Journal of Combinatorial Theory, Series A 117 (2010) 475–482

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Note

Interval partitions and Stanley depth

Csaba Biró [1], David M. Howard, Mitchel T. Keller, William T. Trotter, Stephen J. Young

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

A R T I C L E I N F O

Article history:
Received 20 May 2008
Available online 21 August 2009

Keywords:
Boolean lattice
Partition
Interval
Stanley depth
Monomial ideal

A B S T R A C T

In this paper, we answer a question posed by Herzog, Vladoiu, and Zheng. Their motivation involves a 1982 conjecture of Richard Stanley concerning what is now called the Stanley depth of a module. The question of Herzog et al., concerns partitions of the non-empty subsets of $\{1, 2, \dots, n\}$ into intervals. Specifically, given a positive integer n, they asked whether there exists a partition $\mathcal{P}(n)$ of the non-empty subsets of $\{1, 2, \dots, n\}$ into intervals, so that $|B| \geqslant n/2$ for each interval $[A, B]$ in $\mathcal{P}(n)$. We answer this question in the affirmative by first embedding it in a stronger result. We then provide two alternative proofs of this second result. The two proofs use entirely different methods and yield non-isomorphic partitions. As a consequence, we establish that the Stanley depth of the ideal $(x_1, \dots, x_n) \subseteq K[x_1, \dots, x_n]$ (K a field) is $\lceil n/2 \rceil$.

1. Introduction

In [4], Herzog, Vladoiu, and Zheng established an interesting connection between a long-standing question in commutative algebra and special partitions of partially ordered sets corresponding to algebraic structures. As a result, some natural combinatorial partitioning questions have arisen. In this paper, we address such a question using a purely combinatorial approach; however, we provide a brief description of the connection to algebra to make our motivation clear.

In a 1982 paper [6], Richard P. Stanley defined what is now called the Stanley depth of a $\mathbb{Z}^n$-graded module over a commutative ring S. He conjectured that the Stanley depth was always at least the depth of the module. The question is still largely open, but see [1–3,5].

E-mail addresses: biro@math.sc.edu (C. Biró), dmh@math.gatech.edu (D.M. Howard), keller@math.gatech.edu (M.T. Keller), trotter@math.gatech.edu (W.T. Trotter), young@math.gatech.edu (S.J. Young).

[1] Current address: Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.07.008

Herzog et al. showed in [4] that for a field K, the Stanley depth of a monomial ideal I of $S = K[x_1, \dots, x_n]$ can be computed in finite time (although not efficiently) by looking at partitions of a certain finite subposet of $\mathbb{N}^n$ into intervals. In [4], the authors demonstrate that for the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n) \subseteq K[x_1, \dots, x_n]$, computing the Stanley depth of $\mathfrak{m}$ is equivalent to finding a partition of the non-empty subsets of $[n]$ into intervals with a particular property. They claim that for $n \leqslant 9$, they were able to show $\operatorname{sdepth} \mathfrak{m} = \lceil n/2 \rceil$, raising a combinatorial problem, which the following theorem answers in the affirmative:

Theorem 1.1. *Let n be a positive integer. Then there exists a partition $\mathcal{P}(n)$ of the non-empty subsets of $[n]$ into intervals so that $|Y| \geqslant n/2$ for each interval $[X, Y] \in \mathcal{P}(n)$.*

In fact we will prove an even stronger result that gives a very regular structure to the intervals used in the partition.

The paper begins by precisely stating the relationship between monomial ideals and posets needed to answer the question raised by Herzog et al. We then provide two proofs of our main theorem. The first proof is inductive, while the second is non-inductive and allows for immediate identification of the interval that contains any given subset. The two proofs provide non-isomorphic partitions even for relatively small values of n.

2. Background and notation

2.1. Combinatorics

For a positive integer n, we let $[n] = \{1, 2, \dots, n\}$, and we let $\mathcal{B}(n)$ denote the Boolean algebra consisting of all subsets of $[n]$. For sets $X, Y \subseteq [n]$ with $X \subseteq Y$, we let $[X, Y] = \{Z : X \subseteq Z \subseteq Y\}$. It is customary to refer to $[X, Y]$ as an *interval* in $\mathcal{B}(n)$.

In the remainder of this paper, we will concentrate on the case where n is odd, say $n = 2k + 1$ for some $k \geqslant 0$. The reason is that if n is odd, and we have a partition $\mathcal{P}(n)$ of the non-empty subsets of $[n]$, with $|Y| \geqslant n/2$ for each interval $[X, Y] \in \mathcal{P}(n)$, then $\mathcal{Q}(n) = \mathcal{P}(n) \cup \{[\{n + 1\}, [n + 1]]\}$ is a partition of the non-empty subsets of $[n + 1]$ into intervals and $|Y| \geqslant (n + 1)/2$ for each interval $[X, Y] \in \mathcal{Q}(n)$.

Keeping this remark on parity in mind, it is then clear that Theorem 1.1 follows as an immediate corollary to the following more structured result.

Theorem 2.1. *Let k be a non-negative integer. Then there exists a partition $\mathcal{C}(k)$ of the non-empty subsets of $[2k + 1]$ into intervals so that for each interval $[X, Y] \in \mathcal{C}(k)$, $|X|$ is odd and $|Y| = k + 1 + \lfloor |X|/2 \rfloor$.*

In the next two sections of this paper, we provide alternative proofs of Theorem 2.1. These proofs lead to non-isomorphic partitions when $k \geqslant 3$.

2.2. Algebra

Let K be a field and $S = K[x_1, \dots, x_n]$. For $c \in \mathbb{N}^n$, let x^c denote the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Let $J \subseteq I \subseteq S$ be monomial ideals. Say $I = (x^{a_1}, x^{a_2}, \dots, x^{a_r})$ and $J = (x^{b_1}, x^{b_2}, \dots, x^{b_t})$ where $a_i, b_j \in \mathbb{N}^n$. Let

$$g = \bigvee_i a_i \vee \bigvee_j b_j$$

(the component-wise maximum of the a_i and b_j). Then the characteristic poset of I/J with respect to g, denoted $P^g_{I/J}$, is the induced subposet of $\mathbb{N}^n$ with ground set

$$\{c \in \mathbb{N}^n \mid c \leqslant g, \text{ there is } i \text{ such that } c \geqslant a_i, \text{ and for all } j, c \ngeqslant b_j\}.$$

(Note that such a poset can be defined for any $g \geqslant a_i, b_j$ for all i, j, but it is simply convenient to take g as the join of the a_i and b_j.)

Let $\mathcal{P}$ be a partition of $P^g_{I/J}$ into intervals. For $I = [x, y] \in \mathcal{P}$, define $Z_I := \{i \in [n] \mid y(i) = g(i)\}$. Define the Stanley depth of a partition $\mathcal{P}$ to be

$$\operatorname{sdepth} \mathcal{P} := \min_{I \in \mathcal{P}} |Z_I|$$

and the Stanley depth of the poset $P^g_{I/J}$ to be $\operatorname{sdepth} P^g_{I/J} := \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where the maximum is taken over all partitions $\mathcal{P}$ of $P^g_{I/J}$ into intervals. Herzog et al. showed in [4] that $\operatorname{sdepth} I/J = \operatorname{sdepth} P^g_{I/J}$. By considering all partitions of the characteristic poset, this correspondence provides an algorithm (albeit inefficient) to find the Stanley depth of I/J. Given this setting, it is easy to see the correspondence between $P^{(1,1,\dots,1)}_{\mathfrak{m}}$ and the set of non-empty subsets of $[n]$. (In this context, $|Z_I| = |Y|$.) We will establish, using purely combinatorial techniques, the following theorem as a consequence of Theorem 1.1 along with some elementary counting.

Theorem 2.2. *Let K be a field and $S = K[x_1, \dots, x_n]$. Let $\mathfrak{m}$ be the maximal ideal $(x_1, \dots, x_n) \subseteq S$. Then*

$$\operatorname{sdepth} \mathfrak{m} = \left\lceil \frac{n}{2} \right\rceil.$$

3. An inductive approach to the main theorem

This proof presented here relies on the construction of two auxiliary partitions, to be denoted $\mathcal{A}(k)$ and $\mathcal{B}(k)$, respectively. In contrast to the partition $\mathcal{C}(k)$ we seek to complete the proof of Theorem 2.1, $\mathcal{A}(k)$ and $\mathcal{B}(k)$ will each be partitions of *all* subsets of $[2k+1]$ into intervals. The partition $\mathcal{C}(k)$ will then be constructed from $\mathcal{A}(k)$ and $\mathcal{B}(k)$ using some intervals from one and some intervals from the other.

It will also be important to keep track of the sizes of the intervals in the partitions $\mathcal{A}(k)$ and $\mathcal{B}(k)$.

Size Property:

(1) If $[X, Y]$ is an interval in the partition $\mathcal{A}(k)$, then $|Y| = k + 1 + \lfloor |X|/2 \rfloor$.
(2) If $[X, Y]$ is an interval in the partition $\mathcal{B}(k)$, then $|Y| = k + \lceil |X|/2 \rceil$.

We note that when $[X, Y]$ is an interval in either $\mathcal{A}(k)$ or $\mathcal{B}(k)$ and $|X|$ is odd, say $|X| = 2s+1$, then $|Y| = k + s + 1$.

As we proceed with the construction, we will use the partitions $\mathcal{A}(k)$ and $\mathcal{B}(k)$ to determine functions, denoted A_k and B_k respectively, mapping the subsets of $[2k+1]$ to $\{0, 1\}$, by the following rules:

Coloring Rule: Let $S \subseteq [2k+1]$. Set $A_k(S) = 0$ when S belongs to an interval $[X, Y]$ in the partition $\mathcal{A}(k)$ with $|X|$ even; else set $A_k(S) = 1$. Similarly, set $B_k(S) = 0$ when S belongs to an interval $[X, Y]$ in the partition $\mathcal{B}(k)$ with $|X|$ even; else set $B_k(S) = 1$.

We will maintain the following property inductively:

Coloring Property: For every non-empty subset $S \subseteq [2k+1]$, $A_k(S) = 1 - B_k(S)$.

3.1. Construction of the two sequences

First, set

$$\mathcal{A}(0) = \big\{ \big[\emptyset, \{1\}\big] \big\} \quad \text{and} \quad \mathcal{B}(0) = \big\{ [\emptyset, \emptyset], \big[\{1\}, \{1\}\big] \big\}.$$

Note that these two partitions satisfy the Size Property. Also, note that $A_0(\{1\}) = 0$ and $B_0(\{1\}) = 1$, so the Coloring Property holds as well.

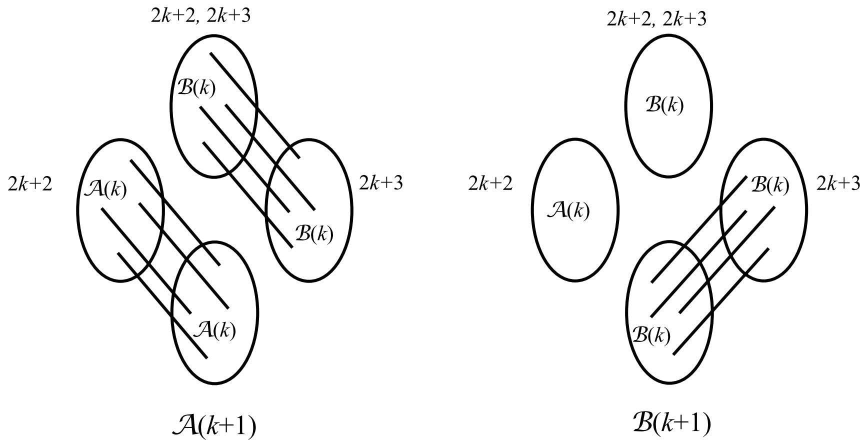

Fig. 1. The inductive construction.

Now suppose that for some $k \geqslant 0$, we have constructed partitions $\mathcal{A}(k)$ and $\mathcal{B}(k)$ of the subsets of $[2k+1]$ into intervals so that both the Size Property and the Coloring Property hold.

Then $\mathcal{A}(k+1)$ is defined by

$$\begin{aligned}\mathcal{A}(k+1) = &\bigl\{\bigl[X, Y \cup \{2k+2\}\bigr]\colon\ [X, Y] \in \mathcal{A}(k)\bigr\}\\ &\cup \bigl\{\bigl[X \cup \{2k+3\}, Y \cup \{2k+2, 2k+3\}\bigr]\colon\ [X, Y] \in \mathcal{B}(k)\bigr\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}(k+1) = &\bigl\{\bigl[X \cup \{2k+2\}, Y \cup \{2k+2\}\bigr]\colon\ [X, Y] \in \mathcal{A}(k)\bigr\}\\ &\cup \bigl\{\bigl[X, Y \cup \{2k+3\}\bigr]\colon\ [X, Y] \in \mathcal{B}(k)\bigr\}\\ &\cup \bigl\{\bigl[X \cup \{2k+2, 2k+3\}, Y \cup \{2k+2, 2k+3\}\bigr]\colon\ [X, Y] \in \mathcal{B}(k)\bigr\}.\end{aligned}$$

We have found it convenient to view these two constructions using the suggestive diagram shown in Fig. 1.

It is straightforward to verify that $\mathcal{A}(k+1)$ and $\mathcal{B}(k+1)$ are partitions of the subsets of $[2k+3]$ into intervals. Also, it is clear that the Size Property holds. We now show that the functions A_{k+1} and B_{k+1} satisfy the Coloring Property. Let S be a non-empty subset of $[2k+3]$. We distinguish four cases and show that $A_{k+1}(S) = 1 - B_{k+1}(S)$ in each case.

Case 1. $S \cap \{2k+2, 2k+3\} = \emptyset$.

Let $[X, Y]$ and $[Z, W]$ be the intervals containing S in $\mathcal{A}(k)$ and $\mathcal{B}(k)$, respectively. Then S is contained in the intervals $[X, Y \cup \{2k+2\}]$ and $[Z, W \cup \{2k+3\}]$ in $\mathcal{A}(k+1)$ and $\mathcal{B}(k+1)$ respectively. This implies that $A_{k+1}(S) = A_k(S)$ and $B_{k+1}(S) = B_k(S)$. Since A_k and B_k satisfy the Coloring Property, we conclude that $A_{k+1}(S) = A_k(S) = 1 - B_k(S) = 1 - B_{k+1}(S)$.

Case 2. $S \cap \{2k+2, 2k+3\} = \{2k+2\}$.

Let $T = S - \{2k+2\}$, and let $[X, Y]$ be the interval in the partition $\mathcal{A}(k)$ containing T. It follows that S is contained in the interval $[X, Y \cup \{2k+2\}]$ in $\mathcal{A}(k+1)$. Thus $A_{k+1}(S) = A_k(T)$. On the other hand, S is contained in the interval $[X \cup \{2k+2\}, Y \cup \{2k+2\}]$ in $\mathcal{B}(k+1)$. Thus $B_{k+1}(S) = 1 - A_k(T)$, so that $A_{k+1}(S) = 1 - B_{k+1}(S)$.

Case 3. $S \cap \{2k+2, 2k+3\} = \{2k+3\}$.

Let $T = S - \{2k+3\}$, and let $[Z, W]$ be the interval in the partition $\mathcal{B}(k)$ containing T. It follows that S is contained in the interval $[Z \cup \{2k+3\}, W \cup \{2k+2, 2k+3\}]$ in $\mathcal{A}(k+1)$. Thus $A_{k+1}(S) =$

$1 - B_k(T)$. On the other hand, S is contained in the interval $[Z, W \cup \{2k+3\}]$ in $\mathcal{B}(k+1)$. Thus $B_{k+1}(S) = B_k(T)$, so that $A_{k+1}(S) = 1 - B_{k+1}(S)$.

Case 4. $S \cap \{2k+2, 2k+3\} = \{2k+2, 2k+3\}$.

Let $T = S - \{2k+2, 2k+3\}$, and let $[Z, W]$ be the interval in $\mathcal{B}(k)$ containing T. It follows that S is in the interval $[Z \cup \{2k+3\}, W \cup \{2k+2, 2k+3\}]$ in $\mathcal{A}(k+1)$. Thus $A_{k+1}(S) = 1 - B_k(T)$. On the other hand, S belongs to the interval $[Z \cup \{2k+2, 2k+3\}, W \cup \{2k+2, 2k+3\}]$ in $\mathcal{B}(k+1)$. Thus $B_{k+1}(S) = B_k(T)$, so that $A_{k+1}(S) = 1 - B_{k+1}(S)$.

It is worth noting that in the last three cases of the preceding argument, we did not have to consider whether the set T was empty or not.

3.2. Merging the partitions

We are now ready to construct the partition $\mathcal{C}(k)$ of the non-empty subsets of $[2k+1]$ satisfying the conclusion of Theorem 2.1. The rule is that an interval $[X, Y]$ belongs to the partition $\mathcal{C}(k)$ if and only if $[X, Y]$ belongs to one of $\mathcal{A}(k)$ and $\mathcal{B}(k)$ and $|X|$ is odd.

The fact that $\mathcal{C}(k)$ is a partition of the non-empty subsets of $[2k+1]$ into intervals is an immediate consequence of the Coloring Property. Also, the cardinality condition follows immediately from our remark just after the Size Property. This completes the proof.

4. A non-inductive approach

Throughout this section, we fix a non-negative integer k and consider the integers in $[2k+1]$ placed in *clockwise natural order* around a circle. We interpret arithmetic cyclically; for example, when $k = 9$, we say that $18 + 5 = 4$, since $18 + 5 = 23 = 19 + 4$.

For each element $i \in [2k+1]$, the remaining $2k$ elements are partitioned into two blocks each of size k, with the *clockwise block* consisting of $\{i+1, i+2, \dots, i+k\}$, and the *counterclockwise block* consisting of $\{i-1, i-2, \dots, i-k\}$. In the discussion to follow, we will denote these two blocks as $\mathrm{cw}(i)$ and $\mathrm{ccw}(i)$ respectively. For example, when $k = 9$,

$$\mathrm{cw}(14) = \{15, 16, 17, 18, 19, 1, 2, 3, 4\}$$

and

$$\mathrm{ccw}(14) = \{13, 12, 11, 10, 9, 8, 7, 6, 5\}$$

Definition 4.1. A non-empty subset $B \subseteq [2k+1]$ is *balanced* if

$$\bigl|B \cap \mathrm{cw}(i)\bigr| = \bigl|B \cap \mathrm{ccw}(i)\bigr| \quad \text{for all } i \in B.$$

Clearly, if B is a balanced set, then $|B|$ is odd, and if $|B| = 2s+1$, then there are s elements in $\mathrm{cw}(i)$ and s elements in $\mathrm{ccw}(i)$, for every $i \in B$.

For example, referring to the circle shown in the left half of Fig. 2, when $k = 9$, the set $B_1 = \{2, 5, 10, 15, 17\}$ is not balanced since $|B_1 \cap \mathrm{cw}(5)| = 1$ and $|B_1 \cap \mathrm{ccw}(5)| = 3$. However, referring to the circle shown in the right half of Fig. 2, the set $B_2 = \{4, 8, 13, 17, 18\}$ is balanced.

Lemma 4.2. *Let s be a non-negative integer, and let B be a balanced subset of $[2k+1]$ with $|B| = 2s+1$. Then for each $j \in [2k+1] - B$, either*

(1) $|\mathrm{cw}(j) \cap B| = s$ *and* $|\mathrm{ccw}(j) \cap B| = s+1$, *or*
(2) $|\mathrm{cw}(j) \cap B| = s+1$ *and* $|\mathrm{ccw}(j) \cap B| = s$.

Proof. Let $j \in [2k+1] - B$. If $\mathrm{cw}(j)$ contains at least $s+2$ elements of B, let i be the element of $B \cap \mathrm{cw}(j)$ that is closest to j. Clearly, $\mathrm{cw}(i)$ contains at least $s+1$ elements of B, which contradicts

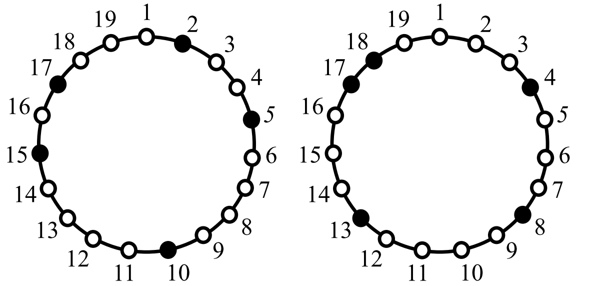

Fig. 2. Clockwise natural order.

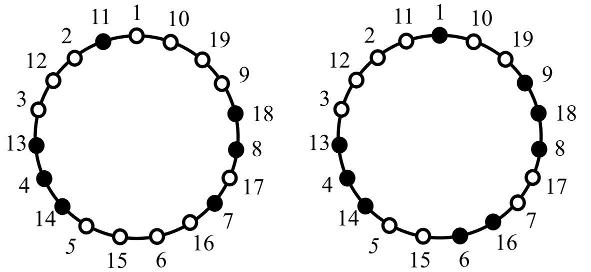

Fig. 3. Clockwise star order.

the fact that B is balanced. Thus $\mathrm{cw}(j)$ contains at most $s+1$ elements of B. Dually, $\mathrm{ccw}(j)$ contains at most $s+1$ elements of B. These two statements together imply the conclusion of the lemma. $\square$

In view of Lemma 4.2, it is natural to partition the elements of $[2k+1]-B$ into two sets L_B and R_B with an element $j \in [2k+1]-B$ belonging to L_B when statement (1) of the lemma holds and R_B when statement (2) holds.

Lemma 4.3. *If B is a balanced set, then $|L_B| = |R_B|$. Furthermore, an element $j \in [2k+1]$ belongs to L_B if and only if $j+k$ belongs to R_B.*

Proof. Suppose first that element j belongs to L_B. If $j+k$ belongs to B, then there are only $s-1$ elements of B in $\mathrm{ccw}(j+k)$. The contradiction implies $j \notin B$. Furthermore, there are s elements of B in $\mathrm{ccw}(j+k)$ so $j+k \in R_B$.

By symmetry, if $j+k \in R_B$, then $j \in L_B$. $\square$

As suggested by Lemma 4.3, there is another useful way to arrange the elements of $[2k+1]$ around a circle in a clockwise manner. We call this alternative order the *clockwise star order*. In this order, integer i is followed by $i+k$. We illustrate this definitions with the circles shown in Fig. 3.

When S is a non-empty subset of $[2k+1]$ and $s \in S$, we let $Z(s,S)$ denote the set (possibly empty) of elements of $[2k+1]-S$ encountered by starting immediately after s and continuing around the circle in clockwise star order until just before another element of S is encountered. Note that when $|S|=1$ and $S=\{s\}$, $Z(s,S)=[2k+1]-S$. Also, note that $Z(s,S)=\emptyset$ when s is followed immediately by another element of S in the clockwise star order.

Referring to the circle in the left half of Fig. 3, note that when $S=\{4,7,8,11,13,14,18\}$, $Z(11,S)=\{1,10,19,9\}$ and while $Z(14,S)=\emptyset$.

Lemma 4.4. *A non-empty subset $B \subseteq [2k+1]$ is balanced if and only if $|Z(b,B)|$ is even for every $b \in B$.*

Proof. Suppose first that B is balanced. Then let $b \in B$ and suppose that $|Z(b, B)|$ is odd. Then $Z(b, B)$ is a non-empty set whose elements must alternate between members of L_B and R_B, starting with an element of L_B, when listed in clockwise star order. By parity, the last element of $Z(b, B)$ in this listing also belongs to L_B. Using Lemma 4.3, this would imply that the element of B following immediately after the last element of $Z(b, B)$ in the clockwise star order belongs to R_B. The contradiction shows that $|Z(b, B)|$ must be even.

Now suppose that B is a non-empty subset of $[2k+1]$ and that $|Z(b, B)|$ is even for every $b \in B$. We show that B is balanced.

Let $b_0 \in B$. As we proceed around the circle in clockwise star order starting immediately after b_0 and continuing full circle through the remaining $2k$ elements of $[2k+1] - \{b_0\}$, note that we alternate between elements of $\mathrm{cw}(b_0)$ and $\mathrm{ccw}(b_0)$, starting with an element of $\mathrm{cw}(b_0)$ and ending with an element of $\mathrm{ccw}(b_0)$. However, since $|Z(b, B)|$ is even for each $b \in B$, the elements of $[2k+1] - B$ are evenly divided between $\mathrm{cw}(b_0)$ and $\mathrm{ccw}(b_0)$. Also, by parity, this also implies that the remaining elements of $B - \{b_0\}$ are evenly divided between $\mathrm{cw}(b_0)$ and $\mathrm{ccw}(b_0)$. This shows that B is balanced. □

We illustrate Lemma 4.4 with the circles shown in Fig. 3. Referring to the circle on the left half of Fig. 3, we see that the set $S = \{4, 7, 8, 11, 13, 14, 18\}$ is not balanced since $Z(13, S) = \{3, 12, 2\}$ so that $|Z(13, S)| = 3$. On the other hand, referring to the circle in the right half of Fig. 3, the set $B = \{1, 4, 6, 8, 9, 13, 14, 16, 18\}$ is balanced.

We state the following elementary fact for emphasis. The proof is an immediate consequence of Lemma 4.3.

Proposition 4.5. *When B is a balanced subset of $[2k+1]$, then for each $b \in B$ with $Z(b, B) \neq \emptyset$, the elements of $Z(b, B)$ alternate between elements of L_B and elements of R_B, starting with an element of L_B, when listed in the clockwise star order.*

Referring again to the right part of Fig. 3, we see that for the balanced set $B = \{1, 4, 6, 8, 9, 13, 14, 16, 18\}$, we have

$$L_B = \{3, 2, 10, 17, 15\} \quad \text{and} \quad R_B = \{12, 11, 19, 7, 5\}.$$

Note that when B is a balanced subset of $[2k+1]$ and $|B| = 2s+1$, then $|L_B| = |R_B| = k - s$. Therefore $|B \cup L_B| = k + s + 1$. Therefore, the interval $[B, B \cup L_B]$ satisfies the cardinality constraint of Theorem 2.1.

We next present the technical lemma that is the heart of this alternate construction.

Lemma 4.6. *Let k be a non-negative integer and let S be a non-empty subset of $[2k+1]$. Then there is a unique balanced set B for which S belongs to the interval $[B, B \cup L_B]$.*

Proof. Let S be a non-empty subset of $[2k+1]$. Then partition S as $S = D \cup U$ where

$$D = \{s \in S\colon |Z(s, S)| \text{ is even}\}$$

and

$$U = S - D = \{s \in S\colon |Z(s, S)| \text{ is odd}\}.$$

Claim 1. D is a balanced subset of $[2k+1]$.

Proof of Claim 1. For each $d \in D$, the cardinality of $Z(d, D)$ is even. This follows from the fact that $Z(d, D)$ is the union of $Z(d, S)$ and a set of blocks of the form $\{s\} \cup Z(s, S)$ where $s \in U$. Using Lemma 4.4, we conclude that D is balanced.

Claim 2. $U \subseteq L_D$.

Proof of Claim 2. From Proposition 4.5, for each element $d \in D$, the elements of $Z(d, D)$ alternate between members of L_D and members of R_D, beginning with a member of L_D. By parity, it follows that all members of U belong to L_D.

We are now ready to complete the proof of the lemma. Let B be a balanced subset of $[2k+1]$ with $B \subseteq S \subseteq B \cup L_B$. We show that $B = D$. First, let $d \in D$. If $d \in L_B$, then the element of S occurring immediately after the last element of $Z(d, S)$ belongs to R_B. The contradiction shows that $d \in B$.

Now let $u \in U$. If $u \in B$, then the next element of D occurring after u in the clockwise star order belongs to R_B. The contradiction shows $u \in L_B$. $\square$

For the sake of completeness, we summarize the contents of this section with the following statement.

Theorem 4.7. *Let k be a non-negative integer, and let $n = 2k+1$. Then*

$$\mathcal{C}(k) = \big\{[B, B \cup L_B]\colon\ B \text{ is a balanced subset of } [2k+1]\big\}$$

is a partition of the non-empty subsets of $[2k+1]$ into intervals. Furthermore, if B is a balanced set and $|B| = 2s+1$, then $|B \cup L_B| = k+s+1$.

5. Conclusions

Returning to the original question of Herzog et al., we see that Theorem 1.1 implies that $\operatorname{sdepth}(x_1, \ldots, x_n) \geqslant \lceil n/2 \rceil$. It remains to show that no partition can have all of the upper bounds of its intervals further up in $\mathcal{B}(n)$. Let n be a positive integer, and let $\mathcal{P}(n)$ be any partition of the non-empty subsets of $[n]$ into intervals so that $|Y| \geqslant \lceil n/2 \rceil$ for every interval $[X, Y] \in \mathcal{P}(n)$. When n is odd, say $n = 2k+1$, then it is easy to see that for each $i = 1, 2, \ldots, 2k+1$, we must have an interval in $\mathcal{P}(n)$ of the form $[\{i\}, Y]$ with $|Y| = k+1$. Furthermore, there are no intervals in $\mathcal{P}(n)$ of the form $[X, Y]$ with $|X| = 2$. However, we do not know whether there are other cardinality constraints of this type that must apply. On the other hand, when n is even, say $n = 2k$, then it is easy to see that there must be at least one $i \in [n]$ for which there is an interval of the form $[\{i\}, Y]$ in $\mathcal{P}(n)$ with $|Y| = k$. Thus, we have completed the proof of Theorem 2.2.

The more general class of poset partitioning questions raised in [4] appears to have the potential for further interesting mathematics. For example, at present the best known algorithm for computing the Stanley depth of a monomial ideal inspects all of the interval partitions of its characteristic poset. It would be interesting to know if there is a general way of identifying the partitions that need to be inspected, providing a more efficient algorithm. It would also be interesting to examine other classes of monomial ideals to see if they give rise to easily-recognizable classes of posets for which combinatorial techniques can find optimal partitions.

References

[1] J. Apel, On a conjecture of R.P. Stanley. I. Monomial ideals, J. Algebraic Combin. 17 (1) (2003) 39–56.
[2] J. Herzog, A.S. Jahan, S. Yassemi, Stanley decompositions and partitionable simplicial complexes, J. Algebraic Combin. 27 (1) (2008) 113–125.
[3] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes, Manuscripta Math. 121 (3) (2006) 385–410.
[4] J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, J. Algebra 322 (9) (2009) 3151–3169.
[5] A.S. Jahan, Prime filtrations of monomial ideals and polarizations, J. Algebra 312 (2) (2007) 1011–1032.
[6] R.P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. 68 (2) (1982) 175–193.