of the original W-equivariant structure on $\mathscr{G}$ with the natural W-action on $H^*(\mathscr{B}, \bar{\mathbb{Q}}_l)$. Since every perverse cohomology of $p_!(\mathscr{K})$ is equal to the Goresky–MacPherson extension of its restriction to T_{rs} it follows that it is enough to check this equality only on T_{rs} as guaranteed by the following lemma.

Lemma 2.6. *Let X be a k-scheme, $U \subset X$—an open subset. Let $\mathscr{F} \in \mathscr{D}(X)$ such that every perverse cohomology of $\mathscr{F}$ is equal to its Goresky–MacPherson extension from U.*

Let σ be an automorphism of $\mathscr{F}$. Assume that $\sigma|_U = \mathrm{id}$. Then $\sigma = \mathrm{id}$.

Proof. To prove that $\sigma = \mathrm{id}$ on an object $\mathcal{F} \in \mathcal{D}(X)$ it is enough to prove that σ acts as identity on every perverse cohomology of $\mathcal{F}$ which follows immediately from the assumption that $\sigma|_U = \mathrm{id}$. $\square$

However, the fact that the above W-equivariant structures coincide follows immediately from the description of the W-equivariant structure on $\alpha_!(\bar{\mathbb{Q}}_l)_{\tilde{G}}[d](\frac{d}{2})|_{\tilde{G}_{\mathrm{rs}}}$ given in Section 2.4 and the following observation:

Choose as before an embedding $T \to G$. Let $N(T)$ denote the normalizer of T in G. Then $W = N(T)/T$ acts on G/T and hence on $H_c^*(G/T, \bar{\mathbb{Q}}_l)$. By identifying $H_c^*(G/T, \bar{\mathbb{Q}}_l)[d](\frac{d}{2})$ with $H^*(\mathcal{B}, \bar{\mathbb{Q}}_l)$ we get an action of W on the latter. This is the standard W-action on $H^*(\mathcal{B}, \bar{\mathbb{Q}}_l)$. $\square$

It follows from Lemma 2.5 that $\chi(\mathcal{F})$ is equal to the Euler characteristic of $(H_c^*(\mathcal{G}) \otimes H^*(\mathcal{B}, \bar{\mathbb{Q}}_l))^W$. However, it is known that $H^*(\mathcal{B}, \bar{\mathbb{Q}}_l)$ lives only in even degrees and when we forget the grading it is isomorphic to the regular representation $\bar{\mathbb{Q}}_l[W]$ of W. Hence

$$\chi(\mathcal{F}) = \chi((H_c^*(T, \mathcal{G}) \otimes H^*(\mathcal{B}, \bar{\mathbb{Q}}_l)[d])^W)$$
$$= \chi((H_c^*(T, \mathcal{G}) \otimes \bar{\mathbb{Q}}_l[W])^W) = \chi(H_c^*(T, \mathcal{G})) = \chi(\mathcal{G})$$

which finishes the proof. $\square$

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