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An optimization algorithm applied to the Morrey conjecture in nonlinear elasticity

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Abstract

For a long time it has been studied whether rank-one convexity and quasiconvexity give rise to different families of constitutive relations in planar nonlinear elasticity. Stated in 1952 the Morrey conjecture says that these families are different, but no example has come forward to prove it. Now we attack this problem by deriving a specialized optimization algorithm based on two ingredients: first, a recently found necessary condition for the quasiconvexity of fourth-degree polynomials that distinguishes between both classes in the three dimensional case, and secondly, upon a characterization of rank-one convex fourth-degree polynomials in terms of infinitely many constraints.

After extensive computational experiments with the algorithm, we believe that in the planar case, the necessary condition mentioned above is also necessary for the rank-one convexity of fourth-degree polynomials. Hence the question remains open.

Keywords: Rank-one convexity; Quasiconvexity; Morrey conjecture; Cutting-plane method; Constitutive relations in nonlinear elasticity

1. Introduction

Different relaxations of convexity have been proposed as constitutive assumptions in the Theory of Nonlinear Elasticity in the framework of the Calculus of Variations, see Ball (1978) or Dacorogna (1989). The purpose is to have a sufficiently large class of functions, i.e., that contains functions which might represent the stored energy function for a wide variety of materials, but one would like to keep the global energy functional being sequentially weakly lower semicontinuous (s.w.l.s.c. for short), to have that weakly convergent sequences of minimizers converge to a minimum. For this latter condition the class of quasiconvex functions is the precise one, unfortunately its definition is extremely difficult to verify. On the other hand from the Hadamard stability condition one obtains that the stored energy function is necessarily rank-one convex.

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It is known that quasiconvex functions are rank-one convex, however the reciprocal has not been proved nor disproved in the planar case and Morrey conjectured that they are different, see Morrey (1952) which was confirmed forty years later, but only for three dimensions, by the famous counterexample of Šverák (1992), consisting of a fourth-degree polynomial. Much analytical work has been done in terms of comparing the two classes in the planar case, see for example Alibert and Dacorogna (1992) where additionally a counterexample was obtained of a fourth-degree quasiconvex polynomial that is not polyconvex, a sufficient condition for quasiconvexity requiring that the function can be written as a convex function of its minors. See also Pedregal (1996), Müller (1999), Kałamajska (2003) and the references therein. Finally in Székelyhidi (2005) the characterization of the rank-one convex hull is studied, leading to an important analytical result.

Some numerical work has been done previously for this problem. In Dacorogna et al. (1990) a particular, albeit important, function was studied, namely the counterexample of Dacorogna and Marcellini (1988) that shows that rank-one convexity does not imply polyconvexity, which is again a fourth-degree polynomial. A very extensive study was done in Dacorogna and Haeberly (1998) covering particular families of functions for which there were detailed analytical results. See also Brighi and Chipot (1994).

The existence of non-quasiconvex rank-one convex functions is linked to a fundamental question in the theory of composite materials, namely whether all composites can be constructed by sequential laminates, see Milton (2002). Furthermore quasiconvex functions have been extensively used in many other subjects in mechanics due to their good behavior in terms of variational principles, as is explained below. Examples of this are: the modelling of phase transitions in solids, see Carstensen (2005), shape optimization, see Pedregal (2005) and fracture mechanics as in Francfort and Marigo (1998), just to cite one reference for each area of research.

The question about the difference between quasiconvexity and rank-one convexity is also connected to a conjecture in the theory of quasiconformal mappings, see Astala et al. (1998) and Iwaniec (2002), related to the norm of the Beurling–Ahlfors transform, however the direction explored here is not helpful for such conjecture since Proposition 2 below gives a trivial result when applied to the functions related to this conjecture.

Here this question is studied through the derivation of a specialized optimization algorithm based on a necessary condition for the quasiconvexity of fourth-degree polynomials contained in Proposition 2 below, that we proved recently see Gutiérrez (2006), and which explains Šverák's example. Therefore the approach followed here is novel since it gives a new way to look at this problem, which by now is known to be very hard analytically.

We report here our numerical efforts, since this might help in the discussion about the most appropriate way to attack numerically this very hard problem, for example by proposing a better optimization algorithm than the one we used. On the other hand, for the more analytically minded researchers it is very useful to have a strong indication that the much sought-for counterexample, most likely lies on a class of functions broader than that of the fourth-degree polynomials and that the condition we found in Gutiérrez (2006) does not seem to be fine enough to distinguish between rank-one convexity and quasiconvexity in the most general case.

2. Rank-one convexity and quasiconvexity

Let m and N be either 2 or 3 and $\Omega \subset \mathbb{R}^N$ be a bounded regular open set, representing the body whose deformation we want to study. $M_{m\times N}$ will denote the space of the $m \times N$ real matrices. If we have a sequence of vector fields with m components which are defined on Ω, it is customary to speak of their gradients as a sequence of $m \times N$-matrix valued fields on Ω.

The system of Nonlinear Elasticity reads as

$$-\operatorname{div}\sigma(x, \nabla u) = f \quad x \in \Omega,$$

where $u : \Omega \to \mathbb{R}^m$ represents the displacement fields, which should also satisfy some boundary conditions, $f : \Omega \to \mathbb{R}^m$ are the external forces and $\sigma : \Omega \times M_{m\times N} \to M_{m\times N}$ gives the internal stresses. Then, assuming that there exists a smooth function W, called the strain-energy density of the body, one has that

$$\sigma_{ij}(x, \nabla u) = \frac{\partial W}{\partial (\nabla u)_{ij}}(x, \nabla u). \tag{1}$$

Eq. (1) is called the stress–strain relation and it represents the constitutive assumption made on the material at position $x \in \Omega$. It corresponds to the generalization of Hooke's law.

One can use the framework of the Direct Method of the Calculus of Variations and try to minimize the total strain energy

$$J(u) = \int_{\Omega} W(x, \nabla u(x))\,\mathrm{d}x.$$

In this approach, however, one needs two ingredients: first one has to generate a minimizing sequence that belongs to a sequentially compact set for weak convergence and secondly J should be s.w.l.s.c. In that form the weak limit of the sequence will be a minimizer of the problem and then, under smoothness assumptions, a solution to the nonlinear elasticity system. The assumptions needed over W to make this approach work, lead to the notion of quasiconvexity, namely W is quasiconvex if for any matrix U in $M_{m\times N}$, any ω measurable subset of Ω whose measure is denoted by $|\omega|$ and any $\varphi \in C_c^{\infty}(\omega, \mathbb{R}^m)$, one has that

$$\int_{\omega} W(U + \nabla\varphi)\,\mathrm{d}x \geqslant |\omega| W(U).$$

Quasiconvexity is equivalent to s.w.l.s.c. for the $W^{1,\infty}$ weak ★ topology. The problem, however, is that to check whether the definition of quasiconvexity holds is extremely difficult, which explains the usefulness of having the more stringent condition of polyconvexity, see Ball (1978), and the more relaxed condition of rank-one convexity. A function $F : M_{m\times N} \to \mathbb{R}$ is said rank-one convex if for any matrix U in $M_{m\times N}$ and any pair $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^N$, the function

$$\phi(t) = F(U + t\eta \otimes \xi)$$

is convex when $t \in (0, 1)$.

A function F that is quasiconvex is necessarily rank-one convex. This has been known at least since the work of Morrey, who also stated in 1952 the conjecture that these two families of functions are different in the vectorial case, i.e., $N, m \geqslant 2$, known as the Morrey conjecture. This conjecture, however, has not been proved neither disproved in the most general case $N, m \geqslant 2$. For $N \geqslant 2$ and $m \geqslant 3$ the conjecture was proved by Šverák (1992), who gave a rank-one convex fourth-degree polynomial that is not quasiconvex. In Gutiérrez (2006) we derived a necessary condition for the quasiconvexity of a polynomial of degree four with vanishing constant and linear terms, see Proposition 2 below. In that paper it was shown that the counterexample of Šverák, which falls into the class for which our proposition holds, does violate the necessary condition we found.

If $F \in C^2(M_{m\times N}, \mathbb{R})$ an equivalent condition to rank-one convexity is the Legendre–Hadamard condition, namely that for any $U \in M_{m\times N}$

$$\sum_{i,k=1}^{m} \sum_{j,l=1}^{N} \frac{\partial^2 F(U)}{\partial U_{ij}\partial U_{kl}} \eta_i \xi_j \eta_k \xi_l \geqslant 0 \quad \forall \eta \in \mathbb{R}^m, \quad \xi \in \mathbb{R}^N.$$

From now on we identify $M_{m\times N}$ with $\mathbb{R}^{Nm}$ by putting the first row of the matrix as the first N entries of the vector, the second row as the following N entries and so on. Then, the Legendre–Hadamard condition for F becomes that for any $d \in \mathbb{R}^{Nm}$, the Hessian $F''(d)$ must be a positive semidefinite matrix over the following cone in $\mathbb{R}^{Nm}$, equivalent to the set of rank-one matrices in $M_{m\times N}$:

$$\Lambda = \{\lambda \in \mathbb{R}^{Nm} : \exists \eta \in \mathbb{R}^m, \xi \in \mathbb{R}^N \setminus \{0\} \text{ such that } \lambda = (\eta_1\xi, \ldots, \eta_m\xi)^{\mathrm{T}}\}.$$

If we restrict F to be a fourth-degree polynomial, then the Hessian matrix of F at any point td, with $t \in \mathbb{R}$ and $d \in \mathbb{R}^{Nm}$, is given by

$$F''(td)(\cdot,\cdot) = F''(0)(\cdot,\cdot) + tF^{(3)}(0)(d,\cdot,\cdot) + \frac{1}{2}t^2 F^{(4)}(0)(d,d,\cdot,\cdot).$$

Thus, F is rank-one convex if and only if for any $\lambda \in \Lambda$, $d \in \mathbb{R}^{Nm}$ and $t \in \mathbb{R}$ one has that

$$F''(0)(\lambda,\lambda) + tF^{(3)}(0)(d,\lambda,\lambda) + \frac{1}{2}t^2 F^{(4)}(0)(d,d,\lambda,\lambda) \geqslant 0,$$

which then gives the following result.

Proposition 1. *Let F be a fourth-degree polynomial in $\mathbb{R}^{Nm}$. Then F is rank-one convex if and only if the following three conditions hold*:

(a) $F''(0)(\lambda,\lambda) \geqslant 0 \; \forall \lambda \in \Lambda$,
(b) $F^{(4)}(0)(d,d,\lambda,\lambda) \geqslant 0 \; \forall d \in \mathbb{R}^{Nm},\; \lambda \in \Lambda$,
(c) $F^{(3)}(0)(d,\lambda,\lambda)^2 - 2F''(0)(\lambda,\lambda)F^{(4)}(0)(d,d,\lambda,\lambda) \leqslant 0 \; \forall d \in \mathbb{R}^{Nm},\; \lambda \in \Lambda.$

Let us define

$$V_0 = \{(\lambda,\xi) \in \mathbb{R}^{Nm} \times (\mathbb{R}^N \setminus \{0\}) : \lambda = (\eta_1\xi,\ldots,\eta_m\xi)^{\mathrm{T}},\; \eta_i \in \mathbb{R},\; i = 1,\ldots,m\}.$$

So when we project V_0 onto $\mathbb{R}^{Nm}$ we get the cone Λ.

The Theory of Compensated Compactness, see Tartar (1979) or Tartar (1993), is a part of Nonlinear Analysis devoted to characterize either sequential continuity under weak convergence or even s.w.l.s.c. when the weakly convergent sequences also satisfy some linear partial differential equations. It has been successfully used to answer several mathematical questions in fields as diverse as: Homogenization, Optimal Design and Systems of Nonlinear Hyperbolic Conservation Laws. Because quasiconvexity is equivalent to s.w.l.s.c. for the $W^{1,\infty}$ weak $\star$ topology, it is this latter characterization of quasiconvexity which has a connection with Compensated Compactness and then, based on this theory, in Gutiérrez (2006) the following result was derived.

Proposition 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded regular open set and F be a fourth-degree polynomial in $\mathbb{R}^{2m}$, satisfying $F(0) = 0$ and such that one has that $V^\infty \geqslant 0$, for any sequence $U^n \in L^\infty(\Omega,\mathbb{R}^{2m})$ for which the following hold*:

(i) $U^n \rightharpoonup 0$ *in* L^∞ *weak* $\star$
(ii) $F(U^n) \rightharpoonup V^\infty$ *in* L^∞ *weak* $\star$
(iii) $\frac{\partial U^n_{2k-1}}{\partial x_2} = \frac{\partial U^n_{2k}}{\partial x_1}$ *for* $k = 1,\ldots,m$ *and* $x \in \Omega$.

Then if the directions ξ^1 and ξ^2 are linearly independent and we take $\xi^3 = \xi^1 + \xi^2$ and choose $(\lambda^i,\xi^i) \in V_0$ for $i = 1,2,3$ and if we call

$$X = \sum_{i=1}^{3} F^{(4)}(0)(\lambda^i,\lambda^i,\lambda^i,\lambda^i) + 4\sum_{i,j=1,i<j}^{3} F^{(4)}(0)(\lambda^i,\lambda^i,\lambda^j,\lambda^j),$$

one necessarily one has that

$$4F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3)^2 \leqslant X\sum_{i=1}^{3} F''(0)(\lambda^i,\lambda^i). \tag{2}$$

Condition (2) is violated by the famous counterexample of Šverák valid for $N = 2$ and $m \geqslant 3$ cited in Section 1. This is verified in Gutiérrez (2006). Therefore, for $N = m = 2$ it is natural to look for a rank-one convex fourth-degree polynomial that violates condition (2). For that purpose we derive in the next section a specialized optimization algorithm.

3. The optimization algorithm

Let us call P_4 the set of the fourth-degree polynomials in $\mathbb{R}^4$ with vanishing constant and linear terms. Bearing in mind Proposition 2, on which our optimization algorithm will be based, we first define the function $\Psi : P_4 \times \Lambda^3 \to \mathbb{R}$ by

$$\Psi(F,\lambda^1,\lambda^2,\lambda^3) = -4F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3)^2 + X\sum_{i=1}^{3} F''(0)(\lambda^i,\lambda^i)$$

and secondly we define the following subset of Λ^3:

$$\Lambda_s = \{(\lambda^1, \lambda^2, \lambda^3) : (\lambda^i, \xi^i) \in V_0 \text{ for } i = 1, 2, 3;\ \xi^1, \xi^2 \text{ are l.i. and } \xi^3 = \xi^1 + \xi^2\}.$$

Let P_4^{R1} be the set of the polynomials in P_4 which are rank-one convex. Then, the goal is to see whether we can find a polynomial in P_4^{R1} which is not quasiconvex.

Now for any $F \in P_4$ and any triplet $\lambda^1, \lambda^2, \lambda^3 \in \mathbb{R}^4$, we first consider $F''(0)(\lambda^i, \lambda)$ as a function of λ, compute its Taylor expansion about λ^k and evaluate it at λ^j to obtain

$$F''(0)(\lambda^i, \lambda^j) = F''(0)(\lambda^i, \lambda^k) + F^{(3)}(0)(\lambda^i, \lambda^k, \lambda^j - \lambda^k) + \frac{1}{2} F^{(4)}(0)(\lambda^i, \lambda^k, \lambda^j - \lambda^k, \lambda^j - \lambda^k)$$
$$\text{for } i, j, k \in \{1, 2, 3\} \text{ with } j \neq k, \tag{3}$$

which is exact since F is a fourth-degree polynomial. Similarly, we consider $F^{(3)}(0)(\lambda^i, \lambda^j, \lambda)$, compute its Taylor expansion at λ^l and evaluate it at λ^k, which now gives

$$F^{(3)}(0)(\lambda^i, \lambda^j, \lambda^k) = F^{(3)}(0)(\lambda^i, \lambda^j, \lambda^l) + F^{(4)}(0)(\lambda^i, \lambda^j, \lambda^l, \lambda^k - \lambda^l) \quad \text{for } i, j, k, l \in \{1, 2, 3\} \text{ with } k \neq l. \tag{4}$$

Equalities (3) and (4) give 27 nonlinear constraints that any $F \in P_4$ must satisfy. We shall enforce them as constraints in the optimization problem we now define.

To get started we formulate the following optimization problem:

$$\min\{\Psi(F, \lambda^1, \lambda^2, \lambda^3) :\ F \in P_4^{R1},\ (\lambda^1, \lambda^2, \lambda^3) \in \Lambda_s,\ (3) \text{ and } (4) \text{ hold}\}. \tag{5}$$

If the optimal value of this problem is negative, we would have a counterexample that shows that both classes of functions are indeed different. On the other hand if the solution is nonnegative, it could be either because the two classes do coincide when intersected with P_4, or because condition (2) is just not precise enough to distinguish one from the other.

The number of coefficients needed to characterize any $F \in P_4$ is 65: there are 10 quadratic, 20 cubic and 35 quartic coefficients. Hence one can identify P_4^{R1} with a set in $\mathbb{R}^{65}$. To characterize a point in Λ_s one uses two vectors in the plane: ξ^1 and ξ^2 and to generate the λ^is we need another three two-dimensional vectors, corresponding to the pairs (η_1, η_2) in the definition of Λ. Therefore to characterize a point in Λ_s we need 10 real variables. To avoid ill scalings we require that these five vectors have unitary euclidean norm and also normalize the coefficients of the polynomial F to be a vector with the square of its euclidean norm being 65. These scalings do not affect the objective function Ψ, which is homogeneous of degree six on the λ's and homogeneous of degree two on F. Similarly, these scalings do not affect constraints (3) and (4).

The main difficulty comes from the fact that the set P_4^{R1} is characterized by infinitely many inequalities, namely those coming from conditions (a), (b) and (c) in Proposition 1. Then we can either use the machinery of semi-infinite programming, see Polak (1997), to handle the infinitely many constraints at all times, or progressively include some of these constraints in the fashion of the cutting-plane method of Kelly. We chose the last option, first because it looks much simpler and because there will be a natural way in which to pick the constraints to be added from one iteration to the next.

Observe that if the condition in (a) is satisfied with strict inequality and (c) holds, then condition (b) is automatically satisfied. Similarly if condition (c) holds as a strict inequality and (a) holds, then again condition (b) will automatically hold. Now, if for the polynomials in P_4 we only impose conditions (a) and (c), we get a set just slightly larger than P_4^{R1} as now condition (b) could be violated, but only if the conditions in (a) and (c) hold with equality. Therefore we only consider conditions (a) and (c) and for that we define two auxiliary functions, $G_1 : P_4 \times \Lambda \to \mathbb{R}$ given by $G_1(F, \lambda) = F''(0)(\lambda, \lambda)$ and $G_2 : P_4 \times \Lambda \times \mathbb{R}^4 \to \mathbb{R}$ being

$$G_2(F, \lambda, d) = 2F''(0)(\lambda, \lambda) F^{(4)}(0)(d, d, \lambda, \lambda) - F^{(3)}(0)(d, \lambda, \lambda)^2.$$

From Proposition 1 we get that $P_4^{R1} \subset \widetilde{P}_4$, where

$$\widetilde{P}_4 = \left\{ F \in P_4 : \min_{\lambda \in \Lambda} G_1(F, \lambda) \geqslant 0 \text{ and } \min_{\lambda \in \Lambda, d \in \mathbb{R}^4} G_2(F, \lambda, d) \geqslant 0 \right\}. \tag{6}$$

Finally we write a relaxation of problem (5) by replacing P_4^{R1} by $\widetilde{P}_4$. The objective function will now be given by a homogeneous fourteenth-degree polynomial and we should, in principle, handle infinitely many nonlinear constraints. Namely the problem to solve is

$$\min\{\Psi(F,\lambda^1,\lambda^2,\lambda^3):\ F\in\widetilde{P}_4;\ (\lambda^1,\lambda^2,\lambda^3)\in\Lambda_s;\ (3)\text{ and }(4)\text{ hold}\}. \tag{7}$$

To solve this problem we implement an algorithm, based on the idea of the cutting-plane method of Kelly, that solves the relaxation of (7) obtained by replacing condition $F\in\widetilde{P}_4$ by a finite number of constraints on the coefficients of F, number which is increased from one iteration to the next by choosing those vectors d and λ for which either condition (a) or condition (c) are most violated by the current F.

The algorithm reads as follows:

- initial step: $k=0$ with I_0 and J_0 finite sets in Λ and $\Lambda\times\mathbb{R}^4$ respectively.
- general step: For k a nonnegative integer we call

$$P_4^k=\left\{F\in P_4:\ \begin{matrix}\min\{G_1(F,\lambda):\lambda\in I_k\}\geqslant 0\\ \min\{G_2(F,\lambda,d):(\lambda,d)\in J_k\}\geqslant 0\end{matrix}\right\}$$

and solve the following relaxation of (7):

$$\Psi(F^k,\lambda_k^1,\lambda_k^2,\lambda_k^3)=\min_{\substack{F\in P_4^k\\(\lambda^1,\lambda^2,\lambda^3)\in\Lambda_s\\(3)\text{ and }(4)}}\Psi(F,\lambda^1,\lambda^2,\lambda^3). \tag{8}$$

- If the minimum value is nonnegative we stop, since all rank-one convex functions will satisfy condition (2) and therefore we cannot find the counterexample.
- If the minimum is strictly negative, we want to check whether F^k does belong to $\widetilde{P}_4$. For this we solve the problems

$$\min_{\lambda\in\Lambda}G_1(F^k,\lambda) \tag{9}$$

and

$$\min_{\lambda\in\Lambda,d\in\mathbb{R}^4}G_2(F^k,\lambda,d). \tag{10}$$

- If both problems have nonnegative minimum values we stop, since then $F^k\in\widetilde{P}_4$ and we would have a good candidate for our counterexample.
- If not, we incorporate the constraints in the characterization (6) that are being most violated by F^k, this is the minimizers of (9) and (10), therefore increasing I_k and J_k into I_{k+1} and J_{k+1}. This will then make P_4^{k+1} closer to $\widetilde{P}_4$. Make $k=k+1$ and iterate once more.

4. Numerical results and final remarks

We ran the algorithm with the hope that only a finite number of constraints will suffice to characterize $\widetilde{P}_4$ well enough in terms of the solution of (7). Hope which was confirmed by the extensive numerical computations we made.

We performed such calculations on our local Computing Center (SECICO) on an ALPHA Compaq DS20 machine running under TRU64 5.1b (OSF1), with 2 GB of main memory and a dual ev7 processor, running at a speed of 666 MHz. We used the AMPL modelling language, the solver LANCELOT for the main problem (8) as well as for the two subproblems (9) and (10). We made several other runs which gave a similar behavior. Each run took about six weeks with an average CPU usage of about 50%. The algorithm stops when it finds three consecutive values above a small positive tolerance, for the objective function of the main problem. The first example made 796 iterations while the second made 1235 iterations.

As a first example of the algorithm, we show in Figs. 1–3 the evolution of the optimal values in terms of the iteration number for the main problem (8) and for the minimization of G_1 and G_2, (9) and (10) respectively.

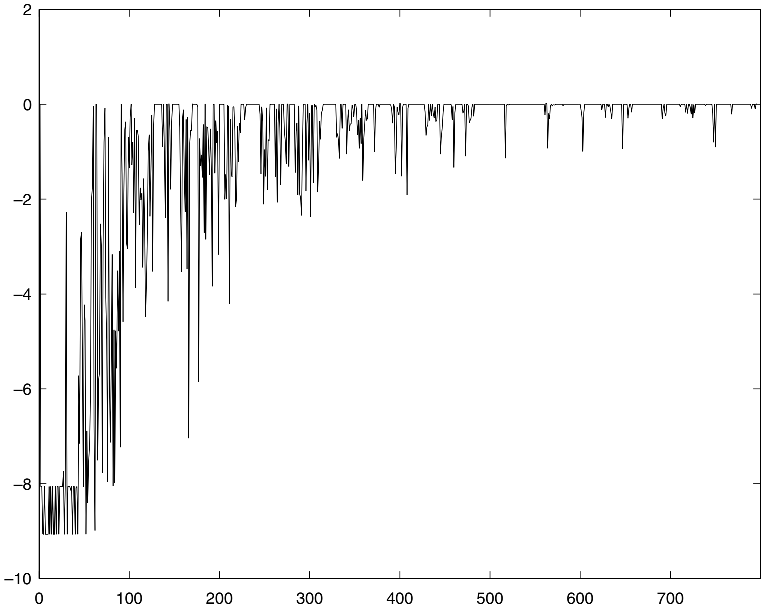

Fig. 1. Objective function for the main problem: first example.

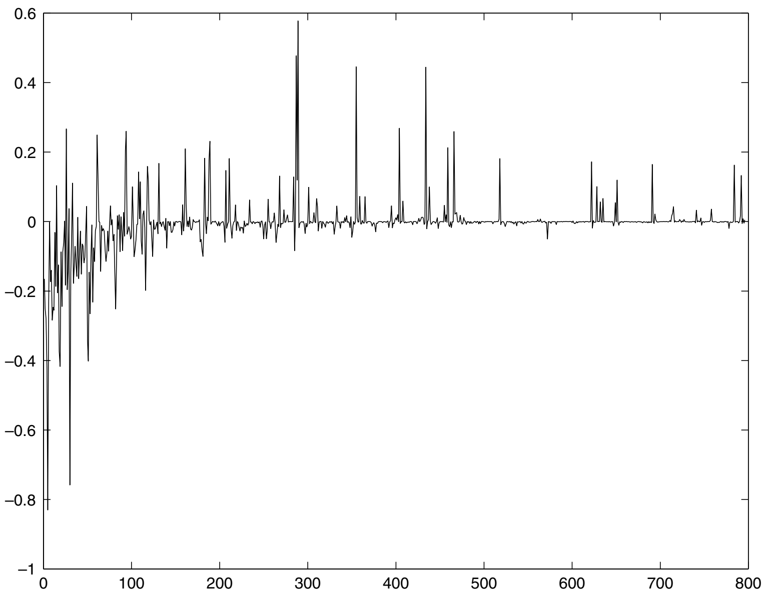

Fig. 2. Objective function for G1: first example.

Similar plots for a second example are shown in Figs. 4–6. From Figs. 1 and 4 we see that the objective function of the main problem goes to zero, meaning that no counterexample was found.

From Fig. 2 we see that as the optimization process advances, it becomes progressively harder to find vectors in Λ to make G_1 negative, meaning that the quadratic form induced by the Hessian matrix of F is becoming close to being positive definite over Λ for $F \in P_4^k$ and then condition (a) is very close to being enforced by iteration 400. The fact that sometimes G_1 cannot be made negative is not very important as it only contributes new constraints to the main problem when it is found to be negative for a certain $\lambda \in \Lambda$. On the other hand Fig. 3 shows that G_2 is progressively being forced to be positive on P_4^k, but much more slowly than G_1, which is

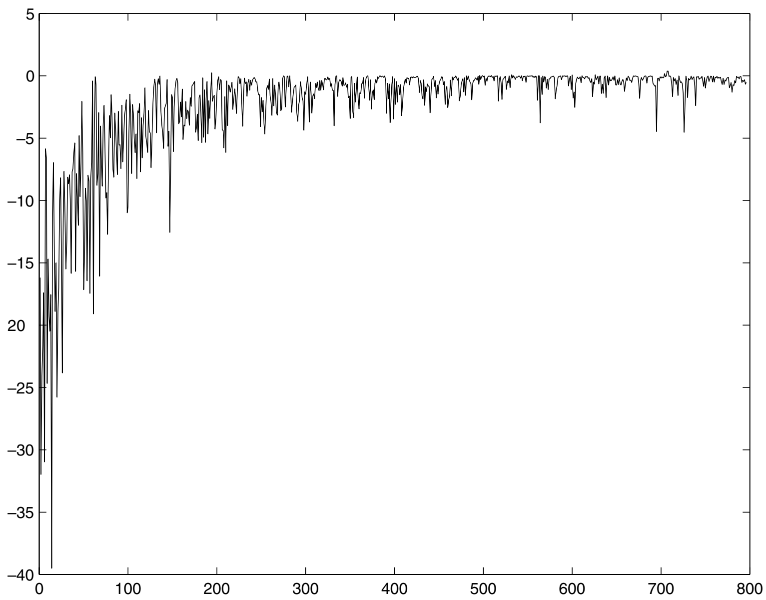

Fig. 3. Objective function for G2: first example.

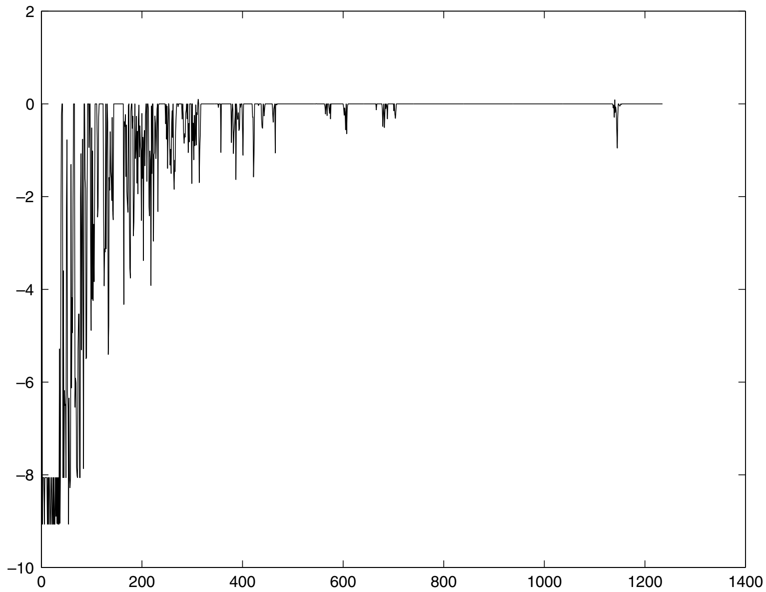

Fig. 4. Objective function for the main problem: second example.

natural since G_2 is far more complex to minimize than G_1. Figs. 1 and 4 resemble each other taking into account of the difference in iterations made. Similarly Figs. 2 and 5 are similar. However Figs. 3 and 6 look different, but this is due to the two very strong negative values found in the second case, quite late in the computation.

The final answer, after several attempts with different starting points, is that the solution of (7) is zero. Therefore the numerical conclusion is that for $N = m = 2$ condition (2), which is necessary for quasiconvexity, is also necessary for rank-one convexity. Furthermore, since only finitely many constraints sufficed to force the objective function of the main problem to be almost zero, it would seem that the necessary condition for quas-

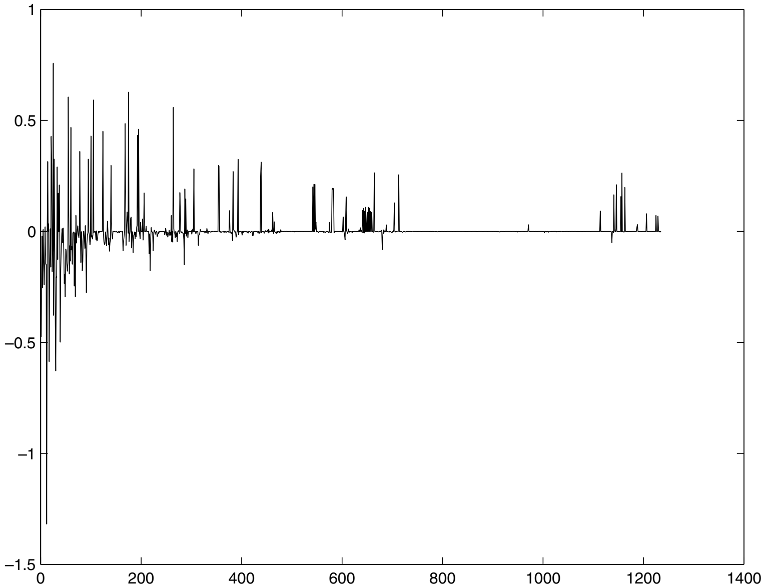

Fig. 5. Objective function for G1: second example.

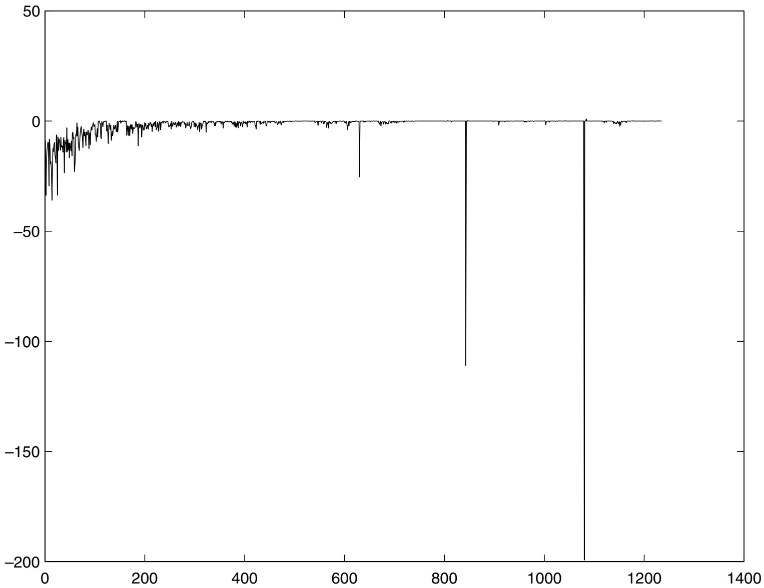

Fig. 6. Objective function for G2: second example.

iconvexity is necessary on a set strictly containing P_4^{R1} and then it cannot be sufficient for the quasiconvexity of a fourth-degree polynomial. Therefore if one searches analytically for the example, it should be on a set larger than the subset of P_4 containing polynomials with nonzero quadratic, cubic and quartic terms.

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