Available online at www.sciencedirect.com
SciVerse ScienceDirect
Journal of Functional Analysis 263 (2012) 782–802
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Symmetries of Gaussian measures and operator colligations

Yury Neretin [a,b,c,*,1]

[a] *Math. Dept., University of Vienna, Nordbergstrasse, 15, Vienna, Austria*
[b] *Institute for Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia*
[c] *Mech. Math. Dept., Moscow State University, Vorob'evy Gory, Moscow, Russia*

Received 6 December 2011; accepted 21 February 2012

Available online 15 May 2012

Communicated by D. Voiculescu

Abstract

Consider an infinite-dimensional linear space equipped with a Gaussian measure and the group $\mathrm{GLO}(\infty)$ of linear transformations that send the measure to equivalent one. Limit points of $\mathrm{GLO}(\infty)$ can be regarded as 'spreading' maps (polymorphisms). We show that the closure of $\mathrm{GLO}(\infty)$ in the semigroup of polymorphisms contains a certain semigroup of operator colligations and write explicit formulas for action of operator colligations by polymorphisms of the space with Gaussian measure.

Keywords: Infinite-dimensional group; Feldman–Hajek theorem; Boundaries of groups

1. Introduction. Polymorphisms, Gaussian measures, and colligations

1.1. The group $\mathrm{Gms}(M)$. Let $M = (M, \mu)$ be a Lebesgue space M with a probability measure μ ([29], see, also [14]), let $L^p(M, \mu)$ be the space of measurable functions on M with norm

$$\|f\|_p = \left(\int_M |f(m)|^p \, d\mu(m)\right)^{\frac{1}{p}}, \quad \text{where } 1 \leqslant p \leqslant \infty.$$

* Correspondence to: Institute for Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia.
E-mail address: neretin@mccme.ru.
URLs: http://www.mat.univie.ac.at/~neretin, http://wwwth.itep.ru/~neretin.
[1] Supported by grants FWF, P22122, P19064, and by RosAtom, contract H.4e.45.90.11.1059.

0022-1236
http://dx.doi.org/10.1016/j.jfa.2012.02.016

Denote by $\mathrm{Gms}(M)$ the group of all bijective a.s. maps $M \to M$ that send the measure μ to an equivalent measure. For $g \in \mathrm{Gms}(M)$ we denote by $g'(m)$ the Radon–Nikodym derivative of g.

Fix $\lambda \in \mathbb{C}$ lying in the strip $0 \leqslant \mathrm{Re}\,\lambda \leqslant 1$,

$$\lambda = \frac{1}{p} + is, \quad \text{where } 1 \leqslant p \leqslant \infty,\ s \in \mathbb{R}. \tag{1.1}$$

For any $g \in \mathrm{Gms}(M)$ we define the linear operator $T_\lambda(g)$ by

$$T_\lambda(g) f(m) = f(mg) g'(m)^\lambda. \tag{1.2}$$

Evidently, the operators $T_\lambda(g)$ form a representation of the group $\mathrm{Gms}(M)$ by isometric operators in the Banach space $L^p(M, \mu)$. For $p = 2$ we get a unitary representation in $L^2(M, \mu)$.

Polymorphisms, which are introduced below, are "limit points" of the group $\mathrm{Gms}(M)$.

1.2. Gaussian measures. Consider $\mathbb{R}$ equipped with the Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$. Let $n = 1, 2, \ldots, \infty$. Denote by $\mathbb{R}^\omega$ the product of n copies of $\mathbb{R}$ equipped with the product measure $\mu_\omega = \mu \times \mu \times \cdots$. We denote elements of $\mathbb{R}^\omega$ by $x = (x_1, x_2, \ldots)$.

Proposition 1.1. *If $\sum b_j^2 < \infty$, then the series $\sum b_j x_j$ converges a.s. on $\mathbb{R}^\infty$ with respect to the measure μ_∞.*

This is a special case of the Kolmogorov–Hinchin theorem about series of independent random variables, see, e.g., [32].

1.3. Groups of symmetries of Gaussian measures. Denote by $\mathrm{O}(\infty)$ the infinite-dimensional orthogonal group, i.e., the group of all infinite real matrices A satisfying the conditions

$$AA^t = A^t A = 1,$$

where ^{t} denotes the transposition.

For an invertible real infinite matrix A we consider the polar decomposition $A = SU$, where $U \in \mathrm{O}(\infty)$, and S is a positive self-adjoint operator. We define the group $\mathrm{GLO}(\infty)$ consisting of matrices $A = SU$ such that $S - 1$ is a Hilbert–Schmidt[2] operator. Equivalently, we can represent A as $A = \exp(T)U$, where $U \in \mathrm{O}(\infty)$ and T is a Hilbert–Schmidt self-adjoint operator.

Thus the set $\mathrm{GLO}(\infty)$ is the product of $\mathrm{O}(\infty)$ and the space of self-adjoint Hilbert–Schmidt matrices. We take the weak operator topology[3] on $\mathrm{O}(\infty)$ and the natural topology on the space of Hilbert–Schmidt matrices.[4] We equip $\mathrm{GLO}(\infty)$ with the topology of product. Then $\mathrm{GLO}(\infty)$ is a topological group with respect to this topology (*the Shale topology*, [30]).

Consider an infinite matrix $A = \{a_{ij}\}$. Apply it to a vector $x \in \mathbb{R}^\infty$,

$$xA = \begin{pmatrix} x_1 & x_2 & \ldots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \Big(\sum x_i a_{i1} \quad \sum x_i a_{i2} \quad \ldots \Big). \tag{1.3}$$

[2] An operator T is Hilbert–Schmidt, if $\sum_{ij} |t_{ij}|^2 < \infty$, see, e.g., [28].

[3] See e.g., [28].

[4] See e.g. [28].

Let A be an operator bounded in the space ℓ_2. By Proposition 1.1 the vector xA is defined for almost all $x \in (\mathbb{R}^\infty, \mu_\infty)$.

Theorem 1.2. a) *For $A \in \mathrm{O}(\infty)$ the map $x \mapsto xA$ preserves measure μ_∞.*

b) *For $A \in \mathrm{GLO}(\infty)$, the map $x \mapsto xA$ is defined a.s. on $(\mathbb{R}^\infty, \mu_\infty)$ and sends the measure μ_∞ to an equivalent measure $\mu(xA)$.*

c) *Let $A = (1+T)U$, where $A \in \mathrm{O}(\infty)$ and T is in the trace class.*[5] *Then the Radon–Nikodym derivative is given by the formula*

$$\frac{d\mu(xA)}{d\mu(x)} = |\det A| \cdot \exp\left(-\frac{1}{2}\langle xA, xA\rangle + \frac{1}{2}\langle x, x\rangle\right)$$
$$:= \left|\det(1+T)\right| \cdot \exp\left(-\langle xT, x\rangle - \frac{1}{2}\langle xT, xT\rangle\right). \tag{1.4}$$

d) *Let $A = 1 + T$, where T is a diagonal matrix with entries $t_j > -1$ satisfying $\sum_j t_j^2 < \infty$. Then the Radon–Nikodym derivative is given by*

$$\prod_{j=1}^{\infty} (1+t_j) e^{-(2t_j + t_j^2) x_j^2/2},$$

the product converges a.s. on $(\mathbb{R}^\infty, \mu_\infty)$.

e) *For $A, B \in \mathrm{GLO}(\infty)$ the identity*

$$(xA)B = x(AB)$$

holds a.s. on $(\mathbb{R}^\infty, \mu)$.

The theorem is a reformulation of the Feldman–Hajek theorem on equivalence of Gaussian measures (see, e.g., [11,4]), the most comprehensive exposition is in [31].

Remark. Consider the group $\mathrm{GLO}_1(\infty)$ consisting of matrices that can be represented in the form $U(1+T)$, where $U \in \mathrm{O}(\infty)$ and T is in the trace class. For $A \in \mathrm{GLO}_1(\infty)$, the absolute value of determinant $|\det(A)| := |\det(1+T)|$ is well-defined (see, e.g, [17]), it satisfies

$$\left|\det(A_1 A_2)\right| = \left|\det(A_1)\right| \cdot \left|\det(A_2)\right|.$$

The $\det(A)$ makes no sense. □

Remark. In our definition the action is defined a.s, and the identity $x(AB) = (xA)B$ also is valid a.s. The removing of "a.s." is impossible, the group $\mathrm{O}(\infty)$ cannot act pointwise by measure preserving transformations, see [8]. □

---

[5] See [28].

1.4. Polymorphisms (spreading maps). For details, see [22,17,20]). Denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers, denote by t the coordinate on $\mathbb{R}^\times$, by $\alpha * \beta$ we denote the convolution of measures on $\mathbb{R}^\times$. Let $M = (M, \mu)$, $N = (N, \nu)$ be Lebesgue spaces with probability measures. A *polymorphism*[6] $\mathfrak{P} : (M, \mu) \rightsquigarrow (N, \nu)$ is a measure $\mathfrak{P} = \mathfrak{P}(m, n, t)$ on $M \times N \times \mathbb{R}^\times$ satisfying two conditions:

a) the projection of $\mathfrak{P}(m, n, t)$ to M is μ;
b) the projection of $t \cdot \mathfrak{P}(m, n, t)$ to N is ν.

We denote by $\mathrm{Pol}(M, N)$ the set of all polymorphisms $(M, \mu) \rightsquigarrow (N, \nu)$.

There is a well defined associative multiplication

$$\mathrm{Pol}(M, N) \times \mathrm{Pol}(N, K) \to \mathrm{Pol}(M, K).$$

1.5. Convergence of polymorphisms. For $\mathfrak{P} \in \mathrm{Pol}(M, N)$ and measurable subsets $A \subset M$, $B \subset N$ we consider the projection $A \times B \times \mathbb{R}^\times \to \mathbb{R}^\times$ and denote by $\mathfrak{p}[A \times B]$ the pushforward of $\mathfrak{P}$ under this projection.

We say that a sequence $\mathfrak{P}_j \in \mathrm{Pol}(M, N)$ *converges* to $\mathfrak{P}$ if for any $A \subset M$, $B \subset N$ we have weak convergences

$$\mathfrak{p}_j[A \times B] \to \mathfrak{p}[A, \times B], \qquad t \cdot \mathfrak{p}_j[A \times B] \to t \cdot \mathfrak{p}[A \times B].$$

Proposition 1.3. *The product of polymorphisms is separately continuous, i.e. if $\mathfrak{P}_j$ converges to $\mathfrak{P}$ in* $\mathrm{Pol}(M, N)$ *and $\mathfrak{Q}_j$ converges to $\mathfrak{Q}$ in* $\mathrm{Pol}(N, K)$*, then $\mathfrak{Q} \diamond \mathfrak{P}_j$ converges to $\mathfrak{Q} \diamond \mathfrak{P}$ and $\mathfrak{Q}_j \diamond \mathfrak{P}$ converges to $\mathfrak{Q} \diamond \mathfrak{P}$.*

Note that there is no joint continuity, generally $\mathfrak{Q}_j \mathfrak{P}_j$ does not converge to $\mathfrak{Q} \diamond \mathfrak{P}$.

1.6. Embedding $\mathfrak{I}$: Gms(M) → Pol(M, M). Now let a measure μ on M be continuous. We consider the embedding

$$\mathfrak{I} : \mathrm{Gms}(M) \to \mathrm{Pol}(M, M) \tag{1.5}$$

given by the following way. Take the map $M \mapsto M \times M \times \mathbb{R}^\times$ given by $m \mapsto (m, g(m), g'(m))$. Then the pushforward of the measure μ is a polymorphism $\mathfrak{I}(g) : M \to M$.

Proposition 1.4. *(See [16,22].) The group* $\mathrm{Gms}(M)$ *is dense in* $\mathrm{Pol}(M, M)$.

1.7. Formulation of problem. We wish to describe the closure of $\mathrm{GLO}(\infty)$ in the semigroup of polymorphisms[7] of $\mathbb{R}^\infty$. Our solution is not final, we show a large semigroup (see the next subsection) in this closure.

---

[6] These objects were introduced in [16], see also [17]. The term was proposed be Vershik [33], who used it for measures on $M \times N$, see also "bistochastic kernels" from [10], see also [5,19]. On some appearances of polymorphisms in variation problems and mathematical hydrodynamics, see [2].

[7] The closure of $\mathrm{O}(\infty)$ gives action of the semigroup of all contractive linear operators by polymorphisms of $\mathbb{R}^\infty$, see Nelson [15].

1.8. Operator colligations. Fix $\omega = 0, 1, \dots, \infty$. Denote by $\mathrm{GLO}(\omega+\infty)$ the group consisting of $(\omega+\infty)\times(\omega+\infty)$ matrices g that are elements of the group GLO (i.e, $\mathrm{GLO}(\omega+\infty)$ is another notation for $\mathrm{GLO}(\infty)$). Consider the subgroup $\mathrm{O}(\infty) \subset \mathrm{GLO}(\omega+\infty)$ consisting of block $(\omega+\infty)\times(\omega+\infty)$ matrices $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$, where u is an orthogonal matrix.

We say that an *operator colligation* is an element g of $\mathrm{GLO}(\omega+\infty)$ defined up to the equivalence

$$g \sim h_1 g h_2, \quad \text{where } h_1, h_2 \in \mathrm{O}(\infty),$$

or, in more details,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \tag{1.6}$$

where u, v are orthogonal matrices. Denote by $\mathrm{Coll}(\omega)$ the set of all operator colligations. In other words, $\mathrm{Coll}(\omega)$ is the double coset space

$$\mathrm{Coll}(\omega) = \mathrm{O}(\infty) \setminus \mathrm{GLO}(\omega+\infty)/\mathrm{O}(\infty).$$

The *product of operator colligations* is defined by the formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ \begin{pmatrix} \varphi & \psi \\ \theta & \varkappa \end{pmatrix} := \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 & \psi \\ 0 & 1 & 0 \\ \theta & 0 & \varkappa \end{pmatrix} = \begin{pmatrix} \alpha\varphi & \beta & \alpha\psi \\ \gamma\varphi & \delta & \gamma\psi \\ \theta & 0 & \varkappa \end{pmatrix}. \tag{1.7}$$

The resulting matrix has size

$$\bigl(\omega + (\infty+\infty)\bigr) \times \bigl(\omega + (\infty+\infty)\bigr) = (\omega+\infty)\times(\omega+\infty),$$

i.e., we again get an element of $\mathrm{Coll}(\omega)$.

Proposition 1.5. *The product $\circ$ is a well defined associative operation on the set* $\mathrm{Coll}(\omega)$.

This can be verified by a straightforward calculation. For a clarification of this operation, see [17, Section IX.5]. Classical operator colligations are matrices determined up to the equivalence

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Colligations, their multiplication, and characteristic functions appeared in the spectral theory of non-self-adjoint operators (M.S. Livshits, V.P. Potapov, 1946–1955, [12,13,27], see survey in [3], see also algebraic version in [7]).

1.9. Results of the paper. First (Theorem 3.2), we prove the following statements:

- The closure of $\mathrm{GLO}(\infty)$ in polymorphisms of $(\mathbb{R}^\infty, \mu_\infty)$ contains the semigroup $\mathrm{Coll}(\infty)$.
- For $n < \infty$ the semigroup $\mathrm{Coll}(n)$ admits a canonical embedding to semigroup of polymorphisms of the space $(\mathbb{R}^n, \mu_n)$.

Our main purpose is to write explicit formulas (Theorems 5.2, 6.1) for this embedding.

Remark. There arises a natural question: is it correct that Coll(n) exhaust the closure? The author does not know answer. Apparently, the complete closure of GLO(∞) in polymorphisms is a semigroup of operator colligations, i.e. a semigroup of matrices defined upto the equivalence (1.6) and with the multiplication (1.7). But it can be slightly larger than our Coll(n).

1.10. A general problem. Many interesting actions of infinite-dimensional groups on spaces with measures are known, see survey [18] and recent 'new' constructions [9,26,21,1]. In all cases there arises the problem of description of closure of the group in polymorphisms, in all the cases this gives semigroups that essentially differ from the initial groups.[8] In this work and in [20] the problem was solved in two the most simple cases (Gaussian and Poisson measures). In both cases we get unusual interesting formulas.

2. Polymorphisms. Preliminaries

First, we need some preliminaries on polymorphisms.

2.1. Measures on $\mathbb{R}^\times$. Denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers, denote by t the coordinate on $\mathbb{R}^\times$, by $\varphi * \psi$ we denote *convolution* of finite measures φ and ψ on $\mathbb{R}^\times$, it defined by

$$\int\limits_{\mathbb{R}^\times} f(t)\, d(\varphi * \psi)(t) = \int\limits_{\mathbb{R}^\times}\int\limits_{\mathbb{R}^\times} f(pq)\, d\psi(p)\, d\varphi(q).$$

Recall that a sequence of finite measures ψ_j on $\mathbb{R}^\times$ *weakly converges* to a measure ψ if for any continuous function f on $\mathbb{R}^\times$ we have the convergence

$$\int\limits_{\mathbb{R}^"} f(t)\, d\psi_j(t) \longrightarrow \int\limits_{\mathbb{R}^\times} f(t)\, d\psi(t).$$

2.2. Product of polymorphisms. Here we give a formal definition of the product of polymorphisms, but actually we use Theorem 2.4 instead of the definition. For details, see [22].

Let p be a function on $M \times N$ taking values in finite measures on $\mathbb{R}^\times$. Such a function determines a measure $\mathfrak{P}$ on a product $M \times N \times \mathbb{R}^\times$,

$$\iiint\limits_{M\times N\times\mathbb{R}^\times} f(m,n,t)\, d\mathfrak{P}(m,n,t) := \iiint\limits_{A\times B}\int\limits_{\mathbb{R}^\times} f(m,n,t)\, dp(m,n)(t)\, d\nu(n)\, d\mu(m).$$

If p satisfies two identities

$$\int\limits_A\int\limits_N\int\limits_{\mathbb{R}^\times} dp(m,n)(t)\, dp(m,n)(t)\, d\nu(n)\, d\mu(m) = \mu(A),$$

[8] This is counterpart of Olshanski problem about weak closure of image of unitary representation, see [24,25]; for a finite-dimensional counterpart, see [6].

$$\int\limits_{M}\int\limits_{B}\int\limits_{\mathbb{R}^\times} t\,dp(m,n)(t)\,dp(m,n)(t)\,d\nu(n)\,d\mu(m)=\nu(B)$$

for any measurable subsets $A\subset M$, $B\subset N$, then $\mathfrak{P}$ is a polymorphism. If $\mathfrak{P}$ has such a form, we say that $\mathfrak{P}$ is absolutely continuous.

Now let $\mathfrak{P}\in \mathrm{Pol}(M,N)$, $\mathfrak{Q}\in \mathrm{Pol}(N,K)$ be absolutely continuous polymorphisms, p, q be the corresponding functions. Then the function r on $M\times K$ is determined by

$$r(a,c)=\int\limits_{N} p(m,n)*q(n,k)\,d\nu(n).$$

The integral is convergent a.s.

Theorem 2.1. *This product admits a unique separately continuous extension to an operation* $\mathrm{Pol}(M,N)\times \mathrm{Pol}(N,K)\to \mathrm{Pol}(M,K)$.

2.3. Involution in the category of polymorphisms. Let $\mathfrak{P}:M\rightsquigarrow N$ be a polymorphism. We define the polymorphism $\mathfrak{P}^\star:N\rightsquigarrow M$ by

$$\mathfrak{P}^\star(n,m,t)=t\cdot\mathfrak{P}\bigl(m,n,t^{-1}\bigr).$$

For any polymorphisms $\mathfrak{P}:M\rightsquigarrow N$, $\mathfrak{Q}:N\rightsquigarrow K$, the following property holds

$$(\mathfrak{Q}\diamond\mathfrak{P})^\star=\mathfrak{P}^\star\diamond\mathfrak{Q}^\star.$$

If $g\in \mathrm{Gms}(M)$, then

$$\mathfrak{I}(g)^\star=\mathfrak{I}\bigl(g^{-1}\bigr).$$

Our next purpose is to extend the operators (1.2) to arbitrary polymorphisms.

2.4. Mellin transform of polymorphisms. Here we present without proof some simple statements from [22]. Notice that below we use Theorem 2.4 and do not refer to the definition of product of polymorphisms.

Fix $\lambda=\frac{1}{p}+is\in\mathbb{C}$ as above (1.1). Let q is defined from $\frac{1}{p}+\frac{1}{q}=1$. For a polymorphism $\mathfrak{P}:M\rightsquigarrow N$ we consider the bilinear form on $L^p(M,\mu)\times L^q(N,\nu)\to\mathbb{C}$ given by

$$S_\lambda(f,g)=\iiint\limits_{M\times N\times\mathbb{R}^\times} f(m)g(n)t^\lambda\,d\mathfrak{P}(m,n,t).$$

Proposition 2.2. *(See [22].)* a)

$$\bigl|S_\lambda(f,g)\bigr|\leqslant\|f\|_{L_p}\cdot\|g\|_{L_q}.$$

b) $\mathfrak{P}$ *is uniquely determined by the family of forms* $S_\lambda(\cdot,\cdot)$.

Corollary 2.3. a) *There exists a unique linear operator*

$$T_\lambda(\mathfrak{P}) : L^p(N,\nu) \to L^p(M,\mu)$$

such that

$$S(f,g) = \int_M f(m) \cdot T_\lambda(\mathfrak{P}) \cdot g(m)\, d\mu(m).$$

b) $\|T_\lambda(\mathfrak{P})\| \leqslant 1$, *where a norm is the norm of an operator* $L^p(N,\nu) \to L^p(M,\mu)$.

c) *A polymorphism* $\mathfrak{P}$ *is uniquely determined by the operator-valued function* $\lambda \mapsto T_\lambda(\mathfrak{P})$, *and, moreover, by its values on each line* $\frac{1}{p} + is$ *for fixed* p.

For $h \in \mathrm{Gms}(M)$, we have

$$T_\lambda\big(\iota(h)\big) = T_\lambda(h),$$

where $T_\lambda(h)$ is defined by (1.2).

Theorem 2.4. T_λ *is a representation of a category, i.e.*

$$T_\lambda(\mathfrak{Q} \diamond \mathfrak{P}) = T_\lambda(\mathfrak{Q}) T_\lambda(\mathfrak{P}). \tag{2.1}$$

2.5. Convergence.

Theorem 2.5. a) $T_\lambda(\mathfrak{P})$ *is weakly continuous, i.e., if* $\mathfrak{P}_j$ *converges to* $\mathfrak{P}$, *then*

$$\int_M f(m) \cdot T_\lambda(\mathfrak{P}_j) g(m))\, d\mu(m) \quad \text{converges to} \quad \int_M f(m) T_\lambda(\mathfrak{P}) g(m)\, d\mu(m) \tag{2.2}$$

for any $f \in L^q(M)$, $g \in L^p(N)$.

b) *Conversely, if* (2.2) *holds for each* λ *in the strip* $0 \leqslant \operatorname{Re}\lambda \leqslant 1$, *then* $\mathfrak{P}_j$ *converges to* $\mathfrak{P}$. *Moreover, it is sufficient to require the convergences on the lines* $\operatorname{Re}\lambda = 0$ *and* $\operatorname{Re}\lambda = 1$.

3. Abstract statement

3.1. Polymorphisms $\mathfrak{l}_n$. Let (M,μ) be a space with measure. Denote by $\Delta(m,m')$ the measure on $M \times M$ supported by the diagonal of $M \times M$ such that the projection of Δ to the first factor M is μ.

Let $\omega = 0, 1, \ldots, \infty$. Consider the space $\mathbb{R}^\omega \times \mathbb{R}^\infty$ equipped with the measure $\mu_{\omega+\infty} = \mu_\omega \times \mu_\infty$. Let x, x' range in $\mathbb{R}^\omega$, y in $\mathbb{R}^\infty$, t in $\mathbb{R}^\times$. Consider the polymorphism

$$\mathfrak{l}_\omega : \big(\mathbb{R}^\omega, \mu_\omega\big) \rightsquigarrow \big(\mathbb{R}^\omega \times \mathbb{R}^\infty, \mu_\omega \times \mu_\infty\big)$$

given by

$$\mathfrak{l}_\omega\big(x'; x, y; t\big) = \Delta\big(x, x'\big) \times \mu_\infty(y) \times \delta(t-1),$$

where δ is the delta-function.

The following statement is straightforward.

Lemma 3.1. a) *For a function f on $\mathbb{R}^\omega$ we have*

$$T_\lambda(\mathfrak{l}_\omega) f(x,y) = f(x).$$

b) *For a function $g(x,y)$ on $\mathbb{R}^{\omega+\infty}$, we have*

$$T_\lambda\big(\mathfrak{l}_\omega^\star\big) g(x) = \int_{\mathbb{R}^\infty} g(x,y)\, d\mu_\infty(y).$$

c) $\mathfrak{l}_\omega^\star \diamond \mathfrak{l}_\omega : \mathbb{R}^\omega \rightsquigarrow \mathbb{R}^\omega$ *is* $\Delta(x,x') \times \delta(t-1)$.

d) *The polymorphism*

$$\mathfrak{t}_\omega := \mathfrak{l}_\omega \diamond \mathfrak{l}_\omega^\star : \mathbb{R}^{\omega+\infty} \rightsquigarrow \mathbb{R}^{\omega+\infty}$$

equals

$$\Delta\big(x,x'\big) \times \mu_\infty(y) \times \mu_\infty\big(y'\big) \times \delta(t-1),$$

where (x,y) is in the first copy of $\mathbb{R}^{\omega+\infty}$ and (x',y') is in the second copy.

e) *The operator corresponding to $\mathfrak{t}_\omega$ is*

$$T_\lambda(\mathfrak{t}_\omega) f(x,y) = \int_{\mathbb{R}^\infty} f(x,z)\, d\mu_\infty(z).$$

In particular, in L^2 this operator is the orthogonal projection to the space of functions independent on y.

f) *Consider a sequence $h_j = \begin{pmatrix} 1 & 0 \\ 0 & u_j \end{pmatrix} \in \mathrm{O}(\infty)$ where u_j weakly converges to 0. Then $\mathfrak{I}(h_j)$ converges to $\mathfrak{t}_\omega = \mathfrak{l}_\omega \diamond \mathfrak{l}_\omega^\star$.*

An example of a sequence u_j is

$$u_j = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \}j \\ \}j \\ \}\infty \end{matrix}.$$

3.2. Action of colligations. Let $\omega = 0, 1, \ldots, \infty$. Let $\mathfrak{a} \in \mathrm{Coll}(\omega)$, let A be its representative in $\mathrm{GLO}(\omega+\infty)$. Consider the polymorphism

$$\tau^{(\omega)}(\mathfrak{a}) : \big(\mathbb{R}^\omega, \mu_\omega\big) \rightsquigarrow \big(\mathbb{R}^\omega, \mu_\omega\big)$$

given by

$$\tau^{(\omega)}(\mathfrak{a}) = \mathfrak{l}_\omega \mathfrak{I}(A) \mathfrak{l}_\omega^\star.$$

Theorem 3.2. *The map $\tau^{(\omega)} : \mathrm{Coll}(\omega) \to \mathrm{Pol}(\mathbb{R}^\omega, \mathbb{R}^\omega)$ is a homomorphism of semigroups.*

Theorem 3.3. *For $\omega = \infty$ the image $\tau^{(\infty)}(\mathrm{Coll}(\infty)) \subset \mathrm{Pol}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is contained in the closure of $\mathfrak{I}(\mathrm{GLO}(\infty))$.*

3.3. Proof of Theorem 3.2. We must verify the identity

$$T_\lambda(\mathfrak{a}_1)T_\lambda(\mathfrak{a}_2) = T_\lambda(\mathfrak{a}_1 \circ \mathfrak{a}_2) \tag{3.1}$$

or, equivalently,

$$T_\lambda(\mathfrak{t}_\omega A_1 \mathfrak{t}_\omega)T_\lambda(\mathfrak{t}_\omega A_2 \mathfrak{t}_\omega) = T_\lambda^{(\omega)}(\mathfrak{t}_\omega A_1 A_2 \mathfrak{t}_\omega).$$

Let ρ be a unitary representation of $\mathrm{GLO}(\omega + \infty) \simeq \mathrm{GLO}(\infty)$ continuous with respect to the Shale topology. Denote by $H(\omega)$ the space of $\mathrm{O}(\infty)$-invariant vectors. Denote by $P(\omega)$ the orthogonal projection on $H(\omega)$. For $A \in \mathrm{GLO}(\omega + \infty)$, we define the operator

$$\rho^{(\omega)}(\mathfrak{a}) := P(\omega)\rho(A) : H(\omega) \to H(\omega). \tag{3.2}$$

It can be easily checked that $\rho^{(\mathfrak{a})}(g)$ depends on a operator colligation $\mathfrak{a}$ and not on A itself.

Theorem 3.4. *We get a representation of the semigroup* $\mathrm{Coll}(\omega)$ *in the space* $H(\omega)$.

$$\rho^{(\omega)}(\mathfrak{a}_1)\rho^{(\omega)}(\mathfrak{a}_2) = \rho^{(\omega)}(\mathfrak{a}_1 \circ \mathfrak{a}_2). \tag{3.3}$$

See [24,17], see a simple proof in [23].

We need this theorem for representations $T_{1/2+is}$ of the group $\mathrm{GLO}(\omega + \infty)$ in $L^2(\mathbb{R}^{\omega+\infty})$, $\mu_{\omega+\infty}$, in this case $P(\omega)$ is $T_{1/2+is}(\mathfrak{t})$,

$$T_{1/2+is}(\mathfrak{a}) = T_{1/2+is}(\mathfrak{t})T_{1/2+is}(A)T_{1/2+is}(\mathfrak{t}),$$

the identity (3.3) can be written as

$$T^{(\omega)}_{1/2+is}(\mathfrak{a}_1)T^{(\omega)}_{1/2+is}(\mathfrak{a}_2) = T^{(\omega)}_{1/2+is}(\mathfrak{a}_1 \circ \mathfrak{a}_2). \tag{3.4}$$

Since T_λ depends holomorphically in λ, we get (3.1).

Remark. Identity (3.4) can be verified by a long straightforward calculation (and in fact this was done in [24]).

3.4. Proof of Theorem 3.3. Let $\mathfrak{a} \in \mathrm{Coll}(\infty)$, let $A \in \mathrm{GLO}(\infty + \infty)$ be its representative. We define the polymorphism

$$\sigma(\mathfrak{a}) : \left(\mathbb{R}^{\infty+\infty}, \mu_{\infty+\infty}\right) \rightsquigarrow \left(\mathbb{R}^{\infty+\infty}, \mu_{\infty+\infty}\right)$$

by

$$\sigma(\mathfrak{a}) = \mathfrak{t}_\infty \diamond \tau(A) \diamond \mathfrak{t}_\infty^\star.$$

By Lemma 3.1(f), the element $\mathfrak{t}_\infty$ is contained in the closure of $\mathrm{O}(\infty)$. By separate continuity of the product, $\mathfrak{t}_\infty \diamond \tau(A) \diamond \mathfrak{t}_\infty^\star$ is contained in the closure of $\mathrm{GLO}(\infty+\infty)$

Next, represent the set of natural numbers $\mathbb{N}$ as a union of two disjoint sets I, J. Consider the monotonic bijections $I \to \mathbb{N}$, $J \to \mathbb{N}$. In this way we identify $\mathbb{R}^\infty$ and $\mathbb{R}^{\infty+\infty}$. Denote by $\sigma(\mathfrak{a}; I) : \mathbb{R}^\infty \rightsquigarrow \mathbb{R}^\infty$ the image of the polymorphism $\sigma(\mathfrak{a})$ under this identification. By construction $\sigma(\mathfrak{a}, I)$ is contained in the closure of $\mathrm{GLO}(\infty)$.

Now take

$$I_k = \{1, 2, 3, \dots, k, k+2, k+4, k+6, \dots\}.$$

Then $\sigma(\mathfrak{a}, I_k)$ converges to $\tau(\mathfrak{a})$. □

3.5. Injectivity. We formulate without proof the following statement.

Theorem 3.5. *The maps* $\mathrm{Coll}(\omega) \to \mathrm{Pol}(\mathbb{R}^\omega, \mathbb{R}^\omega)$ *are injective.*

This is equivalent to the statement: the family of representations $\mathfrak{a} \mapsto P(\omega)T_\lambda(\mathfrak{a})P(\omega)$ separates points of $\mathrm{Coll}(\omega)$.

4. Canonical forms

4.1. Canonical forms. Let $n < \infty$, $\mathfrak{g} \in \mathrm{Coll}(n)$. Let $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ be a representative of $\mathfrak{g}$.

Lemma 4.1. *Assume that rank of g_{12} is maximal. Then $\mathfrak{g}$ has a representative of the form*

$$G = \underbrace{\begin{pmatrix} a & b \\ c & d \\ 0 & H \end{pmatrix}}_{n \quad n+\infty} \begin{matrix} \}n \\ \}n \\ \}\infty \end{matrix} = \underbrace{\begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix}}_{n \quad n \quad \infty} \begin{matrix} \}n \\ \}n \\ \}\infty \end{matrix} \tag{4.1}$$

where h is a diagonal matrix with positive entries h_j, $\sum (h_j - 1)^2 < \infty$.

Lemma 4.2. *Any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{O}(n+\infty)$ admits a representation in the form*

$$g = (1+S)\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix},$$

where S is a Hilbert–Schmidt matrix and $u \in \mathrm{O}(\infty)$.

Proof of Lemma 4.2. The matrix $\delta^t\delta - 1$ is Hilbert–Schmidt and δ is Fredholm of index 0, therefore δ can be represented as

$$\delta = vHu,$$

where $u, v \in \mathrm{O}(\infty)$, and H is a diagonal matrix, the matrix $H - 1$ is Hilbert–Schmidt. Therefore g has the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} \alpha & \beta' \\ \gamma' & H \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

The middle factor is (1+ Hilbert–Schmidt matrix). Finally, we get a desired representation

$$g = \left[\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}\begin{pmatrix}\alpha & \beta'\\ \gamma' & H\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}^{-1}\right]\cdot\left[\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}\right]. \qquad \square$$

Proof of Lemma 4.1. By Lemma 4.2, we can assume that $G - 1$ is a Hilbert–Schmidt matrix. Since $\operatorname{rk} g_{12} = n$, a left multiplication by an orthogonal matrix w can reduce g_{12} to the form $\binom{c}{0}$.

Thus we get a matrix $R' = \begin{pmatrix}a & b\\ c & d\\ 0 & H\end{pmatrix}$ such that $R' - 1$ is Hilbert–Schmidt. We transform R' by

$$\begin{pmatrix}a & b\\ c & d\\ 0 & H\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & u\end{pmatrix}\begin{pmatrix}a & b\\ c & d\\ 0 & H\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & v_{11} & v_{12}\\ 0 & v_{21} & v_{22}\end{pmatrix},$$

where u and $\begin{pmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix}$ are orthogonal matrices. Consider $(n+\infty)\times\infty$ matrix $J = \begin{pmatrix}0 & 1\end{pmatrix}$. Then $H - J$ is a Hilbert–Schmidt operator, therefore the Fredholm index of H equals n. Since G is invertible, $\ker H = 0$, Hence $\operatorname{codim} \operatorname{Im} H = n$. Such H can be reduced to the form $\begin{pmatrix}0 & h\end{pmatrix}$, where h is diagonal. The standard proof of the theorem about singular values (see [28]) can be adapted to this case. $\square$

4.2. Coordinates. Take a colligation reduced to a canonical form (4.1). We pass to *Potapov coordinates* (see [27]) on the space of matrices,

$$\begin{pmatrix}P & Q\\ R & T\end{pmatrix} := \begin{pmatrix}b - ac^{-1}d & -ac^{-1}\\ c^{-1}d & c^{-1}\end{pmatrix}$$

or

$$\begin{pmatrix}P_1 & P_2 & Q\\ R_1 & R_2 & T\end{pmatrix} := \begin{pmatrix}b_1 - ac^{-1}d_1 & b_2 - ac^{-1}d_2 & -ac^{-1}\\ c^{-1}d_1 & c^{-1}d_2 & c^{-1}\end{pmatrix},$$

the size of the block matrices is $(n+\infty+n)\times(n+n)$. Formulas below are written in the terms of P, Q, R, T, and h.

5. Calculations. Finite matrices

5.1. Measures $\Phi[b, M; t]$. Let $M \geqslant 0$, $b \in \mathbb{R}$. We define the measure $\Phi[b, M; t]$ on $\mathbb{R}^\times$ by

– for $b > 0$

$$\Phi[b, M; t] = \begin{cases}\frac{1}{\sqrt{2\pi}} t^{1/b}(-b\ln t)^{-1/2}\cosh\sqrt{-\frac{4M}{b}\ln t}\,\frac{dt}{t} & \text{if } 0 < t < 1;\\ 0 & \text{if } t > 1;\end{cases}$$

– for $b = 0$

$$\Phi[0, M; t] = e^M \delta(t-1);$$

– for $b < 0$,

$$\Phi[b, M; t] = \begin{cases} 0 & \text{if } 0 < t < 1, \\ \frac{1}{\sqrt{2\pi}} t^{-1/b} (4Mb \ln t)^{-1/2} \cosh \sqrt{\frac{4M}{b} \ln t} \, \frac{dt}{t} & \text{if } t > 1. \end{cases}$$

Lemma 5.1.

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^\times} t^\lambda \Phi[b, M; t] = \frac{1}{\sqrt{1 + b\lambda}} \exp\left\{ \frac{M}{1 + b\lambda} \right\}.$$

Proof. To be definite, set $b > 0$. We must evaluate

$$\frac{1}{\sqrt{2\pi}} \int_0^1 t^{\lambda + 1/b} (-b \ln t)^{-1/2} \cosh \sqrt{-\frac{4M}{b} \ln t} \, \frac{dt}{t}.$$

We substitute $y = \ln t$ and get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{(\lambda + 1/b) y} (-by)^{-1/2} \cosh \sqrt{-\frac{4M}{b} y} \, dy.$$

Next, we set $z = -\frac{4M}{b} y$, and come to

$$\frac{1}{\sqrt{2\pi} \cdot \sqrt{4M}} \int_0^\infty e^{-\frac{1}{4M}(b\lambda + 1) z} z^{-1/2} \cosh \sqrt{z} \, dz = \frac{1}{\sqrt{2\pi} \cdot \sqrt{M}} \int_0^\infty e^{-\frac{1}{4M}(b\lambda + 1) u^2} \cosh u \, du.$$

Writing $\cosh u = \frac{1}{2}(e^u + e^{-u})$, we get

$$\frac{1}{\sqrt{2\pi} \cdot 2\sqrt{M}} \int_{-\infty}^\infty e^{-\frac{1}{4M}(b\lambda + 1) u^2} e^u \, du = \frac{1}{\sqrt{1 + b\lambda}} \exp\left\{ \frac{M}{1 + b\lambda} \right\}. \qquad \square$$

5.2. Formula. We consider coordinates on $\mathrm{Coll}(n)$ defined above. For $x, u \in \mathbb{R}^n$ we define the following δ-measure $dN_{x,u}(t)$ on $\mathbb{R}^\times$

$$dN_{x,u}(t) = A(x, u) \delta\big(t - B(x, u)\big),$$

where

$$A(x, u) = |\det T| \exp\left\{ -\frac{1}{2} \|xQ + uT\|^2 - \frac{1}{2} \big\| (xP + uR) H^t \big(1 - HH^t\big)^{-1} \big\|^2 \right\},$$

$$B(x,u) = |\det G| \exp\left\{\frac{1}{2}\left(\|xQ + uT\|^2 - \|x\|^2 + \|u\|^2 - (xP + uR)\left(1 - H^t H\right)^{-1}(xP + uR)^t\right)\right\}, \tag{5.1}$$

where $\|\cdot\|$ is the standard norm in $\mathbb{R}^n$.

Denote by h_j the diagonal entries of the matrix h. Denote by $(\psi_1, \psi_2, \ldots)$ the coordinates of the vector $xP_2 + uR_2$.

Theorem 5.2. *Let* $\mathfrak{g} \in \mathrm{Coll}(n)$ *have a representative*

$$G = \underbrace{\begin{pmatrix} a & b_1 & b_2 & 0 \\ c & d_1 & d_2 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{n \quad n \quad m-n \quad \infty} \begin{matrix} \}n \\ \}n \\ \}m-n \\ \}\infty \end{matrix} \tag{5.2}$$

and $h_j \neq 1$. *Then the polymorphism* $\tau(\mathfrak{a})$ *is given by*

$$\left(N_{x,u}(t) * \mathop{\text{\Huge$*$}}_{j=1}^{m-n} \Phi\left[h_j^2 - 1, \frac{h_j^2 |\psi_j|^2}{2(1-h_j^2)};\ t\right]\right) dx\, du, \tag{5.3}$$

where $*$ *denotes the convolution in* $\mathbb{R}^\times$ *and* $\text{\Huge$*$}$ *is the symbol of multiple convolution with respect to* j.

5.3. Transformation of the determinant. Note that

$$\det G = \det\begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix} = \det\begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix} \cdot \det(h) = \pm \det(c) \det\left(b_1 - ac^{-1} d_1\right) \det(h).$$

Thus

$$|\det G| = \left|\frac{\det(P_1)\det(H)}{\det(T)}\right|.$$

5.4. Calculation. We wish to write explicitly operators (3.2) for the representations $T_\lambda(G)$,

$$T_\lambda^{(n)}(G) = T_\lambda(\mathfrak{l}) T_\lambda(G) T_\lambda\left(\mathfrak{l}^\star\right).$$

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^{m-n}$, $\xi \in \mathbb{R}^\infty$. The operator $T_\lambda(\mathfrak{l}^\star)$ sends a function $f(x)$ on $\mathbb{R}^n$ to the same function $f(x)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\infty$. We apply $T_\lambda(G)$ and come to

$$|\det G|^\lambda f(xa + yc) \exp\left\{-\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} \left(GG^t - 1\right) \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix}\right\}. \tag{5.4}$$

Next, the operator $T_\lambda(\mathfrak{l})$ is the average with respect to variables $(y, z, \xi) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\infty$. Since the function (5.4) is independent on ξ, we take average with respect to (y, z). We come to

$$\begin{aligned} T_\lambda^{(n)}(G) f(x) &= |\det G|^\lambda \iint_{\mathbb{R}^n \times \mathbb{R}^{m-n}} f(xa + yc) \\ &\quad \times \exp\left\{ -\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} \left(GG^t - 1\right) \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right\} d\mu_n(y)\, d\mu_{m-n}(z) \\ &= \frac{|\det(G)|^\lambda}{(2\pi)^{m/2}} \cdot e^{\frac{1}{2}x^2} \iint_{\mathbb{R}^n \times \mathbb{R}^{m-n}} f(xa + yc) \exp\left\{ -\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} GG^t \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right. \\ &\quad \left. + \frac{\lambda - 1}{2} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right\} dy\, dz. \end{aligned} \tag{5.5}$$

We change variable y by u according

$$u = xa + yc, \qquad y = uc^{-1} - xac^{-1}.$$

Then

$$\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} x & u & z \end{pmatrix} S,$$

where

$$S = \begin{pmatrix} 1 & -ac^{-1} & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quadratic form in (5.5) transforms to

$$\left\{ -\frac{\lambda}{2} \begin{pmatrix} x & u & z \end{pmatrix} SGG^t S^t \begin{pmatrix} x^t \\ u^t \\ z^t \end{pmatrix} + \frac{\lambda - 1}{2} \begin{pmatrix} x & u & z \end{pmatrix} SS^t \begin{pmatrix} x^t \\ u^t \\ z^t \end{pmatrix} \right\}.$$

Passing to Potapov coordinates, we get

$$SS^t = \begin{pmatrix} 1 + QQ^t & QT^t & 0 \\ TQ^t & TT^t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$SG = \begin{pmatrix} 0 & P \\ 1 & R \\ 0 & H \end{pmatrix}, \qquad SGG^t S^t = \begin{pmatrix} PP^t & PR^t & PH^t \\ RP^t & 1 + RR^t & RH^t \\ HP^t & HR^t & HH^t \end{pmatrix}.$$

We come to the expression of the form

$$T_\lambda^{(n)}(G)\, f(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,u) f(u)\, du,$$

where the kernel $\mathcal{K}$ is given by

$$\mathcal{K}(x,u) = (2\pi)^{-n/2} \bigl|\det(G)\bigr|^{\lambda} |\det c|^{-1} \exp\bigl\{V(x,u)\bigr\} \int_{\mathbb{R}^{m-n}} \exp\bigl\{U(x,u,z)\bigr\}\, dz,$$

where

$$\begin{aligned}
\exp\bigl\{V(x,u)\bigr\} &= \exp\left\{\frac{1}{2} x x^t + \frac{\lambda-1}{2} \begin{pmatrix} x & u \end{pmatrix} \begin{pmatrix} QQ^t + 1 & QT^t \\ TQ^t & TT^t \end{pmatrix} \begin{pmatrix} x^t \\ u^t \end{pmatrix} \right. \\
&\quad \left. - \frac{\lambda}{2} \begin{pmatrix} x & u \end{pmatrix} \begin{pmatrix} PP^t & PR^t \\ RP^t & RR^t + 1 \end{pmatrix} \begin{pmatrix} x^t \\ u^t \end{pmatrix} \right\} \\
&= \exp\left\{ -\frac{\lambda}{2} \|xP + uR\|^2 + \frac{\lambda - 1}{2} \|xQ + uT\|^2 + \frac{\lambda}{2}\bigl(\|x\|^2 - \|u\|^2\bigr) \right\} \qquad (5.6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{m-n}} \exp\bigl\{U(x,u,z)\bigr\}\, dz \\
&= (2\pi)^{-(m-n)/2} \int_{\mathbb{R}^{m-n}} \exp\left\{\frac{1}{2} z\bigl(-\lambda H H^t + \lambda - 1\bigr) z^t \right\} \exp\bigl\{-\lambda z H\bigl(P^t x^t + R^t u^t\bigr)\bigr\}\, dz \\
&= \det\bigl(\lambda H H^t - \lambda + 1\bigr)^{-1/2} \\
&\quad \times \exp\left\{ \frac{\lambda^2}{2} (xP + uR) H^t \bigl(\lambda H H^t - \lambda + 1\bigr)^{-1} H (xP + yR)^t \right\}. \qquad (5.7)
\end{aligned}$$

We wish to examine the exponential factor in (5.7). Recall that H is an $(m \times n)$ matrix of the form

$$H = \begin{pmatrix} 0 & \dots & 0 & h_1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & h_2 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & h_{m-n} \end{pmatrix}.$$

Therefore HH^t is the diagonal matrix with entries h_j^2 and $H^t(\lambda H H^t - \lambda + 1)^{-1} H$ is the diagonal matrix with entries 0 (n times) and $\frac{h_j^2}{\lambda h_j^2 - \lambda + 1}$. Therefore, (5.7) equals

$$(2\pi)^{n-m} \prod_{j=1}^{m-n} \bigl(1 + \lambda\bigl(h_j^2 - 1\bigr)\bigr)^{-1/2} \exp\left\{ \frac{\lambda^2 h_j^2 |\psi_j|^2}{2(\lambda h_j^2 - \lambda + 1)} \right\}. \qquad (5.8)$$

Next, we write

$$\frac{\lambda^2 h_j^2}{\lambda h_j^2 - \lambda + 1} = \frac{\lambda h_j^2}{h_j^2 - 1} - \frac{h_j^2}{(h_j^2-1)^2} + \frac{h_j^2}{(h_j^2-1)^2} \cdot \frac{1}{\lambda h_j^2 - \lambda + 1} \tag{5.9}$$

and represent the product (5.8) as

$$\exp\left\{-\frac{1}{2}(xP+uR)H^t\left(1-HH^t\right)^{-2}H(xP+uR)^t\right\}$$
$$\times \exp\left\{-\frac{\lambda}{2}(xP+uR)H^t\left(1-HH^t\right)^{-1}H(xP+uR)^t\right\}$$
$$\times \prod_{j=1}^{m-n}\left(\lambda\left(h_j^2-1\right)+1\right)^{-1/2}\exp\left\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2}\cdot\frac{1}{\lambda\left(h_j^2-1\right)+1}\right\}. \tag{5.10}$$

Uniting (5.6) and (5.10), we come to a final expression for the kernel of integral operator

$$\mathcal{K}_\lambda(x,u) = |\det c|^{-1}\exp\left\{-\frac{1}{2}\|xQ+uT\|^2 - \frac{1}{2}\left\|(xP+uR)H^t\left(1-HH^t\right)^{-1}\right\|^2\right\} \tag{5.11}$$
$$\times \left|\det(G)\right|^\lambda \cdot \exp\left\{\frac{\lambda}{2}\left(\|xQ+uT\|^2 + \|x\|^2 - \|u\|^2\right.\right. \tag{5.12}$$
$$\left.\left. - (xP+uR)\left(1-H^tH\right)^{-1}(xP+yR)^t\right)\right\} \tag{5.13}$$
$$\times \prod_{j=1}^{m-n}\left(\lambda\left(h_j^2-1\right)+1\right)^{-1/2}\exp\left\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2}\cdot\frac{1}{\lambda\left(h_j^2-1\right)+1}\right\}. \tag{5.14}$$

Now we must represent the kernel as a Mellin transform of a measure

$$\mathcal{K}_\lambda(x,u) = \int_0^\infty t^\lambda\, dM_{x,u}(t).$$

The expression for $\mathcal{K}_\lambda(x,u)$ is a product, therefore its Mellin transform is a convolution. We must evaluate inverse Mellin transform for all factors. The first factor (5.11) is constant. The second factor (5.12)–(5.13) has the form $e^{\lambda a(x,u)}$, we have

$$e^{\lambda a(x,u)} = \int_0^\infty t^\lambda \delta\left(t - e^{a(x,u)}\right).$$

For factors in (5.14) the inverse Mellin transform was evaluated in Lemma 5.1.

This proves Theorem 5.2.

6. Convergent formula

6.1. Formula. Now consider arbitrary $\mathfrak{g} \in \mathrm{Coll}(n)$ being in the canonical form (4.1),

$$\begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix}.$$

To write a formula that is valid in general case, we rearrange factors in (5.3). First, we define δ-measures on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$dN^\circ_{x,u}(t) = A^\circ(x,u)\delta\big(t - B^\circ(x,u)\big),$$

where

$$A^\circ(x,u) = \det(T)\exp\left\{-\frac{1}{2}\|xQ + uT\|^2\right\}$$

$$B^\circ(x,u) = \frac{|\det P_1|}{|\det T|}\exp\left\{\frac{1}{2}\big(\|xQ + uT\|^2 - \|xP_1 + uR_1\|^2 - \|x\|^2 + \|u\|^2\big)\right\}.$$

In fact, $dN^\circ_{x,u}(t)$ is the measure $dN_{x,u}(t)$ defined for the matrix $\begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}$.

Next, we define the following probability measures $\Xi_j = \Xi[h_j, \psi_j]$ on $\mathbb{R}^\times$:

$$\Xi[h_j,\psi_j] = \exp\left\{-\frac{|\psi_j|^2 h_j^2}{2(1-h_j^2)^2}\right\} \cdot \delta\left(t - h_j \exp\left\{\frac{|\psi_j|^2}{2(1-h_j^2)}\right\}\right)$$
$$* \Phi\left[h_j^2 - 1, \frac{h_j^2|\psi_j|^2}{2(1-h_j^2)^2}; t\right] \tag{6.1}$$

if $h_j \neq 1$. For $h_j = 1$ we set

$$\Xi[1,\psi_j] = \frac{1}{|\psi_j|} e^{-\frac{1}{8}|\psi_j|^2} \exp\left\{-\frac{\ln^2 t}{2|\psi_j|^2}\right\}\frac{dt}{t^{3/2}}, \qquad \Xi[1,0] = \delta(t-1).$$

Theorem 6.1. *Let $\mathfrak{a} \in \mathrm{Coll}(n)$ be arbitrary. Then the polymorphism $\tau(\mathfrak{a})$ is given by*

$$\left(dN^\circ_{x,u}(t) * \mathop{\ast}_{j=1}^{\infty} \Xi[h_j,\psi_j]\right) dx\, du. \tag{6.2}$$

Lemma 6.2. a) *Measures $\Xi[h_j,\psi_j]$ are probabilistic.*

b) *The products*

$$\mathop{\ast}_{j=1}^{\infty} \Xi[h_j,\psi_j], \qquad \mathop{\ast}_{j=1}^{\infty} \big(t \cdot \Xi[h_j,\psi_j]\big) \tag{6.3}$$

weakly converge in the semigroup of measures on $\mathbb{R}^\times$.

Theorem 6.3. a) *For a matrix g denote by denote by $g^{(m)}$ the matrix $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$, where z is the upper left $(n+m)\times(n+m)$ corner of the matrix g. Then the polymorphism $\tau(\mathfrak{g}^{(m)})$ coincides with*

$$\left(dN^{\circ}_{x,u}(t) * \mathop{\ast}_{rj=1}^{m-n} \Xi[h_j,\psi_j]\right) dx\,du. \tag{6.4}$$

b) *The sequence of polymorphisms* (6.4) *converges in semigroup of polymorphisms of* $(\mathbb{R}^n,\mu_n)$ *to* $\tau(\mathfrak{a})$.

6.2. Rearrangement of factors (Lemma 6.3(a)). First, rearrange factors in (5.11)–(5.14):

$$\mathcal{K}_\lambda(x,u) = |\det T|\exp\left\{-\frac{1}{2}\|xQ+uT\|^2\right\}\left(\frac{|\det(P_1)|}{|\det(T)|}\right)^{\lambda} \tag{6.5}$$

$$\times\exp\left\{\frac{\lambda}{2}\left(\|xQ+uT\|^2+\|x\|^2-\|u\|^2-\|xP_1+uR_1\|^2\right)\right\} \tag{6.6}$$

$$\times\prod_{j=1}^{m-n}\left(\exp\left\{\frac{h_j^2|\psi_j|^2}{2(1-h_j^2)^2}\right\}\cdot h_j^{\lambda}\exp\left\{\frac{\lambda|\psi_j|^2}{2(1-h_j^2)}\right\}\right. \tag{6.7}$$

$$\left.\times\left(\lambda\left(h_j^2-1\right)+1\right)^{-1/2}\exp\left\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2}\cdot\frac{1}{\lambda\left(h_j^2-1\right)+1}\right\}\right). \tag{6.8}$$

Factors in the product (6.5)–(6.6) looks as singular near $h_j=1$. But this singularity is artificial, it appears due division in the line (5.9). Returning to the previous line (5.8) of the calculation, we get for $h_j=1$ the following factor

$$\exp\left\{-\frac{1}{2}\lambda|\psi_j|^2+\frac{1}{2}\lambda^2|\psi_j|^2\right\}=\frac{1}{|\psi_j|}e^{-\frac{1}{8}|\psi_j|^2}\int_0^\infty t^\lambda\exp\left\{-\frac{\ln^2 t}{2|\psi_j|^2}\right\}\frac{dt}{t^{3/2}}.$$

6.3. Proof of Lemma 6.3(b).

Lemma 6.4. *The embedding* $\iota:\mathrm{GLO}(\infty)\to\mathrm{Pol}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ *is continuous.*

Proof. According Proposition 2.5(b) it is sufficient to prove that the representations $T_\lambda(g)$ of $\mathrm{GLO}(\infty)$ are weakly continuous for all λ. It is sufficient to take $f=e^{iax}$ and $g=e^{ibx}$ in (2.2) and to verify continuity of the corresponding matrix elements with respect to the Shale topology. □

Let g be of the form (4.1). For finite matrices formulas (5.3) and (6.2) coincide. Denote by $g^{(m)}$ the matrix $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$, where z is the upper left $(n+m)\times(n+m)$ corner of the matrix g. For $g^{(m)}$ the formula (6.4) gives a correct result. Next, $g^{(m)}$ converges to g in the Shale topology. Therefore $\tau(g^{(m)})$ converges to $\tau(g)$ as $g\to\infty$. This proves the last statement of the theorem.

6.4. Proof of Theorem 6.1. We must prove convergence of the infinite convolution in (6.3). The characteristic function of $\Xi[h_j, \psi_j]$ is given by

$$\int_0^\infty t^\lambda \Xi_j[h_j, \psi_j] = h_j^\lambda \bigl(1 + \lambda\bigl(h_j^2 - 1\bigr)\bigr)^{-1/2} \exp\left\{ \frac{\lambda^2 h_j^2 |\psi_j|^2}{2(\lambda h_j^2 - \lambda + 1)} - \frac{\lambda}{2} |\psi_j|^2 \right\}.$$

We have $\sum (h_j - 1)^2 < \infty$, $\sum |\psi_j|^2 < \infty$. Under these conditions we have a convergence of the product in the strip $0 \leqslant \operatorname{Re} \lambda \leqslant 1$. This implies the weak convergence of measures on $\mathbb{R}^\times$.

The convergence is uniform on compacts sets with respect to x, u, and this implies coincidence of (6.2) and limit of (6.4).

References

[1] A. Borodin, G. Olshanski, Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, Ann. of Math. (2) 161 (2005) 1319–1422.

[2] Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, Comm. Pure Appl. Math. 52 (4) (1999) 411–452.

[3] M.S. Brodskij, Unitary operator colligations and their characteristic functions, Russian Math. Surveys 33 (4) (1978) 159–191.

[4] V.I. Bogachev, Gaussian Measures, Amer. Math. Soc., Providence, RI, 1998.

[5] A.I. Danilenko, V.V. Ryzhikov, Mixing constructions with infinite invariant measure and spectral multiplicities, Ergodic Theory Dynam. Systems 31 (3) (2011) 853–873.

[6] C. De Concini, C. Procesi, Complete symmetric varieties, in: Invariant Theory, Montecatini, 1982, in: Lecture Notes in Math., vol. 996, Springer, 1983, pp. 1–44.

[7] H. Dym, Linear Algebra in Action, Amer. Math. Soc., Providence, RI, 2007.

[8] E. Glasner, B. Tsirelson, B. Weiss, The automorphism group of the Gaussian measure cannot act pointwise, Israel J. Math. 148 (2005) 305–329.

[9] S. Kerov, G. Olshanski, A. Vershik, Harmonic analysis on the infinite symmetric group, Invent. Math. 158 (3) (2004) 551–642.

[10] U. Krengel, Ergodic Theorems, Walter de Gruyter, Berlin, 1985, Springer, 1983.

[11] Hui Hsiung Kuo, Gaussian Measures in Banach Spaces, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin–New York, 1975.

[12] M.S. Livshits, On a certain class of linear operators in Hilbert space, Mat. Sb. (N.S.) 19 (61) (1946) 239–262, English transl. in Amer. Math. Soc. Transl. (Ser. 2) 13 (1960) 61–83.

[13] M.S. Livshits, On spectral decomposition of linear nonself-adjoint operators, Mat. Sb. (N.S.) 34 (76) (1954) 145–199, English transl. in Amer. Math. Soc. Transl. (Ser. 2) 5 (1957) 67–114.

[14] N.F.G. Martin, J.W. England, Mathematical Theory of Entropy, Addison–Wesley Publishing Co., Reading, MA, 1981.

[15] E. Nelson, The free Markoff field, J. Funct. Anal. 12 (1973) 211–227.

[16] Yu.A. Neretin, Categories of bistochastic measures and representations of some infinite-dimensional groups, Mat. Sb. 183 (2) (1992) 52–76, English transl. in Sb. Math. 75 (1) (1993) 197–219.

[17] Yu.A. Neretin, Categories of Symmetries and Infinite-Dimensional Groups, Oxford University Press, New York, 1996, Russian transl. in URSS, 1998.

[18] Yu.A. Neretin, Fractional diffusions and quasi-invariant actions of infinite-dimensional groups, Tr. Mat. Inst. Steklova 217 (1997) 135–181, English transl. in Proc. Steklov Inst. Math. 217 (1997) 126–173.

[19] Yu.A. Neretin, Notes on affine isometric actions of discrete groups, in: H. Heyer, J. Marion (Eds.), Analysis on Infinite-Dimensional Lie Groups and Algebras, Marseille, 1997, World Sci. Publ., River Edge, NJ, 1998, pp. 274–320.

[20] Yu.A. Neretin, Spreading maps (polymorphisms), symmetries of Poisson processes, and matching summation, J. Math. Sci. (N. Y.) 126 (2) (2005) 1077–1094.

[21] Yu.A. Neretin, Hua-type integrals over unitary groups and over projective limits of unitary groups, Duke Math. J. 114 (2) (2002) 239–266.

[22] Yu.A. Neretin, On the boundary of the group of transformations leaving a measure quasi-invariant, preprint arXiv:1111.2833.
[23] Yu.A. Neretin, Sphericity and multiplication of double cosets for infinite-dimensional classical groups, Funct. Anal. Appl. 3 (2011).
[24] G.I. Olshanski, Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe, in: Representation of Lie Groups and Related Topics, Adv. Stud. Contemp. Math. 7 (1990) 269–463.
[25] G.I. Olshanski, On semigroups related to infinite-dimensional groups, in: A.A. Kirillov (Ed.), Topics in Representation Theory, in: Adv. in Soviet Math., vol. 2, Amer. Math. Soc., Providence, RI, 1991, pp. 67–101.
[26] D. Pickrell, Measures on infinite-dimensional Grassmann manifolds, J. Funct. Anal. 70 (2) (1987) 323–356.
[27] V.P. Potapov, The multiplicative structure of J-contractive matrix functions, Tr. Mosk. Mat. Obs. 4 (1955) 125–236, English transl. in Amer. Math. Soc. Transl. (Ser. 2) 15 (1960) 131–243.
[28] M. Reed, B. Simon, Methods of Modern Mathematical Physics. I. Functional Analysis, Academic Press, New York–London, 1972.
[29] V.A. Rohlin, On the fundamental ideas of measure theory, Mat. Sb. (N.S.) 25 (67) (1949) 107–150, English transl. in Amer. Math. Soc. Transl. 1952 (71) (1952).
[30] D. Shale, Linear symmetries of free boson fields, Trans. Amer. Math. Soc. 103 (1962) 149–167.
[31] G.E. Shilov, Fan Dyk Tin, Integral Measure and Derivative on Linear Spaces, Nauka, Moscow, 1966.
[32] A.N. Shiryaev, Probability, second ed., Grad. Texts in Math., vol. 95, Springer-Verlag, New York, 1996.
[33] A.M. Vershik, Multivalued mappings with invariant measure (polymorphisms) and Markov operators, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 72 (1977) 26–61, English transl. in J. Sov. Math. 23 (1983) 2243–2266.