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Some stability properties of T. Chan's preconditioner

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Abstract

A matrix is said to be stable if the real parts of all the eigenvalues are negative. In this paper, for any matrix A_n, we give some sufficient and necessary conditions for the stability of T. Chan's preconditioner $c_U(A_n)$.

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1. Introduction

In 1988, T. Chan [5] proposed a circulant preconditioner for solving Toeplitz systems. In 1991, R. Chan et al. [3] showed that T. Chan's preconditioner can be defined not only for Toeplitz matrices but also for general matrices. We begin with the general case. Given a unitary matrix $U \in \mathbb{C}^{n\times n}$, define

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$$\mathscr{M}_U \equiv \{U^* \Lambda_n U \mid \Lambda_n \text{ is any } n\text{-by-}n \text{ diagonal matrix }\}. \tag{1}$$

For any matrix $A_n \in \mathbb{C}^{n\times n}$, T. Chan's preconditioner $c_U(A_n) \in \mathscr{M}_U$ satisfies

$$\|c_U(A_n) - A_n\| = \min_{W_n \in \mathscr{M}_U} \|W_n - A_n\|,$$

where $\|\cdot\|$ is the Frobenius norm. Let F denote the Fourier matrix whose entries are given by

$$(F)_{j,k} = \frac{1}{\sqrt{n}} \mathrm{e}^{2\pi \mathrm{i}(j-1)(k-1)/n}, \quad \mathrm{i} \equiv \sqrt{-1}, \quad 1 \leqslant j,k \leqslant n. \tag{2}$$

If $U = F$ in (1), then $\mathscr{M}_U$ is the set of all circulant matrices, see [6].

It has been proved that T. Chan's preconditioner is a good preconditioner for solving a large class of linear systems, see [4,5,9]. In this paper, we will study some stability properties of T. Chan's preconditioner. The stability property is essential in control theory and dynamical systems [1].

Definition 1. A matrix is said to be stable if the real parts of all the eigenvalues are negative.

Now, we introduce some symbols and review some results. For any matrix $E \in \mathbb{C}^{n\times n}$, let $\lambda_j(E)$ be the jth eigenvalue of E and $\delta(E)$ denote the diagonal matrix whose diagonal is equal to the diagonal of E. For T. Chan's preconditioner, we have the following important lemma, see [3,8,9,12].

Lemma 1. *Let $A_n \in \mathbb{C}^{n\times n}$, $U \in \mathbb{C}^{n\times n}$ be unitary and $c_U(A_n)$ be T. Chan's preconditioner. Then*

(i) *$c_U(A_n)$ is uniquely determined by A_n and is given by*

$$c_U(A_n) \equiv U^* \delta(U A_n U^*) U.$$

(ii) *If A_n is Hermitian, then $c_U(A_n)$ is also Hermitian. Moreover, we have*

$$\min_j \lambda_j(A_n) \leqslant \min_j \lambda_j(c_U(A_n)) \leqslant \max_j \lambda_j(c_U(A_n)) \leqslant \max_j \lambda_j(A_n).$$

From Lemma 1(ii), it is easy to see that if A_n is Hermitian and stable, then so is $c_U(A_n)$. In [11], Jin et al. show that if A_n is normal and stable, then $c_U(A_n)$ is also normal and stable. The result is further generalized in [2] by Cai and Jin. They prove that if A_n is $*$-congruent to a stable diagonal matrix, i.e., $A_n = Q^* D Q$ where Q is a nonsingular matrix and D is a stable diagonal matrix, then $c_U(A_n)$ is stable.

Now, the problem we are facing is that for any given matrix A_n, how to judge its T. Chan's preconditioner $c_U(A_n)$ is stable? In this paper, we try to give some sufficient and necessary conditions for the stability of $c_U(A_n)$.

2. Main results

Let $A_n \in \mathbb{C}^{n\times n}$. A well-known fact [7] is that A_n can be decomposed as

$$A_n = H + \mathrm{i}K,$$

where H and K are Hermitian matrices given by

$$H \equiv \frac{1}{2}(A_n + A_n^*), \quad K \equiv \frac{1}{2\mathrm{i}}(A_n - A_n^*)$$

and are called the Hermitian part and skew-Hermitian part of A_n, respectively. We have the following main results.

Theorem 1. *Let $A_n \in \mathbb{C}^{n\times n}$ and suppose that $A_n = H + \mathrm{i}K$ where H and K are Hermitian. Then T. Chan's preconditioner $c_U(A_n)$ is stable for any unitary matrix $U \in \mathbb{C}^{n\times n}$ if and only if H is negative definite.*

Proof. For any unitary matrix U, we have by Lemma 1(i),

$$c_U(A_n) = U^*\delta(UA_nU^*)U.$$

With $A_n = H + \mathrm{i}K$, we obtain

$$\begin{aligned} c_U(A_n) &= U^*\delta(U(H+\mathrm{i}K)U^*)U \\ &= U^*\delta(UHU^* + \mathrm{i}UKU^*)U \\ &= U^*[\delta(UHU^*) + \mathrm{i}\delta(UKU^*)]U. \end{aligned}$$

As UHU^* and UKU^* are Hermitian matrices, their diagonal elements are real. Thus, we see that the real parts of the eigenvalues of $c_U(A_n)$ are just the diagonal elements of UHU^*. Now if H is negative definite, then, for any unitary matrix U, UHU^* has all the diagonal elements being negative and consequently $c_U(A_n)$ is stable.

Conversely, if H has a nonnegative eigenvalue, by choosing a unitary matrix V such that VHV^* is diagonal, we see that VHV^* has a nonnegative diagonal element and consequently, $c_V(A_n)$ is not stable. □

Theorem 2. *Let $A_n \in \mathbb{C}^{n\times n}$ and suppose that $A_n = H + \mathrm{i}K$ where H and K are Hermitian. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $c_U(A_n)$ is stable if and only if*

$$\mathrm{tr}(H) = \mathrm{Re}[\mathrm{tr}(A_n)] < 0,$$

where $\mathrm{Re}[\cdot]$ *denotes the real part of a complex number and* $\mathrm{tr}(\cdot)$ *denotes the trace of a matrix.*

Proof. "⇒": Suppose $c_U(A_n)$ is stable. Then we see from the proof of Theorem 1 that UHU^* has all the diagonal elements being negative and so

$$\mathrm{Re}[\mathrm{tr}(A_n)] = \mathrm{tr}(H) = \mathrm{tr}(UHU^*) < 0.$$

"$\Leftarrow$": In view of the proof of Theorem 1, it suffices to show that there exists a unitary matrix V such that VHV^* has all diagonal entries being negative, as these diagonal entries are the real parts of the eigenvalues of $c_V(A_n)$.

Note that if

$$D = \mathrm{diag}(\mu_1, \cdots, \mu_n)$$

is a diagonal matrix and U is any unitary matrix, then the diagonal of UDU^*, say

$$[d_{11}, d_{22}, \cdots, d_{nn}]^{\mathrm{T}},$$

is given by

$$\begin{bmatrix} d_{11} \\ \vdots \\ d_{nn} \end{bmatrix} = (U \circ \overline{U}) \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

where "$\circ$" denotes the Hadamard product. We remark that for $P = [p_{ij}] \in \mathbb{C}^{m\times n}$, $Q = [q_{ij}] \in \mathbb{C}^{m\times n}$, their Hadamard product is given by

$$P \circ Q = [p_{ij}q_{ij}] \in \mathbb{C}^{m\times n}.$$

As H is Hermitian, let W be a unitary matrix that diagonalizes H, i.e.,

$$WHW^* = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Let $V = FW$ where F is the Fourier matrix given by (2), and let

$$[h_{11}, h_{22}, \cdots, h_{nn}]^{\mathrm{T}}$$

be the diagonal of VHV^*. Then

$$\begin{bmatrix} h_{11} \\ \vdots \\ h_{nn} \end{bmatrix} = (F \circ \overline{F}) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$
$$= \frac{\sum_{j=1}^n \lambda_j}{n}\mathbf{1}_n = \frac{\mathrm{tr}(H)}{n}\mathbf{1}_n,$$

where $\mathbf{1}_n = [1, 1, \ldots, 1]^{\mathrm{T}}$. As $\mathrm{tr}(H) < 0$, we see that

$$c_V(A_n) = V^*[\delta(VHV^*) + \mathrm{i}\delta(VKV^*)]V$$

is stable. $\square$

We include here an application of Theorem 1. The details can be found in [2]. When solving by a boundary value method scheme the initial value problem associated to a system of first order linear differential equations

$$\mathbf{y}'(t) = J\mathbf{y}(t) + \mathbf{g}(t),$$

one needs to solve a linear system $My = b$. A preconditioner S for the system $My = b$ is defined in term of $c_F(J)$. This preconditioner S is invertible when $c_F(J)$ is stable. In [2], J is assumed to be $*$-congruent to a stable normal matrix to assure the stability of $c_F(J)$. Here, by Theorem 1, one obtains the stability of $c_F(J)$ by assuming negative the eigenvalues of the Hermitian part of J, which is a simplier condition. For the asymptotically stable solution of some system of functional differential equations [10], it is known that the Hermitian part of the system has negative eigenvalues and consequently we may use the preconditioner S to speed up the computation.

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