The Set of Generalized Exponents of Primitive Simple Graphs

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ABSTRACT

The exponent of a primitive digraph is the smallest integer \( k \) such that for each ordered pair of (not necessarily distinct) vertices \( x \) and \( y \) there is a walk of length \( k \) from \( x \) to \( y \). The exponent set (the set of those numbers attainable as exponents of primitive digraphs with \( n \) vertices) and bounds on the exponent have been extensively studied. As a generalization of exponent, R. A. Brualdi and B. Liu introduced three types of generalized exponents for primitive digraphs in 1990. We improve the bounds on these generalized exponents given by B. Liu for primitive simple graphs, and we express explicitly for this class of primitive graphs the exponent sets of all three types of generalized exponents.

1. INTRODUCTION

A directed graph \( D \) is called primitive if there exists a positive integer \( k \) such that for each ordered pair of vertices \( x \) and \( y \) (not necessarily distinct), there is a walk of length \( k \) from \( x \) to \( y \). The smallest such \( k \) is called the exponent of \( D \), denoted by \( \gamma(D) \). It is well known that a directed graph \( D \) is primitive if and only if \( D \) is strongly connected and the greatest common divisor of the lengths of its cycles is 1.

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For convenience, a directed graph will be called a digraph and an undirected graph will be simply called a graph.

There is considerable information known about bounds on the exponent and about the exponent set (the set of those numbers that can be exponents) for general primitive digraphs or for some particular classes of primitive digraphs. For references see [7].

In 1990, R. A. Brualdi and B. Liu [2] generalized the traditional concept of the exponent of a primitive digraph by introducing three new types of exponents. In order to give the definitions of these three types of generalized exponents, we need to define the vertex exponent and the set exponent in a primitive digraph.

**Definition 1.1.** Let \( x \) be a vertex in a primitive digraph \( D \). The vertex exponent \( \gamma_D(x) \) is defined to be the smallest positive integer \( p \) such that there are walks of length \( p \) from \( x \) to all vertices of \( D \).

**Definition 1.2** [2]. Let \( D \) be an primitive digraph of order \( n \). If we choose to order the \( n \) vertices \( v_1, v_2, \ldots, v_n \) of \( D \) in such a way that

\[
\gamma_D(v_1) \leq \gamma_D(v_2) \leq \cdots \leq \gamma_D(v_n),
\]

then we call \( \gamma_D(v_k) \) the \( k \)th first type generalized exponent of \( D \), denoted by \( \exp_D(k) \).

It is obvious that

\[
\exp_D(1) \leq \exp_D(2) \leq \cdots \leq \exp_D(n).
\]

(1.1)

In particular, \( \exp_D(n) = \gamma(D) \), the exponent of \( D \). So the concept of the generalized exponents is a generalization of the traditional concept of the exponents for primitive digraphs.

**Definition 1.3** [2]. Let \( X \) be a vertex subset of a primitive digraph \( D \). The set exponent \( \exp_D(X) \) is defined to be the smallest positive integer \( p \) such that for each vertex \( y \) of \( D \), there exists a walk of length \( p \) from at least one vertex in \( X \) to \( y \).

It is easy to see the following relation between the vertex exponent and set exponent:

\[
\gamma_D(x) = \exp_D(\{x\}) \quad [x \in V(D)].
\]

(1.2)
\textbf{Definition 1.4} [2]. Let $D$ be a primitive digraph of order $n$ and $1 \leq k \leq n - 1$. Then we define

$$f(D, k) = \min \{\exp_D(X) | X \subseteq V(D) \text{ and } |X| = k\}$$

and

$$F(D, k) = \max \{\exp_D(X) | X \subseteq V(D) \text{ and } |X| = k\}.$$  

We call $f(D, k)$ and $F(D, k)$ the $k$th \textit{second type generalized exponent} and the $k$th \textit{third type generalized exponent} of $D$, respectively.

It is easy to see from (1.2) that

$$f(D, 1) = \exp_D(1), \quad F(D, 1) = \exp_D(n).$$

Notice that (because the case $k = n$ is trivial) we only define $f(D, k)$ and $F(D, k)$ for the integers $k$ with $1 \leq k \leq n - 1$.

Generalized exponents also have an interpretation in a model of a memoryless communication system associated with digraphs. For details of this model we refer the reader to R. A. Brualdi and B. Liu [2, pp. 484–485].

In this paper, we consider generalized exponents for primitive simple (undirected) graphs. The primitivity and primitive exponent for a graph can be defined in a same way as for a digraph by simply replacing "a directed graph $D$" by "a graph $G$" in the corresponding definition for digraphs. The three types of generalized exponents, $\exp_G(k)$, $f(G, k)$, and $F(G, k)$, for a primitive graph $G$ can also be defined in the same way as for a primitive digraph $D$. In fact, we define $\exp_G(k)$, $f(G, k)$, and $F(G, k)$ by simply replacing "primitive digraph $D$" by "primitive graph $G$" in Definitions 1.1–1.4.

A graph $G$ can be naturally associated with a symmetric digraph $D_G$ by replacing each undirected edge $[x, y]$ of $G$ by a pair of directed arcs $(x, y)$ and $(y, x)$. It is easy to see that a graph $G$ is primitive if and only if the corresponding digraph $D_G$ is primitive and in this case we have

$$\exp_G(k) = \exp_{D_G}(k), \quad f(G, k) = f(D_G, k), \quad F(G, k) = F(D_G, k).$$

Thus, we can actually identify the graph $G$ with the corresponding symmetric digraph $D_G$ in the study of the generalized exponents.
Since any edge of $G$ corresponds to a directed cycle of length 2 in $D_G$, by the criterion for primitivity of digraphs mentioned earlier, we deduce that $G$ is primitive if and only if $G$ is connected and contains at least one odd cycle; namely, $G$ is a connected nonbipartite graph.

A simple graph is a graph without loops and multiple edges. Since loops often play an important role in the study of exponents and generalized exponents, the study of generalized exponents for primitive simple graphs is, in general, more difficult than the study of generalized exponents for general primitive graphs where loops are permitted.

The exponent set (or generalized exponent set) of a particular class of primitive digraphs (or primitive graphs) is the set of those numbers that can be the exponents (or generalized exponents) of digraphs (or graphs) in this class. The exponent set problem and the upper bound problem are the two main problems in the study of exponents and generalized exponents.

In this paper, we use PG($n$) and PSG($n$) to denote the class of all primitive graphs of order $n$ and the class of all primitive simple graphs of order $n$, respectively. Let

$$E_1(\text{PSG} (n), k) = \{ \exp_G(k) | G \in \text{PSG}(n) \}$$

$$E_2(\text{PSG} (n), k) = \{ f(G, k) | G \in \text{PSG}(n) \}$$

$$E_3(\text{PSG} (n), k) = \{ F(G, k) | G \in \text{PSG}(n) \}$$

be the three generalized exponent sets of the class PSG($n$), and let

$$e_1(\text{PSG} (n), k) = \max \{ \exp_G(k) | G \in \text{PSG}(n) \}$$

$$e_2(\text{PSG} (n), k) = \max \{ f(G, k) | G \in \text{PSG}(n) \}$$

$$e_3(\text{PSG} (n), k) = \max \{ F(G, k) | G \in \text{PSG}(n) \}$$

be the largest numbers in these three exponent sets. The main purpose of this paper is to give the explicit expressions for the sets $E_i(\text{PSG} (n), k)$ and the numbers $e_i(\text{PSG} (n), k)$ ($i = 1, 2, 3$).

For the sake of simplicity, in the remainder of this paper we write

$$E_i(n, k) = E_i(\text{PSG} (n), k) \quad (i = 1, 2, 3)$$

and

$$e_i(n, k) = e_i(\text{PSG} (n), k) \quad (i = 1, 2, 3).$$
In [2], R. A. Brualdi and B. Liu obtained various upper bounds on the three types of generalized exponents for primitive digraphs. In particular, they obtained the explicit expressions for the largest $k$th generalized exponents (of all the three types) for the class $PG(n)$ of primitive graphs of order $n$ as follows (see Theorem 6.2, 6.7, and 6.3 in [2]):

$$\max\{\exp_G(k) | G \in PG(n)\} = n - 2 + k,$$

$$\max\{f(G, k) | G \in PG(n)\} = \left\lfloor \frac{n - k}{(k + 2)/2} \right\rfloor,$$

$$\max\{F(G, k) | G \in PG(n)\} = 2(n - k),$$

where $[a]$ denotes the largest integer not exceeding $a$, and $\lfloor b \rfloor$ denotes the smallest integer not less than $b$.

In [3], B. Liu studied the generalized exponents for some other classes of primitive digraphs such as the class of primitive minimally strongly connected digraphs and the class of primitive tournaments.

In [1], B. Liu further studied the generalized exponents for the class $PSG(n)$ of primitive simple graphs of order $n$ and gave the following expressions for the largest $k$th generalized exponents (of all three types) for the class $PSG(n)$ (see Lemmas 2.1, 2.4 and Theorems 4.5, 3.1 in [1]):

$$e_1(n, k) = \begin{cases} n - 2, & k = 1, 2, \\ n - 4 + k, & 3 \leq k \leq n \end{cases} \text{ (if } n \text{ is even),}$$

$$e_1(n, k) = \begin{cases} n - 1, & k = 1, 2, \\ n - 4 + k, & 3 \leq k \leq n \end{cases} \text{ (if } n \text{ is odd),}$$

$$e_2(n, k) = \left\lfloor \frac{n - k - 1}{((k + 2)/2)} \right\rfloor,$$

$$e_3(n, k) = 2(n - k - 1).$$

In this paper we will show that (1.12) is only true for the case $1 \leq k \leq n/2$ and (1.11) is only true for the case $k = 3$. We will use different methods to prove the following expressions for $e_3(n, k)$ and $e_2(n, k)$ in Theorem 3.1 and
Theorem 4.1:

\[
e_3(n, k) = \begin{cases} 
2(n - k - 1) & \text{if } 1 \leq k \leq \frac{n}{2}, \\
2(n - k) - 1 & \text{if } k = \frac{n + 1}{2} \text{ (and } n \text{ is odd),} \\
2(n - k) & \text{if } \frac{n}{2} + 1 \leq k \leq n - 1
\end{cases}
\]  

and

\[
e_2(n, k) = \begin{cases} 
\left\lceil \frac{n - k}{((k + 2)/2)} \right\rceil & k = 2 \text{ or } 4 \leq k \leq n - 2, \\
\left\lceil \frac{n - 4}{2} \right\rceil & k = 3.
\end{cases}
\]

We will further prove the following explicit expressions for the exponent sets \(E_1(n, k), E_3(n, k),\) and \(E_2(n, k)\) in Theorem 2.3, Theorem 3.2, and Theorem 4.2 (the cases considered in these theorems are all the unknown and nontrivial cases):

\[
\begin{align*}
E_1(n, k) & = \{2, 3, \ldots, e_1(n, k)\} & (n \geq 3, \ 1 \leq k \leq n - 1), \\
E_3(n, k) & = \{1, 2, \ldots, e_3(n, k)\} & (2 \leq k \leq n - 1), \\
E_2(n, k) & = \{1, 2, \ldots, e_2(n, k)\} & (2 \leq k \leq n - 2),
\end{align*}
\]

where the expressions for \(e_i(n, k)\) (\(i = 1, 2, 3\)) are given in (1.9), (1.10), (1.14), and (1.13).

We conclude this introduction by pointing out that the three types of generalized exponents defined above also have corresponding matrix versions. Let \(A\) be the adjacency matrix of the primitive digraph \(D\). Then \(\exp_D(k)\) is the smallest power of \(A\) for which there are \(k\) rows with no zero entry, \(f(D, k)\) is the smallest power of \(A\) such that there is a \(k \times n\) submatrix of this power with no zero column, and \(F(D, k)\) is the smallest power of \(A\) such that any \(k \times n\) submatrix of this power has no zero column. From this point of view, all the results in this paper can be expressed in their corresponding matrix versions.
2. THE EXPONENT SET \( E_1(n, k) \)

In this section we will determine the exponent set \( E_1(n, k) \). Notice that if \( k = n \), then \( \exp_D(n) = \gamma(D) \), the primitive exponent of \( D \). Thus, the exponent set \( E_1(n, n) \) is just the primitive exponent set \( \{ \gamma(D) | D \in \text{PSG}(n) \} \) of the class \( \text{PSG}(n) \), which has already been determined in [5] as follows:

**Theorem A** [5]. *The primitive exponent set of the class \( \text{PSG}(n) \) is*

\[
\{ \gamma(D) | D \in \text{PSG}(n) \} = \{2, 3, \ldots, 2n - 4\} \setminus S,
\]

*where \( S \) is the set of all odd numbers in \( \{n - 2, n - 1, \ldots, 2n - 5\} \).*

Since the case \( k = n \) has already been settled, we will only consider the cases \( 1 \leq k \leq n - 1 \) in this section.

In order to determine the exponent set \( E_1(n, k) \), we will use the following expression for the largest number \( e_1(n, k) \) of the set \( E(n, k) \), which can be obtained by combining Lemma 2.1 and Lemma 2.4 in B. Liu [1].

**Lemma 2.1** (B. Liu [1]). *Let \( n \) and \( k \) be positive integers with \( 1 \leq k \leq n \), and let \( e_1(n, k) \) be the largest number in the set \( E_1(n, k) \). Then*

(1) if \( n \) is even, then

\[
 e_1(n, k) = \begin{cases} 
 n - 2, & k = 1, 2, \\
 n - 4 + k, & 3 \leq k \leq n;
\end{cases}
\]

(2) if \( n \) is odd, then

\[
 e_1(n, k) = \begin{cases} 
 n - 1, & k = 1, 2, \\
 n - 4 + k, & 3 \leq k \leq n.
\end{cases}
\]

In the following, we will prove that for \( n \geq 3 \),

\[
 E_1(n, k) = \{2, 3, \ldots, e_1(n, k)\} \quad (1 \leq k \leq n - 1). \tag{2.1}
\]

Firstly, by taking the complete graph \( K_n \), we obviously have

\[
 2 = \exp_{K_n}(k) \in E_1(n, k) \quad (1 \leq k \leq n). \tag{2.2}
\]
Next, we will construct in Lemma 2.2 and Lemma 2.3 two families of graphs $D_n(m)$ and $D^*_n(m)$ in the class PSG($n$) which will give us a number of exponents in the set $E_1(n, k)$. The proofs of these lemmas can be obtained by direct verification.

**Lemma 2.2.** Let $3 \leq m \leq n - 1$, and let $D = D_n(m)$ be the primitive simple graph in Figure 1. Then we have

$$\exp_D(k) = \begin{cases} 
  m - 1, & k = 1, 2, \\
  m + k - 3, & 3 \leq k \leq m, \\
  2m - 2, & m + 1 \leq k \leq n.
\end{cases} \quad (2.3)$$

**Lemma 2.3.** Let $4 \leq m \leq n - 2$, and let $D^* = D^*_n(m)$ be the primitive simple graph in Figure 2. Then

$$\exp_{D^*}(k) = \begin{cases} 
  m - 1, & k = 1, 2, \\
  m + k - 3, & 3 \leq k \leq m - 1, \\
  2m - 3, & m \leq k \leq n - 1, \\
  2m - 2, & k = n.
\end{cases} \quad (2.4)$$

In order to prove (2.1), we first prove for $3 \leq k \leq n - 1$ that

$$E_1(n, k) = \{2, 3, \ldots, n - 4 + k\} \quad (3 \leq k \leq n - 1).$$

For this purpose, we divide the set $\{2, 3, \ldots, n - 4 + k\}$ into four parts, and will show (in Lemmas 2.4, 2.5, 2.6, and 2.7) that each part is contained in the exponent set $E_1(n, k)$.

![F. 1. The graph $D_n(m)$.](image-url)
Lemma 2.4. Let \( n \) and \( k \) be integers with \( 3 \leq k \leq n - 1 \). Then

\[
\{2k - 3, \ldots, n - 4 + k\} \subseteq E_1(n, k).
\]

Proof. Take an integer \( m \) with \( k \leq m \leq n - 1 \), and let \( D = D_n(m) \) as in Figure 1. Then by Lemma 2.2 we have

\[
m + k - 3 = \exp_D(k) \in E_1(n, k) \quad (k \leq m \leq n - 1)
\]

and so

\[
\{2k - 3, \ldots, n - 4 + k\} = \{m + k - 3 | k \leq m \leq n - 1\} \subseteq E_1(n, k). \]

Lemma 2.5. Let \( n \) and \( k \) be integers with \( 3 \leq k \leq n - 1 \), and let \( A_k \) be the set of even numbers in \([2, 2k - 4]\). Then \( A_k \subseteq E_1(n, k) \).

Proof. Take an integer \( m \) with \( 3 \leq m \leq k - 1 \), and let \( D = D_n(m) \) as in Figure 1. Then by Lemma 2.2 we have

\[
2m - 2 = \exp_D(k) \in E_1(n, k) \quad (3 \leq m \leq k - 1).
\]

So

\[
\{4, 6, \ldots, 2k - 6, 2k - 4\} = \{2m - 2 | 3 \leq m \leq k - 1\} \subseteq E_1(n, k) \quad (2.5)
\]

Combining (2.5) and (2.2), we obtain \( A_k \subseteq E_1(n, k) \).

Lemma 2.6. Let \( n \) and \( k \) be integers with \( 3 \leq k \leq n - 1 \). Let \( B_k \) be the set of odd numbers in \([5, 2k - 5]\). Then \( B_k \subseteq E_1(n, k) \).
Proof. The set $[5,2k-5]$ is empty for $k = 3$ and $k = 4$, so we may assume $k \geq 5$. Take integer $m$ with $4 \leq m \leq k-1$, and let $D^* = D^*_n(m)$ as in Figure 2. Then by Lemma 2.3 we have

$$2m - 3 = \exp_{D^*}(k) \in E_1(n,k) \quad (4 \leq m \leq k-1).$$

So

$$B_k = \{2m - 3| 4 \leq m \leq k-1\} \subseteq E_1(n,k).$$

**Lemma 2.7.** Let $n$ and $k$ be integers with $3 \leq k \leq n-1$. Then $3 \in E_1(n,k)$.

Proof. Let $D_1(n)$, $D_2$, $D_3$, $D_4$ be the primitive graphs in Figure 3. Then it is not difficult to verify that

1. $\exp_{D_1(n)}(k) = 3$ for $n \geq 6$ and $1 \leq k \leq n$,
2. $\exp_{D_2}(k) = 3$ for $n = |V(D_2)| = 5$ and $k = 4$,
3. $\exp_{D_3}(k) = 3$ for $n = |V(D_3)| = 5$ and $k = 3$,
4. $\exp_{D_4}(k) = 3$ for $n = |V(D_4)| = 4$ and $k = 3$.

Thus $3 \in E_1(n,k)$ for all $3 \leq k \leq n-1$.

Combining Lemmas 2.4, 2.5, 2.6, and 2.7, we can determine the exponent set $E_1(n,k)$ in the case $3 \leq k \leq n-1$.

**Theorem 2.1.** Let $n$ and $k$ be integers with $3 \leq k \leq n-1$. Then

$$E_1(n,k) = \{2, 3, \ldots, n-4+k\}. \quad (2.6)$$

Proof. From Lemmas 2.4, 2.5, 2.6, and 2.7, we have

$$\{2, 3, \ldots, n-4+k\} \subseteq E_1(n,k). \quad (2.7)$$

On the other hand, by Lemma 2.1 and because $1 \notin E_1(n,k)$ (since simple graphs contain no loops), we also have

$$E_1(n,k) \subseteq \{2, 3, \ldots, n-4+k\}. \quad (2.8)$$

The result now follows from (2.7) and (2.8).
Now we determine the exponent set $E_1(n,k)$ for the case $k = 1, 2$.

**Theorem 2.2.** Let $n$ and $k$ be integers with $n \geq 3$ and $1 \leq k \leq 2$. Then

$$E_1(n,k) = \begin{cases} 
\{2, 3, \ldots, n-1\} & \text{if } n \text{ is odd}, \\
\{2, 3, \ldots, n-2\} & \text{if } n \text{ is even}.
\end{cases}$$

**Proof.** Take an integer $m$ with $3 \leq m \leq n-1$, and let $D = D_n(m)$ as in Figure 1. Then by Lemma 2.2 we have for $k = 1, 2$ that

$$m - 1 = \exp_D(k) \in E_1(n,k) \quad (k = 1, 2).$$

So

$$\{2, 3, \ldots, n-2\} = \{m - 1 | 3 \leq m \leq n - 1\} \subseteq E_1(n,k). \quad (2.9)$$

It follows from Lemma 2.1 and $1 \notin E_1(n,k)$ that $E_1(n,k) = \{2, 3, \ldots, n-2\}$ when $n$ is even (and $k = 1, 2$). If $n$ is odd, let $C_n$ be the cycle of order $n$. Then it is easy to see that

$$n - 1 = \exp_{C_n}(k) \in E_1(n,k) \quad (1 \leq k \leq n).$$

Combining this with (2.9) and Lemma 2.1, we obtain

$$E_1(n,k) = \{2, 3, \ldots, n-1\} \quad (n \text{ odd}, \ k = 1, 2).$$

This completes the proof of the theorem.

\[\square\]
Now it is straightforward to obtain (2.1) from Theorems 2.1, 2.2 and Lemma 2.1; namely, we have:

**Theorem 2.3.**

\[ E_i(n, k) = \{2, 3, \ldots, e_i(n, k)\} \quad (n \geq 3, \ 1 \leq k \leq n - 1). \]

3. THE EXPONENT SET \( E_3(n, k) \)

Let \( X \) and \( Y \) be two subsets of vertices of a digraph (or a graph) \( D \). We say that \( X \) \( d \)-covers \( Y \) provided that for any vertex \( y \in Y \) there exists a walk of length \( d \) from at least one vertex of \( X \) to the vertex \( y \). If \( D_1 \) is a subdigraph (or a subgraph) of \( D \), then we also say that \( X \) \( d \)-covers \( D_1 \) if \( X \) can \( d \)-cover \( V(D_1) \).

As in the case for \( E_i(n, k) \), we first need to determine the largest number \( e_3(n, k) \) in the exponent set \( E_3(n, k) \). It was stated in [1, Theorem 3.1] that \( e_3(n, k) = 2(n - k - 1) \), but we will show that this result is only true in the case \( 1 \leq k \leq n/2 \) and is not true in general. In this section we will use methods different from those used in [1] to prove that

\[
e_3(n, k) = \begin{cases} 
2(n - k - 1) & \text{if } 1 \leq k \leq \frac{n}{2}, \\
2(n - k) - 1 & \text{if } k = \frac{n + 1}{2} \text{ (and } n \text{ is odd)}, \\
2(n - k) & \text{if } \frac{n}{2} + 1 \leq k \leq n - 1 
\end{cases}
\]  

(3.1)

and then determine that the exponent set \( E_3(n, k) \) is

\[ E_3(n, k) = \{1, 2, \ldots, e_3(n, k)\} \quad (2 \leq k \leq n - 1). \]  

(3.2)

Notice that \( F(D, 1) = \exp_D(n) \), so the exponent set \( E_3(n, 1) = E_1(n, n) \), which was already determined in [5] (also see Theorem A in Section 2 of this paper). Thus we only need to consider the case \( 2 \leq k \leq n - 1 \) for \( E_3(n, k) \).

We first prove the case \( n/2 + 1 \leq k \leq n - 1 \) of (3.1).
Lemma 3.1. Let \( n \) and \( k \) be integers with \( n/2 + 1 \leq k \leq n - 1 \). Then

\[
e_3(n, k) = 2(n - k).
\]  

(3.3)

Proof. Firstly, we know from [2, Theorem 6.3] [which was cited as (1.8) in this paper] that \( e_3(n, k) \leq 2(n - k) \).

Next, we consider the primitive simple graph \( \Gamma \) in Figure 4. Take a vertex subset \( X_0 \) of \( \Gamma \) as follows:

\[
X_0 = V(\Gamma) \setminus \{v_{2i} | 1 \leq i \leq n - k \}.
\]

Then \( |X_0| = k \). Take any vertex \( y \in X_0 \). We will show that there is no walk of length \( 2(n - k) - 1 \) from \( y \) to the vertex \( v_1 \).

Case 1: \( y = v_j \) with \( j \geq 2(n - k) + 1 \). By the hypothesis \( n/2 + 1 \leq k \), we have \( 2(n - k) \leq n - 2 \), which implies \( d_\Gamma(v_{2(n-k)+1}, v_1) = 2(n - k) \), and so \( d_\Gamma(v_j, v_1) \geq d_\Gamma(v_{2(n-k)+1}, v_1) = 2(n - k) \).

Case 2: \( y = v_{2i-1} \) with \( 1 \leq i \leq (n-k) \). Let \( W \) be any walk of odd length from \( v_{2i-1} \) to \( v_1 \). Then \( W \) must contain the (unique) odd cycle (of length 3) in \( \Gamma \), so its length satisfies

\[
|W| \geq 2i - 2 + 2(n - 2 - 2i + 1) + 3 = 2n - 2i - 1 \geq 2k - 1
\]

\[
> 2(n - k) - 1.
\]

Combining case 1 and case 2, we know that there is no walk of length \( 2(n - k) - 1 \) from any vertex \( y \) of \( X_0 \) to the vertex \( v_1 \). This implies that

\[
e_3(n, k) \geq F(\Gamma, k) \geq \exp_\Gamma(X_0) \geq 2(n - k).
\]

Combining this with the inequality \( e_3(n, k) \leq 2(n - k) \) obtained earlier, we have \( e_3(n, k) = 2(n - k) \), as desired.

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Fig. 4. The graph \( \Gamma \).
Before proving the case $1 \leq k \leq n/2$ of (3.1), we need to use the graphs $D_n(m)$ and $D_n^*(m)$ constructed in Section 2 to obtain suitable exponents in the set $E_3(n, k)$. These exponents will be used in Lemma 3.3, Lemma 3.6, and Lemma 3.7 later.

**Lemma 3.2.** Let $D = D_n(m)$ and $D^* = D_n^*(m)$ be the primitive simple graphs as in Figure 1 and Figure 2. Then

1. we have
   \[ F(D, k) = 2m - 2 \quad (1 \leq k \leq n - m); \]

2. we have
   \[ F(D^*, k) = 2m - 3 \quad (2 \leq k \leq n - m). \]

**Proof.** (1): Let $X$ be any $k$-vertex subset of $D$, and let $x \in X$. Then by using Lemma 2.2 we have
   \[ \exp_D(X) \leq \gamma_D(x) \leq 2m - 2. \]
   Thus
   \[ F(D, k) \leq 2m - 2. \]
   On the other hand, take $X_0 = \{m + 1, \ldots, m + k\}$. Let $W$ be a walk of odd length from some vertex $x \in X_0$ to the vertex $m + 1$. Then $W$ must contain the odd cycle of $D$. Hence $|W| \geq 3 + 2(m - 2) = 2m - 1$. This shows there is no walk of length $2m - 3$ from any vertex of $X_0$ to the vertex $m + 1$, so we have
   \[ F(D, k) \geq \exp_D(X_0) \geq 2m - 2. \]
   Combining the above two relations, we obtain (1).

(2): Let $X$ be any $k$-vertex subset of $D^*$. Since $k \geq 2$, there exists some vertex $x \in X$ with $x \neq n$. Thus using Lemma 2.3 we have
   \[ \exp_{D^*}(X) \leq \gamma_{D^*}(x) \leq 2m - 3 \]
   and so
   \[ F(D^*, k) \leq 2m - 3. \]
On the other hand, take $X_0 = \{m, m+1, \ldots, m+k-1\}$. Then it is not difficult to verify that there is no walk of length $2m - 4$ from any vertex of $X_0$ to the vertex $n$. Thus we have

$$F(D^*, k) \geq \exp_D(X_0) \geq 2m - 3.$$ Combining the above two relations, we obtain (2).

Now we prove the case $1 \leq k \leq n/2$ of (3.1).

**Lemma 3.3.** Let $n$ and $k$ be positive integers with $1 \leq k \leq n/2$. Then

$$e_3(n, k) = 2(n - k - 1). \quad (3.4)$$

**Proof.** Firstly take the primitive simple graph $D_n(m)$ in Lemma 3.2 with $m = n - k$. Then we know from (1) of Lemma 3.2 that

$$F(D_n(n - k), k) = 2(n - k - 1). \quad (3.5)$$

Next let $D$ be any primitive simple graph of order $n$. We will show that:

$$F(D, k) \leq 2(n - k - 1).$$

Take any vertex $v_0 \in V(D)$. Let

$$X = V(v_0, 2(n - k - 1))$$

$$= \{x \in V(D) | \text{there is a walk of length } 2(n - k - 1) \text{ from } x \text{ to } v_0 \text{ in } D\}.$$ We want to show that

$$|X| \geq n - k + 1. \quad (3.6)$$

If $X = V(D)$, then (3.6) holds trivially. So we may assume that $X \neq V(D)$ and thus there exists a vertex $u$ in $V(D)$ such that $u \notin X$.

Let $P$ be a path between $u$ and $v_0$ with length $|P|$, then $|P| \leq n - 1 < 2(n - k)$ (by the hypothesis $k \leq n/2$). If $|P|$ is even, then $|P| \leq 2(n - k - 1)$, and we can obtain a walk of length exactly $2(n - k - 1)$ from $u$ to $v_0$ by oscillating along an edge adjacent to $v_0$. This contradicts the condition $u \notin X$. Thus the length of the path $P$ is odd.
Now let $C$ be an odd cycle of $D$ with odd length $r > 3$ (which exists, since $D$ is a primitive simple graph). If the cycle $C$ and the path $P$ have two or more common vertices, then by using the two parts of $C$ divided by two common vertices we will obtain two paths between $u$ and $v_0$ with different parity. Thus we will get a path of even length between $u$ and $v_0$, contradicting that the length of every path between $u$ and $v_0$ is odd. Thus $C$ and $P$ have at most one common vertex.

Let $Q$ be a shortest path between $C$ and $P$. Then we have the subgraph of $D$ shown in Figure 5, where $z$ ($y$) is the common vertex of $Q$ and $C$ (of $Q$ and $P$), and $P_1$ ($P_2$) is the subpath of $P$ between $v_0$ and $y$ (between $y$ and $u$). In the case where $C$ and $P$ have one common vertex, then $z = y$ and $Q$ is the path of length zero.

Now take the walk

$$W = P_1 + Q + C + Q + P_2$$

between $v_0$ and $u$. Then its length $|W|$ is even (since $|P| = |P_1| + |P_2|$ is odd and $|C|$ is odd). If $|W| < 2(n - k - 1)$, then by the same arguments used above we will obtain a walk of length exactly $2(n - k - 1)$ from $v_0$ to $u$, again contradicting $u \notin X$. Thus we have $|W| \geq 2(n - k)$.

Now let the sequence of vertices in the direction from $v_0$ to $u$ of the walk $W$ be

$$v_0, v_1, v_2, \ldots, v_{2(n-k)}, \ldots, v_{|W|}.$$ 

Notice that any closed subwalk of $W$ contains the odd cycle $C$ exactly once, so the length of any closed subwalk of $W$ is odd. Therefore the following $n - k$ vertices of the walk $W$ are distinct:

$$v_0, v_2, v_4, \ldots, v_{2(n-k-1)},$$ 

(3.7)

![Fig. 5.](image-url)
for otherwise \( W \) will contain a closed subwalk of even length, which is a contradiction. Also there is a walk of length \( 2(n - k - 1) \) from any vertex in (3.7) to the vertex \( v_0 \), so we have

\[
\{v_0, v_2, v_4, \ldots, v_{2(n-k-1)}\} \subseteq X.
\]

On the other hand, \( |P_l| + |Q| \leq n - |C| = n - r \leq n - 3 \leq 2(n - k - 1) - 1 \), and \( C \) is an odd cycle of length \( r \geq 3 \), so at least one of the vertices in (3.7) is on \( C \setminus \{z\} \). Let \( v_{2j} \) be the first vertex in (3.7) which is on \( C \setminus \{z\} \) (here \( 0 \leq j \leq n - k - 1 \)). Let \( w \) be the vertex symmetric to \( v_{2j} \) on \( C \) with respect to the vertex \( z \). Then \( w \not\in \{v_0, v_2, v_4, \ldots, v_{2(n-k-1)}\} \), since \( r \) is odd and \( r \geq 3 \). Also, there is a walk of length \( 2j \) from \( w \) to \( v_0 \), where \( 0 \leq j \leq n - k - 1 \), so there is also a walk of length \( 2(n - k - 1) \) from \( w \) to \( v_0 \). Thus \( w \in X \), and \( w, v_0, v_2, v_4, \ldots, v_{2(n-k-1)} \) are \( n - k + 1 \) distinct vertices in \( X \). Thus \( |X| \geq n - k + 1 \), and (3.6) holds.

Now let \( A \) be any \( k \)-vertex subset of \( D \), and let \( v_0 \) be any vertex of \( D \) as before. Since \( |A| + |V(v_0, 2(n - k - 1))| \geq k + (n - k + 1) = n + 1 \), we have \( A \cap V(v_0, 2(n - k - 1)) \neq \emptyset \), so there is a walk of length \( 2(n - k - 1) \) from some vertex of \( A \) to the vertex \( v_0 \). But \( v_0 \) is an arbitrary vertex of \( D \), so we have actually shown that \( A \) can \( 2(n - k - 1) \)-cover \( V(D) \). Thus \( \exp(D, A) \leq 2(n - k - 1) \). But \( A \) is also an arbitrary \( k \)-vertex subset of \( D \), so we have

\[
F(D, k) = \max\{\exp(D, A) | A \subseteq V(D) \text{ and } |A| = k\} \leq 2(n - k - 1).
\]

This holds for any primitive simple graph \( D \) of order \( n \). Combining this with (3.5), we obtain \( e_3(n, k) = 2(n - k - 1) \), as desired.

Next we prove the case \( k = (n + 1)/2 \) (where \( n \) is odd) of (3.1).

**Lemma 3.4.** Let \( n \) be an odd number with \( n \geq 3 \), and let \( k = (n + 1)/2 \). Then

\[
e_3(n, k) = 2(n - k) - 1 = n - 2.
\]

**Proof.** We first show

\[
F(D, k) \leq 2(n - k) - 1
\]
for any primitive simple graph $D$ of order $n$. If $D \cong C_n$ is a cycle of length $n$, then any two vertices can $(n - 2)$-cover $V(D)$, since there is a walk of length $n - 2$ from a vertex $u$ to any vertex different from $u$ in $C_n$. Also $k = (n + 1)/2 \geq 2$, so we have

$$F(D, k) \leq F(C_n, 2) \leq n - 2 = 2(n - k) - 1.$$  

Now we assume $D \not\cong C_n$. Let $C$ be a shortest odd cycle of $D$. Then there exists some vertex $v^* \not\in V(C)$ such that the graph $D^* = D \setminus \{v^*\}$ is also connected, and hence is a primitive simple graph of order $n - 1$, because $D^*$ also contains the odd cycle $C$ (for example, we can take $v^*$ to be the vertex farthest from the cycle $C$). Let $u$ be a vertex adjacent to the vertex $v^*$ in $D$.

Let $X$ be any $k$-vertex subset of $D$, and let $Y = X \setminus \{v^*\}$. Then $Y$ is a vertex subset of $D^*$.

If $v^* \not\in X$, then $|Y| = k - 1 = (n - 1)/2 = \frac{1}{2}|V(D^*)|$. We can use Lemma 3.3 to obtain

$$\exp_{D^*}(Y) \leq F(D^*, k - 1) \leq 2\left(|V(D^*)| - (k - 1) - 1\right) = 2(n - k - 1).$$

If $v^* \not\in X$, then $|Y| = k = (n + 1)/2 = \frac{1}{2}|V(D^*)| + 1$, and we can use Lemma 3.1 to obtain

$$\exp_{D^*}(Y) \leq F(D^*, k) \leq 2\left(|V(D^*)| - k\right) = 2(n - k - 1).$$

Thus in any case we have

$$\exp_{D^*}(Y) \leq 2(n - k - 1) < 2(n - k) - 1.$$  

Hence $Y$ can $2(n - k - 1)$-cover $V(D^*)$ and $2(n - k) - 1$-cover $V(D^*)$, which implies that there is a walk of length $2(n - k - 1)$ from some vertex (say, a vertex $y$) of $Y$ to the vertex $u$, and so there is also a walk of length $2(n - k) - 1$ from $y$ to $v^*$. This shows that $Y$ can also $2(n - k) - 1$-cover $V(D)$, and so

$$\exp_{D}(X) \leq \exp_{D}(Y) \leq 2(n - k) - 1.$$  

Now $X$ is an arbitrary $k$-vertex subset of $D$, so we have $F(D, k) \leq 2(n - k) - 1$, and (3.9) is proved.

On the other hand, let $\Gamma$ be the graph in Figure 4. Let

$$X_0 = V(\Gamma) \setminus \{v_1, v_3, \ldots, v_{2(n-k)-1}\},$$
where \(2(n - k) - 1 = n - 2\). Then \(|X_0| = k\), and there is no walk of length \(2(n - k) - 2 = n - 3\) from any vertex of \(X_0\) to the vertex \(v_1\). Thus

\[
F(\Gamma, k) \geq \exp_r(X_0) \geq 2(n - k) - 1. \tag{3.10}
\]

Combining (3.10) and (3.9), we obtain \(e_3(n, k) = 2(n - k) - 1\), as desired.

Now our expression (3.1) for \(e_3(n, k)\) follows directly from Lemmas 3.1, 3.3, and 3.4.

**Theorem 3.1.** Let \(n\) and \(k\) be integers with \(1 \leq k \leq n - 1\). Then we have

\[
e_3(n, k) =\begin{cases} 
2(n - k - 1) & \text{if } 1 \leq k \leq \frac{n}{2}, \\
2(n - k) - 1 & \text{if } k = \frac{n + 1}{2} \text{ (and } n \text{ is odd),} \\
2(n - k) & \text{if } \frac{n}{2} + 1 \leq k \leq n - 1. 
\end{cases} \tag{3.1}
\]

**Proof.** Combine Lemmas 3.1, 3.3, and 3.4.

Now that \(e_3(n, k)\) [the largest number in the exponent set \(E_3(n, k)\)] is determined, we can start to determine the set \(E_3(n, k)\). In order to prove the expression (3.2) for the exponent set \(E_3(n, k)\), we first show that the integers 1, 2, and 3 are all in the set \(E_3(n, k)\).

**Lemma 3.5.** Let \(n\) and \(k\) be positive integers. Then

\[
1 \in E_3(n, k) \quad (2 \leq k \leq n - 1), \tag{3.11}
\]

\[
2 \in E_3(n, k) \quad (2 \leq k \leq n - 1), \tag{3.12}
\]

\[
3 \in E_3(n, k) \quad (2 \leq k \leq n - 2, \quad n \geq 5). \tag{3.13}
\]
Proof. (3.11): Let $K_n$ be the complete graph of order $n$. Then for $2 \leq k \leq n - 1$ we have

$$1 = F(K_n, k) \in E_3(n, k) \quad (2 \leq k \leq n - 1).$$

(3.12): Let $G_n$ be the primitive simple graph in Figure 6(a). Then

$$2 = F(G_n, k) \in E_3(n, k) \quad (2 \leq k \leq n - 1).$$

(3.13): Let $H_n$ be the primitive simple graph in Figure 6(b). Let

$$X_0 = \{v_1, v_2, \ldots, v_{n-3}, v_{n-1}\}.$$

Then there is no walk of length 2 from any vertex of $X_0$ to vertex $v_n$. Thus for $2 \leq k \leq n - 2$, we have

$$F(H_n, k) \geq F(H_n, n - 2) \geq \exp_{H_n}(X_0) \geq 3.$$

Also it is not difficult to verify that $F(H_n, k) \leq 3$ for $3 \leq k \leq n - 2$, so

$$3 = F(H_n, k) \in E_3(n, k) \quad (3 \leq k \leq n - 2). \quad (3.14)$$

Now for the case $k = 2$, consider the graph $F_n = H_n + e$, where $e$ is the edge $[v_{n-3}, v_n]$. Then there is no walk of length 2 from $v_{n-1}$ to $v_n$ and from $v_{n-3}$ to $v_n$, so

$$F(F_n, 2) \geq \exp_F([v_{n-1}, v_{n-3}]) \geq 3.$$
Also it is not difficult to verify that $F(F_n, 2) \leqslant 3$, so we have

$$3 = F(F_n, 2) \in E_3(n, 2) \quad (3.15)$$

Combining (3.14) and (3.15), we obtain (3.13).

Using Lemma 3.5 and the exponents of the graphs $D_n(m)$ and $D_n^*(m)$ given by taking particular values of the parameter $m$ in Lemma 3.2, we can further show that for $2 \leqslant k \leqslant n - 2$, all the integers between 1 and $2(n - k - 1)$ are in the exponent set $E_3(n, k)$.

**Lemma 3.6.** Let $n$ and $k$ be integers with $2 \leqslant k \leqslant n - 2$. Then

$$\{1, 2, \ldots, 2(n - k - 1)\} \subseteq E_3(n, k). \quad (3.16)$$

**Proof.** The case $n = 4$ follows from (3.11) and (3.12), so we may assume $n \geqslant 5$.

Firstly, by Lemma 3.5 we have

$$\{1, 2, 3\} \subseteq E_3(n, k) \quad (2 \leqslant k \leqslant n - 2). \quad (3.17)$$

Secondly, take an integer $m$ with $3 \leqslant m \leqslant n - k$, and let $D = D_n(m)$ as in Figure 1. We have by Lemma 3.2 that

$$2m - 2 = F(D, k) \in E_3(n, k),$$

so

$$\{4, 6, \ldots, 2(n - k - 1)\} = \{2m - 2|3 \leqslant m \leqslant n - k\} \subseteq E_3(n, k). \quad (3.18)$$

Thirdly, take an integer $m$ with $4 \leqslant m \leqslant n - k$, and let $D^* = D_n^*(m)$ as in Figure 2. We have by Lemma 3.2 that

$$2m - 3 = F(D^*, k) \in E_3(n, k),$$
Combining (3.17), (3.18), and (3.19), we obtain (3.16).

Comparing Lemma 3.6 with the expression (3.1) for \( e_3(n, k) \), we can already see that (3.2) holds when \( 2 \leq k \leq n/2 \). For the second case \( k = (n + 1)/2 \) (where \( n \) is odd), we have \( e_3(n, k) = 2(n - k) - 1 \) by (3.1), so

\[
\{1, 2, \ldots, e_3(n, k)\} = \{1, 2, \ldots, 2(n - k - 1)\} \cup \{e_3(n, k)\}.
\]

By definition \( e_3(n, k) \) must be in \( E_3(n, k) \), so (3.2) also holds in this case. For the third case \( n/2 + 1 \leq k \leq n - 1 \), we also have \( \{1, 2, \ldots, 2(n - k - 1)\} \subseteq E_3(n, k) \) and \( 2(n - k) = e_3(n, k) \in E_3(n, k) \). So all that remains to be proved is \( 2(n - k) - 1 \in E_3(n, k) \) in the case \( n/2 + 1 \leq k \leq n - 1 \). The following Lemma 3.7 will settle this case for \( n/2 + 1 \leq k \leq n - 2 \) [notice that the case \( k = n - 1 \) follows from (3.11) in Lemma 3.5].

**Lemma 3.7.** Let \( n \) and \( k \) be positive integers with \( n/2 + 1 \leq k \leq n - 2 \), and let \( D = D_n^*(2n - 2k) \), where \( D_n^*(m) \) is the graph defined in Figure 2. Then we have

\[
F(D, k) = 2(n - k) - 1.
\]  

Proof. Let

\[
X_0 = V(D) \setminus \{3, 5, \ldots, 2n - 2k - 3, 2n - 2k - 1, n\}.
\]

Then \( |X_0| = k \), and there is no walk of length \( 2(n - k) - 2 \) from any vertex of \( X_0 \) to the vertex \( n \). Thus we have

\[
F(D, k) \geq \exp_D(X_0) \geq 2(n - k) - 1.
\]  

On the other hand, let \( X \) be any \( k \)-vertex subset of \( D \), and let \( Z = \{2n - 2k, 2n - 2k + 1, \ldots, n - 3, n - 2, n\} \) be the set of pendant vertices of \( D \). We consider two cases.
Case 1: There exists some vertex \( u \in Z \) such that \( u \notin X \). Consider the primitive graph \( D_1 = D \setminus \{u\} \). Then \( X \) is a \( k \)-vertex subset of \( D_1 \), and so by Theorem 3.1 we have

\[
\exp_{D_1}(X) \leq F(D_1, k) \leq 2(n - 1 - k).
\]

Now since \( D = D_1 + u \), we have

\[
\exp_D(X) \leq \exp_{D_1}(X) + 1 \leq 2(n - k) - 1. \tag{3.22}
\]

Case 2: \( Z \subseteq X \). Let \( Y = X \setminus Z \). Then \( |Y| = |X| - |Z| = k - (2k - n) = n - k \geq 2 \). Let \( D_2 = D \setminus Z \). Then \( D_2 \) is a primitive graph of order \( 2n - 2k \). Now \( |Y| = \frac{1}{2}|V(D)| \), so by Lemma 3.3 we have

\[
\exp_{D_2}(Y) \leq F(D_2, n - k) \leq 2(2n - 2k - (n - k) - 1) = 2(n - k - 1).
\]

But any vertex of \( Z \) is adjacent to some vertex of \( D_2 \), so we have

\[
\exp_D(Y) \leq \exp_{D_2}(Y) + 1 \leq 2(n - k) - 1,
\]

which implies

\[
\exp_D(X) \leq \exp_D(Y) \leq 2(n - k) - 1. \tag{3.23}
\]

Combining (3.21), (3.22), and (3.23), we obtain (3.20). \( \blacksquare \)

Now we can completely determine the exponent set \( E_3(n, k) \) for \( 2 \leq k \leq n - 1 \).

**Theorem 3.2.** Let \( n \) and \( k \) be integers with \( 2 \leq k \leq n - 1 \). Then

\[
E_3(n, k) = \{1, 2, \ldots, e_3(n, k)\} \quad (2 \leq k \leq n - 1). \tag{3.24}
\]

**Proof.** By the definition of \( e_3(n, k) \) we obviously have \( E_3(n, k) \subseteq \{1, 2, \ldots, e_3(n, k)\} \). So in the following we only need to prove that

\[
\{1, 2, \ldots, e_3(n, k)\} \subseteq E_3(n, k). \tag{3.25}
\]
Case 1: $2 \leq k \leq n/2$. Then (3.25) follows from Lemma 3.6.

Case 2: $k = (n + 1)/2$ (where $n$ is odd). Then (3.25) follows from Lemma 3.6 and the fact that $e_3(n, k) \in E_3(n, k)$.

Case 3: $n/2 + 1 \leq k \leq n - 2$. Then (3.25) follows from Lemma 3.6, Lemma 3.7, and the fact that $e_3(n, k) \in E_3(n, k)$.

Case 4: $k = n - 1$. Then $e_3(n, k) = 2(n - k) = 2$, and (3.25) follows from (3.11) and (3.12) in Lemma 3.5.

4. THE EXPONENT SET $E_2(n, k)$

In this section we will determine the exponent set $E_2(n, k)$. If $k = 1$, then by (1.5) we have $f(D, 1) = \exp_D(1)$, so $E_2(n, 1) = E_1(n, 1)$, which was already determined in Theorem 2.2. If $k = n - 1$, then it is not difficult to verify that

$$E_2(n, n - 1) = \{1\} \quad (n \geq 3).$$

Therefore, in this section we will only consider the remaining case $2 \leq k \leq n - 2$.

As in the cases for $E_1(n, k)$ and $E_3(n, k)$, we need first to determine the largest number $e_2(n, k)$ in the exponent set $E_2(n, k)$. We will prove in Theorem 4.1 that

$$e_2(n, k) = \begin{cases} \left\lfloor \frac{n - k}{(k + 2)/2} \right\rfloor, & k = 2 \text{ or } 4 \leq k \leq n - 2, \\ \left\lfloor \frac{n - 4}{2} \right\rfloor, & k = 3, \end{cases} \quad (4.1)$$

which means that the upper bound given by R. A. Brualdi and B. Liu in [2, Theorem 6.7] [also see (1.7) in Section 1 of this paper] for primitive graphs can actually be achieved by primitive simple graphs, except in the case $k = 3$. Then we will determine in Theorem 4.2 that the exponent set $E_2(n, k)$ is

$$E_2(n, k) = \{1, 2, \ldots, e_2(n, k)\} \quad (2 \leq k \leq n - 2). \quad (4.2)$$

In order to prove (4.1), we consider the cases $k = 2, k = 3,$ and $4 \leq k \leq n - 2$ separately in the following Lemmas 4.1, 4.2, and 4.3. Firstly we consider the case $k = 2$ of (4.1).
Lemma 4.1. Let $n$ be an integer with $n \geq 4$. Then

$$e_2(n, 2) = \left\lceil \frac{n - 2}{2} \right\rceil.$$

Proof. From [2, Theorem 6.7 or Lemma 6.5] we have $e_2(n, 2) \leq [(n - 2)/2]$. To prove the equality, we consider two cases:

Case 1: $n$ is even. Write $n = 2d + 2$. Take the primitive simple graph $G_n = P_n + e$ as in Figure 7, where $P_n = v_1 v_2 \cdots v_{2d+2}$ is a path of order $n$ and $e$ is the edge $[v_{d+1}, v_{d+3}]$. Then it can be verified that $G_n$ cannot be $(d - 1)$-covered by any set of two vertices, so we have

$$e_2(n, 2) \geq f(G_n, 2) \geq d = \left\lceil \frac{n - 2}{2} \right\rceil.$$

From this we deduce that $e_2(n, 2) = [(n - 2)/2]$.

Case 2: $n$ is odd. Let $C_n$ be a cycle of length $n$. Then $C_n$ is a primitive simple graph of order $n$, since $n$ is odd. Let $X = \{u, v\}$ be any 2-vertex subset of the graph $C_n$. Then $u$ can reach at most $(n - 1)/2$ vertices of $C_n$ by walks of length $(n - 3)/2$, and the same is the case for $v$. Thus $\{u, v\}$ can reach at most $n - 1$ vertices of $C_n$ by walks of length $(n - 3)/2$, and so $X = \{u, v\}$ cannot $(n - 3)/2$-cover $C_n$. This shows that

$$e_2(n, 2) \geq f(C_n, 2) \geq \frac{n - 1}{2} = \left\lceil \frac{n - 2}{2} \right\rceil,$$

and therefore we conclude $e_2(n, 2) = [(n - 2)/2]$.

\[\text{FIG. 7. The graph } G_n.\]
Next we consider the case $k = 3$ of (4.1).

**Lemma 4.2.** Let $n$ be an integer with $n \geq 5$. Then

$$e_2(n, 3) = \left\lfloor \frac{n - 4}{2} \right\rfloor.$$  \hspace{1cm} (4.3)

**Proof.** We consider the following two cases.

**Case 1: $n$ is odd.** From [2, Theorem 6.7] [also see (1.7) of this paper] we have

$$e_2(n, 3) \leq \left\lfloor \frac{n - 3}{2} \right\rfloor = \frac{n - 3}{2} = \left\lfloor \frac{n - 4}{2} \right\rfloor.$$  \hspace{1cm} (4.4)

On the other hand, we write $n = 2d + 3$ and take the primitive simple graph $\Gamma_n = P_n + e$ as in Figure 8, where $P_n = v_1v_2 \cdots v_{2d+3}$ is a path of order $n$ and $e$ is the edge $[v_{d+1}, v_{d+3}]$. It can be verified that $\Gamma_n$ can not be $(d - 1)$-covered by any set of three vertices, so

$$e_2(n, 3) \geq f(\Gamma_n, 3) \geq d = \frac{n - 3}{2} = \left\lfloor \frac{n - 4}{2} \right\rfloor.$$  \hspace{1cm} (4.5)

Combining (4.4) and (4.5) we obtain (4.3).

**Case 2: $n$ is even.** We write $n = 2d + 4$. Let $G$ be any primitive simple graph of order $n = 2d + 4$. Then $G$ contains a spanning tree $T$ which is not a path (since $G \neq C_n$ and $G$ contains no loops). Let $Q$ be a longest path of $T$ with length $|Q|$. Then $|Q| \leq n - 2 = 2d + 2$. Let $u, v, w$ be three consecu-

![Fig. 8. The graph $\Gamma_n$.](image-url)
tive central vertices of $Q$. Then $(u, v, w)$ can $d$-cover $G$, so

$$f(G, 3) \leq \exp_G(\{u, v, w\}) \leq d.$$  

But this holds for any primitive simple graph $G$ of order $n = 2d + 4$, so we have

$$e_2(n, 3) \leq d = \frac{n - 4}{2} = \left\lfloor \frac{n - 4}{2} \right\rfloor. \tag{4.6}$$

On the other hand, take the primitive simple graph $H_n = P_n + e$ as in Figure 9, where $P_n = v_1v_2 \cdots v_{2d+4}$ is a path of order $n$ and $e$ is the edge $[v_{d+1}, v_{d+3}]$. Then it can be verified that $H_n$ can not be $(d - 1)$-covered by any set of three vertices, so

$$e_2(n, 3) \geq f(H_n, 3) \geq d = \frac{n - 4}{2} = \left\lfloor \frac{n - 4}{2} \right\rfloor. \tag{4.7}$$

Thus (4.3) follows from (4.6) and (4.7).

Now we consider the case $4 \leq k \leq n - 2$ of (4.1).

**Lemma 4.3.** Let $n$ and $k$ be integers with $4 \leq k \leq n - 2$. Then we have

$$e_2(n, k) = \left\lfloor \frac{n - k}{((k + 2)/2)} \right\rfloor \quad (4 \leq k \leq n - 2). \tag{4.8}$$

*Fig. 9. The graph $H_n$.***
Proof. From [2, Theorem 6.7] [also see (1.7) of this paper] we know that

$$f(D, k) \leq \left\lfloor \frac{n - k}{[(k + 2)/2]} \right\rfloor$$

holds for any primitive graph $D$ of order $n$, and hence also holds for any primitive simple graph $D$ of order $n$. Thus we obtain

$$e_2(n, k) \leq \left\lfloor \frac{n - k}{[(k + 2)/2]} \right\rfloor. \quad (4.9)$$

To prove the equality, we write

$$d = d(n, k) = \left\lfloor \frac{n - k}{[(k + 2)/2]} \right\rfloor.$$

Then

$$\left\lfloor \frac{k + 2}{2} \right\rfloor d - \left\lfloor \frac{k}{2} \right\rfloor \leq n - k \leq \left\lfloor \frac{k + 2}{2} \right\rfloor d,$$

so we can write

$$n = \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) d + k - \left\lfloor \frac{k}{2} \right\rfloor + j$$

$$= \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) d + \left\lfloor \frac{k + 1}{2} \right\rfloor + j \quad (0 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor).$$

Take the graph $D(k, d, j)$ (for $k \geq 4$ and $0 \leq j \leq [k/2]$) as in Figure 10. The graph $D(k, d, j)$ consists of the $[k/2] + 1$ paths $P_0, P_1, \ldots, P_{[k/2]}$ with a common end vertex at (the center) $u$ plus an edge joining the vertex adjacent to $u$ in $P_1$ and the vertex adjacent to $u$ in $P_2$ (Notice here we need
$k \geq 4$ to ensure $[k/2] \geq 2$.) The lengths of the paths $P_0, P_1, \ldots, P_{[k/2]}$ are

$$|P_i| = \begin{cases} a_i, & i = 0, \\ a_i + 1, & 1 \leq i \leq j, \\ a_i, & j < i \leq [k/2], \end{cases}$$

where the numbers $a_i$ ($i = 0, 1, \ldots, [k/2]$) are defined as

$$a_i = \begin{cases} d, & i = 0, \\ d + \left[\frac{k + 1}{2}\right] - \left[\frac{k}{2}\right], & i = 1, \\ d + 1, & 2 \leq i \leq [k/2]. \end{cases}$$

Now the number of vertices of the graph $D(k, d, j)$ is

$$|V(D(k, d, j))| = \sum_{i=0}^{[k/2]} |P_i| + 1 = \sum_{i=0}^{[k/2]} a_i + j + 1$$

$$= \left(\left[\frac{k}{2}\right] + 1\right)d + \left(\left[\frac{k}{2}\right] - 1\right) + \left(\left[\frac{k+1}{2}\right] - \left[\frac{k}{2}\right]\right) + j + 1$$

$$= \left(\left[\frac{k}{2}\right] + 1\right)d + \left[\frac{k+1}{2}\right] + j = n.$$
Also $D(k, d, j)$ is connected and contains a cycle of length 3, so $D(k, d, j)$ is a primitive simple graph of order $n$. Let

$$P_i = uv_i v_{i+1} \cdots v_{i+|P_i|}, \quad (0 \leq i \leq \lfloor k/2 \rfloor),$$

and suppose a vertex subset $Y \subseteq V(D(k, d, j))$ can $(d - 1)$-cover $D(k, d, j)$. Then $Y$ must contain at least two vertices different from $u$ in each path $P_i$ ($2 \leq i \leq \lfloor k/2 \rfloor$) to $(d - 1)$-cover the vertices $v_{i+|P_i|-1}$ and $v_{i+|P_i|}$. Also $Y$ must contain at least two vertices in $P_0$ to $(d - 1)$-cover $v_{0(d-1)}$ and $v_{0d}$, and $Y$ must contain $1 + \lfloor (k + 1)/2 \rfloor - \lfloor k/2 \rfloor$ vertices different from $u$ in $P_1$ to $(d - 1)$-cover $v_{1|P_i|-1}$ and $v_{1|P_i|}$. Thus we have

$$|Y| \geq 2 \left\lfloor \frac{k}{2} \right\rfloor + \left(1 + \left\lfloor \frac{k + 1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k + 1}{2} \right\rfloor + 1 = k + 1.$$

This shows that $D(k, d, j)$ cannot be $(d - 1)$-covered by any $k$-vertex subset of it, so we obtain

$$e_2(n, k) \geq f(D(k, d, j), k) \geq d = \left\lfloor \frac{n - k}{\lfloor (k + 2)/2 \rfloor} \right\rfloor \quad (4 \leq k \leq n - 2)$$

Together with (4.9) this gives us

$$e_2(n, k) = \left\lfloor \frac{n - k}{\lfloor (k + 2)/2 \rfloor} \right\rfloor \quad \text{for} \quad 4 \leq k \leq n - 2. \quad \blacksquare$$

Combining Lemmas 4.1, 4.2, and 4.3, we obtain the expression (4.1) for the number $e_2(n, k)$ in the case $2 \leq k \leq n - 2$. We state this again as the following theorem.

**Theorem 4.1.** Let $n$ and $k$ be integers with $2 \leq k \leq n - 2$. Then we have

$$e_2(n, k) = \begin{cases} \left\lfloor \frac{n - k}{\lfloor (k + 2)/2 \rfloor} \right\rfloor, & k = 2 \text{ or } 4 \leq k \leq n - 2, \\ \left\lfloor \frac{n - 4}{2} \right\rfloor, & k = 3. \end{cases}$$

(4.10)
Now we need to prove the following "ascending property" of the exponent set $E_2(n, k)$, which will be used in the determination of the set $E_2(n, k)$.

**Lemma 4.4.** Let $n$ and $k$ be integers with $1 \leq k \leq n - 1$. Then we have

$$E_2(n, k) \subseteq E_2(n + 1, k).$$

**(4.11)**

**Proof.** Let $D$ be a primitive simple graph of order $n$. Take $v \in V(D)$ and $v^* \notin V(D)$. We construct a primitive simple graph $D^*$ of order $n + 1$ as follows:

$$V(D^*) = V(D) \cup \{v^*\},$$

$$E(D^*) = E(D) \cup \{[u, v^*] | [u, v] \in E(D)\}.$$

Namely, the new vertex $v^*$ in $D^*$ is a copy of $v$ with respect to adjacency. We will prove that $f(D^*, k) = f(D, k)$.

Suppose $f(D, k) = m$. Then there exists a $k$-vertex subset $X_0 \subseteq V(D)$ such that $X_0$ can $m$-cover $V(D)$ in $D$. It is easy to see that $X_0$ can also $m$-cover $V(D^*)$ in $D^*$ (since a walk of length $m$ from some vertex $x \in X_0$ to the vertex $v$ in $D$ will give a walk of length $m$ from $x$ to $v^*$ in $D^*$), so

$$f(D^*, k) \leq \exp_{D^*}(X_0) \leq m = f(D, k).$$

**(4.12)**

On the other hand, suppose $f(D^*, k) = p$. Then there exists a $k$-vertex subset $Y_0 \subseteq V(D^*)$ which can $p$-cover $V(D^*)$ in $D^*$. Now let

$$X_1 = \begin{cases} Y_0 & \text{if } v^* \notin Y_0, \\ (Y_0 \setminus \{v^*\}) \cup \{v\} & \text{if } v^* \in Y_0. \end{cases}$$

Then $X_1$ is a vertex subset of $V(D)$ and $|X_1| \leq k$. It is easy to see that $X_1$ can also $p$-cover $V(D)$ in $D$, so

$$f(D, k) \leq \exp_D(X_1) \leq p = f(D^*, k).$$

**(4.13)**

Combining (4.12) and (4.13), we obtain $f(D^*, k) = f(D, k)$. 

Now take any $m \in E_2(n, k)$, and suppose $m = f(D, k)$ for some primitive simple graph $D$ of order $n$. Then by the above argument we know there exists a primitive simple graph $D^*$ of order $n + 1$ such that $f(D^*, k) = f(D, k) = m$. Thus $m \in E_2(n + 1, k)$. This shows $E_2(n, k) \subseteq E_2(n + 1, k)$ and completes the proof of the lemma.

Now we can determine the exponent set $E_2(n, k)$.

**Theorem 4.2.** Let $n$ and $k$ be integers with $2 \leq k \leq n - 2$. Then we have

$$E_2(n, k) = \{1, 2, \ldots, e_2(n, k)\},$$

(4.14)

where the expression for $e_2(n, k)$ is given in Theorem 4.1.

**Proof.** By definition we know that $e_2(n, k) \in E_2(n, k)$. Also we can use Lemma 4.4 to obtain for $k + 2 \leq m \leq n$ that

$$e_2(m, k) \in E_2(m, k) \subseteq E_2(n, k) \quad (k + 2 \leq m \leq n).$$

Thus we have

$$\{e_2(m, k)|k + 2 \leq m \leq n\} \subseteq E_2(n, k).$$

(4.15)

Notice that the left hand side of (4.15) is a set of consecutive integers from 1 to $e_2(n, k)$ [we can see this from the expression for $e_2(m, k)$ in (4.10)], so (4.15) can be rewritten as

$$\{1, 2, \ldots, e_2(n, k)\} \subseteq E_2(n, k).$$

Also we have $E_2(n, k) \subseteq \{1, 2, \ldots, e_2(n, k)\}$ by the definition of $e_2(n, k)$. Therefore (4.14) holds, and the proof of the theorem is completed.

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**References**


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