JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 30, 113–127 (1970)

Singularly Perturbed Differential Equations in a Hilbert Space

L. E. Bobisud

AND

James Calvert

University of Idaho, Moscow, Idaho 83843

Submitted by N. Coburn

Introduction

Let A be a self-adjoint (not necessarily bounded) operator in a Hilbert space H with norm $\|\cdot\|$. We shall consider, for $t \geqslant 0$ and small $\epsilon > 0$, the differential equation

$$L_\epsilon[u_\epsilon(t)] \equiv \sum_{j=1}^{m} \epsilon^j a_{(n+j)} u_\epsilon^{(n+j)}(t) + \sum_{k=1}^{n} a_k u_\epsilon^{(k)}(t) + Au_\epsilon(t) = 0, \qquad (1)$$

where the a_p are constants and $m \geqslant 1$, $n \geqslant 1$, along with the initial conditions

$$u_\epsilon^{(k)}(0) = x_k \qquad (k = 0, 1,..., m + n - 1), \qquad (2)$$

where the x_k are elements of a certain dense subset of H (namely, the domain of e^{A^2}). We shall also consider the *reduced* or *degenerate* problem

$$L_0[U(t)] \equiv \sum_{k=1}^{n} a_k U^{(k)}(t) + AU(t) = 0, \qquad (3)$$

$$U^{(k)}(0) = x_k \qquad (k = 0, 1,..., n - 1). \qquad (4)$$

We show, under simple assumptions on the coefficients a_k, that these problems have unique solutions in H and that $\| u_\epsilon(t) - U(t)\| \to 0$ as $\epsilon \to 0$, uniformly for $t \in [0, T]$ for any finite $T > 0$.

The case $m = n = 1$, A nonnegative, was considered by Kisyński [1], who used somewhat different methods than we shall use. Smoller ([2], [3]) extended these results to the equation

$$\epsilon u_\epsilon'''(t) + u_\epsilon'(t) + Au_\epsilon(t) = 0,$$

409/30/1-8

and showed that $\| u_\epsilon(t) - U(t)\| \to 0$ is not true for initial data in *any* dense subset of H for a perturbed equation of the form

$$\epsilon u_\epsilon^{(1+p)}(t) + u_\epsilon'(t) + Au_\epsilon(t) = 0,$$

for $p \geqslant 3$. Latil [4] extended the results of Smoller to the higher-order equation

$$\epsilon u_\epsilon^{(k+p)}(t) + u_\epsilon^{(k)}(t) + a_{k-1}u_\epsilon^{(k-1)}(t) + \cdots + a_1 u_\epsilon'(t) + Au_\epsilon(t) = 0,$$

showing that $\| u_\epsilon(t) - U(t)\| \to 0$ on a dense subset of H if, and only if, $p = 1$ or $p = 2$; he also showed that $\|u_\epsilon^{(n)}(t) - U^{(n)}(t)\| \to 0$ for certain n and $t > 0$ provided $p = 1$ or 2.

Our results contain those of Smoller and Latil for $p = 1$, but for $p \geqslant 2$ the form of the equation considered here is different from that considered by these authors. The basic outline of the proof presented here follows that of Smoller and Latil, but we obtain the necessary estimates by representing the solution as an integral in the complex plane, thus avoiding use of determinants. Recently Friedman [7] has studied the degeneration of (1) with variable coefficients a_i for both the Cauchy and boundary-value problems. For the constant-coefficient case our assumptions are weaker than his in that we do not require the roots ν_i defined below to be distinct.

A Special Case; Preliminary Estimates

We first consider the case $H = R$, the real line, obtaining estimates which will be used later with the functional calculus for the operator A to extend the results to a general Hilbert space. We thus consider the differential operators

$$L_\epsilon \equiv \sum_{j=1}^{m} \epsilon^j a_{n+j} \left(\frac{d}{dt}\right)^{n+j} + \sum_{k=0}^{n} a_k \left(\frac{d}{dt}\right)^k,$$

$$L_0 \equiv \sum_{k=0}^{n} a_k \left(\frac{d}{dt}\right)^k,$$

where $a_0 \equiv \lambda$, a real parameter. *We assume*, as we may without loss of generality, *that* $a_{m+n} = 1$, $a_n \neq 0$. We denote the characteristic polynomial of L_ϵ by

$$P(\mu, \epsilon, \lambda) \equiv \sum_{j=1}^{m} \epsilon^j a_{j+n} \mu^{j+n} + \sum_{k=0}^{n} a_k \mu^k,$$

and set, for convenience,

$$Q(\sigma, \epsilon, \lambda) = \epsilon^n P\left(\frac{\sigma}{\epsilon}, \epsilon, \lambda\right) = \sum_{j=1}^{m} a_{j+n}\sigma^{j+n} + \sum_{k=0}^{n} \epsilon^{n-k} a_k \sigma^k.$$

We denote the zeros of the characteristic polynomial $P(\mu, 0, \lambda)$ of L_0 by $\mu_1, \ldots, \mu_n$; these zeros are functions of λ but not of ϵ. Similarly, we denote the *nonzero* roots of $Q(\sigma, 0, \lambda) = 0$ by $\nu_1, \ldots, \nu_m$; these roots depend on neither λ nor ϵ. We allow any of these zeros to be multiple, and, in the case of the μ_i, these zeros may be multiple for only certain values of λ or identically in λ. One of the chief advantages of our approach lies in the fact that we do not have to distinguish between single and multiple roots in the estimates to follow.

We can now state our main assumption on the coefficients $a_n, \ldots, a_{n+m}$:

Fundamental Assumptions

The roots ν_i satisfy

$$\operatorname{Re}(\nu_i) < 0 \qquad (i = 1, \ldots, m).$$

Thus there exists $\beta > 0$ such that $\operatorname{Re}(\nu_i) < -\beta$ $(i = 1, \ldots, m)$; henceforth we assume this is satisfied, without explicit mention. Notice that the fundamental assumption concerns not the polynomial $P(\mu, \epsilon, \lambda)$, of degree $n + m$, but rather the polynomial $\sigma^{-n}Q(\sigma, 0, \lambda)$, of degree m.

The following lemma on the roots of $P(\mu, \epsilon, \lambda)$ is basic to our approach:

Lemma 1. *The zeros of the polynomial $P(\mu, \epsilon, \lambda)$ can be labeled $\bar{\mu}_i$ $(i = 1, \ldots, n)$ and $\bar{\nu}_{j/\epsilon}$ $(j = 1, \ldots, m)$ in such a way that*

$$\bar{\mu}_i = \mu_i + 0(\epsilon) \qquad (i = 1, \ldots, n) \tag{5}$$

uniformly in λ and

$$\bar{\nu}_j = \nu_j + 0(\epsilon) + 0(\epsilon^n \lambda) \qquad (j = 1, \ldots, m). \tag{6}$$

The proof of this lemma differs from the proof of the corresponding result of Visik and Lyusternik ([5], p. 252, 262-3) only in the attention paid to the parameter λ, and will, therefore, be omitted. We henceforth assume the roots of $P(\mu, \epsilon, \lambda)$ indexed as in the lemma.

Lemma 2. *All zeros of $Q(\sigma, \epsilon, \lambda)$ lie in the disk*

$$|\sigma| < 1 + N + \epsilon^n |\lambda|,$$

where

$$N = \max(|a_{n+m-1}|, \ldots, |a_1|),$$

for $0 \leqslant \epsilon \leqslant 1$. *The roots* μ_i *lie in the disk*

$$|\mu| < 1 + M + \frac{1}{|a_n|}|\lambda|,$$

where

$$M = \frac{1}{|a_n|}\max(|a_{n-1}|,\dots,|a_1|).$$

This lemma follows at once from Theorem 27, 2 of Marden ([6], p. 96).

We define functions $s_\alpha(t, \epsilon, \lambda)$ by

$$s_\alpha(t, \epsilon, \lambda) = \frac{\epsilon^\alpha}{2\pi i}\oint \frac{Q_\alpha(\mu, \epsilon)}{Q(\mu, \epsilon, \lambda)} e^{(\mu/\epsilon)t}\, d\mu \qquad (\alpha = 0,\dots, m+n-1), \tag{7}$$

where the positively oriented path of integration includes in its interior all zeros of $Q(\mu, \epsilon, \lambda)$ and where

$$Q_\alpha \equiv \mu^{n-\alpha}\sum_{k=1}^{m} \mu^{k-1}a_{n+k} + \sum_{k=1}^{n-\alpha} \epsilon^{k-1}a_{n+1-k}\mu^{n-\alpha-k} \qquad (\alpha = 0, 1,\dots, n-1),$$

$$Q_\alpha \equiv \sum_{k=1}^{m+n-\alpha} \mu^{k-1}a_{\alpha+k} \qquad (\alpha = n,\dots, m+n-1). \tag{8}$$

$Q_\alpha(\mu, \epsilon)$ has degree $n + m - \alpha - 1$ and consists of the product of $\mu^{-\alpha-1}$ and the leading $m + n - \alpha$ terms of $Q(\mu, \epsilon, \lambda)$.

LEMMA 3. *For* $\epsilon > 0$ *and for each* $\alpha = 0, 1,\dots, m+n-1$, $L_\epsilon[s_\alpha] = 0$ *and*

$$\left(\frac{d}{dt}\right)^j s_\alpha(0, \epsilon, \lambda) = \delta_\alpha{}^j; \tag{9}$$

the solution of the initial-value problem $L_\epsilon[u_\epsilon] = 0$, $u_\epsilon^{(k)}(0) = x_k \in R$ $(k = 0, 1,\dots, m+n-1)$ *is given by*

$$u_\epsilon = \sum_{\alpha=0}^{m+n-1} s_\alpha(t, \epsilon, \lambda)\, x_\alpha .$$

Proof. That $L_\epsilon[s_\alpha] = 0$ follows from the computation

$$L_\epsilon[s_\alpha] = \frac{\epsilon^\alpha}{2\pi i}\oint \frac{Q_\alpha(\mu, \epsilon)}{Q(\mu, \epsilon, \lambda)} L_\epsilon[e^{(\mu/\epsilon)t}]\, d\mu = \frac{\epsilon^{\alpha-n}}{2\pi i}\oint Q_\alpha(\mu, \epsilon)\, e^{(\mu/\epsilon)t}\, d\mu = 0.$$

The third statement follows from the first two and the linearity of L_ϵ; it thus remains to establish the validity of (9). To this end we note the identity

$$\mu^{\alpha+1}Q_\alpha - Q = \begin{cases} \sum_{j=1}^{\alpha-n} a_{n+j}\mu^{n+j} + \sum_{k=0}^{n} \epsilon^{n-k} a_k \mu^k & (\alpha = n,\dots, m+n-1) \\ \sum_{k=0}^{\alpha} \epsilon^{n-k} a_k \mu^k & (\alpha = 0,\dots, n-1); \end{cases} \tag{10}$$

i.e., $\mu^{\alpha+1}Q_\alpha = Q + p_\alpha(\mu, \epsilon, \lambda)$, where p_α is a polynomial in μ of deg α. Thus

$$\left(\frac{d}{dt}\right)^j s_\alpha(0, \epsilon, \lambda) = \frac{\epsilon^{\alpha-j}}{2\pi i} \oint \frac{\mu^j Q_\alpha(\mu, \epsilon)}{Q(\mu, \epsilon, \lambda)} d\mu$$
$$= \frac{\epsilon^{\alpha-j}}{2\pi i} \oint \mu^{j-\alpha-1}\, d\mu + \frac{\epsilon^{\alpha-j}}{2\pi i} \oint \frac{\mu^{j-\alpha-1} p_\alpha(\mu, \epsilon, \lambda)}{Q(\mu, \epsilon, \lambda)} d\mu = \delta_\alpha^{\ j},$$

where we see that the final integral is zero by expanding the contour of integration to infinity, observing that the numerator of the integrand is a polynomial in μ of degree $j - 1 \leqslant m + n - 2$, hence, at least two lower than the degree of the denominator.

For the degenerate problem we define, in a similar fashion,

$$w_\alpha(t, \lambda) = \frac{1}{2\pi i} \oint \frac{P_\alpha(\mu)\, e^{\mu t}}{P(\mu, 0, \lambda)} d\mu \qquad (\alpha = 0, 1,\dots, n-1),$$

where

$$P_\alpha(\mu) = \sum_{k=\alpha+1}^{n} a_k \mu^{k-\alpha-1},$$

and the contour of integration encircles all zeros of $P(\mu, 0, \lambda)$. In the manner of Lemma 3 we have

LEMMA 4.

$$L_0[w_\alpha] = 0, \qquad \left(\frac{d}{dt}\right)^j w_\alpha(0, \lambda) = \delta_\alpha^{\ j} \qquad (\alpha = 0,\dots, n-1);$$

the solution of the initial-value problem $L_0[U] = 0$, $U^{(k)} = x_k$ $(k = 0,\dots, n-1)$ *is given by*

$$U = \sum_{\alpha=0}^{n-1} w_\alpha(t, \lambda)\, x_\alpha .$$

THEOREM 1. *Let* $u_\epsilon(t, \lambda)$ *and* $U(t, \lambda)$ *be defined as in Lemmas* 3, 4. *Then for each fixed* λ *and any finite* $T > 0$,

$$\sup_{t\in[0,T]} | u_\epsilon(t, \lambda) - U(t, \lambda)| \to 0$$

as $\epsilon \to 0$.

For the case of distinct roots ν_i, μ_i, this is a known result [5]. We give a proof which is valid regardless of the multiplicities of these roots. In view of Lemmas 3 and 4, the theorem follows at once from the following lemma.

LEMMA 5. *For* $\alpha = n,..., m + n - 1$ *and fixed* λ,

$$\sup_{t\in[0,T]} | s_\alpha(t, \epsilon, \lambda)| \to 0$$

as $\epsilon \to 0$. *For* λ *fixed and* $\alpha = 0,..., n - 1$,

$$\sup_{t\in[0,T]} | s_\alpha(t, \epsilon, \lambda) - w_\alpha(t, \lambda)| \to 0$$

as $\epsilon \to 0$.

Proof. First we consider s_α for $\alpha \geqslant n$. Observing Lemma 2 and the definition of $Q(\mu, \epsilon, \lambda)$, we take for contour of integration the union of the circles

$$C_0(\epsilon): \qquad | \mu | = 2\epsilon \left(2 + M + \frac{1}{| a_n |} | \lambda |\right),$$

$$C_i: \qquad | \mu - \nu_i | = r \qquad (i = 1,..., m),$$

where we choose $r \leqslant 3\beta/4$ small enough that these latter circles are either coincident (if any ν_i are multiple) or disjoint. We stipulate that ϵ be small enough that

$$| \bar{\mu}_i - \mu_i | < 1, \qquad | \bar{\nu}_i - \nu_i | < \frac{r}{2} \qquad \text{[cf. (5), (6)]},$$

$$2\epsilon \left(2 + M + \frac{1}{| a_n |} | \lambda |\right) < \frac{\beta}{4}.$$

Then on $C_0(\epsilon)$ we have

$$| Q(\mu, \epsilon, \lambda)| = \prod_{j=1}^{m} | \mu - \bar{\nu}_i | \cdot \prod_{k=1}^{n} | \mu - \epsilon\bar{\mu}_k |$$

$$\geqslant \left(\frac{r}{2}\right)^m \epsilon^n \left(2 + M + \frac{1}{| a_n |} | \lambda |\right)^n,$$

since μ lies outside C_i,

$$|\mu - \nu_i| > r, \qquad \text{and} \qquad |\mu - \bar{\nu}_i| \geqslant |\mu - \nu_i| - |\nu_i - \bar{\nu}_i| > \frac{r}{2},$$

and since $|\mu - \epsilon\bar{\mu}_k| \geqslant |\mu| - \epsilon|\bar{\mu}_k|$ is

$$2\epsilon\left(2 + M + \frac{1}{|a_n|}|\lambda|\right) - \epsilon\left(1 + M + \frac{1}{|a_n|}|\lambda|\right) \geqslant \epsilon\left(2 + M + \frac{1}{|a_n|}|\lambda|\right),$$

by Lemma 2. The above requirements on ϵ imply that $\epsilon|\bar{\mu}_k| < \beta/8$, whence for μ on C_i, $|\mu - \epsilon\bar{\mu}_k| \geqslant |\mu| - \epsilon|\bar{\mu}_k| \geqslant \beta/8$, and $|\mu - \bar{\nu}_i| > r/2$; thus on C_i

$$|Q(\mu, \epsilon, \lambda)| \geqslant \left(\frac{r}{2}\right)^m \left(\frac{\beta}{8}\right)^n.$$

Denote the union of $C_0(1)$ and its interior by $\overline{C_0(1)}$, and let M_α denote a bound for $|Q_\alpha|$ on $\overline{C_0(1)} \cup C_1 \cup \cdots \cup C_m$. Then

$$\begin{aligned} |s_\alpha(t, \epsilon, \lambda)| &\leqslant \frac{\epsilon^\alpha}{2\pi} \oint_{C_0(\epsilon)} \left|\frac{Q_\alpha(\mu, \epsilon)}{Q(\mu, \epsilon, \lambda)}\right| e^{\mathrm{Re}[(\mu/\epsilon)t]} |d\mu| \\ &\quad + \sum_{i=1}^m \frac{\epsilon^\alpha}{2\pi} \oint_{C_i} \left|\frac{Q_\alpha}{Q}\right| e^{\mathrm{Re}[(\mu/\epsilon)t]} |d\mu| \\ &\leqslant \frac{2\epsilon^\alpha M_\alpha \exp\left[2\left(2 + M + \frac{1}{|a_n|}|\lambda|\right)t\right]}{\left(\frac{r}{2}\right)^m \epsilon^{n-1}\left(2 + M + \frac{1}{|a_n|}|\lambda|\right)^{n-1}} + \sum_{i=1}^m \frac{r\epsilon^\alpha M_\alpha e^{-(\beta t/4\epsilon)}}{\left(\frac{r}{2}\right)^m \left(\frac{\beta}{8}\right)^n} \\ &= \epsilon^{\alpha+1-n} \cdot \exp\left[2\left(2 + M + \frac{1}{|a_n|}|\lambda|\right)t\right] \cdot \text{const} + \epsilon^\alpha \cdot \text{const}. \end{aligned}$$

Since $\alpha \geqslant n$, the first part of the lemma follows.

To finish the proof, we show that for $0 \leqslant \alpha < n$

$$\epsilon^\alpha \oint_{C_0(\epsilon)} \frac{Q_\alpha(\sigma, \epsilon)}{Q(\sigma, \epsilon, \lambda)} e^{(\sigma/\epsilon)t} \, d\sigma - \oint_{|\mu|=K} \frac{P_\alpha(\mu)}{P(\mu, \lambda)} e^{\mu t} \, d\mu \to 0 \tag{11}$$

as $\epsilon \to 0$, where for convenience we set $K = 2(2 + M + [|\lambda|/|a_n|])$, and we show that

$$\epsilon^\alpha \oint_{C_i} \frac{Q_\alpha(\sigma, \epsilon)}{Q(\sigma, \epsilon, \lambda)} e^{(\sigma/\epsilon)t} \, d\sigma \to 0 \qquad (i = 1, \ldots, m) \tag{12}$$

as $\epsilon \to 0$. To prove (11), we set $\sigma = \epsilon\mu$ in the first integral, observing that

$$\epsilon^{\alpha+1}Q_\alpha(\epsilon\mu, \epsilon) = \epsilon^n \left[\sum_{k=1}^{m} \epsilon^k \mu^{n+k-\alpha-1} a_{n+k} + \sum_{k=1}^{n-\alpha} \mu^{n-\alpha-k} a_{n+1-k}\right]$$

$$\equiv \epsilon^n Z_\alpha(\mu, \epsilon),$$

where clearly $Z_\alpha(\mu, \epsilon) \to P_\alpha(\mu)$ as $\epsilon \to 0$, uniformly in μ for $|\mu|$ bounded. Since also $Q(\epsilon\mu, \epsilon, \lambda) = \epsilon^n P(\mu, \epsilon, \lambda)$, the left side of (11) is bounded by

$$\oint_{|\mu|=K} \left| \frac{Z_\alpha(\mu, \epsilon)}{P(\mu, \epsilon, \lambda)} - \frac{P_\alpha(\mu)}{P(\mu, \lambda)} \right| | e^{\mu t} | \, | d\mu |,$$

which converges to zero as $\epsilon \to 0$ since the integrand does so uniformly. To prove (12) we observe from (8) and the definition of Q that

$$\frac{Q_\alpha(\sigma, \epsilon)}{Q(\sigma, \epsilon, \lambda)} \to \frac{1}{\sigma^{\alpha+1}}$$

uniformly in σ for σ on C_i, so

$$\oint_{C_i} \frac{Q_\alpha(\sigma, \epsilon)}{Q(\sigma, \epsilon, \lambda)} e^{(\sigma/\epsilon)t} \, d\sigma \to \oint_{C_i} \frac{e^{(\sigma/\epsilon)t}}{\sigma^{\alpha+1}} \, d\sigma = 0,$$

since the integrand of the last integral has no poles inside C_i. This finishes the proof of the Lemma and of Theorem 1.

We now establish two lemmas which will be needed in the abstract Hilbert-space problem.

LEMMA 6. *For each* $\alpha = 0, 1, \ldots, m + n - 1$

$$| s_\alpha(t, \epsilon, \lambda)|^2 \leqslant f_\alpha(t, \epsilon) \, e^{\lambda^2},$$

where f_α *is bounded on* $[0, T] \times (0, \epsilon_0]$ *for some* $\epsilon_0 > 0$ *and each* $T > 0$.

Proof. Let $q \leqslant 1$ be a number small enough that the term denoted $0(\epsilon) + 0(\epsilon^n\lambda)$ in (6) is less than $\beta/2$ in magnitude for $\epsilon \leqslant q$, $\epsilon^n |\lambda| \leqslant q$. Then we have

$$\text{Re}(\bar{\nu}_i) \leqslant -\frac{\beta}{2} < 0 \qquad (i = 1, \ldots, m).$$

Moreover, by reducing q if necessary, we have from Lemmas 1 and 2 that

$$| \epsilon\bar{\mu}_i | \leqslant \epsilon \left(2 + M + \frac{1}{|a_n|} |\lambda|\right).$$

Thus all the zeros of $Q(\mu, \epsilon, \lambda)$ are located in the half-plane

$$\operatorname{Re}(\mu) \leqslant \epsilon \left(2 + M + \frac{1}{|a_n|} |\lambda|\right)$$

provided $0 < \epsilon \leqslant q$, $|\epsilon^n \lambda| \leqslant q$.

We first estimate s_α for $n \leqslant \alpha \leqslant m + n - 1$, $0 < \epsilon \leqslant q$, $|\epsilon^n \lambda| \leqslant q$. For the contour of integration in the definition of s_α we take the boundary of the intersection of the circle $|\mu| \leqslant 2 + N + q$ and the half-plane

$$\operatorname{Re}(\mu) \leqslant 2\epsilon \left(2 + M + \frac{1}{|a_n|} |\lambda|\right).$$

Let

$$M_\alpha = \sup_{|\mu| \leqslant 2+N+q} |Q_\alpha(\mu, \epsilon)|;$$

M_α is independent of λ and ϵ since Q_α is. We have, on the contour of integration,

$$|Q(\mu, \epsilon, \lambda)| \geqslant \left(\min\left[1, \frac{1}{2}\beta\right]\right)^m \epsilon^n \left(2 + M + \frac{1}{|a_n|} |\lambda|\right)^n \geqslant \epsilon^n \cdot \text{const.}$$

Using these bounds on $|1/Q|$ and $|Q_\alpha|$ and the bound

$$|e^{(\mu/\epsilon)t}| \leqslant \exp\left[2\left(2 + M + \frac{1}{|a_n|} |\lambda|\right) t\right]$$

in the definition of s_α, it is easy to derive the estimate

$$|s_\alpha(t, \epsilon, \lambda)| \leqslant \text{const}\ \epsilon^{\alpha-n} e^{2(2+M)t} e^{[(2/|a_n|)|\lambda| t]},$$

whence

$$\begin{aligned} |s_\alpha(t, \epsilon, \lambda)|^2 &\leqslant \text{const}\ \epsilon^{2\alpha-2n} e^{4(2+M)t} e^{(2t/|a_n|)^2} e^{\lambda^2} \\ &= \bar{f}_\alpha(t, \epsilon)\, e^{\lambda^2}, \end{aligned} \tag{13}$$

using $e^{2ab} \leqslant e^{a^2+b^2}$. $\bar{f}_\alpha(t, \epsilon)$ is bounded on $[0, T] \times (0, q]$ since $\alpha \geqslant n$.

We now estimate s_α for $0 \leqslant \alpha < n$, $0 < \epsilon \leqslant q$, $|\epsilon^n \lambda| \leqslant q$; the chief difficulty here stems from the fact that $\epsilon\lambda$ is not necessarily small unless $n = 1$. We introduce a new contour of integration composed of the positively oriented boundary of the union of the following regions of the complex μ plane:

1. $|\mu| < 2\epsilon \left(2 + M + \frac{1}{|a_n|} |\lambda|\right)$
2. $|\mu - \nu_i| < r$ $(i = 1, \ldots, m)$,

where $r \leqslant 3\beta/4$ is independent of ϵ and λ and is chosen small enough that these m circles are either disjoint or coincident (if any of the ν_i are multiple). From (6) of Lemma 1 we have $\tilde{\nu}_j - \nu_j = 0(\epsilon) + 0(\epsilon^n\lambda) = 0(q)$, so we may reduce q further if necessary to insure that

$$| \tilde{\nu}_i - \nu_i | \leqslant \frac{r}{2}. \tag{14}$$

The contour just defined is the disjoint union of circles with centers at certain of the ν_i and a "circle with bumps" centered at the origin. Denote this last component of the contour by C_0, the other components by C_i, $i = 1,..., s \leqslant m$; s as well as C_0 depends on ϵ and λ. Set

$$k = 2\epsilon \left(2 + M + \frac{1}{| a_n |} | \lambda |\right)$$

for convenience, and consider μ on C_0. C_0 is the boundary of the union of the disk 1 above and those disks 2 whose centers ν_i satisfy $r + k \geqslant | \nu_i |$. Since $| \nu_i | \geqslant \beta \geqslant 4r/3$, we have $r \leqslant 3k$ for these disks. Thus if σ is any point on C_0, we have $| \sigma | \leqslant 2r + k \leqslant 7k = 14\epsilon(2 + M + [| \lambda |/| a_n |])$.

For μ on C_i we have, using (14) and geometric arguments in the complex μ plane,

$$| \mu - \tilde{\nu}_i | \geqslant \frac{r}{2}, \qquad | \mu - \epsilon\bar{\mu}_i | \geqslant \frac{\beta}{8},$$

so

$$| Q | \geqslant \left(\frac{r}{2}\right)^m \left(\frac{\beta}{8}\right)^n.$$

Thus

$$\left| \frac{\epsilon^\alpha}{2\pi i} \oint_{C_i} \frac{e^{(\mu/\epsilon)t} Q_\alpha}{Q} \, d\mu \right| \leqslant \frac{\epsilon^\alpha}{2\pi} e^{-(\beta/4\epsilon)t} \frac{M_{i,\alpha}}{\left(\frac{r}{2}\right)^m \left(\frac{\beta}{8}\right)^n} 2\pi r \leqslant c_{i,\alpha}\epsilon^\alpha,$$

where

$$M_{i,\alpha} = \sup_{\substack{|\mu - \nu_i| = r \\ 0 \leqslant \epsilon \leqslant q}} | Q_\alpha |$$

and $c_{i,\alpha}$ is a constant depending only on i and α. Noting that the contour is composed of disjoint circles if both ϵ and λ are sufficiently small, we see that the constants $c_{i,\alpha}$ are defined for $i = 1,..., m$.

For μ on C_0 we have, using (14) again,

$$| Q | \geqslant \left(\frac{r}{2}\right)^m \left(2 + M + \frac{1}{| a_n |} | \lambda |\right)^n \epsilon^n.$$

Thus, using (10),

$$\begin{aligned}&\left| \frac{\epsilon^\alpha}{2\pi i} \oint_{C_0} \frac{e^{(\mu/\epsilon)t} Q_\alpha}{Q} \, d\mu \right| \\ &\leqslant \frac{\epsilon^\alpha}{2\pi} \left| \oint_{C_0} \frac{e^{(\mu/\epsilon)t}}{\mu^{\alpha+1}} \, d\mu \right| + \frac{\epsilon^\alpha}{2\pi} \left| \oint_{C_0} \frac{e^{(\mu/\epsilon)t} \sum_{k=0}^{\alpha} \epsilon^{n-k} a_k \mu^k}{\mu^{\alpha+1} Q} \, d\mu \right| \\ &\leqslant \left[\frac{t^\alpha}{\alpha!} + \text{const}\, e^{2(2+m)t} \right] \exp \left\{ \frac{2}{|a_n|} |\lambda| \, t \right\} = f(t) \exp \left\{ \frac{2}{|a_n|} |\lambda| \, t \right\}.\end{aligned}$$

Finally,

$$\begin{aligned}| s_\alpha | &\leqslant \sum_{i=1}^{s} \left| \frac{\epsilon^\alpha}{2\pi} \oint_{C_i} \frac{e^{(\mu/\epsilon)t} Q_\alpha}{Q} \, d\mu \right| + \left| \frac{\epsilon^\alpha}{2\pi} \oint_{C_0} \frac{e^{(\mu/\epsilon)t} Q_\alpha}{Q} \, d\mu \right| \\ &\leqslant \epsilon^\alpha \sum_{i=1}^{m} c_{i,\alpha} + f(t) \exp \left(\frac{2}{|a_n|} |\lambda| \, t \right),\end{aligned}$$

from which we easily conclude that

$$| s_\alpha |^2 \leqslant g(t, \epsilon) \, e^{\lambda^2} \tag{15}$$

or $0 \leqslant \alpha < n$, $0 < \epsilon \leqslant q$, $0 \leqslant \epsilon^n |\lambda| \leqslant q$, where g is bounded on $[0, T] \times (0, q]$ for any $T > 0$.

We now turn to estimating s_α for $0 < \epsilon \leqslant q$, $\epsilon^n |\lambda| > q$. For this we use as contour of integration the circle $|\mu| = K\epsilon^n |\lambda|$, where $K \equiv q^{-1}(2 + N + q)$ so that $K\epsilon^n |\lambda| \geqslant 2 + N + \epsilon^n |\lambda|$. On this circle $|Q| \geqslant 1$ (see Lemma 2). Hence,

$$| s_\alpha(t, \epsilon, \lambda)| \leqslant \epsilon^\alpha e^{K\epsilon^{n-1}|\lambda| t} M_\alpha (K\epsilon^n |\lambda|)^{m+n-\alpha},$$

where M_α depends only on Q_α, i.e., only on α. Thus

$$\begin{aligned}| s_\alpha |^2 &\leqslant \epsilon^{2\alpha} (K\epsilon^n)^{2(m+n-\alpha)} M_\alpha^{\,2} e^{2K^2\epsilon^{2n-2}t^2} |\lambda|^{2(m+n-\alpha)} e^{\lambda^2/2} \\ &\leqslant \bar{g}(t, \epsilon) \, e^{\lambda^2},\end{aligned} \tag{16}$$

where $\bar{g}$ is bounded on $[0, T] \times (0, q]$ for any $T > 0$. Here we have used the fact that $|\lambda|^b e^{-\lambda^2/2}$ is bounded for any constant $b > 0$.

Together, the estimates (13), (15), (16) imply the validity of the lemma.

LEMMA 7. *For each* $\alpha = 0, 1, \ldots, m + n - 1$ *and* $j = 1, 2, \ldots,$ *and for each* $\epsilon > 0$, $T > 0$, *there exists a constant* $f_{j,\alpha}(T, \epsilon)$ *such that*

$$\left| \left(\frac{d}{dt} \right)^j s_\alpha(t, \epsilon, \lambda) \right|^2 \leqslant f_{j,\alpha}(T, \epsilon) \, e^{\lambda^2}$$

holds for $t \in [0, T]$.

The function $f_{j,\alpha}$ of the lemma is not necessarily well behaved as $\epsilon \to 0$. This estimate is easily made in the manner of the proof of Lemma 6, since each t derivative of s_α results simply in a factor μ/ϵ in the integrand of the integral defining s_α.

By methods similar to those employed above, we have the following result for the degenerate operator.

LEMMA 8. *For each* $\alpha = 0, 1,..., n - 1$ *and* $j = 0, 1, 2,\cdots$ *we have the estimate*

$$\left| \left(\frac{d}{dt}\right)^j w_\alpha(t, \lambda) \right|^2 \leqslant g_{j,\alpha}(t)\, e^{\lambda^2},$$

where, for each $T > 0$, $g_{j,\alpha}$ *is bounded on* $[0, T]$.

THE PROBLEM IN HILBERT SPACE

Since A is self-adjoint, there is a resolution of the identity $\{E_\lambda\}$ such that A has the spectral representation

$$A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda .$$

For any $\epsilon > 0$ and $t \geqslant 0$, define $S_\alpha(t, \epsilon)$ by

$$S_\alpha(t, \epsilon) = \int_{-\infty}^{\infty} s_\alpha(t, \epsilon, \lambda)\, dE_\lambda \qquad (\alpha = 0, 1,..., m + n - 1).$$

Letting D be the dense domain of the operator $\exp(A^2)$, we see that D is contained in the domain of $S_\alpha(t, \epsilon)$ because, for $x \in D$,

$$\| S_\alpha(t, \epsilon)\, x \|^2 = \int_{-\infty}^{\infty} | s_\alpha(t, \epsilon, \lambda)|^2\, d \| E_\lambda x \|^2$$

$$\leqslant \int_{-\infty}^{\infty} f_\alpha(t, \epsilon)\, e^{\lambda^2}\, d \| E_\lambda x \|^2 < \infty.$$

Also, if $x_\alpha \in D$ $(\alpha = 0,..., m + n - 1)$ and

$$u_\epsilon(t) = \sum_{\alpha=0}^{m+n-1} S_\alpha(t, \epsilon)\, x_\alpha ,$$

then $u_\epsilon(t)$ is in the domain of A, as is shown by the calculation

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda^2 d \,\| E_\lambda u_\epsilon \|^2 &= \int_{-\infty}^{\infty} \lambda^2 d \,\| E_\lambda \sum_{\alpha=0}^{m+n-1} \int_{-\infty}^{\infty} s_\alpha(t, \epsilon, \mu)\, dE_\mu x_\alpha \|^2 \\ &\leqslant \sum_{\alpha=0}^{m+n-1} \int_{-\infty}^{\infty} \lambda^2 d \int_{-\infty}^{\lambda} f_\alpha(t, \epsilon)\, e^{\mu^2} d \,\| E_\mu x_\alpha \|^2 \\ &= \sum_{\alpha=0}^{m+n-1} \int_{-\infty}^{\infty} f_\alpha(t, \epsilon)\, \lambda^2 e^{\lambda^2} d \,\| E_\lambda x_\alpha \|^2 \\ &\leqslant \sum_{\alpha=0}^{m+n-1} f_\alpha(t, \epsilon) \int_{-\infty}^{\infty} e^{2\lambda^2} d \,\| E_\lambda x_\alpha \|^2 < \infty.\end{aligned}$$

LEMMA 9. *u_ϵ is the unique solution of the Cauchy problem*

$$L_\epsilon[u_\epsilon] = 0, \qquad u_\epsilon^{(\alpha)}(0) = x_\alpha \qquad (\alpha = 0,..., m + n - 1)$$

for $x_\alpha \in D$.

LEMMA 10. *U is the unique solution of the Cauchy problem*

$$L_0[U] = 0, \qquad U^{(\alpha)}(0) = x_\alpha \qquad (\alpha = 0,..., m - 1)$$

for $x_\alpha \in D$, where

$$U(t) = \sum_{j=0}^{n-1} W_\alpha(t)\, x_\alpha\,, \qquad W_\alpha(t) \equiv \int_{-\infty}^{\infty} w_\alpha(t, \lambda)\, dE_\lambda\,.$$

Proof. We prove Lemma 9; the proof of Lemma 10 is similar. In view of Lemma 3, we need only verify that for $x \in D$,

$$\left(\frac{d}{dt}\right)^j S_\alpha(t, \epsilon)\, x = \int_{-\infty}^{\infty} \left(\frac{d}{dt}\right)^j s_\alpha(t, \epsilon, \lambda)\, dE_\lambda x \qquad (\alpha = 0, 1,..., m + n - 1).$$

We check this for $j = 1$; the result then follows by a simple induction. In view of the definition of S_α, we have

$$\frac{S_\alpha(t + h, \epsilon)\, x - S_\alpha(t, \epsilon)\, x}{h} = \int_{-\infty}^{\infty} \frac{d}{dt} s_\alpha(t', \epsilon, \lambda)\, dE_\lambda x$$

for some t' between h and $t + h$. Since

$$\left\| \int_{-\infty}^{\infty} \frac{d}{dt} s_\alpha(t', \epsilon, \lambda)\, dE_\lambda x - \int_{-\infty}^{\infty} \frac{d}{dt} s_\alpha(t, \epsilon, \lambda)\, dE_\lambda x \right\|^2$$

$$= \int_{-\infty}^{\infty} \left| \frac{d}{dt} s_\alpha(t', \epsilon, \lambda) - \frac{d}{dt} s_\alpha(t, \epsilon, \lambda) \right|^2 d\, \| E_\lambda x \|^2,$$

the result will follow, on letting $h \to 0$, by the Lebesgue dominated convergence theorem if we show that $| d/dt\, [s_\alpha(t', \epsilon, \lambda) - d/dt\, s_\alpha(t, \epsilon, \lambda)] |^2$ is dominated by a function integrable with respect to $d \| E_\lambda x \|$. For this choose $T > 0$ and h small enough that t and $t + h$ are both less than T. From Lemma 7 we now have

$$\left| \frac{d}{dt} s_\alpha(t', \epsilon, \lambda) \right|^2 \leqslant f_{1,\alpha}(T, \epsilon)\, e^{\lambda^2}, \qquad \left| \frac{d}{dt} s_\alpha(t, \epsilon, \lambda) \right|^2 \leqslant f_{1,\alpha}(T, \epsilon)\, e^{\lambda^2},$$

whence the result follows since e^{λ^2} is integrable.

LEMMA 11. *If B is a bounded operator on a Hilbert space and $(d/dt)\, x(t) = Bx(t)$, $t > 0$, with $x(0) = 0$, then $x(t) \equiv 0$.*

This is well known; see Smoller [2].

LEMMA 12. *The solutions u_ϵ and U are unique.*

Proof. We will deal with u_ϵ; the proof for U is similar. In the usual fashion we write (1) as a system

$$\frac{d}{dt} X(t) = BX(t), \tag{17}$$

where $X(t) \in H^{m+n}$. Let A_k be the bounded self-adjoint operator $A_k = \int_{-k}^{k} \lambda\, dE_\lambda$ and let B_k be the matrix obtained from B by replacing A with A_k; then B_k is a bounded operator on H^{m+n}. Suppose that u_ϵ is a solution for $x_\alpha = 0$ ($\alpha = 0, 1, \ldots, m + n - 1$), and let $v_k(t) = (E_k - E_{-k})\, u_\epsilon(t)$. Let $V_k(t)$ be the vector $[v_k(t), v_k'(t), \ldots, v_k^{(n+m-1)}(t)]$, so V_k solves $V_k' = BV_k$, $V_k(0) = 0$. By Lemma 11, $V_k(t) \equiv 0$, whence $v_k(t) \equiv 0$. Hence,

$$u_\epsilon(t) = \lim_{k\to\infty} v_k(t) = 0.$$

THEOREM 2. *For the unique solutions u_ϵ and U of* (1)-(4) *with $x_\alpha \in D$ ($\alpha = 0, 1, \ldots, m + n - 1$),*

$$\lim_{\epsilon\to 0} \| u_\epsilon(t) - U(t) \| = 0,$$

where the convergence is uniform in t for $t \in [0, T]$, any $T > 0$.

Proof. It will suffice to show that

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} | s_\alpha(t, \epsilon, \lambda) - w_\alpha(t, \lambda)|^2 \, d \| E_\lambda x \|^2 = 0,$$

where $x \in D$. This follows by the Lebesgue Dominated Convergence Theorem, Lemmas 5 and 6, and

$$| s_\alpha(t, \epsilon, \lambda) - W_\alpha(t, \lambda)|^2 \leqslant 2(| s_\alpha(t, \epsilon, \lambda)|^2 + | w_\alpha(t, \lambda)|^2)$$

$$\leqslant 2(f_\alpha(t, \epsilon) + g_\alpha(t)) e^{\lambda^2} \leqslant \text{const. } e^{\lambda^2}$$

for $\epsilon < \epsilon_0$ (where ϵ_0 depends on T). Since e^{λ^2} is integrable with respect to $d \| E_\lambda x \|^2$, we are done.

REMARK. Estimates similar to those in the proofs of Lemmas 5, 6, and 7 can be used to prove that, for any $\delta > 0$ and $T > \delta$,

$$\lim_{\epsilon\to 0} \left\| \left(\frac{d}{dt}\right)^j u_\epsilon(t) - \left(\frac{d}{dt}\right)^j U(t) \right\| = 0 \qquad (j = 1, 2,...)$$

uniformly in t for $t \in [\delta, T]$.

REFERENCES

1. J. KISYŃSKI, Sur les équations hyperboliques avec petit paramètre, *Colloq. Math.* **10** (1963), 331–343.
2. J. A. SMOLLER, Singular perturbations and a theorem of Kisyński, *J. Math. Anal. Appl.* **12** (1965), 105–114.
3. J. A. SMOLLER, Singular perturbations of Cauchy's problem, *Comm. Pure Appl. Math.* **18** (1965), 665–677.
4. B. LATIL, Singular perturbations of Cauchy's problem, *J. Math. Anal. Appl.* **23** (1968), 683–698.
5. M. I. VĬSIK AND L. A. LYUSTERNIK, Regular degeneration and boundary layer for linear differential equations with small parameter, *Uspehi Mat. Nauk.* (N. S.) 12 (1957), 3–122; *Amer. Math. Soc. Transl.* (2) **20** (1962), 239–364.
6. M. MARDEN, The Geometry of the Zeros of a Polynomial in a Complex Variable. Math. Surveys, No. 3, AMS 1949.
7. A. FRIEDMAN, Singular perturbations for the Cauchy problem and for boundary value problems, *Differential Equations* **5** (1969), 226–261.