Journal of Differential Equations **159**, 447–484 (1999)
Article ID jdeq.1999.3629, available online at http://www.idealibrary.com on IDEAL®

Riemann Problems for a Class of Coupled Hyperbolic Systems of Conservation Laws

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Received June 18, 1998; revised December 28, 1998

The Riemann problems are solved constructively. The Riemann solutions exactly include two kinds: delta-shock wave solutions and vacuum solutions. Under the generalized Rankine–Hugoniot relation and entropy condition, all of the existence, uniqueness, and stability of solutions to viscous perturbations are proved. Two typical examples are presented finally. © 1999 Academic Press

Key Words: coupled hyperbolic system; delta-shock waves; generalized Rankine–Hugoniot relation; entropy condition; measure solution.

1. INTRODUCTION

In this paper we consider the Riemann problem for the system

$$\begin{cases} v_t + (vf(u))_x = 0, \\ (vu)_t + (vuf(u))_x = 0, \end{cases} \tag{1.1}$$

with initial data

$$(v, u)(0, x) = (v_\pm, u_\pm) \qquad (\pm x > 0), \tag{1.2}$$

where $f(u)$ is given to be a smooth and strictly monotone function, and the sign of v is assumed to be unchanging.

The model (1.1) can be reduced from Zheng [24] in which a class of systems of conservation laws having repeated linearly degenerate

0022-0396/99 $30.00

eigenvalues was presented, while the existence and uniqueness of solutions of Riemann problems were not solved completely. Moreover, it is a coupled system which coincides with the one-dimensional zero-pressure gas dynamics when $f(u) \equiv u$, $v \geqslant 0$, and u are thought of as the density and velocity variables, respectively.

One main feature of system (1.1) is that v and $\partial u/\partial x$ blow up simultaneously in a finite time even starting from smooth initial data (see Section 2 below). Therefore, we have to understand (1.1) in the sense of measures and introduce delta-shock waves in the piecewise smooth solutions of (1.1). For delta-shock waves, we refer the reader to papers [12, 14, 8, 11, 20, 21, 19, 16, 24] for more details.

The aim of the present paper is to extend and improve the results and proofs in [19, 16]. With characteristic method, we constructively solve the Riemann problem (1.2) for system (1.1). There are only two kinds of solutions: the one involves delta-shock waves, the other involves vacuums. By definition of measure solutions to (1.1), we propose the generalized Rankine–Hugoniot relation for a delta-shock of (1.1), which describes the relationship among the location, propagation speed, weight, and assignment of u on its discontinuity relative to the delta-shock. This relation applied to the Riemann problem (1.1) and (1.2) can be reduced to a function equation, especially a quadratic equation of one variable for the zero-pressure gas dynamics. Thus, the existence and uniqueness of solutions involving delta-shocks are equivalent to studying the existence and uniqueness of solutions of the function equation, which are solved completely under the entropy condition. Furthermore, with the vanishing viscosity method, we check the generalized Rankine–Hugoniot relation above and prove that delta-shock waves are w^*-limits in L^1 of solutions to viscous perturbations (see (4.1) below). Here, we point out that assignment of u on its discontinuity plays two roles: the first is to overcome the problem of multiplying a delta-function in v with a discontinuous function u; the second is for needs of the concentration in v to travel at the speed of the delta-shock. There is an extensive literature on the ideas of choice of u; see [22, 14, 8, 20, 21, 19, 24]. Finally, we give two typical examples to present an application for the results of the paper. We mention also that the ideas of this paper could be applied to other systems of conservation laws with the behavior of delta-shocks in solutions [12, 14, 11, 20].

The plan of the paper is as follows. In Section 2, we present some preliminary knowledge about system (1.1) and construct the Riemann solutions by characteristic method. Section 3 contains the result of existence and uniqueness of solutions involving delta-shock waves. In Section 4, we consider the existence of solutions to viscous system (4.1). Section 5 discusses the limit of solutions to viscous system (4.1) as the viscosity vanishes. Section 6 presents two typical examples.

2. PRELIMINARIES AND SOLUTIONS OBTAINED WITH CHARACTERISTIC METHOD

2.1. *Preliminaries*

We consider the Riemann problem (1.2) for system (1.1). The system has a double eigenvalue and only one right eigenvector

$$\lambda = f(u), \qquad r = (1, 0)^T, \tag{2.1}$$

and

$$\nabla\lambda \cdot r \equiv 0. \tag{2.2}$$

Thus (1.1) is nonstrictly hyperbolic and λ is linearly degenerate. The system is also not diagonalizable in v. The characteristic equations can be written as

$$\begin{cases} \dfrac{dx}{dt} = f(u), \\ \dfrac{du}{dt} = 0, \\ \dfrac{dv}{dt} = -v\,\dfrac{\partial}{\partial x}\,(f(u)), \end{cases} \tag{2.3}$$

which means that the characteristic lines are straight, and u keeps constant along each of them.

As usual, we should seek the self-similar solution

$$(v, u)(t, x) = (v, u)(\xi), \qquad \xi = \frac{x}{t},$$

for which (1.1) becomes

$$\begin{cases} -\xi v_\xi + (vf(u))_\xi = 0, \\ -\xi(vu)_\xi + (vuf(u))_\xi = 0, \end{cases} \tag{2.4}$$

and the initial condition (1.2) changes to the boundary condition

$$(v, u)(\pm\infty) = (v_\pm, u_\pm). \tag{2.5}$$

Any smooth solution of (2.4) satisfies

$$AU_\xi = 0, \tag{2.6}$$

where

$$A=\begin{pmatrix} f(u)-\xi & vf'(u) \\ u(f(u)-\xi) & v(uf'(u)+f(u)-\xi) \end{pmatrix}, \qquad U=\begin{pmatrix} v \\ u \end{pmatrix}.$$

It provides either the general solution (constant state)

$$(v,u)(\xi)=\text{constant} \qquad (v\neq 0), \tag{2.7}$$

or the singular solution

$$\begin{cases} v=0, \\ f(u)=\xi, \end{cases} \tag{2.8}$$

called the vacuum state (see [19]), where $u(\xi)$ is an arbitrary smooth function. Thus the smooth solutions of system (1.1) only contain constants and vacuum solutions.

For a bounded discontinuity at $\xi=\sigma$, the Rankine–Hugoniot condition holds,

$$\begin{cases} -\sigma[v]+[vf(u)]=0, \\ -\sigma[vu]+[vuf(u)]=0, \end{cases} \tag{2.9}$$

where $[p]=p_l-p_r$, the jump of p across the discontinuity. Solving (2.9), we obtain that

$$\xi=\sigma=f(u_l)(=\lambda_l)=f(u_r)(=\lambda_r). \tag{2.10}$$

It is a contact discontinuity, denoted by J, which is just the characteristic line for both sides in (t,x)-plane. Two states (v_l,u_l) and (v_r,u_r) can be connected by a contact discontinuity J if and only if $f(u_l)=f(u_r)$, namely, $u_l=u_r$ because of the monotonicity of $f(u)$; that is, these two states are located on the line $u=u_l=u_r$ in the (v,u)-phase plane. The contact discontinuity J in (t,x)-plane is characterized by $x/t=f(u_l)=f(u_r)$ (see Fig. 1).

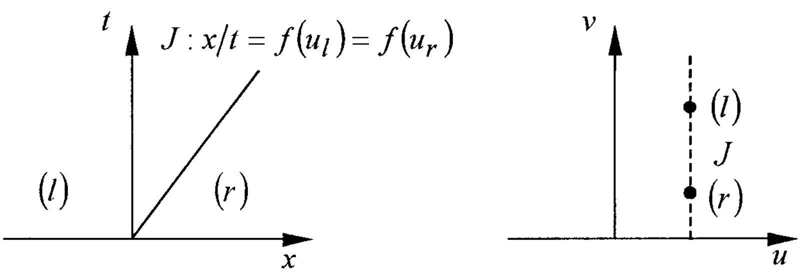

FIGURE 1

2.2. *Structure of Riemann Solutions*

With constants, vacuum, and contact discontinuity, one can construct solutions of the Riemann problem (1.1)–(1.2) in the following two cases:

$$f(u_-) < f(u_+), \qquad \text{and} \qquad f(u_-) > f(u_+).$$

It is easy to observe if

$$f(u_-) > f(u_+), \tag{2.11}$$

then the system (1.1) has shocks. Thus for convenience, we assume

$$f'(u) > 0, \qquad v \geqslant 0 \tag{2.12}$$

throughout this paper. The rest of the cases can be discussed in a similar way. Now we proceed to construct Riemann solutions of (1.1) by two cases.

In the case $u_- < u_+$, let us in the (v, u)-phase plane draw lines $u = u_-$ and $u = u_+$ from (v_-, u_-) and (v_+, u_+), respectively, these two lines intersect the u-axis at $(0, u_-)$ and $(0, u_+)$. Thus we obtain the solution which consists of two contact discontinuities and a vacuum state besides two constant states (see Fig. 2). The solution can be expressed as

$$(v, u)(\xi) = \begin{cases} (v_-, u_-), & -\infty < \xi < f(u_-), \\ (0, u(\xi)), & f(u_-) \leqslant \xi \leqslant f(u_+), \\ (v_+, u_+), & f(u_+) < \xi < +\infty, \end{cases} \tag{2.13}$$

where $u(\xi)$ satisfies that $u(f(u_-)) = u_-$ and $u(f(u_+)) = u_+$.

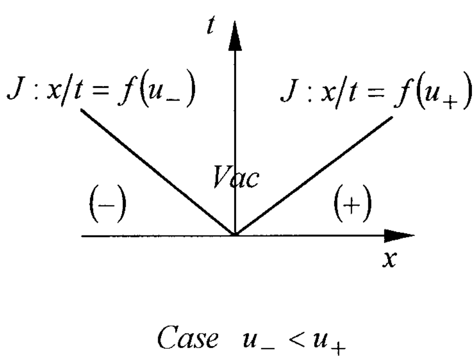

FIGURE 2

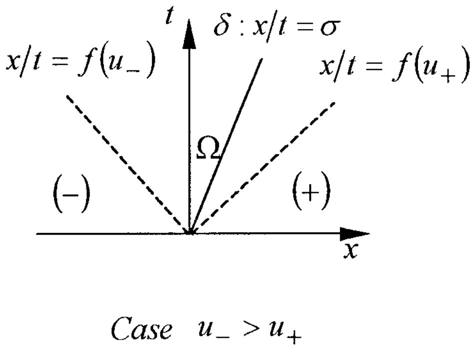

FIGURE 3

In the case $u_- > u_+$, the characteristics from initial data will overlap in the domain Ω as shown in Fig. 3. So, the singularity of solutions must develop in Ω. We assert that there is no classical weak solutions, that is, no solutions exist in BV (bounded variation) space. In fact, for smooth solution, (1.1) is equivalent to the system

$$\begin{cases} v_t + (vf(u))_x = 0, \\ u_t + f(u)\, u_x = 0. \end{cases} \tag{2.14}$$

We consider (1.1) and (2.14) with sufficiently smooth initial data $u(0, x) = u_0(x)$ satisfying $u_0'(x) < 0$. From (2.1), the characteristic passing through any given point $(0, a)$ on the x-axis is

$$x = a + f(u_0(a))\, t, \tag{2.15}$$

on which the solution u takes the constant value $u = u_0(a)$. Notice that

$$\frac{dv}{dt} = -vf'(u)\, \frac{\partial u}{\partial x}, \tag{2.16}$$

again differentiating the second equation of (2.14) with respect to x gives

$$\left(\frac{\partial}{\partial t} + f(u)\, \frac{\partial}{\partial x}\right) \frac{\partial u}{\partial x} = -f'(u) \left(\frac{\partial u}{\partial x}\right)^2,$$

i.e.,

$$\frac{d\left(f'(u)\, \dfrac{\partial u}{\partial x}\right)}{dt} = -\left(f'(u)\, \frac{\partial u}{\partial x}\right)^2, \tag{2.17}$$

which is a type of the Riccatic equation. Hence, along the characteristic (2.15), it follows from (2.16) and (2.17) that

$$v(t,x) = \frac{v(a)}{1+f'(u_0(a))\,u_0'(a)\,t}, \tag{2.18}$$

and

$$\frac{\partial u(t,x)}{\partial x} = \frac{u_0'(a)}{1+f'(u_0(a))\,u_0'(a)\,t}. \tag{2.19}$$

Then one obtains

$$\lim_{t\to -(f'(u_0(a))\,u_0'(a))^{-1}} \left(v, \frac{\partial u}{\partial x}\right) = (\infty, \infty) \tag{2.20}$$

along the characteristic (2.15) from below. This shows that the solution v and $\partial u/\partial x$ must blow up simultaneously at a finite time. Define

$$E = \{(t,x) \mid x = a + f(u_0(a))t,\ t = -(f'(u_0(a))\,u_0'(a))^{-1}\}. \tag{2.21}$$

Then the set E is the envelope of all characteristics, on which v and $\partial u/\partial x$ all become singular. Therefore, the smooth solution can be define by

$$(v,u)(t,x) = \left(\frac{v(a)}{1+f'(u_0(a))\,u_0'(a)\,t},\ u_0(a)\right) \tag{2.22}$$

only for $t < -(f'(u_0(a))\,u_0'(a))^{-1}$. The above discussion shows the mechanism of occurrence of delta-shock waves.

Thus, motivated by [20, 19], for the case $u_- > u_+$, using the delta-shock wave, we can construct the solution which consists of a delta-shock wave besides two constant states (see Fig. 3), where σ is the propagation speed of the delta-shock wave.

In the forthcoming several sections, we will study in more detail the existence and uniqueness of solutions involving delta-shock waves and the stability of the above Riemann solutions to viscous perturbations.

3. GENERALIZED RANKINE–HUGONIOT RELATIONS OF DELTA-SHOCKS

In view of the above section, we find that v and $\partial u/\partial x$ may develop simultaneously singular measures in a finite time even though the initial data $(v, u)(0, x)$ are smooth enough. Then it is natural and convenient to seek solutions in Borel measure space.

In the space of bounded Borel measures BM, the definition of a measure solution of (1.1) can be given as follows.

DEFINITION 3.1. A pair (v, u) is called a measure solution of (1.1) if (v, u) satisfies

$$v \in L^{\infty}([0, \infty), BM(R^1)) \cap C([0, \infty), H^{-s}(R^1)), \tag{3.1}$$

$$u \in L^{\infty}([0, \infty), L^{\infty}(R^1)) \cap C([0, \infty), H^{-s}(R^1)), \qquad s > 0 \tag{3.2}$$

$$u \text{ is measurable with respect to } v \text{ at almost all } t \geqslant 0, \tag{3.3}$$

and

$$\begin{cases} \int_0^{\infty} \int_{R^1} (\phi_t + f(u)\phi_x)\, dv\, dt = 0, \\ \int_0^{\infty} \int_{R^1} u(\phi_t + f(u)\, \phi_x)\, dv\, dt = 0, \end{cases} \tag{3.4}$$

hold in the sense of measures for all $\phi \in C_0^{\infty}([0, \infty) \times R^1)$.

Remark that the continuity conditions in $C([0, \infty), H^{-s}(R^1))$ may be used to give an interpretation for (v, u) to take on initial values(see [24]).

DEFINITION 3.2. A two-dimensional weighted delta function $w(s)\, \delta_L$ supported on a smooth curve L parametrized as $t = t(s)$, $x = x(s)(c \leqslant s \leqslant d)$ is defined by

$$\langle w(s) \delta_L, \phi \rangle = \int_c^d w(s)\, \phi(t(s), x(s))\, ds \tag{3.5}$$

for all $\phi \in C_0^{\infty}(R^2)$.

Now we propose to find a solution with discontinuity $x = x(t)$ of (1.1) of the form

$$(v, u)(t, x) = \begin{cases} (v_l, u_l)(t, x), & x < x(t), \\ (w(t)\, \delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (v_r, u_r)(t, x), & x > x(t), \end{cases} \tag{3.6}$$

where (v_l, u_l) and $(v_r, u_r)(t, x)$ are piecewise smooth solutions of (1.1), $x(t) \in C^1$, and $\delta(x)$ is the standard Dirac measure. If (3.6) satisfies the relation

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\ \dfrac{dw}{dt} = -[v]\sigma + [vf(u)], \\ \dfrac{dwu_\delta}{dt} = -[vu]\sigma + [vuf(u)], \end{cases} \tag{3.7}$$

and

$$\sigma = f(u_\delta), \tag{3.8}$$

then the solution $(v, u)(t, x)$ defined in (3.6) satisfies (1.1) in the sense of measures. The proof is easy and similar to that in [16]; we omit it.

The relation (3.7)–(3.8) is called the generalized Rankine–Hugoniot relation. It describes the relationship among the location, propagation speed, weight, and assignment of u on its discontinuity.

In addition to the generalized Rankine–Hugoniot relation (3.7)–(3.8), to guarantee uniqueness, the discontinuity satisfies

$$\lambda(u_r) < \sigma(u_r, u_l) < \lambda(u_l),$$

namely,

$$f(u_r) < \sigma = f(u_\delta) < f(u_l). \tag{3.9}$$

Condition (3.9) is called the entropy condition which means that all characteristics on both sides of the discontinuity are in-coming. It is equivalent, under assumption (2.12), to

$$u_r < u_\delta < u_l. \tag{3.10}$$

A discontinuity satisfying (3.7)–(3.8) and (3.9) will be called a delta-shock wave, symbolized by δ.

In what follows, the generalized Rankine–Hugoniot relation will be applied in particular to the Riemann problem (1.1)–(1.2) for the case

$u_- > u_+$. At this moment, the Riemann problem is reduced to solving (3.7)–(3.8) with the initial data

$$t=0: \qquad x(0)=0, \quad w(0)=0, \quad u_\delta(0)=0. \tag{3.11}$$

In view of knowledge concerning delta-shock waves in [21, 19, 16, 23], we find that σ and u_δ are constants and $w(t)$ is a linear function of t. So a delta-shock of (1.1)–(1.2) can be assumed to take the form

$$\delta: \qquad x(t)=\sigma t, \quad w(t)=w_0 t, \quad u_\delta(t)=u_\delta \tag{3.12}$$

satisfying (3.11), where σ, w_0, and u_δ are to be determined constants. Substituting (3.12) into (3.7)–(3.8) we obtain that

$$\begin{cases} w_0 = -[v]\sigma + [vf(u)], \\ w_0 u_\delta = -[vu]\sigma + [vuf(u)], \\ \sigma = f(u_\delta). \end{cases} \tag{3.13}$$

From (3.13) it follows that

$$([v]\, u_\delta - [vu])\, f(u_\delta) - [vf(u)]\, u_\delta + [vuf(u)] = 0. \tag{3.14}$$

Under the entropy condition (3.10) which is, in this situation,

$$u_+ < u_\delta < u_-, \tag{3.15}$$

we consider solutions of function equation (3.14). Set

$$G(u_\delta) = \text{the left side of (3.14)}. \tag{3.16}$$

One can calculate that

$$\begin{aligned} G(u_+) &= ([v]u_+ - [vu])\, f(u_+) - [vf(u)]u_+ + [vf(u)] \\ &= v_-(u_- - u_+)(f(u_-) - f(u_+)) \\ &= v_-[u][f(u)], \end{aligned} \tag{3.17}$$

and similarly,

$$G(u_-) = -v_+[u][f(u)]. \tag{3.18}$$

Then

$$G(u_+) \cdot G(u_-) = -v_- v_+ [u]^2 \, [f(u)]^2 < 0 \tag{3.19}$$

because of assumption (2.12). Furthermore, noting that differentiating $G(u_\delta)$ in (3.16) with respect to u_δ yields

$$\begin{aligned} G'(u_\delta) &= [v] \; f(u_\delta) + ([v] \, u_\delta - [vu]) \; f'(u_\delta) - [vf(u)] \\ &= v_- (f(u_\delta) - f(u_-)) + v_+ (f(u_+) - f(u_\delta)) \\ &\qquad + (v_- (u_\delta - u_-) + v_+ (u_+ - u_\delta)) \; f'(u_\delta) \\ &< 0 \end{aligned} \tag{3.20}$$

because of assumption (2.12). Therefore, by zero point theorem in mathematical analysis, there exists one and only one zero point of function $G(u_\delta)$ in (u_+, u_-). That is, Eq. (3.14) has a unique solution, denoted by u_δ, under the entropy condition (3.15). Returning to the relation (3.13), we solve the σ and w_0 uniquely. Thus we obtain the following theorem.

THEOREM 3.3. *Under assumption* (2.12), *let* $u_- > u_+$. *Then the Riemann problem* (1.1)–(1.2) *admits one and only one entropy measure solution of the form*

$$(v, u)(t, x) = \begin{cases} (v_-, u_-), & x < \sigma t, \\ (w(t) \, \delta(x - \sigma t), u_\delta), & x = \sigma t, \\ (v_+, u_+), & x > \sigma t, \end{cases} \tag{3.21}$$

where $w(t) = w_0 t$ *and all of the three constants* σ, w_0, *and* u_δ *are determined uniquely by* (3.13) *and* (3.15) (*see Fig.* 3).

Remark that Theorem 3.3 is true for the case $f'(u) > 0$, $v \leqslant 0$. Besides, by the same arguments as used in the above proof with only few modifications, one can obtain the results similar to Theorem 3.3 for the cases $f'(u) < 0$, $v \geqslant 0$ and $f'(u) < 0$, $v \leqslant 0$. Therefore, associating with the contents in Section 2, we can conclude the following theorem.

THEOREM 3.4. *Assume that* $f(u)$ *is a smooth and strictly monotone function, and assume that the sign of* v *is unchanging. Then there exists a unique measure solution of the Riemann problem* (1.1)–(1.2) *which contains a vacuum state in the case* $f(u_-) < f(u_+)$ *and a delta-shock wave in the case* $f(u_-) > f(u_+)$.

4. EXISTENCE OF SOLUTIONS TO THE VISCOUS SYSTEM (4.1)

In this section we consider the existence of solutions for the viscous system

$$\begin{cases} v_t + (vf(u))_x = 0, \\ (vu)_t + (vuf(u))_x = \varepsilon t u_{xx}, \end{cases} \tag{4.1}$$

with initial data

$$(v, u)(0, x) = (v_\pm, u_\pm), \qquad (\pm x > 0). \tag{4.2}$$

Performing the self-similar transformation $\xi = x/t$, we get

$$\begin{cases} -\xi v_\xi + (vf(u))_\xi = 0, \\ -\xi (vu)_\xi + (vuf(u))_\xi = \varepsilon u_{\xi\xi}, \end{cases} \tag{4.3}$$

and

$$(v, u)(\pm\infty) = (v_\pm, u_\pm). \tag{4.4}$$

This is a two-point boundary value problem of high-order ordinary differential equations with the boundary value in the infinity, for which we have the following main result of existence.

THEOREM 4.1. *There exists a weak solution*

$$(v, u) \in L^1(-\infty, +\infty) \times C^2(-\infty, +\infty)$$

for the boundary value problem (4.3)–(4.4).

To prove this theorem, we first consider the existence of solutions of system (4.3) in the interval $[-R, R]$, where R is a sufficiently large real number, with boundary condition

$$(v, u)(\pm R) = (v_\pm, u_\pm). \tag{4.5}$$

The main idea is to use Shauder fixed point theorem. For this purpose, we take

$$B = C^2[-R, R], \qquad K = \{U \mid U \in B,\ U(\pm R) = u_\pm,\ U \text{ is monotone}\}.$$

Obviously, K is a bounded convex closed set in B, a Banach space.

LEMMA 4.2. *For any $U \in K$, the problem*

$$\begin{gathered} -\xi v_\xi + (vf(U))_\xi = 0, \\ v(\pm R) = v_\pm, \end{gathered} \tag{4.6}$$

possesses a weak solution $v \in L^1[-R, R]$.

(i) *When $u_- > u_+$,*

$$v(\xi) = \begin{cases} v_1(\xi), & -R \leqslant \xi < \xi_\sigma, \\ v_2(\xi), & \xi_\sigma < \xi \leqslant R, \end{cases} \tag{4.7}$$

where ξ_σ is a unique solution of equation

$$f(U(\xi_\sigma)) = \xi_\sigma, \tag{4.8}$$

$v_1(\xi)$ is increasing in $(-R, \xi_\sigma)$, while $v_2(\xi)$ is decreasing in (ξ_σ, R).

(ii) *When $u_- < u_+$,*

$$v(\xi) = \begin{cases} v_1(\xi), & -R \leqslant \xi < \xi_{\sigma_1}, \\ 0, & \xi_{\sigma_1} \leqslant \xi \leqslant \xi_{\sigma_2}, \\ v_2(\xi), & \xi_{\sigma_2} < \xi \leqslant R, \end{cases} \tag{4.9}$$

where $\xi_{\sigma_1} \leqslant \xi_{\sigma_2}$ satisfying

$$\xi_{\sigma_1} = \min\{\xi \mid f(U(\xi)) = \xi\}, \qquad \xi_{\sigma_2} = \max\{\xi \mid f(U(\xi)) = \xi\} \tag{4.10}$$

and

$$\lim_{\xi \to \xi_{\sigma_1}^-} v_1(\xi) = \lim_{\xi \to \xi_{\sigma_2}^+} v_2(\xi) = 0, \tag{4.11}$$

$v_1(\xi)$ is decreasing in $(-R, \xi_{\sigma_1})$, while $v_2(\xi)$ is increasing in (ξ_{σ_2}, R).

Specifically, $v_1(\xi)$ and $v_2(\xi)$ in (4.7) and (4.9) can be formulated as

$$v_1(\xi) = v_- \exp\left(\int_{-R}^{\xi} \frac{-(f(U(s)))'}{f(U(s)) - s}\, ds\right), \tag{4.12}$$

and

$$v_2(\xi) = v_+ \exp\left(\int_{\xi}^{R} \frac{(f(U(s)))'}{f(U(s)) - s}\, ds\right), \tag{4.13}$$

where $' = d/d\xi$.

Proof. The problem (4.6) is a two-point boundary value one for the first order degenerate ordinary equation. The singularity point of (4.6) is given by the solution of Eq. (4.8).

(i) When $u_- > u_+$, $U(\xi)$ is decreasing. From (2.12) we know that the singularity point is unique, denoted by ξ_σ. Equation (4.6) can be rewritten as

$$(f(U(\xi)) - \xi)\, v_\xi + v(f(U(\xi)))_\xi = 0. \tag{4.14}$$

Integrating (4.14) from $v(-R) = v_-$ or $v(+R) = v_+$, respectively, one can find the solution (4.7) with formulas (4.12) and (4.13), which immediately show the monotonicity of $v_1(\xi)$ and $v_2(\xi)$. Furthermore, one easily obtains

$$\lim_{\xi \to \xi_\sigma^-} v_1(\xi) = +\infty, \qquad \lim_{\xi \to \xi_\sigma^+} v_2(\xi) = +\infty.$$

We proceed to prove that $v(\xi)$ is a weak solution of (4.6) and $v \in L^1[-R, R]$. Integrating (4.6) on $[-R, \xi]$ for $-R < \xi < \xi_\sigma$, we get

$$(f(U(\xi)) - \xi)\, v_1(\xi) - (f(u_-) + R) v_- + \int_{-R}^{\xi} v_1(r)\, dr = 0. \tag{4.15}$$

Set

$$p(\xi) = \int_{-R}^{\xi} v_1(r)\, dr, \qquad A_1 = (f(u_-) + R) v_-, \; a(\xi) = f(U(\xi)) - \xi.$$

Then Eq. (4.15) can be written as

$$\begin{cases} a(\xi)\, p'(\xi) + p(\xi) = A_1, \\ p(-R) = 0. \end{cases} \tag{4.16}$$

It follows that

$$p(\xi) = A_1 \left\{ 1 - \exp\left(-\int_{-R}^{\xi} \frac{dr}{a(r)} \right) \right\}.$$

Noting that $a(\xi) > 0$ and $a(\xi) = O(|\xi - \xi_\sigma|)$ as $\xi \to \xi_\sigma{}^-$, we obtain

$$\lim_{\xi \to \xi_\sigma^-} \int_{-R}^{\xi} v_1(r)\, dr = A_1. \tag{4.17}$$

Hence

$$\lim_{\xi \to \xi_\sigma^-} (f(U(\xi)) - \xi)\, v_1(\xi) = 0. \tag{4.18}$$

Similarly, one can get

$$\lim_{\xi \to \xi_\sigma^+} \int_R^\xi v_2(r)\, dr = A_2, \tag{4.19}$$

$$\lim_{\xi \to \xi_\sigma^+} (f(U(\xi)) - \xi)\, v_2(\xi) = 0, \tag{4.20}$$

where $A_2 = (f(u_+) - R)\, v_+$. The equalities (4.17) and (4.19) imply that $v(\xi) \in L^1[-R, R]$.

For arbitrary $\phi \in C_0^\infty[-R, R]$, we verify that

$$-\int_{-R}^{R} (f(U) - \xi)\, v\phi'\, d\xi + \int_{-R}^{R} v\phi\, d\xi = 0. \tag{4.21}$$

Indeed, for any ξ_1, ξ_2, such that $-R < \xi_1 < \xi_\sigma < \xi_2 < R$,

$$\begin{aligned} I &= -\int_{-R}^{R} (f(U) - \xi)\, v\phi'\, d\xi + \int_{-R}^{R} v\phi\, d\xi \\ &= \left(\int_{-R}^{\xi_1} + \int_{-\xi_1}^{\xi_2} + \int_{\xi_2}^{R} \right) (-(f(U) - \xi)\, v\phi' + v\phi)\, d\xi \\ &= I_1 + I_2 + I_3 . \end{aligned}$$

From (4.18) and (4.20),

$$\begin{aligned} |I_1| &= \left| -(f(U(\xi_1)) - \xi_1)\, v_1(\xi_1)\, \phi(\xi_1) + \int_{-R}^{\xi_1} (((f(U) - \xi) v)'\, \phi + v\phi)\, d\xi \right| \\ &= |(f(U(\xi_1)) - \xi_1)\, v_1(\xi_1)\, \phi(\xi_1)| \to 0, \qquad \text{as} \quad \xi_1 \to \xi_\sigma^-, \\ |I_3| &= |(f(U(\xi_2)) - \xi_2)\, v_2(\xi_2)\, \phi(\xi_2)| \to 0, \qquad \text{as} \quad \xi_2 \to \xi_\sigma^+, \end{aligned}$$

and since $v \in L^1[-R, R]$,

$$|I_2| \leqslant \int_{\xi_1}^{\xi_2} |-(f(U) - \xi)\phi' + \phi|\, |v|\, d\xi \to 0, \qquad \text{as} \quad \xi_1 \to \xi_\sigma^-, \quad \xi_2 \to \xi_\sigma^+.$$

Notice that I is independent of ξ_1 and ξ_2, so $I = 0$, i.e., (4.21) holds. Therefore, $v(\xi)$ defined in (4.7) is a weak solution of (4.6).

(ii) When $u_- < u_+$, $U(\xi)$ is increasing, so we can get $\xi_{\sigma_1} \leqslant \xi_{\sigma_2}$. Thus one can find the solution $v(\xi)$ of (4.6) to be (4.9) with (4.12) and (4.13). Since

$$\begin{aligned}
\int_{-R}^{\xi} \frac{(f(U(s)))'}{f(U(s))-s}\, ds &= (f(U(\zeta)))' \int_{-R}^{\xi} \frac{ds}{f(U(s))-s} \\
&\geqslant (f(U(\zeta)))' \int_{-R}^{\xi} \frac{ds}{f(U(\xi))-s} \\
&= -(f(U(\zeta)))' \ln \left(\frac{f(U(\xi))-\xi}{f(U(\xi))+R} \right) \\
&\to +\infty, \qquad \text{as} \quad \xi \to \xi_{\sigma_1}^-,
\end{aligned}$$

where $-R < \xi < \xi_{\sigma_1}$, $-R \leqslant \zeta \leqslant \xi$, we get the first half of (4.11). The second half can be obtained in the similar way. Moreover, themonotonicity of $v_1(\xi)$ and $v_2(\xi)$ is obvious. The proof of Lemma 4.2 is completed. ∎

Define an operator $T\colon K \to B$ as follows: for any $U \in K$, $u = TU$ is the unique solution of the boundary value problem

$$\begin{cases} \varepsilon u'' = v(U, \xi)(f(U) - \xi)\, u', & (4.22)\\ u(\pm R) = u_\pm, & (4.23) \end{cases}$$

where $v(U, \xi)$ is defined in (4.7) or (4.9). Indeed, the solution of this problem can be found to be

$$u(\xi) = \frac{(u_+ - u_-) \int_{-R}^{\xi} \exp(\int_{-R}^{r} (v(f(U)-s)/\varepsilon)\, ds)\, dr}{\int_{-R}^{R} \exp(\int_{-R}^{r} (v(f(U)-s)/\varepsilon)\, ds)\, dr} + u_-. \quad (4.24)$$

LEMMA 4.3. *$T\colon K \to K$ is a continuous operator in B.*

Proof. Take $U_n \to U$ (in B) $(n \to \infty)$, U_n, $U \in K$. Then

$$u_n = TU_n, \qquad u = TU$$

satisfy (4.22)–(4.23), and we have

$$\begin{cases} \varepsilon(u_n - u)'' = v_n(f(U_n) - \xi)(u_n - u)' + (v_n(f(U_n) - \xi) - v(f(U) - \xi))\, u', & (4.25)\\ (u_n - u)(\pm R) = 0. & (4.26) \end{cases}$$

It can be solved to give

$$(u_n-u)'(\xi)=-\frac{\int_{-R}^{R}\int_{-R}^{t} q_n(r)\exp(\int_r^t p_n(s)\,ds)\,dr\,dt}{\int_{-R}^{R}\exp(\int_{-R}^{r} p_n(s)\,ds)\,dr}$$

$$\times\exp\left(\int_{-R}^{\xi} p_n(s)\,ds\right)$$

$$+\int_{-R}^{\xi} q_n(r)\exp\left(\int_r^{\xi} p_n(s)\,ds\right)dr, \tag{4.27}$$

$$(u_n-u)(\xi)=-\frac{\int_{-R}^{R}\int_{-R}^{t} q_n(r)\exp(\int_r^t p_n(s)\,ds)\,dr\,dt}{\int_{-R}^{R}\exp(\int_{-R}^{r} p_n(s)\,ds)\,dr}$$

$$\times\int_{-R}^{\xi}\exp\left(\int_{-R}^{r} p_n(s)\,ds\right)dr$$

$$+\int_{-R}^{\xi}\int_{-R}^{t} q_n(r)\exp\left(\int_r^t p_n(s)\,ds\right)dr\,dt, \tag{4.28}$$

where $p_n=v_n(f(U_n)-\xi)$, $q_n=(v_n(f(U_n)-\xi)-v(f(U)-\xi))u'$. From Eq. (4.6), we have

$$(v(f(U)-\xi))'=-v<0, \qquad (v_n(f(U_n)-\xi))'=-v_n<0 \quad (n=1,2,...).$$

Then, $v(f(U)-\xi)$ and $v_n(f(U_n)-\xi)$ $(n=1,2,...)$ are monotone decreasing and continuous functions. Because the sequence of monotone functions (continuous or discontinuous) which converges to a continuous function must converge uniformly, we get that $q_n(\xi)$ converges to zero uniformly. Thus it follows from (4.25) and (4.27)–(4.28) that

$$u_n\to u \ (\text{in } B), \qquad \text{as} \quad n\to\infty.$$

Therefore T: $K\to B$ is continuous in B.

Again, from Eq. (4.24), we have

$$u'(\xi)=\frac{(u_+-u_-)\exp(\int_{-R}^{\xi}(v(f(U)-s)/\varepsilon)\,ds)}{\int_{-R}^{R}\exp(\int_{-R}^{r}(v(f(U)-s)/\varepsilon)\,ds)\,dr}, \tag{4.29}$$

which implies that $u=TU$ is monotone. So we get $TK\subset K$. The proof of this lemma is completed. ∎

LEMMA 4.4. *TK is precompact in B.*

Proof. According to the continuity of T and the Ascoli–Arzela theorem [1], it remains to show the boundedness of TK in B.

When $u_- > u_+$, for any $U \in K$, we have

$$u'(\xi) = u'(-R) \exp\left(\int_{-R}^{\xi} \frac{v(f(U)-s)}{\varepsilon}\, ds\right).$$

By Lemma 4.2 and when $s < \xi_\sigma$,

$$0 < v(f(U)-s) = v_-(f(u_-)+R) - \int_{-R}^{s} v(r)\, dr < v_-(f(u_-)+R).$$

When $s > \xi_\sigma$,

$$0 > v(f(U)-s) = v_+(f(u_+)-R) + \int_{s}^{R} v(r)\, dr > v_+(f(u_+)-R).$$

Thus it suffices to consider the uniform boundedness of $u'(-R)$. From Eq. (4.22), one can deduce that

$$u''(\xi) < 0, \qquad \xi \in [-R, \xi_\sigma).$$

Then

$$u'(\xi) < u'(-R) < 0, \qquad \xi \in [-R, \xi_\sigma),$$

and

$$\begin{aligned} u_- - u_+ &> u(-R) - u(\xi_\sigma) = u'(\zeta)(-R - \xi_\sigma) \\ &> u'(\zeta)(-R - f(u_+)), \qquad \zeta \in (-R, \xi_\sigma). \end{aligned}$$

So

$$0 > u'(-R) > u'(\zeta) > -\frac{u_- - u_+}{R + f(u_+)}.$$

These above imply that $u'(\xi)$ is uniformly bounded for K.

When $u_- < u_+$, from (4.22) and (4.24) one can obtain

$$0 < u'(\xi) < u'(\xi_{\sigma_1}) \leqslant 1, \qquad -R \leqslant \xi < \xi_{\sigma_1},$$

and

$$0 < u'(\xi) < u'(\xi_{\sigma_2}) \leqslant 1, \qquad \xi_{\sigma_2} < \xi \leqslant R.$$

Therefore, $u(\xi)$, $u'(\xi)$, and $u''(\xi)$ are all uniformly bounded for K, that is, TK is a bounded set in B. ∎

From the above lemmas, by virtue of Shauder fixed point theorem, we get the following result.

THEOREM 4.5. *There exists a weak solution*

$$(v, u) \in L^1[-R, R] \times C^2[-R, R]$$

for the system (4.3) *with boundary value* (4.5).

The next step is to extend the solution of (4.3) in $[-R, R]$ to the whole interval $(-\infty, +\infty)$. To this end, we need the following lemma.

LEMMA 4.6. *The solution* $(v, u)(\xi)$ *of the system* (4.3) *with boundary value* (4.5) *satisfies*

(i) $u(\xi)$ *and* $u'(\xi)$ *have uniform bounds independent of* R;

(ii) $|u''(\xi)| \leqslant C(\varepsilon)$, $\xi \in [-R, R]$, *where* $C(\varepsilon)$ *is a constant only dependent of* ε;

(iii) $v_R(u, \xi)$ *converges as* $R \to +\infty$.

Proof. We only consider the case $u_- > u_+$; for the case $u_- < u_+$, one can prove it similarly. At this moment, we have

$$f(u_+) < \xi_\sigma < f(u_-).$$

(i) Take $-R < \xi_1 < f(u_+)$. From the second equation of (4.3) it follows that

$$u'(\xi) = u'(\xi_1) \exp\left(\int_{\xi_1}^{\xi} \frac{v(f(u) - s)}{\varepsilon}\, ds\right).$$

Noting that

$$u''(\xi) < 0, \qquad \xi \in (-R, \xi_\sigma),$$

we have

$$0 > u'(\xi_1) > u'(\xi), \qquad \xi \in (\xi_1, \xi_\sigma).$$

Since

$$u_- - u_+ > u(\xi_1) - u(\xi_\sigma) = u'(\zeta)(\xi_1 - \xi_\sigma) > u'(\zeta)(\xi_1 - f(u_+)),$$

where $\zeta \in (\xi_1, \xi_\sigma)$, we get

$$u'(\zeta) > \frac{u_- - u_+}{\xi_1 - f(u_+)}, \qquad \zeta \in (\xi_1, \xi_\sigma).$$

It follows that

$$0 > u'(\xi_1) > \frac{u_- - u_+}{\xi_1 - f(u_+)}.$$

In addition,

$$\begin{aligned} v(\xi_1) &= v_- \exp\left(\int_{-R}^{\xi_1} \frac{-(f(u(s)))'}{f(u(s)) - s}\, ds\right) \\ &= v_- \exp\left(\int_{-R}^{\xi_1} \frac{-(f(u(s)) - s)' - 1}{f(u(s)) - s}\, ds\right) \\ &= v_- \frac{f(u_-) + R}{f(u(\xi_1)) - \xi_1} \exp\left(\int_{-R}^{\xi_1} \frac{-ds}{f(u(s)) - s}\right) \\ &\leqslant v_- \frac{f(u_-) + R}{f(u(\xi_1)) - \xi_1} \exp\left(\int_{-R}^{\xi_1} \frac{-ds}{f(u_-) - s}\right) \\ &= \frac{v_-(f(u_-) - \xi_1)}{f(u(\xi_1)) - \xi_1}. \end{aligned}$$

When $\xi < \xi_1$,

$$\exp\left(\int_{\xi_1}^{\xi} \frac{v(f(u) - s)}{\varepsilon}\, ds\right) < 1.$$

When $\xi_1 < \xi < \xi_\sigma$, note that

$$\begin{aligned} v(f(u) - \xi) &= v(\xi_1)(f(u(\xi_1)) - \xi_1) - \int_{\xi_1}^{\xi} v\, dr \\ &\leqslant v(\xi_1)(f(u(\xi_1)) - \xi_1) \leqslant v_-(f(u_-) - \xi_1), \end{aligned}$$

and we obtain

$$\exp\left(\int_{\xi_1}^{\xi} \frac{v(f(u) - s)}{\varepsilon}\, ds\right) \leqslant \exp \frac{v_-(f(u_-) - \xi_1)^2}{\varepsilon}.$$

When $\xi > \xi_\sigma$,

$$\int_{\xi_1}^{\xi} \frac{v(f(u)-s)}{\varepsilon}\,ds = \int_{\xi_1}^{\xi_\sigma} \frac{v(f(u)-s)}{\varepsilon}\,ds + \int_{\xi_\sigma}^{\xi} \frac{v(f(u)-s)}{\varepsilon}\,ds$$
$$< \int_{\xi_1}^{\xi_\sigma} \frac{v(f(u)-s)}{\varepsilon}\,ds.$$

Therefore, $u'(\xi)$ and $u(\xi)$ are uniformly bounded independent of R.

(ii) From Eq. (4.22) and (i), we can easily get this result.

(iii) Completely similar to the estimate on $v(\xi_1)$ in (i), we can obtain that

$$v_1(\xi) \leqslant \frac{v_-(f(u_-)-\xi)}{f(u(\xi))-\xi}, \qquad \xi \in (-R, \xi_\sigma),$$

and

$$v_2(\xi) \leqslant \frac{v_+(f(u_+)-\xi)}{f(u(\xi))-\xi}, \qquad \xi \in (\xi_\sigma, R),$$

which shows that $v_R(u, \xi)$ converges as $R \to +\infty$. ∎

From the above discussions, for any $L > 0$, $\{u_R(\xi)\}$ is a compact set in $C^1[-L, L]$ if $R > L$. Hence there exists a subsequence $\{u_{R_i}(\xi)\}$ such that

$$\lim_{R_i \to +\infty} u_{R_i}(\xi) = u(\xi), \qquad \lim_{R_i \to +\infty} u'_{R_i}(\xi) = u'(\xi), \qquad \xi \in [-L, L].$$

By the Helly selecting principle, we get a subsequence, also denoted by $\{u_{R_i}(\xi)\}$ such that

$$\lim_{R_i \to +\infty} u_{R_i}(\xi) = u(\xi), \qquad \lim_{R_i \to +\infty} u'_{R_i}(\xi) = u'(\xi), \qquad \xi \in (-\infty, +\infty).$$

THEOREM 4.7. *Let $\varepsilon \leqslant \varepsilon_0$. Then $u(\xi)$ satisfies*

$$\begin{cases} \varepsilon u'' = v(u, \xi)(f(u)-\xi)u', \\ u(\pm\infty) = u_\pm, \end{cases} \tag{4.30}$$

and

$$v(\xi) = \begin{cases} v_1(\xi), & -\infty < \xi < \xi_\sigma, \\ v_2(\xi), & \xi_\sigma < \xi < +\infty, \end{cases} \tag{4.31}$$

when $u_- > u_+$, *while*

$$v(\xi)=\begin{cases} v_1(\xi), & -\infty<\xi<\xi_{\sigma_1}, \\ 0, & \xi_{\sigma_1}\leqslant \xi \leqslant \xi_{\sigma_2}, \\ v_2(\xi), & \xi_{\sigma_2}<\xi<+\infty, \end{cases} \tag{4.32}$$

when $u_- < u_+$, *where*

$$\begin{cases} v_1(\xi)=v_-\exp\left(\displaystyle\int_{-\infty}^{\xi}\frac{-(f(u(s)))'}{f(u(s))-s}\,ds\right), \\ v_2(\xi)=v_+\exp\left(\displaystyle\int_{\xi}^{+\infty}\frac{(f(u(s)))'}{f(u(s))-s}\,ds\right), \end{cases} \tag{4.33}$$

and ξ_σ, ξ_{σ_1}, *and* ξ_{σ_2} *satisfying*

$$\xi_{\sigma_1}=\min\{\xi_\sigma \mid f(u(\xi_\sigma))=\xi_\sigma\},\ \xi_{\sigma_2}=\max\{\xi_\sigma \mid f(u(\xi_\sigma))=\xi_\sigma\}.$$

Proof. Denote $(v_R, u_R)(\xi)$ by the solution of (4.3) and (4.5). When $u_- > u_+$, integrating (4.22) from ξ_2 to ξ, ξ_2 is a fixed point. Then from above lemmas we always have

$$\varepsilon(u_R'(\xi)-u_R'(\xi_2))=v_R(\xi)(f(u_R(\xi))-\xi)\,u_R(\xi) \\ -v_R(\xi_2)(f(u_R(\xi_2))-\xi_2)\,u_R(\xi_2)+\int_{\xi_2}^{\xi} v_R u_R\,dr,$$

whenever ξ_σ is between ξ_2 and ξ or not. Letting $R\to+\infty$, by the Lebesgue Convergence Theorem it follows that

$$\varepsilon(u'(\xi)-u'(\xi_2))=v(\xi)(f(u(\xi))-\xi)\,u(\xi)-v(\xi_2)(f(u(\xi_2))-\xi_2)\,u(\xi_2) \\ +\int_{\xi_2}^{\xi} vu\,dr. \tag{4.34}$$

When $u_- < u_+$, we can obtain (4.34) easily. Because the right side of (4.34) is continuous, we get

$$u'\in C^1(-\infty,+\infty).$$

Differentiating (4.34) with respect to ξ yields

$$\varepsilon u''=v(f(u)-\xi)\,u',$$

and from (4.24) we have

$$u(-\infty)=u_-,\qquad u(+\infty)=u_+.$$

The formulas of $v(\xi)$ in (4.31)–(4.33) can be obtained from above lemmas. This completes the proof. ∎

The theorem virtually finishes the proof of Theorem 4.1.

5. THE LIMIT SOLUTIONS OF (4.1)–(4.2) AS VISCOSITY VANISHES

This section concerns the behavior of solutions of (4.3)–(4.4) as $\varepsilon \to 0^+$ and establishes the stability of solutions, including delta-shocks and vacuums.

Case 1. $u_- > u_+$.

LEMMA 5.1. *Let ξ_σ^ε be the unique point satisfying*

$$\xi_\sigma^\varepsilon = f(u^\varepsilon(\xi_\sigma^\varepsilon)), \qquad \xi_\sigma = \lim_{\varepsilon \to 0^+} \xi_\sigma^\varepsilon$$

(*pass to a subsequence if necessary*). *Then for any $\eta > 0$,*

$$\lim_{\varepsilon \to 0^+} u_\xi^\varepsilon(\xi) = 0, \qquad \textit{for} \quad |\xi - \xi_\sigma| \geqslant \eta,$$

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(\xi) = \begin{cases} u_-, & \textit{for} \quad \xi \leqslant \xi_\sigma - \eta, \\ u_+, & \textit{for} \quad \xi \geqslant \xi_\sigma + \eta \end{cases}$$

uniformly in the above intervals.

Here and after, denote u^ε, v^ε as u, v when there is no confusion.

Proof. Take $\xi_3 = \xi_\sigma + \eta/2$, and let ε be so small that $\xi_\sigma < \xi_3 - \eta/4$. Integrating the second equation of (4.3) twice on $[\xi_3, \xi]$, we get

$$u(\xi_3) - u(\xi) = -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{r} \frac{v(f(u) - s)}{\varepsilon}\, ds \right) dr.$$

When $\xi > \xi_\sigma$,

$$\begin{aligned} v(\xi) &= v_+ \exp\left(\int_\xi^{+\infty} \frac{(f(u(s)))'}{f(u(s)) - s}\, ds \right) \\ &= \lim_{R \to +\infty} v_+ \exp\left(\int_\xi^{R} \frac{(f(u(s) - s)' + 1}{f(u(s)) - s}\, ds \right) \end{aligned}$$

$$\leqslant \lim_{R\to+\infty} v_+ \frac{f(u_+)-R}{f(u(\xi))-\xi} \exp\left(\int_\xi^R \frac{ds}{f(u_+)-s}\right)$$
$$= \lim_{R\to+\infty} v_+ \frac{f(u_+)-R}{f(u(\xi))-\xi} \frac{f(u_+)-\xi}{f(u_+)-R}$$
$$= \frac{v_+(f(u_+)-\xi)}{f(u(\xi))-\xi},$$

we have

$$v(f(u)-\xi) \geqslant v_+(f(u_+)-\xi), \qquad \xi \in (\xi_\sigma, +\infty). \tag{5.1}$$

Then

$$u(\xi_3)-u(\xi) \geqslant -u'(\xi_3)\int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{r} \frac{v_+(f(u_+)-s)}{\varepsilon}\, ds\right) dr$$
$$= -u'(\xi_3)\int_{\xi_3}^{\xi} \exp\left(\frac{v_+}{\varepsilon}\left((f(u_+)-\xi_3)(r-\xi_3)-\frac{1}{2}(r-\xi_3)^2\right)\right) dr$$
$$= -u'(\xi_3)\int_{0}^{\xi-\xi_3} \exp \frac{v_+}{\varepsilon}\left((f(u_+)-\xi_3)r-\frac{1}{2}r^2\right) dr.$$

Letting $\xi \to +\infty$, it follows that

$$u_- - u_+ \geqslant -u'(\xi_3)\int_0^{+\infty} \exp \frac{v_+}{2\varepsilon}(2(f(u_+)-\xi_3)r-r^2)\, dr$$
$$\geqslant -u'(\xi_3)\sqrt{\varepsilon}A_3,$$

where A_3 is a constant independent of ε. Thus

$$|u'(\xi_3)| \leqslant \frac{u_- - u_+}{\sqrt{\varepsilon}A_3}.$$

So

$$|u'(\xi)| \leqslant \frac{u_- - u_+}{\sqrt{\varepsilon}A_3} \exp\left(\int_{\xi_3}^{\xi} \frac{v(f(u)-s)}{\varepsilon}\, ds\right).$$

Again note that when $\xi > \xi_3$,

$$v(\xi) = v_+ \exp\left(\int_\xi^{+\infty} \frac{(f(u(s)))'}{f(u(s))-s}\, ds\right)$$
$$= \lim_{R\to+\infty} v_+ \exp\left(\int_\xi^{R} \frac{(f(u(s)-s)'+1}{f(u(s))-s}\, ds\right)$$

$$\begin{aligned}
&\geqslant \lim_{R\to+\infty} v_+ \frac{f(u_+)-R}{f(u(\xi))-\xi} \exp\left(\int_\xi^R \frac{ds}{f(u(\xi_3))-s}\right)\\
&= \lim_{R\to+\infty} v_+ \frac{f(u_+)-R}{f(u(\xi))-\xi}\, \frac{f(u(\xi_3))-\xi}{f(u(\xi_3))-R}\\
&= \frac{v_+(f(u(\xi_3))-\xi)}{f(u(\xi))-\xi},
\end{aligned}$$

we have

$$v(f(u)-\xi) \leqslant v_+(f(u(\xi_3))-\xi), \qquad \text{for} \quad \xi > \xi_3. \tag{5.2}$$

Then

$$|u'(\xi)| \leqslant \frac{u_- - u_+}{\sqrt{\varepsilon A_3}} \exp\left(-\frac{v_+}{\varepsilon}\int_{\xi_3}^{\xi} (s - f(u(\xi_3)))\, ds\right),$$

which implies that

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi) = 0, \qquad \text{uniformly for} \quad \xi \geqslant \xi_\sigma + \eta.$$

Next, we pick ξ_4 such that $\xi > \xi_4 \geqslant \xi_\sigma + \eta$. From

$$u(\xi_4) - u(\xi) = -u'(\xi_4) \int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{r} \frac{v(f(u)-s)}{\varepsilon}\, ds\right) dr,$$

we get

$$\begin{aligned}
|u(\xi_4)-u(\xi)| &\leqslant |u'(\xi_4)| \int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{r} \frac{-A_4}{2\varepsilon}\, ds\right) dr\\
&\leqslant \frac{2\varepsilon}{A_4}\, |u'(\xi_4)| \left\{1 - \exp\left(\frac{A_4}{2\varepsilon}(\xi_4-\xi)\right)\right\},
\end{aligned}$$

where $A_4 = 2v_+(\xi_4 - f(u(\xi_4)))$. Letting $\xi \to +\infty$, we arrive at

$$|u(\xi_4) - u_+| \leqslant \frac{2\varepsilon}{A_4}\, |u'(\xi_4)|,$$

which implies that

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_+, \qquad \text{uniformly for} \quad \xi \geqslant \xi_\sigma + \eta.$$

The results for $\xi \leqslant \xi_\sigma$ can be obtained analogously. ∎

LEMMA 5.2. *For any* $\eta > 0$,

$$\lim_{\varepsilon \to 0^+} v^\varepsilon(\xi) = \begin{cases} v_-, & \text{for} \quad \xi < \xi_\sigma - \eta, \\ v_+, & \text{for} \quad \xi > \xi_\sigma + \eta \end{cases}$$

uniformly.

Proof. From (5.1) and (5.2) it follows that for any $\xi > \xi_5 > \xi_\sigma + \eta$,

$$\frac{v_+(f(u(\xi_5)) - \xi)}{f(u(\xi)) - \xi} \leqslant v(\xi) \leqslant \frac{v_+(f(u_+) - \xi)}{f(u(\xi)) - \xi},$$

which yields

$$\lim_{\varepsilon \to 0^+} v^\varepsilon(\xi) = v_+, \qquad \text{uniformly for} \quad \xi > \xi_\sigma + \eta.$$

The rest part can be obtained similarly. This completes the proof. ∎

In what follows, we study in more detail the limiting behavior of v^ε in the neighborhood of $\xi = \xi_\sigma$ as $\varepsilon \to 0^+$. Denote

$$\sigma = \xi_\sigma = \lim_{\varepsilon \to 0^+} \xi_\sigma^\varepsilon = \lim_{\varepsilon \to 0^+} f(u^\varepsilon(\xi_\sigma^\varepsilon)) = f(u(\sigma)). \tag{5.3}$$

Then

$$f(u_+) < \sigma < f(u_-). \tag{5.4}$$

Now we take $\xi_1 < \sigma < \xi_2$, $\phi \in C_0^\infty[\xi_1, \xi_2]$ such that $\phi(\xi) \equiv \phi(\sigma)$ for ξ in a neighborhood Ω of $\xi = \sigma$ (ϕ is called a sloping test function). When $0 < \varepsilon < \varepsilon_0$, $\xi_\sigma^\varepsilon \in \Omega$. From (4.3) we have

$$-\int_{\xi_1}^{\xi_2} v^\varepsilon(f(u^\varepsilon) - \xi)\phi' \, d\xi + \int_{\xi_1}^{\xi_2} v^\varepsilon \phi \, d\xi = 0, \tag{5.5}$$

$$-\int_{\xi_1}^{\xi_2} v^\varepsilon(f(u^\varepsilon) - \xi)\, u^\varepsilon \phi' \, d\xi + \int_{\xi_1}^{\xi_2} v^\varepsilon u^\varepsilon \phi \, d\xi = \varepsilon \int_{\xi_1}^{\xi_2} u^\varepsilon \phi'' \, d\xi. \tag{5.6}$$

Concerning (5.5), since

$$\int_{\xi_1}^{\xi_2} v^\varepsilon(f(u^\varepsilon) - \xi)\, \phi' \, d\xi = \int_{\xi_1}^{\alpha_1} v^\varepsilon(f(u^\varepsilon) - \xi)\, \phi' \, d\xi + \int_{\alpha_2}^{\xi_2} v^\varepsilon(f(u^\varepsilon) - \xi)\, \phi' \, d\xi,$$

where $\alpha_1, \alpha_2 \in \Omega$ s.t. $\alpha_1 < \sigma < \alpha_2$, we immediately obtain from Lemmas 5.1–5.2 that

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} v^\varepsilon (f(u^\varepsilon) - \xi)\, \phi'\, d\xi$$

$$= \int_{\xi_1}^{\alpha_1} v_-(f(u_-) - \xi)\, \phi'\, d\xi + \int_{\alpha_2}^{\xi_2} v_+(f(u_+) - \xi)\, \phi'\, d\xi$$

$$= (-\sigma[v] + [vf(u)])\, \phi(\sigma) + \int_{\xi_1}^{\xi_2} H(\xi - \sigma)\, \phi(\xi)\, d\xi,$$

where

$$H(x) = \begin{cases} v_-, & x < 0, \\ v_+, & x > 0. \end{cases}$$

Returning to (5.5), we get

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (v^\varepsilon - H(\xi - \sigma))\, \phi(\xi)\, d\xi = (-\sigma[v] + [vf(u)])\, \phi(\sigma) \tag{5.7}$$

for all sloping test functions $\phi \in C_0^\infty[\xi_1, \xi_2]$. For an arbitrary $\psi \in C_0^\infty[\xi_1, \xi_2]$, we take a sloping test function ϕ such that $\phi(\sigma) = \psi(\sigma)$ and

$$\max_{[\xi_1, \xi_2]} |\phi - \psi| < \mu, \qquad \mu > 0.$$

Recalling that $v^\varepsilon \in L^1[\xi_1, \xi_2]$ uniformly, we find that

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (v^\varepsilon - H(\xi - \sigma))\, \psi(\xi)\, d\xi$$

$$= \lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (v^\varepsilon - H(\xi - \sigma))\, \phi(\xi)\, d\xi + O(\mu)$$

$$= (-\sigma[v] + [vf(u)])\, \phi(\sigma) + O(\mu)$$

$$= (-\sigma[v] + [vf(u)])\, \psi(\sigma) + O(\mu).$$

Sending $\mu \to 0$, we find that (5.7) holds for all $\phi \in C_0^\infty[\xi_1, \xi_2]$.

In the similar way, from (5.6) we can obtain

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (v^\varepsilon u^\varepsilon - \tilde{H}(\xi - \sigma))\, \phi(\xi)\, d\xi = (-\sigma[vu] + [vuf(u)])\, \phi(\sigma) \tag{5.8}$$

for all $\phi \in C_0^\infty[\xi_1, \xi_2]$, where

$$\tilde{H}(x) = \begin{cases} v_- u_-, & x < 0, \\ v_+ u_+, & x > 0. \end{cases}$$

Thus, v^ε and $v^\varepsilon u^\varepsilon$ converge in the weak star topology of $C_0^\infty(R^1)$, and the limit functions are all sum of a step function and a Dirac delta function with strengths which are $-\sigma[v] + [vf(u)]$ and $-\sigma[vu] + [vuf(u)]$, respectively.

If we take the test function as $\phi/(\tilde{u}^\varepsilon + \nu)$ in Eq. (5.6), where $\tilde{u}^\varepsilon$ is a modified function satisfying $u^\varepsilon(\sigma)$ in Ω and u^ε outside Ω, and let $\nu \to 0$, then we can find the other formula as

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (v^\varepsilon - H(\xi - \sigma))\, \phi(\xi)\, d\xi \cdot u(\sigma) = (-\sigma[vu] + [vuf(u)])\, \phi(\sigma), \tag{5.9}$$

where $\phi \in C_0^\infty[\xi_1, \xi_2]$.

Let w_0 be the strength of Dirac delta function in v, and denote

$$u_\delta = \lim_{\varepsilon \to 0^+} u^\varepsilon(\xi_\sigma^\varepsilon) = u(\sigma).$$

Then from (5.3), (5.7), and (5.9) it follows that

$$\begin{cases} \sigma = f(u_\delta), \\ w_0 = -\sigma[v] + [vf(u)], \\ w_0 u_\delta = -\sigma[vu] + [vuf(u)], \end{cases} \tag{5.10}$$

which is just the same as the relation (3.13). Under the entropy condition (5.4) which is equivalent, under assumption (2.12), to

$$u_+ < u_\delta < u_-, \tag{5.11}$$

it has been proved that the system (5.10) admits a unique solution (σ, w_0, u_δ). Therefore, we have the following theorem.

THEOREM 5.3. *Under assumption* (2.12), *let* $(v^\varepsilon, u^\varepsilon)$ *be the solution of* (4.3)–(4.4) *and* $u_- > u_+$. *Then*

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(\xi) = \begin{cases} u_-, & \xi < \sigma, \\ u_\delta, & \xi = \sigma, \\ u_+, & \xi > \sigma. \end{cases} \tag{5.12}$$

v^ε and $v^\varepsilon u^\varepsilon$ converge in the weak star topology of $C_0^\infty(R^1)$, and the limit functions are all the sum of a step function and a Dirac delta function supported on $\xi=\sigma$ with strengths w_0 and $w_0 u_\delta$, respectively, where σ, w_0, and u_δ are determined uniquely by (5.10) *and* (5.11).

Theorem 5.3 exhibits the structure of delta-shocks of (1.1). Also, it is noticed that the difference between the strength w_0 and weight $w(t)$ in (3.21) is due to the result of self-similarity of viscosity.

Case 2. $u_- < u_+$.

LEMMA 5.4. *For any* $\eta>0$,

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi)=0, \qquad \textit{for} \quad \xi\leqslant f(u_-)-\eta \quad \textit{or} \quad \xi\geqslant f(u_+)+\eta,$$

$$\lim_{\varepsilon\to 0^+} (v^\varepsilon, u^\varepsilon)(\xi)=\begin{cases}(v_-, u_-), & \textit{for} \quad \xi<f(u_-)-\eta,\\ (0,\xi), & \textit{for} \quad f(u_-)-\eta\leqslant\xi\leqslant f(u_+)+\eta,\\ (v_+, u_+), & \textit{for} \quad \xi>f(u_+)+\eta\end{cases}$$

uniformly in the above intervals.

Proof. The proof of this lemma is basically similar to that of Lemmas 5.1–5.2; we present it just for completeness.

Take $\xi_3=f(u_+)+\eta$. Integrating the second equation of (4.3) twice on $[\xi_3,\xi]$, we get

$$\begin{aligned} u(\xi)-u(\xi_3)&=u'(\xi_3)\int_{\xi_3}^{\xi}\exp\left(\int_{\xi_3}^{r}\frac{v(f(u)-s)}{\varepsilon}\,ds\right)dr\\ &>u'(\xi_3)\int_{\xi_3}^{\xi}\exp\left(\int_{\xi_3}^{r}\frac{v_+(f(u_-)-s)}{\varepsilon}\,ds\right)dr\\ &=u'(\xi_3)\int_{\xi_3}^{\xi}\exp\left(\frac{v_+}{\varepsilon}\left((f(u_-)-\xi_3)(r-\xi_3)-\frac{1}{2}(r-\xi_3)^2\right)\right)dr\\ &=u'(\xi_3)\int_{0}^{\xi-\xi_3}\exp\frac{v_+}{2\varepsilon}(2(f(u_-)-\xi_3)r-r^2)\,dr. \end{aligned}$$

Letting $\xi\to+\infty$, it follows that

$$\begin{aligned} u_+-u_-&\geqslant u'(\xi_3)\int_0^{+\infty}\exp\frac{v_+}{2\varepsilon}(2(f(u_-)-\xi_3)r-r^2)\,dr\\ &\geqslant u'(\xi_3)\sqrt{\varepsilon}A_5, \end{aligned}$$

where A_5 is a constant independent of ε. Thus

$$|u'(\xi_3)| \leqslant \frac{u_+ - u_-}{\sqrt{\varepsilon A_5}}.$$

So

$$|u'(\xi)| \leqslant \frac{u_+ - u_-}{\sqrt{\varepsilon A_5}} \exp\left(\int_{\xi_3}^{\xi} \frac{v(f(u)-s)}{\varepsilon}\, ds\right).$$

Noting that for $\xi > \xi_\sigma$,

$$\begin{aligned}
v(\xi) &= v_+ \exp\left(\int_{\xi}^{+\infty} \frac{(f(u(s)))'}{f(u(s))-s}\, ds\right) \\
&= \lim_{R\to+\infty} v_+ \exp\left(\int_{\xi}^{R} \frac{(f(u(s))-s)'+1}{f(u(s))-s}\, ds\right) \\
&\geqslant \lim_{R\to+\infty} v_+ \frac{f(u_+)-R}{f(u(\xi))-\xi} \exp\left(\int_{\xi}^{R} \frac{ds}{f(u_+)-s}\right) \\
&= \frac{v_+(f(u_+)-\xi)}{f(u(\xi))-\xi},
\end{aligned}$$

we have

$$v(f(u)-\xi) \leqslant v_+(f(u_+)-\xi), \qquad \xi > \xi_\sigma. \tag{5.13}$$

Therefore

$$\begin{aligned}
|u'(\xi)| &\leqslant \frac{u_+ - u_-}{\sqrt{\varepsilon A_5}} \exp\left(\int_{\xi_3}^{\xi} \frac{v_+(f(u_+)-s)}{\varepsilon}\, ds\right) \\
&= \frac{u_+ - u_-}{\sqrt{\varepsilon A_5}} \exp\left(-\frac{v_+}{2\varepsilon}\left((f(u_+)-\xi)^2 - (f(u_+)-\xi_3)^2\right)\right),
\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi) = 0, \qquad \text{uniformly for} \quad \xi \geqslant f(u_+)+\eta.$$

Next, we take ξ_4 such that $\xi > \xi_4 \geqslant f(u_+)+\eta$. From

$$u(\xi) - u(\xi_4) = u'(\xi_4)\int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{r} \frac{v(f(u)-s)}{\varepsilon}\, ds\right) dr,$$

we get

$$|u(\xi)-u(\xi_4)| \leqslant |u'(\xi_4)| \int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{r} \frac{v_+(f(u_+)-s)}{\varepsilon}\,ds\right) dr$$
$$= |u'(\xi_4)| \int_{\xi_4}^{\xi} \exp \frac{v_+}{2\varepsilon}\,(2(f(u_+)-\xi_4)(r-\xi_4)-(r-\xi_4)^2)\,dr$$
$$= |u'(\xi_4)| \int_{0}^{\xi-\xi_4} \exp \frac{v_+}{2\varepsilon}\,(2(f(u_+)-\xi_4)r-r^2)\,dr.$$

Letting $\xi \to +\infty$, it follows that

$$|u(\xi_4)-u_+| \leqslant |u'(\xi_4)|\,\sqrt{\varepsilon}A_6,$$

where A_6 is a constant independent of ε, which implies that

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi)=u_+, \qquad \text{uniformly for} \quad \xi \geqslant f(u_+)+\eta.$$

Furthermore, from Lemma 4.2(ii) and (5.13) we get that for $\xi > f(u_+)+\eta$,

$$v_+ \geqslant v(\xi) \geqslant \frac{v_+(f(u_+)-\xi)}{f(u(\xi))-\xi} \to v_+, \qquad \text{as} \quad \varepsilon \to 0^+.$$

Thus

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(\xi)=v_+, \qquad \text{uniformly for} \quad \xi > f(u_+)+\eta.$$

Analogously, we can obtain the results for $\xi < f(u_-)-\eta$.

Let us now consider the limit solution on $[f(u_-), f(u_+)]$. Set

$$F(\xi)=f(u(\xi))-\xi.$$

Then from Lemma 4.2 (ii) we can find that

$$F'(\xi)=(f(u(\xi))-\xi)'=(f(u(\xi))'-1 \leqslant 0,$$

where $\xi \in [f(u_-), f(u_+)]$. Hence

$$F(f(u_+)+\eta) \leqslant F(\xi) \leqslant F(f(u_-)-\eta),$$

namely

$$f(u(f(u_+)+\eta))-(f(u_+)+\eta) \leqslant f(u(\xi))-\xi$$
$$\leqslant f(u(f(u_-)-\eta))-(f(u_-)-\eta),$$

which yields

$$-\eta \leqslant \lim_{\varepsilon \to 0^+} (f(u(\xi)) - \xi) \leqslant \eta.$$

Since η is arbitrary, we conclude that

$$\lim_{\varepsilon \to 0^+} (f(u(\xi)) - \xi) = 0.$$

This immediately shows that

$$\lim_{\varepsilon \to 0^+} v^\varepsilon(\xi) = 0, \qquad \text{uniformly for} \quad f(u_-) - \eta \leqslant \xi \leqslant f(u_+) + \eta.$$

Then we obtain a vacuum solution with vacuum curve $v = 0$ $(f(u_-) \leqslant \xi \leqslant f(u_+))$. ∎

Thus we have the follow theorem.

THEOREM 5.5. *Under assumption* (2.12), *let* $(v^\varepsilon, u^\varepsilon)$ *be the solution of* (4.3)–(4.4) *and* $u_- < u_+$. *Then*

$$\lim_{\varepsilon \to 0^+} (v^\varepsilon, u^\varepsilon)(\xi) = \begin{cases} (v_-, u_-), & \xi < f(u_-), \\ (0, \xi), & f(u_-) \leqslant \xi \leqslant f(u_+), \\ (v_+, u_+), & \xi > f(u_+). \end{cases} \tag{5.14}$$

The theorem shows that the vacuum solution is stable under viscous perturbations.

Remark that all results in Sections 4–5 are valid for the case $f'(u) > 0$, $v \leqslant 0$. In addition, by the same arguments as used in the proofs of Sections 4–5 with only a few modifications, we can obtain the similar results for the cases $f'(u) < 0$, $v \geqslant 0$ and $f'(u) < 0$, $v \leqslant 0$. Then we get the following theorem.

THEOREM 5.6. *Under assumptions of Theorem* 3.4, *let* $(v^\varepsilon, u^\varepsilon)(t, x)$ *be the self-similar solution of* (4.1)–(4.2). *Then the limit*, $\lim_{\varepsilon \to 0^+} (v^\varepsilon, u^\varepsilon)(t, x) = (v, u)(t, x)$ *exists in the measure sense and* $(v, u)(t, x)$ *solves* (1.1)–(1.2). $(v, u)(t, x)$ *can be given explicitly by the following formula:*

(i) *When* $f(u_-) > f(u_+)$, *then*

$$(v, u)(t, x) = \begin{cases} (v_-, u_-), & x < \sigma t, \\ (w_0 t \delta(x - \sigma t), u_\delta), & x = \sigma t, \\ (v_+, u_+), & x > \sigma t, \end{cases}$$

where σ, w_0, *and* u_δ *are determined uniquely by* (5.10) *and* (5.4);

(ii) *When* $f(u_-) < f(u_+)$, *then*

$$(v, u)(t, x) = \begin{cases} (v_-, u_-), & x < f(u_-)t, \\ (0, x/t), & f(u_-)\,t \leqslant x \leqslant f(u_+)t, \\ (v_+, u_+), & x > f(u_+)t. \end{cases}$$

6. TWO TYPICAL EXAMPLES

To illustrate application of our results and proofs in the above sections, this section presents only two typical examples. In these two models, we focus our attention on delta-shocks, the most interesting topic.

EXAMPLE 1. Consider the system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{6.1}$$

which is well known zero-pressure gas dynamics, where $\rho \geqslant 0$ and u represent the density and velocity, respectively. In this situation, $\lambda = f(u) = u$, $f'(u) = 1 > 0$. The Riemann initial data are

$$(\rho, u)(0, x) = (\rho_\pm, u_\pm) \qquad (\pm x > 0), \tag{6.2}$$

with $u_- > u_+$.

Let $(\rho, u, \sigma, w, u_\sigma)$ denote the delta-shock solution of (6.1)–(6.2) of the form (3.21). Then the generalized Rankine–Hugoniot relation holds,

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\[2ex] \dfrac{dw}{dt} = -[\rho]\sigma + [\rho u], \\[2ex] \dfrac{dwu_\delta}{dt} = -[\rho u]\sigma + [\rho u^2], \end{cases} \tag{6.3}$$

and

$$\sigma = u_\delta, \tag{6.4}$$

which is equivalent to

$$\begin{cases} w_0 = -[\rho]u_\delta + [\rho u], \\ w_0 u_\delta = -[\rho u]u_\delta + [\rho u^2], \end{cases} \tag{6.5}$$

where $w(t) = w_0 t$, $\sigma = u_\delta$, and w_0 are to be determined constants. From (6.5) we get

$$[\rho]u_\delta^2 - 2[\rho u]u_\delta + [\rho u^2] = 0, \tag{6.6}$$

which is a quadratic equation of one variable. Set

$$G(u_\delta) = \text{the left side of (6.6)}. \tag{6.7}$$

Then under the entropy condition

$$u_+ < u_\delta < u_-, \tag{6.8}$$

an easy calculation gives

$$G(u_+) \cdot G(u_-) = -\rho_- \rho_+ [u]^4 < 0,$$
$$G'(u_\delta) = 2(\rho_-(u_\delta - u_-) + \rho_+(u_+ - u_\delta)) < 0.$$

Thus Eq. (6.6) has a unique solution $u_\delta \in (u_+, u_-)$. Of course, we can directly solve (6.6) to get

$$u_\delta = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma. \tag{6.9}$$

Then

$$w(t) = w_0 t = \sqrt{\rho_- \rho_+}\,(u_- - u_+)\,t. \tag{6.10}$$

Therefore, the unique delta-shock solution of (6.1)–(6.2) is

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < u_\delta t, \\ (w(t)\,\delta(x - u_\delta t), u_\delta), & x = u_\delta t, \\ (\rho_+, u_+), & x > u_\delta t, \end{cases} \tag{6.11}$$

where $u_\delta = \sigma$ and $w(t) = w_0 t$ is as in (6.9) and (6.10), respectively.

EXAMPLE 2. Consider the Riemann problem

$$\begin{cases} v_t + \left(\dfrac{vU}{\sqrt{1+U^2}} \right)_x = 0, \\ (vU)_t + \left(\dfrac{vU^2}{\sqrt{1+U^2}} \right)_x = 0, \end{cases} \tag{6.12}$$

and

$$(v, U)(0, x) = (v_\pm, U_\pm), \qquad (\pm x > 0), \tag{6.13}$$

with $U_- > U_+, \quad v_- v_+ > 0$.

The system (6.12) is obtained by performing the transformation $U = u/v$, i.e., $u = vU$ for the system

$$\begin{cases} u_t + \left(\dfrac{u^2}{\sqrt{u^2+v^2}} \right)_x = 0, \\ v_t + \left(\dfrac{uv}{\sqrt{u^2+v^2}} \right)_x = 0, \end{cases} \tag{6.14}$$

which arises in nonlinear geometric optics.

For the system (6.12),

$$\lambda = f(U) = \frac{U}{\sqrt{1+U^2}}, \qquad f'(U) = \frac{1}{(\sqrt{1+U^2})^3} > 0.$$

Let $(v, U, \sigma, w, U_\delta)$ denote the delta-shock solution of (6.12)–(6.13). Then the generalized Rankine–Hugoniot relation holds,

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\ \dfrac{dw}{dt} = -[v]\sigma + \left[\dfrac{vU}{\sqrt{1+U^2}} \right], \\ \dfrac{dwU_\delta}{dt} = -[vU]\sigma + \left[\dfrac{vU^2}{\sqrt{1+U^2}} \right], \end{cases} \tag{6.15}$$

and

$$\sigma = \frac{U_\delta}{\sqrt{1+U_\delta^2}}, \tag{6.16}$$

which can be reduced to

$$\begin{cases} w_0 = -[v]\sigma + \left[\dfrac{vU}{\sqrt{1+U^2}}\right], \\ w_0 U_\delta = -[vU]\sigma + \left[\dfrac{vU^2}{\sqrt{1+U^2}}\right], \\ \sigma = \dfrac{U_\delta}{\sqrt{1+U_\delta^2}}, \end{cases} \tag{6.17}$$

where $w(t) = w_0 t$, σ, w_0, and U_δ are to be determined constants.

Besides, the entropy condition is

$$\frac{U_+}{\sqrt{1+U_+^2}} < \frac{U_\delta}{\sqrt{1+U_\delta^2}} < \frac{U_-}{\sqrt{1+U_-^2}}, \tag{6.18}$$

which is equivalent to

$$U_+ < U_\delta < U_-. \tag{6.19}$$

From (6.17) it follows that

$$([v]\, U_\delta - [vU])\, \frac{U_\delta}{\sqrt{1+U_\delta^2}} - \left[\frac{vU}{\sqrt{1+U^2}}\right] U_\delta + \left[\frac{vU^2}{\sqrt{1+U^2}}\right] = 0, \tag{6.20}$$

which is a quartic equation of one variable. Set

$$G(U_\delta) = \text{the left side of (6.20)}. \tag{6.21}$$

Then under the entropy condition (6.19), one can calculate that

$$G(U_+)\cdot G(U_-) = -v_- v_+ [U]^2 \left[\frac{U}{\sqrt{1+U^2}}\right]^2 < 0,$$

$$\begin{aligned} G'(U_\delta) &= v_- \left(\frac{U_\delta}{\sqrt{1+U_\delta^2}} - \frac{U_-}{\sqrt{1+U_-^2}}\right) \\ &\quad + v_+ \left(\frac{U_+}{\sqrt{1+U_+^2}} - \frac{U_\delta}{\sqrt{1+U_\delta^2}}\right) \\ &\quad + (v_-(U_\delta - U_-) + v_+(U_+ - U_\delta))\, \frac{1}{(\sqrt{1+U_\delta^2})^3} \\ &= \begin{cases} <0, & \text{as} \quad v_-, v_+ > 0, \\ >0, & \text{as} \quad v_-, v_+ < 0. \end{cases} \end{aligned}$$

Thus Eq. (6.20) possesses a unique solution $U_\delta \in (U_+, U_-)$. Returning to (6.17) one can solve σ and w_0 uniquely. Therefore, the unique delta-shock solution of (6.12)–(6.13) can be expressed as

$$(v, U)(t, x) = \begin{cases} (v_-, U_-), & x < \sigma t, \\ (w(t)\,\delta(x - \sigma t), U_\delta), & x = \sigma t, \\ (v_+, U_+), & x > \sigma t, \end{cases} \tag{6.22}$$

where $w(t) = w_0 t$ and constants σ, w_0, and U_δ are determined uniquely by (6.17) and (6.19).

The formula (6.22) shows that there is a weighted Dirac delta function only in v for the system (6.12). With the aid of the transformation $u = Uv$, we conjecture that weighted Dirac delta functions may appear simultaneously in the state variables u and v for the system (6.14). We plan to explore it with the help of numerical computation in the future.

ACKNOWLEDGMENT

The author thanks Professor Tong Zhang, his advisor, for numerous enlightening discussions and many suggestions.

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