On pseudo 2-factors

Siham Bekkai a, Mekkia Kouider b,∗

a LAID3, U.S.T.H.B, Faculté de Mathématiques, BP. 32 El-Alia, Bab Ezzouar, 16111 Alger, Algérie
b LRI, Université Paris-Sud, UMR 8623, Bât. 490, 91405 Orsay Cedex, France

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We show that a graph with minimum degree δ, independence number α ≥ δ and without isolated vertices, possesses a partition by vertex-disjoint cycles and at most α − δ + 1 edges or vertices.

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1. Introduction

Throughout this paper, we consider only finite simple graphs $G = (V, E)$. We denote by δ the minimum degree of the considered graph and by α its independence number. Let $C$ be a cycle with a prescribed orientation. Let $u$ and $v$ be two vertices on the cycle $C$, we denote by $[u, v]_C$ the segment of $C$, following the orientation and delimited by $u$ and $v$, and $u$ and $v$ excluded. If it does not matter whether $u$ and $v$ are included or not then we replace the braces by brackets. We denote by $d_C(u, v)$ the distance between $u$ and $v$ on the cycle $C$. The join of two disjoint graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$ and is the graph obtained by joining each vertex of $G_1$ to each vertex of $G_2$. For a positive integer $p$, the graph $pG$ consists of $p$ vertex-disjoint copies of $G$. For concepts not defined here we refer to [2].

A covering of a graph $G$ is a family of elementary cycles of $G$ such that each vertex of $G$ lies in at least one cycle of this family. In the literature there are many results dealing with coverings of graphs, particularly by disjoint cycles. A summary of results on independent cycles can be found in [5,7]. In particular, there are some results, involving degree conditions for the existence of $k$ disjoint cycles and $s$ edges, where $k$ and $s$ are fixed [1] or $k$ disjoint cycles and a prescribed forest of size $s$ [9,4].

We define a pseudo 2-factor of $G$ as a partition of $V$ by a family of vertex disjoint cycles, edges or vertices. The cardinality of this family will be called the size of the pseudo 2-factor. These two notions as different as they appear generalize in some sense the same concept, namely that of 2-factors. Recall that a 2-factor of $G$ is a 2-regular spanning subgraph of $G$. Clearly, if the cycles taken in a covering of $G$ are vertex-disjoint then this covering is a 2-factor, and, if a pseudo 2-factor of $G$ contains only cycles then it is a 2-factor. This case occurs when the independence number of $G$ is at most $\delta - 1$ (see [8]). In [8], Niessen has also showed that graphs with independence number $\alpha = \delta$ containing no 2-factor are the graphs $H + \delta K_2$, where $H$ is a graph of order $\delta - 1$. We check easily that such graphs possess a pseudo 2-factor (of size at most $\alpha$) in which all the components are cycles but one.

∗ Corresponding author.
E-mail addresses: siham.bekkai@gmail.com (S. Bekkai), km@lri.fr (M. Kouider).

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Our work was inspired by Kouider’s paper [6] and motivated by the desire to answer the following question: What is the number of components which are edges or vertices in a pseudo 2-factor of a graph with $\alpha > \delta$?

We investigate relation between the minimum degree, the independence number and the number of edges or vertices in a pseudo 2-factor. The main result of this paper is the following, answering thereby the set question, and including the case $\alpha = \delta$ too:

**Theorem 1.** Let $G$ be a graph without isolated vertices, with minimum degree $\delta$ and independence number $\alpha \geq \delta$, then there exists a pseudo 2-factor of $G$ with at most $\alpha - \delta + 1$ components that are edges or vertices.

The bound given in the theorem above is best possible. To see that, consider the graph $G = H + pK_2$ where $p \geq |H| + 1$ (whatever the graph $H$ is). This graph has minimum degree $\delta = |H| + 1 \geq 2$, independence number $\alpha = p$ and possesses a pseudo 2-factor with exactly $\alpha - \delta + 1$ edges and without isolated vertices. It is easy to check that no pseudo 2-factor with less edges or vertices can be found for such a graph. There also exists graphs with $\delta = 1$ for which the bound is reached. As an example, take a graph $H$ of order $n$ and a independent set of order $n$. Attach exactly a vertex of $H$ to exactly a vertex of this independent set. The graph obtained has independence number $\alpha = n$ and possesses no pseudo 2-factor with less than $n$ edges or vertices.

Niessen’s result for graphs with $\alpha = \delta$ derives naturally from the theorem above.

**Corollary 1 ([8]).** Let $G$ be a graph with independence number $\alpha$ and minimum degree $\delta$ such that $\alpha = \delta$. Then, $G$ possesses a pseudo 2-factor containing at most one component which is an edge or a vertex.

In addition, Theorem 1 gives a lower bound for the number of vertices that are covered by vertex disjoint cycles.

**Corollary 2.** Let $G$ be a graph with independence number $\alpha$ and minimum degree $\delta \leq \alpha$. Then at least $\max(2\delta - 2, n + 2\delta - 2(\alpha + 1))$ vertices of $G$ can be covered by vertex disjoint cycles.

The bound given above is reached for the graphs of type $H + pK_2$ with $p \geq |H| + 1$ defined above.

### 2. Pseudo-factors, minimum degree and independence number

We begin by the simplest case which is when the graph $G$ has minimum degree $\delta$ at most $1$. In this case, the theorem above is a consequence of the following proposition which has already been established, particularly by Bondy [3].

**Proposition 1.** Let $G$ be a graph with independence number $\alpha$, then $G$ possesses a pseudo 2-factor of size at most $\alpha$.

**Proof.** The proof is done by induction on $\alpha$.

For $\alpha = 1$, it is true.

Suppose that $\alpha \geq 2$, let $P$ be a longest path in $G$ and let $x$ be an end-vertex of $P$.

(1) If $x$ has degree 1, then if we remove $x$ and its neighbor $x'$ we get $\alpha(G - \{x, x'\}) \leq \alpha - 1$. By induction hypothesis, $G - \{x, x'\}$ possesses a pseudo 2-factor containing at most $\alpha - 1$ cycles edges or vertices and adding $\{x, x'\}$ (and the edge joining them) we obtain a pseudo 2-factor of $G$ of size at most $\alpha$.

(2) If $x$ has degree at least 2, then consider $y$ the farthest neighbor of $x$ on $P$ and let $C$ be the cycle formed by the segment $[x, y]_P$ and the edge $e = (x, y)$. We have that $\alpha(G - C) \leq \alpha - 1$. By induction hypothesis, $G - C$ has a pseudo 2-factor of size at most $\alpha - 1$ and it follows that there exists a pseudo 2-factor of $G$ containing at most $\alpha$ cycles, edges or vertices.

From now, let $G$ be a graph with minimum degree $\delta \geq 2$ and independence number $\alpha \geq \delta$. Let $\mathcal{F}$ be a family $C_1, \ldots, C_t$ of vertex disjoint cycles of $G$. Denote by $F$ the smallest component of $G - \bigcup_{i=1}^{t} C_i$, set $W = G - (F \cup (\bigcup_{i=1}^{t} C_i))$ and choose a family $\mathcal{F}'$ of cycles for which:

(a) $\alpha(G - \bigcup_{i=1}^{t} C_i)$ as small as possible;
(b) subject to (a), $r$ as small as possible;
(c) subject to (a) and (b), $F$ as small as possible.

Notice that a family of cycles satisfying the conditions above exists. Indeed, since $\delta \geq 2$, then there exists at least a cycle $C$ such that $\alpha(G - C) < \alpha$. The cycle $C$ can be obtained using the construction with longest paths described in the proof of Proposition 1.

Furthermore each component of $W \cup F$ has minimum degree at most 1. Indeed if a component $A$ of $W \cup F$, has minimum degree $\delta_A$ at least 2, then a longest path $P$ in $A$ provides a cycle $C$ which verifies $\alpha(A - C) < \alpha(A)$ and $\alpha(G - \bigcup_{i=1}^{t} C_i) > \alpha(G - (\bigcup_{i=1}^{t} C_i \cup C))$. This contradicts (a) in the definition of $\mathcal{F}$.

We also remark that under conditions (a) and (b), each cycle of the family $\mathcal{F}$ verifies: $\alpha(W \cup F \cup C_i) > \alpha(W \cup F)$, for $i = 1, \ldots, r$. Indeed, if for some $k$ ($1 \leq k \leq r$), we have $\alpha(W \cup F \cup C_k) = \alpha(W \cup F)$, then the family $\mathcal{F}'$ of cycles $[C_i]_{i \neq k}$ would verify condition (a) and would contain less cycles than $\mathcal{F}$, contradicting condition (b) and thus the choice of $\mathcal{F}$.

Moreover, we shall show that if all the cycles of $\mathcal{F}$ are added to $W \cup F$ then the independence number of this latter will increase by at least $\delta - 1$. More precisely, we show the following result:
\textbf{Theorem 2.} Let $G$ be a graph with minimum degree $\delta \geq 2$ and independence number $\alpha \geq \delta$. Then there exists a pseudo 2-factor of $G$ such that $C_1, \ldots, C_t$ are the cycles of this pseudo 2-factor with

$$\alpha\left(G - \bigcup_{i=1}^{r} C_i\right) \leq \alpha - (\delta - 1).$$

This implies Theorem 1.

\textbf{Proof.} We need some further notations. Denote by $C_1, \ldots, C_t$, the cycles of $F$ on which $F$ possesses at least two neighbors, by $C_{t+1}, \ldots, C_s$ those on which $F$ possesses exactly one neighbor and by $C_{s+1}, \ldots, C_n$ those on which $F$ has no neighbor. Denote by $c_i$ the neighbor of $F$ on a cycle $C_i$, for $r_1 + 1 \leq i \leq r_2$ and with respect to a specific orientation of $C_i$, for each $i, 1 \leq i \leq r_1$, denote by $c_1^i, \ldots, c_{m_i}^i$ the neighbors of $F$, in this order, on $C_i$.

\textbf{Lemma 1.} Let $k$ and $l$ be two integers with $1 \leq k \leq l \leq r_2$. Let $C'$ be a cycle which contains the neighbors of $F$ on $C_k \cup C_l$, at least a vertex of $F$, and such that $V(C') \subset V((C_i \cup C_k) \cup F \cup W)$. Set $W' = G - (\bigcup_{j \neq k} C_j \cup F \cup C')$. Then $\alpha(W') > \alpha(W)$.

\textbf{Proof of Lemma 1.} Set $F_0 = F - C'$ and let $F'$ be the family of cycles $\{C_i \cup F, C'\}$.

(1) If $k \neq l$, then $F'$ contains less cycles than $F$ and hence must not verify condition (a), so:

$$\alpha(G - (\bigcup_{j \neq k} C_j \cup C')) > \alpha(G - (\bigcup_{i=1}^{r} C_i)) = \alpha(G) + \alpha(F).$$

On the other hand, $\alpha(G - (\bigcup_{j \neq k} C_j \cup C')) = \alpha(W') + \alpha(F_0)$, because $F$ has no neighbor in $(C_k \cup C_l) - C'$. It follows that $\alpha(W') > \alpha(W)$ (because $\alpha(F) \geq \alpha(F_0)$).

(2) If $k = l$, then $F'$ and $F$ have the same number of cycles. Two cases may occur.

(i) If $F_0 = \emptyset$, then by condition (a) on $F$, we have:

$$\alpha(W') = \alpha(G - (\bigcup_{j \neq k} C_j \cup C')) \geq \alpha(G - (\bigcup_{i=1}^{r} C_i)) = \alpha(G) + \alpha(F) = \alpha(W') \geq \alpha(W) + 1.$$

(ii) If $F_0 \neq \emptyset$, then $F_0$ is smaller than $F$. The family of cycles $F'$ verifies (b) and consequently do not verify condition (a), otherwise we get a contradiction with condition (c), hence:

$$\alpha(W') + \alpha(F_0) = \alpha(G - (\bigcup_{j \neq k} C_j \cup C')) > \alpha(G - (\bigcup_{i=1}^{r} C_i)) = \alpha(W) + \alpha(F).$$

As $\alpha(F_0) \leq \alpha(F)$ thus $\alpha(W') > \alpha(W)$.

Let $V$ be an interval on a cycle $C_k$, for $1 \leq k \leq r_2$.

We say that the interval $V$ has property $\Theta$ if and only if $\alpha(W \cup F \cup V) = \alpha(W \cup F)$. We say that two different intervals $V$ and $V'$ are path-independent if there exists no path internally disjoint from $\bigcup_{i=1}^{r_2} C_i \cup F$ joining a vertex of $V$ to a vertex of $V'$. We say that $t$ intervals are path-independent if they are pairwise path-independent. The following lemma will be intensively used:

\textbf{Lemma 2.} Let $V$ and $V'$ be two different intervals not neighbors of $F (V \cup V' \subset C_k \cup C_l, 1 \leq k \leq l \leq r_2)$. Suppose that both $V$ and $V'$ have property $\Theta$, then

(1) If $V$ and $V'$ are path-independent then $V \cup V'$ has property $\Theta$.

(2) $V$ and $V'$ are path-independent.

(3) More generally, if $t$ disjoint intervals $V^{(1)}, V^{(2)}, \ldots, V^{(t)} (t \geq 2)$ have property $\Theta$ then $V^{(1)} \cup V^{(2)} \cup \ldots \cup V^{(t)}$ has property $\Theta$.

\textbf{Proof of Lemma 2.} (1) Let $H$ (respectively $H'$) be the union of the components of $W$ that have a neighbor on $V$ (respectively on $V'$). By hypothesis, as $V$ and $V'$ are path independent, $H \cap H' = \emptyset$, hence $W = (H \cup H')$, $H \cup V$, $H' \cup V'$ form a partition of $W \cup V \cup V'$ and it follows that $\alpha(W \cup V \cup V') = \alpha(W - (H \cup H')) + \alpha(H \cup V) + \alpha(H' \cup V')$. Furthermore, because of property $\Theta$, $\alpha(H \cup V) = \alpha(H)$ and $\alpha(H' \cup V') = \alpha(H')$. So $\alpha(W \cup V \cup V') = \alpha(W - (H \cup H')) + \alpha(H') + \alpha(H) = \alpha(W)$.

(2) First, denote by $P_{ij}^k$ a path with internal vertices in $F$ joining the vertices $c_i^j$ and $c_i^j$ belonging to a same cycle $C_i$ ($1 \leq i \leq r_1$), or simply $P_{ij}^k$ if the vertices joined belong to different cycles ($1 \leq i \leq r_1$). Suppose that there exists a path internally disjoint from $\bigcup_{i=1}^{r_2} C_i \cup F$ joining $V$ and $V'$. It implies that either there is a path with internal vertices in $W$ joining a vertex in $V$ to a vertex in $V'$ or that a vertex in $V$ is adjacent to a vertex in $V'$. We distinguish two cases according to the fact that $V$ and $V'$ are on the same cycle or not.

(i) Suppose that $V$ and $V'$ belong to a same cycle $C_k$, $1 \leq k \leq r_1$. Put $V = [c_k^i, v_k]^c_k$ and $V' = [c_k^{j'}, v_k^{j'}]^c_k$ (with $1 \leq j < j' \leq m_k$). Let $x \in V$ and $x' \in V'$ be two vertices joined by a path of $W \cup \{x \} \cup \{x' \}$ and chosen so as to minimize the sum of lengths $d_{c_k}(c_k^i, x)$ and $d_{c_k}(c_k^{j'}, x')$. Note that by this choice the segments $[c_k^i, x]_{c_k}$ and $[c_k^{j'}, x']_{c_k}$ are path-independent, furthermore they do both have property $\Theta$ (as they are, respectively, included in $V$ and $V'$). So by (1), we have:

$$\alpha(W \cup [c_k^i, x]_{c_k} \cup [c_k^{j'}, x']_{c_k}) = \alpha(W)$$

- If $x$ and $x'$ are adjacent, then taking $C' = c_k^{P_{ij}^k} [c_k^i, x]_{c_k} (x, x')[c_k^{j'}, c_k^i]_{c_k}$ in Lemma 1, we obtain $\alpha(W) < \alpha(W \cup [c_k^i, x]_{c_k} \cup [c_k^{j'}, x']_{c_k})$ and hence a contradiction with ($\ast$).

- If $x$ and $x'$ are not adjacent, then taking $C' = c_k^{P_{ij}^k} [c_k^i, x]_{c_k} (x, x')[c_k^{j'}, c_k^i]_{c_k}$ in Lemma 1, we obtain $\alpha(W) < \alpha(W \cup [c_k^i, x]_{c_k} \cup [c_k^{j'}, x']_{c_k})$ and hence a contradiction with ($\ast$).
Lemma 2

There is no path internally disjoint fromLemma 1

and setting $\alpha(W)$ gives a contradiction with the definition of the family $F$.

Proof of Lemma 4.

1. Let $V(1), V(2), \ldots, V(t)$ be $t (t \geq 2)$ different intervals having property $\Theta$.

2. By (2) of Lemma 2, they are path-independent. By induction on $t$ we show that property $\Theta$ is conserved in $V(1) \cup V(2) \cup \cdots \cup V(t)$. We set $V_1 = V(1) \cup V(2) \cup \cdots \cup V(t-1)$, $V_2 = V(t)$ and the result follows by the proof of Lemma 2(1).

We already know that by conditions on the chosen family of cycles, the addition of a cycle $C_i$ of $F$ to $W$ if $F$ increases the independence number of $W \cup F$ by at least 1. We show now that this augmentation can be more significant if $F$ possesses more than a neighbor on the added cycle, in other words if $1 \leq i \leq r_1$. To see that, it suffices to consider the segments of $C_i (1 \leq i \leq r_1)$ included between two consecutive neighbors of $F$. They are of type $|c_i|, |c_i|$, where $j$ is taken modulo $m_i$ ($1 \leq j \leq m_i$). We claim that these segments do not have property $\Theta$. Let $P'_j$ and $P'_j$ be as defined in the proof of Lemma 2.

Lemma 3. For all $i, j, 1 \leq i \leq r_1$, and, $1 \leq j \leq m_i$ where $j$ is taken modulo $m_i$, we have

1. $|c_i|, |c_i| \neq \emptyset$.
2. $\alpha(W \cup |c_i|, |c_i|) > \alpha(W)$.

Proof of Lemma 3. (1) Suppose to the contrary that a segment $|c_i|, |c_i| = \emptyset$ on some cycle $C_i$. Then, $C' = |c_i|, |c_i| \cup \emptyset$ gives a contradiction with the definition of the family $F$.

(2) Setting $C' = |c_i|, |c_i|$, in Lemma 1, we obtain $\alpha(W') = \alpha(W \cup |c_i|, |c_i|) > \alpha(W)$.

Let $u'_i$ be the first vertex in $|c_i|, |c_i|$ such that $\alpha(W \cup |c_i|, u'_i) > \alpha(W)$.

Lemma 4. There is no path internally disjoint from $\cup_{i=1}^{r_1} C_i \cup F$ joining a segment $|c_i|, u'_i$ to a segment $|c_k|, u'_k$, where $1 \leq k \leq l \leq r_1$, $1 \leq j \leq m_i$ and $1 \leq j' \leq m_k$.

Proof of Lemma 4. Suppose that there is a path internally disjoint from $\cup_{i=1}^{r_1} C_i \cup F$ joining a vertex $v \in |c_i|, u'_i$ to a vertex $v' \in |c_k|, u'_k$, and choose $v$ and $v'$ so that the sum of the lengths of $|c_i|, v|c_i|$ and $|c_k|, v'|c_k|$ is minimum. Two cases are to be taken under consideration:

Case $k = l$

(1) If $v$ and $v'$ are adjacent. Setting $C' = |c_i|, |c_i| \cup v, v'_i|c_i|$ in Lemma 1, we obtain: $\alpha(W \cup |c_i|, v|c_i|, v'|c_i|) > \alpha(W)$. (*)

On the other hand, by the choice of $v$ and $v'$, $|c_i|, v|c_i|$ and $|c_i|, v'|c_i|$ are path-independent, as they both have property $\Theta$ (by the choice of $u'_i$ and $u'_k$) then byLemma 2, we get $\alpha(W \cup |c_i|, v|c_i|, v|c_i|) = \alpha(W)$, which contradicts (*).

(2) If $v$ and $v'$ are joined by a path $Q^{l}_{ij}$ with internal vertices in $W$. Then taking $C' = |c_i|, |c_i| \cup v, v|c_i|$ in Lemma 1, and setting $W_0 = W - Q^{l}_{ij}$, we obtain: $\alpha(W \cup |c_i|, v|c_i|, v|c_i|) \geq \alpha(W_0 \cup |c_i|, v|c_i|, v|c_i|) > \alpha(W)$. (***)

On the other hand by Lemma 2 and because $|c_i|, v|c_i|$ and $|c_i|, v|c_i|$ are path-independent, and they both have property $\Theta$, we have $\alpha(W \cup |c_i|, v|c_i|, v|c_i|, v|c_i|) = \alpha(W)$, so we get a contradiction with the inequality (***)

Case $k \neq l$

(1) If $v$ is adjacent to $v'$. Setting $C' = |c_i|, |c_i| \cup v, v|c_i|$ in Lemma 1, we obtain: $\alpha(W \cup |c_i|, v|c_i|, v|c_i|) > \alpha(W)$. Furthermore, using Lemma 2 and the fact that $|c_i|, v|c_i|$ and $|c_i|, v|c_i|$ are path-independent and have property $\Theta$, we get $\alpha(W \cup |c_i|, v|c_i|, v|c_i|) = \alpha(W)$ and hence a contradiction.

(2) If $v$ and $v'$ are joined by a path $Q^{l}_{ij}$ with internal vertices in $W$. Then taking $C' = |c_i|, |c_i| \cup Q^{l}_{ij}, v|c_i|$ in Lemma 1, and setting $W_0 = W - Q^{l}_{ij}$, we obtain: $\alpha(W \cup |c_i|, v|c_i|, v|c_i|, v|c_i|) > \alpha(W)$. (****)

On the other hand, $|c_i|, v|c_i|$ and $|c_i|, v|c_i|$ are path independent (by the choice of $v$ and $v'$) and have property $\Theta$ (by the choice of $u'_i$ and $u'_k$) so Lemma 2 gives: $\alpha(W) = \alpha(W \cup |c_i|, v|c_i|, v|c_i|)$ which contradicts (****).
Lemma 5. Let $r + 1 + 1 \leq k \leq r_2$ and let $C_k$ be a cycle such that $C_k \setminus \{c_k\}$ does not have property $\Theta$. Let $u_k \in C_k \setminus \{c_k\}$ be the first vertex such that $|c_k, u_k|_{C_k}$ does not have property $\Theta$ then

(a) $|c_k, u_k|_{C_k}$ is path-independent from each other segment $[c_l, u_l]_{C_l}$ for $r_1 + 1 \leq l \leq r_2$ verifying the same hypothesis.

(b) $|c_k, u_k|_{C_k}$ is path-independent from each segment $|c_l, u_l|_{C_l}$ for $1 \leq l \leq r_1$ and $1 \leq j \leq m_l$.

Proof of Lemma 5. The proof is similar to the proof of Lemma 4. □

Now if $C_k \setminus \{c_k\}$ has property $\Theta$ then we distinguish two cases,

Case 1: $\alpha(W \cup F \cup \{c_k\}) = \alpha(W \cup F)$. Following the orientation of $C_k$, let $u_k$ be a vertex of $C_k \setminus \{c_k\}$, the nearest to $c_k$ such that $\alpha(W \cup F \cup \{c_k, u_k[c_k]\}) > \alpha(W \cup F)$.

Case 2: $\alpha(W \cup F \cup \{c_k\}) > \alpha(W \cup F)$.

We show that

Lemma 6. In case 1, $|c_k, u_k|_{C_k}$ is path-independent from any segment $[c_l, u_l]_{C_l}$ for $r_1 + 1 \leq l \leq r_2$ or $|c_l, u_l|_{C_l}$ for $1 \leq l \leq r_1$ and $1 \leq j \leq m_l$.

In case 2, $|c_k|_{C_k}$ is path-independent from any $|c_l|_{C_l}$ for $r_1 + 1 \leq l \leq r_2$ verifying $\alpha(W \cup F \cup \{c_k\}) > \alpha(W \cup F)$ and from any interval of the form $[c_l, u_l|_{C_l}$ for $r_1 + 1 \leq l \leq r_2$, $|c_l, u_l|_{C_l}$ for $1 \leq l \leq r_1$, $1 \leq j \leq m_l$.

Proof of Lemma 6. The proofs are very similar to the proof of Lemma 4 or of Lemma 5.

In case 1, to show that $|c_k, u_k|_{C_k}$ is path-independent from any segment $|c_l, u_l|_{C_l}$ or $|c_l, u_l|_{C_l}$ or $|c_l, u_l|_{C_l}$, we suppose to the contrary that a vertex $v \in [c_k, u_k]_{C_k}$ is joined to a vertex $v' \in [c_l, u_l]_{C_l}$ or $[c_l, u_l]_{C_l}$ or $[c_l, u_l]_{C_l}$. In any case and reasoning the same way as Lemma 4, we get a contradiction.

In case 2, we show that $|c_k|_{C_k}$ is path-independent from any $|c_l|_{C_l}$ verifying the same hypothesis of case 2 or $|c_l, u_l|_{C_l}$ or $|c_l, u_l|_{C_l}$ or $|c_l, u_l|_{C_l}$. In this case, notice that by hypothesis $[c_k^+, c_k^-]$ has property $\Theta$ and is by Lemma 2 path-independent from any segment $[c_l^+, c_l^-]$ of the same type. And the same proof of Lemma 4 gives the desired result. □

In Lemma 6, for $r_1 + 1 \leq k \leq r_2$, both $[c_k, u_k]_{C_k}$ and $[c_k]_{C_k}$ contain a neighbor of $F$. For technical reasons, we are not going to consider all the cycles $C_k \setminus \{c_k\}$ of which $F$ has exactly one neighbor but only those cycles on which a fixed vertex $x_0 \in F$ has a neighbor. We choose $x_0$ such that $d_F(x_0) = \delta_F$. We label these cycles from $r_1 + 1$ to $r_3$ ($r_3 \leq r_2$). We observe that

Observation 1. Let $k$ be an integer such that $r_1 + 1 \leq k \leq r_3$ and such that $C_k \setminus \{c_k\}$ has property $\Theta$. Then $z_0$ is the only neighbor of $c_k$ in $F$.

Proof. Suppose to the contrary that $|I'_F(c_k)| \geq 2$, where $I'_F(c_k)$ is the neighborhood of $c_k$ in $F$. Let $x \in F, x \neq z_0$ be another neighbor of $c_k$ and let $P$ be the path with internal vertices in $F$ joining $x$ and $z_0$. Then taking $C' = c_kxPz_0c_k$ in Lemma 1 and the fact that $\alpha(W \cup C_k \setminus \{c_k\}) = \alpha(W)$ gives a contradiction. ■

Observation 2. If $z_0$ belongs to every maximum independent set $S$ of $F$, then for $k, r_1 + 1 \leq k \leq r_3$ there does not exist segments $l_i$ of type $[c_k, u_k]_{C_k}$ or $[c_k]_{C_k}$ (as defined above) such that $\alpha(W \cup F \cup l_i) \geq \alpha(W \cup F) + 1$.

Proof. Suppose that $z_0$ is contained in every maximum independent set $S$ of $F$ and that there exists a segment $l_i$ of type $[c_k, u_k]_{C_k}$ or $[c_k]_{C_k}$ such that $\alpha(W \cup F \cup l_i) \geq \alpha(W \cup F) + 1$. Clearly by the minimality of $l_i$, $c_k$ is contained in a maximum independent set of $W \cup F \cup l_i$.

(1) If $l_i = [c_k, u_k]_{C_k}$ for some $k, r_1 + 1 \leq k \leq r_3$.

Let $s_{max}(W \cup F \cup l_i)$ be a maximum independent set of $W \cup F \cup l_i$. $s_{max}(W \cup F \cup l_i)$ contains either $c_k$ or $z_0$ but not both. So $s_{max}(W \cup F \cup l_i) = s_{max}(W \cup F \cup l_i \setminus [c_k]) = s_{max}(W \cup F \cup l_i \setminus [c_k])$ hence $\alpha(W \cup F \cup l_i \setminus [c_k]) = \alpha(W \cup F \cup l_i) > \alpha(W \cup F)$ and this is a contradiction because $l_i \setminus [c_k] \subset C_k \setminus \{c_k\}$ and by hypothesis an interval $l_i = [c_k, u_k]_{C_k}$ is chosen when $C_k \setminus \{c_k\}$ verifies property $\Theta$.

(2) If $l_i = [c_k]$ for some $k, r_1 + 1 \leq k \leq r_3$. Here again either $c_k$ or $z_0$, but not both, belong to a maximum independent set of $W \cup F \cup l_i$, so $\alpha(W \cup F) = \alpha(W \cup F \cup l_i)$ and this gives a contradiction with $\alpha(W \cup F) = \alpha(W \cup F \cup l_i)$. ■
To summarize, we have shown that on every cycle \( C_k \), for \( 1 \leq k \leq r_3 \), there is a segment \( I_1 \) or \( m_k \) segments \( I_i \), which if added will increase the independence number of \( W \cup F \). We have showed that these segments are pairwise path-independent. To achieve the proof of Theorem 2, we look at two cases:

1. If every maximum independent set \( S \) of \( F \) contains \( z_0 \), then by Observation 2, we have only segments \( I_i \) of type \( \{c_k, u_k\}_c_k \) (for \( r_1 + 1 \leq k \leq r_3 \)) or \( \{c_j, u_{j-k}\}_c_j \) (for \( 1 \leq k \leq r_1, 1 \leq j \leq m_k \)) that verify \( \alpha(W \cup F \cup I_i) > \alpha(W \cup F) \). Notice that these segments are independent from \( F \) and pairwise path-independent (by Lemmas 4 and 5). Set \( m = \sum_{k=1}^{r_1} m_k \) and let \( H^I \) be the union of the components of \( W \) that contain a neighbor of \( I_i \). We have:

\[
\alpha(W \cup F \cup \bigcup_{k=1}^{r_1} C_k) \geq \alpha(W \cup F \cup \bigcup_{i=1}^{m+r_3-r_1} I_i) = \alpha(W - \bigcup_{i=1}^{m+r_3-r_1} H^I) + \sum_{i=1}^{m+r_3-r_1} \alpha(H^I \cup I_i) + \alpha(F) \geq \alpha(W - \bigcup_{i=1}^{m+r_3-r_1} H^I) + \sum_{i=1}^{m+r_3-r_1} \alpha(H^I) + \alpha(F) + m + r_3 - r_1 = \alpha(W \cup F) + m + r_3 - r_1.
\]

2. If there exists a maximum independent set \( S \) of \( F \) such that \( z_0 \notin S \), then put \( F' = F - \{z_0\} \). We have \( \alpha(F) = \alpha(F') \). Notice that using Observation 1, any segment \( I_i \) is independent from \( F' \). Furthermore, by Lemmas 4–6, all the segments \( I_i \) are pairwise path-independent. Replacing \( F \) by \( F' \) in the proof of the previous case, we get

\[
\alpha \geq \alpha(W \cup F' \cup \bigcup_{i=1}^{r_1} C_i) \geq \alpha(W \cup F') + r_3 - r_1 + m = \alpha(W \cup F) + r_3 - r_1 + m.
\]

Finally, by the choice of \( z_0 \), we have that \( r_3 - r_1 + m \geq d(z_0) - 1 \geq \delta - 1 \) and it follows that in both case (1) and case (2) we have

\[
\alpha \geq \alpha(W \cup F \cup \bigcup_{i=1}^{r_1} C_i) \geq \alpha(W \cup F) + \delta - 1.
\]

So \( \alpha(G - \bigcup_{i=1}^{r_1} C_i) \leq \alpha - \delta + 1 \) and this completes the proof of Theorem 2.

Proof of Theorem 1. According to Proposition 1, \( G - \bigcup_{i=1}^{r_1} C_i \) can be covered by at most \( \alpha(G - \bigcup_{i=1}^{r_1} C_i) \) vertex-disjoint components that are cycles, edges or vertices. Denote by \( \mathcal{E} \) the set of these components. By Theorem 2, the number of these components is at most \( \alpha - \delta + 1 \). Finally, \( \mathcal{F} \cup \mathcal{E} \) is a pseudo 2-factor of \( G \) with at most \( \alpha - \delta + 1 \) components that are edges or vertices and this completes the proof of Theorem 1. 

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References