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Cardinalities of some Lindelöf and ω_1-Lindelöf T_1/T_2-spaces ☆

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Abstract

We show that every first-countable countably paracompact Lindelöf T_1-space has cardinality at most c; every first-countable ω_1-Lindelöf Hausdorff space has cardinality at most 2^c; every realcompact first-countable ω_1-Lindelöf space has cardinality at most c. In all these results, first countability can be replaced by countable tightness plus either countable or countable closed pseudocharacter. We also show that the Lindelöf number of every ω_1-Lindelöf regular space of countable tightness is at most c.

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1. Introduction

In [1], Arhangel'skii solved a half-century old problem of Alexandroff by proving the following inequality:

$$|\textit{First countable Lindelöf } T_2 \textit{ space}| \leqslant c.$$

In this paper we are exploring possibilities of relaxing the conditions in the left side of the above inequality. First we go along an old road trying to reduce T_2 to T_1. Gryzlov proved in [6] that every T_1 compactum of countable pseudocharacter has cardinality at most c. We use the Gryzlov's argument to show that every countably paracompact Lindelöf T_1-space of countable pseudocharacter and countable tightness has cardinality at most c. We also

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present a very short proof of the fact that every first-countable countably paracompact Lindelöf T_1-space has cardinality at most c. The reason we present a shorter proof for a weaker result is that it reveals a very interesting effect of countable paracompactness on T_1-spaces. As it is understood from the abstract we do not reach the final goal in this direction. So we move to a parallel road of relaxing Lindelöfness. A successful attempt in this direction was earlier made in [2], where the authors proved that the cardinality of a first-countable linearly Lindelöf Tychonov space does not exceed c.

In the third section we are trying to relax Lindelöfness to ω_1-Lindelöfness and obtain some partial results. While under CH every first-countable ω_1-Lindelöf Hausdorff space is simply Lindelöf there exists a consistent example (constructed by Koszmider [7]) of a first-countable initially ω_1-compact not compact normal space. In addition to many other credentials this space is ω_1-Lindelöf not linearly Lindelöf, and therefore, not Lindelöf.

A space X is called *ω_1-Lindelöf* if every open cover of X of cardinality ω_1 contains a countable subcover. This is equivalent to the condition that every subset of X of cardinality ω_1 has a complete accumulation point in X.

The *Lindelöf number* of X (denoted by $l(X)$) is the smallest cardinal number τ such that every open cover of X contains a subcover of cardinality not exceeding τ.

A space X is said to have *countable tightness* if for every set $A \subset X$ and every $x \in \overline{A} \setminus A$ there exists a countable $B \subset A$ whose closure contains x.

If $A \subset Y \subset X$, by $\overline{A}$ and $\mathrm{cl}_Y(A)$ we denote the closures of A in X and Y, respectively.

In the rest of notation and terminology we will be consistent with [5].

Throughout the paper we will often use Arhangel'skii's closure argument developed by him to prove the inequality in question.

2. Countably paracompact Lindelöf T_1-spaces

In [6], Gryzlov proved that every T_1 compactum of countable pseudocharacter has cardinality at most c. It is still an open question whether in Arhangel'skii inequality T_2 can be replaced by T_1. Moreover it is not even known if cardinalities of such T_1-spaces have an upper bound. Using Gryzlov's argument we will prove the main result of this section (Theorem 2.7).

We would like to start with a shorter proof of a weaker version of Theorem 2.7 that utilizes an unusual effect of countable paracompactness on T_1-spaces. For both proofs we will need the following definition.

Definition 2.1. Let X be a topological space. A set $Y \subset X$ is called ω-closed in X if the following condition is met: for every family $\{C_n \subset Y\colon |C_n| \leqslant \omega\}$, if $\bigcap_n \mathrm{cl}_Y(C_n) = \emptyset$ then $\bigcap_n \overline{C}_n = \emptyset$.

Observe that every closed set is ω-closed. The following lemma about ω-closed sets is extracted from the argument of Gryzlov [6].

Lemma 2.2. *Let X have countable tightness. Let $Y \subset X$ be ω-closed in X. Let F_n be closed in Y for each n and $\bigcap_n F_n = \emptyset$. Then $\bigcap_n \overline{F}_n = \emptyset$.*

Proof. Assume there exists $x \in \bigcap_n \overline{F}_n$. Since X has countable tightness, for each n there exists countable $C_n \subset F_n$ with $x \in \overline{C}_n$. Then $\bigcap_n \mathrm{cl}_Y(C_n) \subset \bigcap_n F_n = \emptyset$ while $\bigcap_n \overline{C}_n \neq \emptyset$ which contradicts ω-closeness of Y in X. □

A shorter proof: For $x \in X$, the *anti-Hausdorff component* $H_x \subset X$ of x is defined as follows: $y \in H_x$ iff $x \in \overline{O}_y$ for every open neighborhood O_y of y.

Lemma 2.3. *Let X be countably paracompact and Lindelöf. Then H_x is a closed compact subspace of X for every $x \in X$.*

Proof. Take any $z \in X \setminus H_x$. There exists an open neighborhood O_z of z such that $x \notin \overline{O}_z$. Therefore any $y \in O_z$ does not belong to H_x, hence $X \setminus H_x$ is open and H_x is closed.

If H_x is not compact then Lindelöfness of X implies that there exists a discrete closed in X set $\{x_n\colon n \in \omega\} \subset H_x$. Due to countable paracompactness, there exist open sets W_n's such that $\{x_k\colon k > n\} \subset W_n$ and $\bigcap_n \overline{W}_n = \emptyset$. Therefore, there exists k such that $x \notin \overline{W}_k$. Then x_{k+1} cannot be in H_x, a contradiction. □

Theorem 2.4. *Let X be a first-countable Lindelöf T_1-space. If X is countably paracompact then $|X| \leqslant c$.*

Proof. For Arhangel'skii's argument to work in our case, it suffices to show that the closure of any countable subset in X has cardinality at most c. Therefore, we may assume that X is separable. Starting from a countable dense subset of X after ω_1 steps we can build a set Y of cardinality at most c which is dense and ω-closed in X.

Take an arbitrary $x \in \overline{Y} \setminus Y$. Let us show that $x \in H_y$ for some $y \in Y$. Let B_n's be base neighborhoods at x. Let $F = \bigcap_n (\overline{B}_n \cap Y)$. The set F cannot be empty due to Lemma 2.2. Then $y \in F$ is the point we need. Hence, $X = \overline{Y} = \bigcup_{y \in Y} H_y$. Each H_y is compact (Lemma 2.3) and therefore has cardinality at most c by Gryzlov's theorem. Hence, $|X| \leqslant c$. □

A longer proof: The next two lemmas are based on ideas due to Gryzlov [6].

Lemma 2.5. *Let X be a Lindelöf T_1-space of countable pseudocharacter and countable tightness. Let Y be ω-closed in X. Let $\mathcal{F}$ be a maximal family of closed in Y sets with finite intersection property. Then*

(1) $\bigcap_{F \in \mathcal{F}} \overline{F} = \emptyset$;
(2) *There exist $F_1, \ldots, F_n, \ldots \in \mathcal{F}$ such that $\bigcap_n \overline{F}_n = \emptyset$.*

Proof. Assume there exists $x \in \bigcap_{F \in \mathcal{F}} \overline{F}$. Let B_n be open neighborhoods of x such that $\bigcap_n B_n = \{x\}$. By maximality of $\mathcal{F}$ there exists $F_n \in \mathcal{F}$ such that $F_n \subset B_n \cap Y$. Then $\bigcap_n F_n \subset \bigcap_n (B_n \cap Y) = \emptyset$. By Lemma 2.2, $\bigcap_n \overline{F}_n = \emptyset$ contradicting the assumption.

Statement (2) follows from (1) and Lindelöfness of X. □

Lemma 2.6. *Let X be a Lindelöf T_1-space of countable pseudocharacter and countable tightness. Let Y be ω-closed in X. If X is countably paracompact then Y is Lindelöf.*

Proof. Assume the contrary. Then there exists a free countably complete filter $\mathcal{F}$ of closed in Y sets. Since X is Lindelöf there exists $x \in \bigcap_{F\in\mathcal{F}} \overline{F}$.

Let $\mathcal{F}'$ be the maximal family of closed in Y sets such that $x \in \overline{F}$ for every $F \in \mathcal{F}'$. Clearly, $\mathcal{F} \subset \mathcal{F}'$.

By Lemma 2.2, $\mathcal{F}'$ has finite (even countable) intersection property. Let $\mathcal{F}''$ be a maximal family of closed in Y sets with finite intersection property such that $\mathcal{F}' \subset \mathcal{F}''$. Then by Lemma 2.5, there exist $F_1, \ldots, F_n, \ldots \in \mathcal{F}''$ such that $\bigcap_n \overline{F}_n = \emptyset$. We may assume that $F_{n+1} \subset F_n$.

Countable paracompactness of X implies that there exist open W_n's in X such that $\overline{F}_n \subset W_n$ and $\bigcap_n \overline{W}_n = \emptyset$. The set $Y \setminus W_n$ is closed in Y. Since $\mathcal{F}''$ has finite intersection property, $Y \setminus W_n$ is not in $\mathcal{F}''$ and therefore not in $\mathcal{F}'$ either. That is, $x \notin \overline{Y \setminus W_n}$. Since $x \in \overline{Y}$, we have $x \in \overline{W}_n$. The latter inclusion contradicts the fact that $\bigcap_n \overline{W}_n = \emptyset$. $\square$

The proof of the next statement is the classical argument of Arhangel'skii. To avoid repetition we will outline only the most important steps.

Theorem 2.7. *Let X be a Lindelöf T_1-space of countable pseudocharacter and countable tightness. If X is countably paracompact then $|X| \leqslant c$.*

Proof. For each $x \in X$ let $\{V_n(x)\colon n \in \omega\}$ be a collection of open neighborhoods of x such that $\bigcap_n V_n(x) = \{x\}$. Construct a sequence $\{Y_\alpha\colon \alpha < \omega_1\}$ of subsets of X such that for all α:

(1) $|Y_\alpha| \leqslant c$, and $Y_\beta \subset Y_\alpha$ if $\beta < \alpha$;
(2) If $\mathcal{V} \subset \{V_n(x)\colon x \in \bigcup_{\beta<\alpha} Y_\beta,\ n \in \omega\}$ is countable and is not a cover of X then $Y_\alpha \setminus \bigcup \mathcal{V} \neq \emptyset$;
(3) If $\{C_n\colon n \in \omega\}$ is a family of countable subsets of $\bigcup_{\beta<\alpha} Y_\beta$ and $\bigcap_n \overline{C}_n \neq \emptyset$ then $\bigcap_n \mathrm{cl}_{Y_\alpha}(C_n) \neq \emptyset$.

Let $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$. By (3), Y is ω-closed, hence Lindelöf (see Lemma 2.6). By (2), $Y = X$. By (1), $|X| \leqslant c$. $\square$

3. ω_1-Lindelöfness

As we mentioned in the introduction section, a first-countable ω_1-Lindelöf space need not be Lindelöf. Therefore, it is interesting to know if Arhangel'skii's inequality holds in class of ω_1-Lindelöf spaces.

Let us start with the following technical statement that will allow us to derive several important corollaries.

Lemma 3.1. *Let X be an ω_1-Lindelöf space of countable tightness and $l(\overline{A}) \leqslant c$ for every countable $A \subset X$. Then $l(X) \leqslant c$.*

Proof. Let $\mathcal{U}$ be an arbitrary open cover of X. For each $\alpha < \omega_1$ we will define a countable set $A_\alpha \subset X$ and use these sets to choose a subcover of a desired cardinality.

Step 0. Put $A_0 = \emptyset$.
Step $\alpha < \omega_1$. Since $\overline{\bigcup_{\beta<\alpha} A_\beta}$ is separable, by the lemma's hypothesis there exists a cover $\mathcal{U}_\alpha \subset \mathcal{U}$ of $\overline{\bigcup_{\beta<\alpha} A_\beta}$ of cardinality not exceeding c.
Pick an arbitrary point $a_\alpha \in X \setminus [\bigcup_{\beta\leqslant\alpha}(\bigcup \mathcal{U}_\beta)]$. If no such point exists then stop inductive definition. Otherwise, put $A_\alpha = (\bigcup_{\beta<\alpha} A_\beta) \cup \{a_\alpha\}$.

Let us show that at some step $\alpha < \omega_1$ our process must stop. Assume the contrary. Then $A = \bigcup_\alpha \overline{A}_\alpha$ is closed being an ω_1-long increasing sequence of closed sets in a space of countable tightness. Since A is a closed set of an ω_1-Lindelöf space, there exists $\alpha < \omega_1$ such that $\bigcup_{\beta\leqslant\alpha} \mathcal{U}_\beta$ is a cover of A which contradicts the fact that $a_\alpha \in X \setminus [\bigcup_{\beta\leqslant\alpha}(\bigcup \mathcal{U}_\beta)]$.

Therefore, our process stops at some countable step α and $\bigcup_{\beta\leqslant\alpha} \mathcal{U}_\beta$ is a subcover of X of cardinality not exceeding c (recall that each $\mathcal{U}_\beta$ has cardinality at most c). □

If in the proof of Lemma 3.1 we assume that for every closed separable set $Y \subset X$ there exists $U_Y \in \mathcal{U}$ containing Y, then at each step α a cover $\mathcal{U}_\alpha$ can be replaced by a single element of $\mathcal{U}$ and we obtain a countable subcover. Thus, a simple repetition of the above proof results in the following statement to be used later in this section.

Lemma 3.2. *Let X be an ω_1-Lindelöf space of countable tightness. And let $\mathcal{U}$ be an open cover of X such that every separable closed $Y \subset X$ is contained in some $U \in \mathcal{U}$. Then $\mathcal{U}$ contains a countable subcover.*

If we assume that X is regular then the closure of every countable set has weight of cardinality c. Therefore every open cover of a separable closed set admits a subcover of cardinality at most c. Applying Lemma 3.1 we get the following.

Corollary 3.3. *Let X be an ω_1-Lindelöf regular space of countable tightness. Then $l(X) \leqslant c$.*

This fact implies that under CH every ω_1-Lindelöf regular space of countable tightness is Lindelöf. This observation is related to an earlier result of Dow [3], where he proves that under CH every initially ω_1-compact Hausdorff space of countable tightness is compact. This result together with our corollary motivates the following question.

Question 3.4. *Let X be an ω_1-Lindelöf Hausdorff (or T_1) space of countable tightness. Is then $l(X) \leqslant c$?*

Another simple corollary to Lemma 3.1 is that if X is an ω_1-Lindelöf space of countable tightness and the closure of any countable set in X is Lindelöf then X is Lindelöf. This fact was proved in [2] only in class of Tychonov spaces while our version has no restrictions on separation axioms.

Recall that the closure of any countable subset in a first-countable ω_1-Lindelöf Hausdorff space has cardinality at most c. Thus, using Lemma 3.1 we obtain an estimate for Lindelöf number of first-countable ω_1-Lindelöf spaces.

Corollary 3.5. *Let X be a first-countable ω_1-Lindelöf Hausdorff space. Then $l(X) \leqslant c$.*

Using this estimate and the argument of Arhangel'skii's inequality, we arrive at the following.

Theorem 3.6. *Let X be a first-countable ω_1-Lindelöf Hausdorff space. Then $|X| \leqslant 2^c$.*

Note that in the above theorem we can safely replace first-countability by countable tightness plus countable closed pseudocharacter. However we do not know if countable closed pseudocharacter can be replaced by countable pseudocharacter since we do not know an answer to the following question.

Question 3.7. *Let X be a separable ω_1-Lindelöf Hausdorff space of countable tightness and countable pseudocharacter. Is it true that $|X| \leqslant c$?*

For our further discussion let $\mathrm{Ch}(X)$ be defined as the minimum cardinal number κ such that $\beta X \setminus X$ can be written as the union of at most κ compact sets.

In our last result in this section (Theorem 3.10) we will use the strategy developed in [2]. To prove Theorem 3.10 we will need the following two statements.

Theorem 3.8 (Dow [4]). *Suppose κ is a cardinal and Y is a subspace of a Tychonov space X such that $\mathrm{Ch}(Y)$ and $|Y|$ are at most κ and, for each $y \in Y$, $\psi(y, X) \leqslant \kappa$, then Y is a G_κ-set in X.*

The original version of the above theorem has $\chi(y, X)$ instead of $\psi(y, X)$. However the proof uses only the "$\psi(y, X) \leqslant \kappa$" assumption (confirmed with the author of the theorem).

Lemma 3.9. *Let X be a realcompact space and Y a closed separable subset of X of cardinality at most c and $\psi(y, X) \leqslant c$ for every $y \in Y$. Then Y is a G_c-set in X.*

Proof. Since Y is separable, βY has a base of cardinality at most c. Since Y is realcompact, every $z \in \beta Y \setminus Y$ is contained in a compactum $C_z \subset \beta Y \setminus Y$ which is a G_δ-set in βY. Since the weight is at most c, the set of all closed G_δ sets in βY does not exceed c. Therefore, $\beta Y \setminus Y$ can be covered by c many compact subsets of $\beta Y \setminus Y$. The conclusion follows from Dow's theorem. □

In the next theorem we will repeat Arhangel'skii's argument with a tiny change, namely, we replace neighborhoods of points by neighborhoods of closed separable sets. And the rest of Arhangel'skii's argument works smoothly due to Lemma 3.2.

Theorem 3.10. *Let X be a realcompact ω_1-Lindelöf space of countable tightness and countable pseudocharacter. Then $|X| \leqslant c$.*

Proof. For each closed separable $Y \subset X$ fix a c-sized family $\mathcal{U}_Y$ of open sets such that $Y = \bigcap \mathcal{U}_Y$. This can be done by Lemma 3.9 since the closure of any countable set in X has cardinality at most c due to countable tightness, countable pseudocharacter and regularity.

For each $\alpha < \omega_1$, we will define $X_\alpha \subset X$ of cardinality at most c so that X will be $\bigcup_\alpha X_\alpha$.

Definition of X_α. *Let $Z_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ and $\mathcal{W}_\alpha = \bigcup\{\mathcal{U}_Y\colon Y \subset Z_\alpha$ is closed and separable$\}$. For every countable family $\mathcal{U} \subset \mathcal{W}_\alpha$ that does not cover X, fix $x_\mathcal{U} \in X \setminus \bigcup \mathcal{U}$. Put $X_\alpha = \overline{Z_\alpha \cup \{\text{all fixed } x_\mathcal{U}\}}$.*

The set $\bigcup_\alpha X_\alpha$ has cardinality at most c since the number of new points added at step α depends on the number of separable closed subsets of Z_α, which is at most c. Let us show that $X = \bigcup_\alpha X_\alpha$. The set $\bigcup_\alpha X_\alpha$ is closed due to countable tightness. Assume there exists an $x \in X \setminus \bigcup_\alpha X_\alpha$. For each separable closed $Y \subset \bigcup_\alpha X_\alpha$, choose $U_Y \in \mathcal{U}_Y$ that does not contain x. By Lemma 3.2, there exist separable closed $Y_1, \ldots, Y_n, \ldots \subset \bigcup_\alpha X_\alpha$ such that $\bigcup_n U_{Y_n}$ covers $\bigcup_\alpha X_\alpha$. All U_{Y_n}'s are in $\mathcal{W}_\alpha$ for some $\alpha < \omega_1$. Therefore $x \in X_\alpha$, a contradiction. □

In the above theorem the only good we have from realcompactness is writing the remainder of a separable closed subset as the union of c many compacta. Therefore, if we replace realcompactness with local compactness or Čech completeness, the theorem still holds.

Theorem 3.10 as well as our result for Hausdorff case give a hope that the following question might have a positive answer.

Question 3.11. *Let X be a first-countable ω_1-Lindelöf Hausdorff space. Is it true that $|X| \leqslant c$? What if X is regular or Tychonov?*

We do not know an answer to this question for initially ω_1-compact spaces either although the latter are well investigated. For T_1 case we will not be so optimistic and state the question in a rather different way.

Question 3.12. *Is there an example of a first countable ω_1-Lindelöf T_1-space of cardinality greater than c (or even greater than 2^c)?*

And let us finish with a questions standing rather aside yet related to our study.

Question 3.13 (Arhangel'skii). *Let X be hereditarily separable and ω_1-Lindelöf. Is then X Lindelöf?*

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