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# THE HOPF RING FOR COMPLEX COBORDISM

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In [38] Thom defined the unoriented and oriented cobordism rings, soon generalized to complex cobordism by Milnor [19] and Novikov [22]. These geometric constructions were later shown to give rise to generalized homology and cohomology theories [39] by Atiyah [3]. These theories have received a great deal of attention in recent years.

In this paper we offer three new things. First, we obtain unstable homotopy theoretic information from formal group laws. Second, we make essential use of the concept of Hopf rings both in the description of our results and in the proofs. Third, we give a detailed analysis of the homology structure of the (unstable) classifying spaces for complex cobordism, including a completely algebraic construction which contains total information about the unstable complex cobordism operations. Some of our results were announced in [29].

## 0. Introduction

Since the introduction of formal groups into cobordism theory they have been applied to obtain many useful stable homotopy results. Quillen's results [23, 24, 1] are among them, in particular, his direct computation of the complex cobordism ring [23] and his description of the operation algebra for Brown–Peterson cohomology [24]. Later came Hazewinkel's construction of canonical generators for the Brown–Peterson coefficient ring [7, 8]. The results of Quillen and Hazewinkel have made it possible to compute effectively. More recently, the construction of the Morava stabilizers [20, 25] has led [21, 26, 16, 18] to a great deal of new information about the stable homotopy of spheres [9, 17, 27]. However, although formal group laws for homology theories are defined unstably, this fact

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has never really been exploited to obtain unstable information. We rectify that matter.

If $E^*(-)$ is a multiplicative cohomology theory with $E^*\mathbb{C}P^\infty \simeq E^*[[x]]$ (the power series), $x \in E^2\mathbb{C}P^\infty$, we define $a_{ij} \in E^{-2(i+j-1)} \simeq E_{2(i+j-1)}$ by use of the unstable $H$-space product

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty;$$

$$x \to \sum_{i,j\geq 0} a_{ij}x^i \otimes x^j \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty).$$

The formal group law is given by

$$F(y,z) = y +_F z = \sum_{i,j\geq 0} a_{ij}y^iz^j.$$

Dual to $x^n$ is $\beta_n \in E_{2n}\mathbb{C}P^\infty$ with coproduct $\beta_n \to \sum_{i=0}^n \beta_{n-i} \otimes \beta_i$. Let $G^*(-)$ be another such cohomology theory and let $\boldsymbol{G}_* = \{G_k\}_{k\in Z}$ be an $\Omega$-spectrum representing it. Then $x^G$ is a map $\mathbb{C}P^\infty \to \boldsymbol{G}_2$. We let $(x^G)_*(\beta_n) = b_n \in E_{2n}\boldsymbol{G}_2$. The loop and multiplicative structures on $\boldsymbol{G}_*$ induce two products $*$ and $\circ$ respectively on $E_*\boldsymbol{G}_*$. Elements $v \in G^*$ give rise to elements $[v] \in E_0\boldsymbol{G}_*$. Let $b(s) = \sum_{i\geq 0} b_is^i$ and $y +_{[F]} z = *_{i,j\geq 0}[a_{ij}] \circ y^{\circ i} \circ z^{\circ j}$. Our main unstable relation is:

**Theorem 3.8.** *In $E_*\boldsymbol{G}_*[[s,t]]$*

$$b(s +_{F_E} t) = b(s) +_{[F]_G} b(t).$$

This follows from our relation:

**Theorem 3.4.** *In $E_*\mathbb{C}P^\infty[[s,t]]$*

$$\beta(s)\beta(t) = \beta(s +_F t).$$

If we specialize to $E = BP$ where $BP_* \simeq Z_{(p)}[v_1, v_2, \ldots]$, Theorem 3.4 gives up very explicit information.

**Theorem 3.12.** *In $QBP_*\mathbb{C}P^\infty \bmod (p)$,*

$$\sum_{i=1}^{n} v_i^{p^{n-i}} \beta_{p^{n-i}} = 0.$$

Theorem 3.8 is particularly useful when applied to $G = MU$ or $BP$. In fact, it allows one to describe $E_*\boldsymbol{MU}_*$. $E_*\boldsymbol{MU}_*$ has a coproduct and the two products $*$ and $\circ$ turn it into a ring object in the category of coalgebras, i.e. a Hopf ring. We only consider the even spaces for $MU$, so $x^{MU} : \mathbb{C}P^\infty \to \boldsymbol{MU}_1$. Let the Hopf ring $E_*^R\boldsymbol{MU}_*$ be constructed completely algebraically from the elements $[v]$, $v \in MU^*$, $b_i$, the relations from Theorem 3.8 and the general properties of Hopf rings. We then have:

**Corollary 4.7.** *There is an isomorphism of Hopf rings*: $E_*^R\boldsymbol{MU}_* \simeq E_*\boldsymbol{MU}_*$.

Specializing to the case $E = MU$, this result gives a completely algebraic construction for $MU_*\boldsymbol{MU}_*$ which includes both products, the coproduct and the $MU_*$ module structure. The dual of this, $MU^*\boldsymbol{MU}_*$, is the algebra of unstable complex cobordism operations.

Specializing to $H_*(\boldsymbol{MU}_*; Z)$ we see there is no torsion because there are no odd degree elements in $H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p)$. For $H_*(\boldsymbol{MU}_*; \mathbb{F}_p)$ we can even compute the coaction of the dual of the Steenrod algebra because it is known on the $[v]$'s and the $b$'s [37].

As we see from Theorem 4.7, the spaces $\boldsymbol{MU}_*$ have a very rich structure. A study of their homotopy type [41] has led to useful applications [10, 34]. In [34], use is made of Theorem 5.3 below as well as the results of [41].

There are similar results to those above for $BP$, and, specializing to $H_*(\boldsymbol{BP}_*; \mathbb{F}_p)$ (where again $\boldsymbol{BP}_k$ is the $2k$ space in the $\Omega$-spectrum for $BP$) we can do better than produce abstract constructions and isomorphisms. Here we can given explicit formulas. First, as another corollary of Theorem 3.8, we have:

**Theorem 3.14.** *In* $QH_*(\boldsymbol{BP}_*; \mathbb{F}_p)/I^2QH_*(\boldsymbol{BP}_*; \mathbb{F}_p)$

$$\sum_{i=1}^{n} [v_i] \circ b_{p^{n-i}}^{\circ p^i} = 0$$

*where* $I = ([v_1], [v_2], \ldots)$.

We can now give a detailed description of the Hopf ring $H_*(\boldsymbol{BP}_*; \mathbb{F}_p)$. Denote $b_{p^i}$ by $b_{(i)}$ and define

$$v^I b^J = [v_1^{i_1} v_2^{i_2} \cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots.$$

Let $\boldsymbol{BP}_*'$ be the zero components of $\boldsymbol{BP}_*$.

**Theorem 5.3.**

(a) $H_*(\boldsymbol{BP}_*'; \mathbb{F}_p)$ *is a (bi)-polynomial Hopf algebra.*

(b) *A basis for* $QH_*(\boldsymbol{BP}_*'; \mathbb{F}_p)$ *is given by all* $v^I b^J$ $(J \neq 0)$ *such that if*

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + \cdots + p^n\Delta_{k_n} + J'$$

*where* $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ *and* $J'$ *is another sequence of non-negative numbers, then* $i_n = 0$.

(c) *A basis for* $PH_*\boldsymbol{BP}_*'$ *is given by all* $v^I b^J \circ b_1$ *where* $v^I b^J$ *(J possibly zero) satisfies the condition in* (b).

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Section 1 is a detailed account of graded ring objects over a category. Here we completely describe what we mean by a Hopf ring. Some of the properties are unnecessary for this paper but we will need them in a future paper where we hope to compute the Morava $K$-theories [11] of Eilenberg–MacLane spaces [30]. We also discuss graded spaces and $E_*G_*$ as a Hopf ring. At the end of the section we construct certain Hopf rings we need later.

Section 2 deals with the special graded spaces associated with $MU$, in particular we develop the geometry necessary for our main geometrical corollary (4.12).

In Section 3 we prove Theorems 3.8, 3.4, 3.14 and 3.12, give other applications of 3.8 and some examples of how to compute with it. In Section 4 we state and prove our main isomorphisms 4.7 and in Section 5 we do Theorem 5.3. Section 6 states what is known about homology operations and relates a geometric problem of interest.

Each section has its own introduction.

## 1. Hopf rings

In this section we define a graded ring object over a general category. Specializing to the category of coalgebras we call such an object a Hopf ring and in Lemma 1.12 we write down an explicit description of all of the defining formulas. We then show how the generalized homology of the spaces in an $\Omega$-spectrum often give rise to Hopf rings and we give the basic properties of a Hopf ring which comes about in this way. Later on in the paper we will construct Hopf rings purely algebraically. In order to do this we need the notion of a free Hopf ring which we develope at the end of this section. The main purpose of this section is to establish the necessary permanent reference for the precise details of a Hopf ring.

We would like to thank J.C. Moore, K. Sinkinson and R.W. Thomason for helpful discussions about the material in this section. We are particularly grateful to H.R. Miller for completely changing our perspective by showing us the categorical possibilities when he informed us that the algebraic monstrosity we were dealing with was just a ring in the category of coalgebras.

Let $\mathscr{C}$ be a category with finite products ($\Pi$). We assume our products are chosen in such a way as to be functorial (and associative). We let $\mathscr{C}(X, Y)$ denote the morphisms (maps) from $X$ to $Y$ in $\mathscr{C}$. We let $1_X \in \mathscr{C}(X,X)$ be the identity morphism. A *terminal object* $N$ is an object $N$ of $\mathscr{C}$ such that $\mathscr{C}(X,N)$ contains exactly one morphism, $\varepsilon_X = \varepsilon$, for all $X \in \mathscr{C}$. We will assume our category $\mathscr{C}$ has a terminal object.

An *abelian group object* of $\mathscr{C}$ is an object $X \in \mathscr{C}$ and maps $\eta \in \mathscr{C}(N,X)$ (abelian group unit, i.e. zero), $* \in \mathscr{C}(X\Pi X, X)$ (addition) and $\chi \in \mathscr{C}(X,X)$ (inverse) such that the following diagrams commute:

**1.1**
$$\begin{array}{ccc} N\Pi X & \xrightarrow{p_2} & X \\ {\scriptstyle \eta\Pi 1_X}\downarrow & & \downarrow{\scriptstyle 1_X} \\ X\Pi X & \xrightarrow{*} & X \end{array}$$

1.2
$$\begin{array}{ccc} X\Pi X & \xrightarrow{\ *\ } & X \\ {\scriptstyle (p_2,p_1)}\downarrow & \nearrow{\scriptstyle *} & \\ X\Pi X & & \end{array}$$

1.3
$$\begin{array}{ccc} X\Pi X\Pi X & \xrightarrow{1_X\Pi *} & X\Pi X \\ {\scriptstyle *\Pi 1_X}\downarrow & & \downarrow{\scriptstyle *} \\ X\Pi X & \xrightarrow{\ *\ } & X \end{array}$$

1.4
$$\begin{array}{ccc} X & \xrightarrow{(1_X,\chi)} & X\Pi X \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle *} \\ N & \xrightarrow{\ \eta\ } & X \end{array}$$

The diagrams give the standard abelian group properties: 1.1, addition by zero; 1.2, commutativity: 1.3, associativity; 1.4, inverses.

The *category of graded objects of* $\mathscr{C}$, $G\mathscr{C}$, has as objects $X_* = \{X_n\}_{n\in Z}$ where $X_n \in \mathscr{C}$ and morphisms, $G\mathscr{C}(X_*, Y_*)$, all $f_* = \{f_n\}_{n\in Z}$, $f_n \in \mathscr{C}(X_n, Y_n)$. We also have the category of nonnegatively graded objects of $\mathscr{C}$, $G_+\mathscr{C}$, and the category of evenly graded objects of $\mathscr{C}$, $G_2\mathscr{C}$.

A *commutative graded ring object with unit over* $\mathscr{C}$ (henceforth (*graded*) *ring object*) is an abelian group object $X_* \in G\mathscr{C}$, i.e. each $X_n$ is an abelian group object of $\mathscr{C}$ with inverse $\chi_n = \chi$, addition $*_n = *$ and zero $\eta_n = \eta$. Furthermore, we have maps $e \in \mathscr{C}(N, X_0)$ (multiplicative unit) and $\circ_{ij} = \circ \in \mathscr{C}(X_i\Pi X_j, X_{i+j})$ (multiplication) such that the following diagrams commute:

$$\psi_X = (1_X, 1_X) \in \mathscr{C}(X, X\Pi X)$$

1.5
$$\begin{array}{ccc} X_i\Pi X_j\Pi X_k & \xrightarrow{1_{X_i}\Pi\circ} & X_i\Pi X_{j+k} \\ {\scriptstyle \circ\Pi 1_{X_k}}\downarrow & & \downarrow{\scriptstyle \circ} \\ X_{i+j}\Pi X_k & \xrightarrow{\ \circ\ } & X_{i+j+k} \end{array}$$

1.6
$$\begin{array}{ccc} X_i\Pi X_j & \xrightarrow{\ \circ\ } & X_{i+j} \\ {\scriptstyle (p_2,p_1)}\downarrow & & \downarrow{\scriptstyle \chi^{ij}} \\ X_j\Pi X_i & \xrightarrow{\ \hat{\circ}\ } & X_{i+j} \end{array}$$

1.7
$$\begin{array}{ccccc} X_i\Pi X_j\Pi X_j & \xrightarrow{\psi\Pi 1_{X_j\Pi X_j}} & X_i\Pi X_i\Pi X_j\Pi X_j & \xrightarrow{(p_1,p_3,p_2,p_4)} & X_i\Pi X_j\Pi X_i\Pi X_j \\ {\scriptstyle 1_{X_i}\Pi *}\downarrow & & & & \downarrow{\scriptstyle \circ\Pi\circ} \\ X_i\Pi X_j & \xrightarrow{\qquad \circ \qquad} & X_{i+j} & \xleftarrow{\qquad * \qquad} & X_{i+j}\Pi X_{i+j} \end{array}$$

**1.8**
$$\begin{array}{ccc} N\Pi X_i & \xrightarrow[\simeq]{p_2} & X_i \\ {\scriptstyle e\Pi 1_{X_i}}\downarrow & & \downarrow{\scriptstyle 1_{X_i}} \\ X_0\Pi X_i & \xrightarrow{\circ} & X_i \end{array}$$

**1.9**
$$\begin{array}{ccc} N\Pi X_j & \xrightarrow{\varepsilon} & N \\ {\scriptstyle \eta\Pi 1_{X_j}}\downarrow & & \downarrow{\scriptstyle \eta} \\ X_i\Pi X_j & \xrightarrow{\circ} & X_{i+j} \end{array} .$$

The diagrams give the standard graded ring properties: 1.5, associative multiplication; 1.6, commutativity; 1.7, distributivity; 1.8, multiplication by the unit; 1.9, multiplication by zero.

For $G_+\mathscr{C}$ and $G_2\mathscr{C}$ we have the concepts of nonnegatively graded and evenly graded ring objects respectively. Moreover, if $X_*$ is a ring object then $X_* = \{X_n\}_{n\geq 0}$ and $X_{2*} = \{X_{2n}\}_{n\in Z}$ are ring objects in $G_+\mathscr{C}$ and $G_2\mathscr{C}$ respectively. The concepts of maps of graded ring objects and the category of graded ring objects over $\mathscr{C}$ are the obvious ones.

Let $\mathscr{D}$ be a category with finite products and a terminal object $N_{\mathscr{D}}$. Let $\mathscr{F}$ be a product preserving functor from $\mathscr{C}$ to $\mathscr{D}$, i.e. $\mathscr{F}(N_{\mathscr{C}}) = N_{\mathscr{D}}$ and there is a natural equivalence of functors of $\mathscr{C}\times\mathscr{C}$ to $\mathscr{D}$,

**1.10** $\qquad \mathscr{F}(-)\Pi\mathscr{F}(-) \simeq \mathscr{F}(-\Pi-).$

$\mathscr{F}$ induces an obvious functor $\mathscr{F}: G\mathscr{C}\to G\mathscr{D}$ by $\mathscr{F}(X_*) = \{\mathscr{F}(X_n)\}_{n\in Z}$ and $\mathscr{F}(f_*) = \{\mathscr{F}(f_n)\}_{n\in Z}$.

**Lemma 1.11.** *Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{F}$ be as above. If $X_*\in G\mathscr{C}$ is a graded ring object over $\mathscr{C}$, then $\mathscr{F}(X_*)\in G\mathscr{D}$ is a graded ring object over $\mathscr{D}$.*

**Proof.** Just apply $\mathscr{F}$ and 1.10 to all of the defining diagrams.

Let $R$ be a graded (associative, commutative) ring (with unit). We let $\mathscr{D}$ = $\text{CoAlg}_R$ be the category of graded cocommutative coassociative coalgebras with counit over $R$, henceforth *coalgebras*. For each object $C$ we have a coproduct $\psi_C: C\to C\otimes_R C$ and a unit $\varepsilon_C: C\to R$. Morphisms are maps of coalgebras with unit. $R$ is in the category in a natural way and is a terminal object. The unique map from $C$ to $R$ is $\varepsilon_C$. The product in the category, $C\Pi D$, is given by $C\otimes_R D = C\otimes D$, where $1_C\otimes\varepsilon_D: C\otimes D\to C$ and $\varepsilon_C\otimes 1_D: C\otimes D\to D$ are the projections. If $f: B\to C$ and $g: B\to D$ are given, then the map $(f,g): B\to C\otimes D$ is $(f\otimes g)\psi_B$, i.e. $(f,g)(b) = \Sigma f(b')\otimes g(b'')$ where $\psi(b) = \Sigma b'\otimes b''$. We will call a ring object over $\text{CoAlg}_R$ a *(graded) Hopf ring*. The term, "Hopf ring" was first used in [15]. In this context a Hopf algebra should be called a Hopf group. Since the name Hopf algebra is here to stay it presents problems in the naming of a ring object in the category of coalgebras. "Hopf bialgebra" is a name used by some [14]. However, a bialgebra means something distinctly different to algebraists. An appropriate name would be "coalgebraic ring" but we have decided to stick with "Hopf ring" because of its aesthetic value.

We collect the basic facts about Hopf rings in the following lemma. Observe that there are Hopf rings with similar properties for $G_2\mathrm{CoAlg}_R$ and $G_+\mathrm{CoAlg}_R$. In $G_2\mathrm{CoAlg}_R$ the signs which involve $\chi$ go away. If $R$ is concentrated in degree zero (or even degrees) the signs involving it disappear as well. In this paper we will work in $G_2\mathrm{CoAlg}_R$ and it turns out that for our objects $H(*)\in G_2\mathrm{CoAlg}_R$, $H_*(n)$ is evenly graded for all $n$ so the signs never enter into our consideration. However, we will need the signs in a planned sequel to this paper [30].

**Lemma 1.12.** *Let* $H(*)=\{H_*(n)\}_{n\in Z}\in G\mathrm{CoAlg}_R$ *be a Hopf ring. Let* $a\in H_i(n)$, $b\in H_j(k)$, $c\in H_q(k)$. *Define* $\deg x$ *by* $x\in H_{\deg x}(m)$.

(a) *Each* $H_*(n)\in \mathrm{CoAlg}_R$.

(i) *There is a coassociative cocommutative coproduct for all n.*

$$\psi: H_*(n)\to H_*(n)\otimes H_*(n)$$
$$\psi(a)=\sum a'\otimes a''=\sum(-1)^{\deg a'\deg a''}a''\otimes a'.$$

(ii) *There is a counit,* $\varepsilon: H_*(n)\to R$ *such that*

$$H_*(n)\xrightarrow{\psi} H_*(n)\otimes H_*(n)\xrightarrow{1_{H_*(n)}\otimes\varepsilon} H_*(n)\otimes_R R\simeq H_*(n)$$

*is the identity, i.e.* $a=\sum a'\varepsilon(a'')$.

(b) *Each* $H_*(k)$ *is an abelian group object of* $\mathrm{CoAlg}_R$, *i.e. a bicommutative biassociative Hopf algebra with unit, counit and conjugation*:

(i) *There is a product*

$$*: H_*(k)\otimes H_*(k)\to H_*(k)$$

*which is associative and commutative,*

$$b*c=(-1)^{jq}c*b\in H_{j+q}(k).$$

(ii) *The map* $*$ *is in* $\mathrm{CoAlg}_R$

$$\psi(b*c)=\psi(b)*\psi(c)=\sum(b'\otimes b'')*(c'\otimes c'')$$
$$=\sum(-1)^{\deg c'\deg b''}(b'*c')\otimes(b''*c'').$$

(iii) *The abelian group object unit, zero, is* $\eta: R\to H_*(k)$. *We define* $[0_k]=\eta(1)\neq 0$,

$$[0_k]*b=b.$$

(iv) *The conjugation* $\chi: H_*(k)\to H_*(k)$ *has* $\chi\chi=\text{identity}$ *and* $\eta\varepsilon(b)=\sum b'*\chi(b'')$.

(c) *There are associative maps*

$$\circ : H_*(n) \otimes H_*(k) \to H_*(n+k)$$

*such that*:

(i) *The map* $\circ$ *is in* $\mathrm{CoAlg}_R$

$$\psi(a \circ b) = \psi(a) \circ \psi(b) = \sum (a' \otimes a'') \circ (b' \otimes b'')$$

$$= \sum (-1)^{\deg a'' \deg b'} (a' \circ b') \otimes (a'' \circ b'').$$

(ii) *Multiplication by zero*

$$[0_n] \circ b = \eta\varepsilon(b).$$

(iii) *There is a unit map* $e : R \to H_*(0)$. *Define* $e(1) = [1] \in H_0(0)$, *then* $[1] \circ b = b$.

(iv) *Define* $\chi([1]) = [-1] \in H_0(0)$. *Then*

$$\chi(a) = [-1] \circ a$$

*and*

$$\chi(a \circ b) = \chi(a) \circ b = a \circ \chi(b).$$

(v) *Commutativity*

$$a \circ b = (-1)^{ij} [-1]^{\circ nk} \circ b \circ a = (-1)^{ij} \chi^{nk} (b \circ a) \in H_{i+j}(n+k).$$

(vi) *Distributivity*

$$a \circ (b * c) = \sum (-1)^{\deg a'' \deg b} (a' \circ b) * (a'' \circ c).$$

(vii) *Let* $[n] = [1]^{*n} = [1 + 1 + \cdots + 1]$, *then*

$$[n] \circ b = \sum b' * b'' * \cdots * b^{(n)}.$$

**Proof.** Everything follows directly from the definition of a ring object except (c) (iv). This actually holds for a ring object but we will give a direct proof here.

| | |
|---|---|
| $[-1] \circ a = ([-1] * [0_0]) \circ a$ | (b)(iii) |
| $= \sum ([-1] \circ a') * ([0_0] \circ a'')$ | (c)(vi) |
| $= \sum ([-1] \circ a') * \eta\varepsilon(a'')$ | (c)(ii) |
| $= \sum ([-1] \circ a') * a'' * \chi(a''')$ | (b)(iv) and coassociativity |

$$= \sum ([-1]\circ a')*([1]\circ a'')*\chi(a''') \quad \text{(c)(iii)}$$

$$= \sum (([-1]*[1])\circ a')*\chi(a'') \quad \text{(c)(vi) and coassociativity}$$

$$= \sum ([0_0]\circ a')*\chi(a'') \quad \text{(b)(iv)}$$

$$= \sum \eta\varepsilon(a')*\chi(a'') \quad \text{(c)(ii)}$$

$$= \chi\left(\sum \eta\varepsilon(a')*(a'')\right) \quad \chi \text{ restricted to image of } \eta \text{ is the identity}$$

$$= \chi(a). \quad \text{(a)(ii)}$$

Let $\mathscr{C}$ be some full subcategory (with appropriate products) of the homotopy category of topological spaces, *H Top*. The objects of H Top are topological spaces and the morphisms are homotopy classes of continuous functions $[X, Y]$. Cartesian product is a product in H Top and the one point space is a terminal object. A graded ring object over $\mathscr{C}$ will be called a *(graded) ring space.*

Let $\mathscr{C}^0$ be the homotopy category of topological spaces having the same homotopy type as countable CW complexes. Let $E_*(-)$ be an associative commutative multiplicative unreduced generalized homology theory with unit and let $G^*(-)$ be a similar cohomology theory, both defined on $\mathscr{C}^0$. Let $E_*$ and $G^*$ denote the two coefficient rings. Let $G^*(-)$ have a representing $\Omega$-spectrum [4] $\boldsymbol{G}_* = \{\boldsymbol{G}_n\}_{n\in Z} \in G\mathscr{C}^0$, i.e. $G^n(X) \simeq [X, \boldsymbol{G}_n]$ naturally and $\Omega \boldsymbol{G}_{n+1} \simeq \boldsymbol{G}_n$. In general we let $[X, \boldsymbol{G}_*] \simeq \{[X, \boldsymbol{G}_n]\}_{n\in Z} \simeq G^*(X)$. For $X_* \in G\mathscr{C}^0$ we let $E_*X_* = \{E_*X_n\}_{n\in Z}$ is the graded category of $E_*$ modules. We collect some basic facts in the following lemmas.

**Lemma 1.13.** *Let $\mathscr{C} \subset \mathscr{C}^0$ be some full subcategory with appropriate product such that exterior multiplication*

$$E_*(X) \otimes_{E_*} E_*(Y) \to E_*(X \Pi Y)$$

*induces a Künneth isomorphism for all $X, Y \in \mathscr{C}$, then for $\boldsymbol{G}_* \in G\mathscr{C}$ as above,*

(a) *$\boldsymbol{G}_*$ is a ring space.*

(b) *$E_*\boldsymbol{G}_*$ is a Hopf ring over $E_*$.*

**Proof.** (a) The very definition of a multiplicative $\Omega$-spectrum $\boldsymbol{G}_*$ is that $\Omega \boldsymbol{G}_{n+1} \simeq \boldsymbol{G}_n$ and $\boldsymbol{G}_*$ be a ring space. We are given that $E_*(-)$ satisfies the Künneth isomorphism so the diagonal induces a coproduct and $E_*(X) \in \text{CoAlg}_{E_*}$ for all $X \in \mathscr{C}$. Lemma 1.11 applies.

Let $x \in G^n$ in the coefficient ring, then $x \in G^n \simeq$ [point, $G_n$] and so we have a map $E_* \to E_* G_n$. We define $[x] \in E_0 G_n$ to be the image of $1 \in E_*$ under the map induced by $x$. In general note $G^* \simeq$ [point, $G_*$].

**Lemma 1.14.** *Let $z \in G^n$, $x, y \in G^k$, then*

(a) *for the zero element $0_n \in G^n$, $[0_n]$ corresponds to the $[0_n]$ of* Lemma 1.12 (b)(iii).

(b) $[z] \circ [x] = [zx] = [(-1)^{nk} xz] = [-1]^{\circ nk} \circ [x] \circ [z]$.

(c) $[x] * [y] = [x + y] = [y + x] = [y] * [x]$ $(\neq [x] + [y])$.

(d) $\psi([z]) = [z] \otimes [z]$.

(e) *The sub-Hopf algebra of $E_* G_n$ generated by all $[x]$ with $x \in G^n$ is the group ring of $G^n$ over $E_*$, i.e. $E_*[G^n]$ (using* (c)*).*

(f) *The sub-Hopf ring of $E_* G_*$ generated by all $[x]$ where $x \in G^*$ is the "ring-ring" of $G^*$ over $E_*$, i.e. $E_*[G^*]$ (using* (b) *and* (c)*).*

The proofs are straightforward.

The Künneth isomorphism always holds for singular homology with coefficients in a field $k$; $H_*(-;k)$. The Künneth isomorphism holds for singular homology with integer coefficients, $H_*(-;Z)$, complex bordism, $MU_*(-)$, and Brown–Peterson homology, $BP_*(-)$, on the full subcategory of spaces with no torsion in $H_*(-;Z)$, *torsion free spaces* [13]. So we have:

**Corollary 1.15.** (a) *For $G_*$ as above, $H_*(G_*;k)$ is a Hopf ring over k.*

(b) *For $G_*$ as above with each $G_n$ in the category of torsion free spaces, then $H_*(G_*;Z)$, $H_*(G_*;Z_{(p)})$, $MU_*G_*$, and $BP_*G_*$ are Hopf rings over $Z$, $Z_{(p)}$, $MU_*$ and $BP_*$ respectively.*

**Remark 1.16.** Not all ring spaces are $\Omega$-spectra. An example along the lines of our interests is $X_0 =$ integers, $X_n = \Omega^n \text{MSO}(n)$, $n > 0$, the $n$th loops on the Thom complex for SO($n$). The $*$ product comes from the loops and the $\circ$ product can be obtained from the maps $\text{MSO}(n) \wedge \text{MSO}(k) \to \text{MSO}(n+k)$ which are induced by the Whitney sum. We leave the details to the interested reader.

Let $R$ and $S$ be graded rings with $R[S]$ the "ring–ring" as in Lemma 1.14(f). $R[S]$ is a Hopf ring over $R$. We say a Hopf ring $H$ over $R$ is an *$R[S]$-Hopf ring* if there is a given map of Hopf rings $R[S] \to H$. We let *Supp CoAlg$_R$* be the category of supplemented coalgebras over $R$, i.e. each coalgebra $C$ is equipped with a map $\eta : R \to C$ such that $\varepsilon\eta =$ identity on $R$. We define $[0] = \eta(1)$.

We now construct the *free $R[S]$ Hopf ring on* $C(*) \in G$ SuppCoAlg$_R$. Identify the $[0_n] \in C_0(n)$ with the $[0_n] \in R[S]$ and take all possible $*$ and $\circ$ products of $C(*)$

with itself and $R[S]$ (a sub-Hopf ring) subject to the restraints of Lemma 1.12. This gives a functor

1.17 $\quad \mathrm{FHR}: G \text{ SuppCoAlg}_R \to R[S]\text{-Hopf rings}$

with the following universal property. There is a canonical map $C(*) \to \mathrm{FHR}C(*)$ in $\text{SuppCoalg}_R$ such that if $H$ is an $R[S]$-Hopf ring and we are given a map $C(*) \to H$ in $\text{SuppCoAlg}_R$, then there is a unique map of $R[S]$-Hopf rings $\mathrm{FHR}C(*) \to H$ making the following diagram commute:

1.18
$$\begin{array}{ccc} & C(*) & \\ \swarrow & & \searrow \\ \mathrm{FHR}C(*) & \dashrightarrow & H \end{array}$$

1.19. The only free Hopf ring we will be concerned with is $\mathrm{FHR}C(*)$ when $C(2)$ is the $R$ free coalgebra on $b_i \in C_{2i}(2)$, $i \geq 0$, with $\psi(b_n) = \sum_{i=0}^{n} b_i \otimes b_{n-i}$ and $b_0 = [0_2]$. $C(k) =$ the $R$ free coalgebra on $[0_k]$, $k \neq 2$.

## 2. The space $MU_*$

In this section we give some basic facts about $MU$ and $BP$. Readers with some familiarity with $MU$ and $BP$ may wish to skip it entirely. All they will need to know is that $\boldsymbol{MU}_*$ and $\boldsymbol{BP}_*$ are the evenly graded spaces made up from the even spaces in the $\Omega$-spectra for $MU$ and $BP$ respectively, i.e. $\boldsymbol{MU}_n$ is the $2n$ space in the $\Omega$-spectrum for $MU$. We will also use the elementary Proposition 2.4.

Most of the next section where we prove the main relations is independent of any awareness of $MU$ or $BP$ as well. It is only when we specialize to these cases to get explicit formulas are they important. The purpose of most of this section is to set up the geometry necessary to obtain our geometric corollaries of our main theorem, a computation of the complex bordism of the spaces in the $\Omega$-spectrum for $MU$.

Let $MU_n$ denote the Thom space of the unitary group $U_n$. Note that $MU_n$ is $(2n-1)$-connected. The inclusion map $U_n \to U_{n+1}$ induces a map $S^2MU_n \to MU_{n+1}$. The nonnegatively evenly graded space $\{MU_n\}$ together with these maps give the spectrum $MU$. (For a general cobordism reference see [36].) We are interested in the $\Omega$-spectrum representing $MU$. The adjoint of the above map is a map $MU_n \to \Omega^2 MU_{n+1}$. Applying the iterated loop functor gives a map $\Omega^k MU_n \to \Omega^{k+2} MU_{n+1}$ which allows us to define

$$\boldsymbol{MU}_n = \varinjlim \Omega^{2i} MU_{n+i}.$$

$\boldsymbol{MU}_n$ (called $M_{2n}$ in [40]) is an infinite loop space (by construction) and is defined for every integer $n$. $\boldsymbol{MU}_n$ is $(2n-1)$-connected for $n > 0$ and $\boldsymbol{MU}_n = \Omega^2 \boldsymbol{MU}_{n+1}$. The spaces $\{\boldsymbol{MU}_n, \Omega\boldsymbol{MU}_n\}$ together with the appropriate maps constitute the $\Omega$-spectrum associated with $MU$. We will restrict our attention to the even spaces $\boldsymbol{MU}_n$ although in the course of our study the spaces $\Omega\boldsymbol{MU}_n$ will also be described.

We define $\boldsymbol{MU}_*$ to be the evenly graded space $X_{2n} = \boldsymbol{MU}_n$. $MU$ is a ring spectrum, the multiplication being induced by Whitney sums, so $\boldsymbol{MU}_*$ is a ring space by 1.13(a) and the comments after 1.9. We let $MU_*(-)$ and $MU^*(-)$ denote the generalized homology and cohomology theories respectively (complex bordism and complex cobordism) arising from $MU$. We have $MU^{2*}(X) \simeq [X, \boldsymbol{MU}_*]$. In particular, the coefficient rings are given by

$$Z[x_2, x_4, \dots] \simeq \pi_*^S(MU) = MU_* = MU^{-*} \simeq [\text{point}, \boldsymbol{MU}_{-*}]$$

where $Z[x_2, x_4, \dots]$ is a polynomial algebra over $Z$ on positive even dimensional generators.

After localizing at a prime $p$, the study of $MU$ reduces to the study of the Brown–Peterson spectrum [5] which is also a ring spectrum [1, 24]. We can then define the analogous graded ring space for $BP$. Let $\boldsymbol{BP}_n$ be the $2n$ space in the $\Omega$-spectrum for $BP$ ($BP_{2n}$ in [40, 41]), then we let $\boldsymbol{BP}_*$ be the evenly graded space $X_{2n} = \boldsymbol{BP}_n$. Let $BP_*(-)$ and $BP^*(-)$ denote the homology and cohomology theories associated with $BP$. Then [1, 24]

$$Z_{(p)}[v_1, v_2, \dots] \simeq \pi_*^S(BP) = BP^{-*} \simeq [\text{point}, \boldsymbol{BP}_{-*}]$$

where $Z_{(p)}$ is the integers localized at $p$ and the degree of the polynomial generator $v_n$ is $2(p^n - 1)$.

Quillen has constructed a multiplicative idempotent $MU_{(p)} \to BP \to MU_{(p)}$. He obtains:

**Theorem 2.1** (Quillen [24]).

$$MU_*(X)_{(p)} \simeq MU_{*(p)} \otimes_{BP_*} BP_*(X)$$

$$BP_*(X) \simeq BP_* \otimes_{MU_*} MU_*(X).$$

We now turn to a geometric interpretation of the space $\boldsymbol{MU}_*$. Our only need for this is to derive our geometric corollaries from our main theorem later on. However, this description may help clarify the products $*$ and $\circ$. We do need Proposition 2.4 in later sections but it is elementary and can be derived directly if desired.

Let $M_i^m, N_i^n$ $(i = 1, 2)$ be almost complex manifolds of dimensions $m$ and $n$ respectively, and let $f_i : M_i^m \to N_i^n$ be a map which induces a complex linear map on the stable tangent bundles. We say that $f_1$ and $f_2$ are cobordant if there exists a similar map $f : U^{m+1} \to V^{n+1}$ of manifolds with boundary such that $\partial U^{m+1} = M_1^m - M_2^m$, $\partial V^{n+1} = N_1^n - N_2^n$ and $f \mid M_i^m = f_i$. Define the codimension of $f_i$ to be $n - m$.

This cobordism of complex maps is an equivalence relation and the set of equivalence classes is a group under disjoint union. By arguments similar to those made by Stong [35] for the orientable case, one sees that the cobordism group of complex maps of codimension $2n$ is the complex bordism group $MU_*\underline{MU}_n$ and $MU_*\underline{MU}_*$ is the cobordism group of all maps with even codimension.

The additive and multiplicative products in $\underline{MU}_*$ induce products in $MU_*\underline{MU}_*$ which can be described geometrically. First observe that the additive product is the $H$-space structure on $\underline{MU}_*$ which arises from the fact that it is a loop space; the multiplicative product is induced by the Whitney sum maps $MU_m \wedge MU_n \to MU_{m+n}$. Then we have

**Proposition 2.2.** *Let* $f_i : M_i \to N_i$, $i = 1,2$ *represent two elements of* $MU_*\underline{MU}_*$, *then their multiplicative and additive products are represented by* $f_1 \times f_2 : M_1 \times M_2 \to N_1 \times N_2$ *and* $f_1 \times 1 \amalg 1 \times f_2 : M_1 \times N_2 \amalg N_1 \times M_2 \to N_1 \times N_2$ *respectively.*

**Proof.** In order to get a map to a bordism element we lift to an embedding $f_i' : M_i \hookrightarrow S^{2k_i} \times N_i$ which determines a map $S^{2k_i} \times N_i \to MU_{q_i}$, which in turn determines $N_i \to \Omega^{2k_i}MU_{q_i} \to \underline{MU}_{q_i - k_i}$ which represents the bordism element corresponding to $f_i$. For the multiplicative product we have

$$M_1 \times M_2 \xrightarrow{f_1' \times f_2'} S^{2k_1} \times N_1 \times S^{2k_2} \times N_2 \longrightarrow MU_{q_1} \times MU_{q_2} \longrightarrow MU_{q_1+q_2}.$$

In the last map the inverse image of the zero section in $MU_{q_1+q_2}$ is precisely the product of the zero sections of $MU_{q_1}$ and $MU_{q_2}$, so its inverse image in $S^{2k_1} \times N_1 \times S^{2k_2} \times N_2$ is precisely $M_1 \times M_2$, and the statement about the multiplicative product follows.

For the statement about additive products, assume for simplicity that $k_1 = k_2 = k$. Thus since $q_1 - k_1 = q_2 - k_2$ we have $q_1 = q_2 = q$ as well. Let $w : S^{2k} \to S^{2k} \times S^{2k}$ be the composition $S^{2k} \to S^{2k} \vee S^{2k} \hookrightarrow S^{2k} \times S^{2k}$. Then the additive product of $f_1$ and $f_2$ is represented by the adjoint of

$$\begin{array}{ccc} S^{2k} \times N_1 \times N_2 & \xrightarrow{w} & S^{2k} \times S^{2k} \times N_1 \times N_2 \\ & & \| \\ & & (S^{2k} \times N_1) \times (S^{2k} \times N_2) \\ & & \downarrow \\ & & MU_q \times MU_q \end{array}$$

and the inverse image under $w$ of $M_1 \times M_2 \subset (S^{2k} \times N_1) \times (S^{2k} \times N_2)$ is $M_1 \times M_2 \amalg N_1 \times M_2$.

We have the following easy facts whose proof we leave to the reader.

**Proposition 2.3.** *Let* $V$ *represent* $v \in \pi_*MU \simeq MU^{-*} = MU_*$ *and let* $x \in MU_*\underline{MU}_*$ *be represented by* $f : M \to N$, *then*

(a) *The map* $V \to$ *point represents*

$$[v] \in H_0 \boldsymbol{MU}_* \simeq MU_0 \boldsymbol{MU}_*;$$

(b) *the map* $1: V \to V$ *represents*

$$v[1] \in MU_* \boldsymbol{MU}_0;$$

(c) *the map* $1 \times f: V \times M \to V \times N$ *represents* $vx$ *in* $MU_* \boldsymbol{MU}_*$.

Let us consider the special maps $b_n : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ and equivalently $T_n : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty \simeq \boldsymbol{MU}_1 \to \boldsymbol{MU}_1$. Let $\beta_n \in H_{2n}(\mathbb{C}P^n; \mathbb{F}_p)$ be the fundamental class. We define $(T_n)_*(\beta_n) = b_n \in H_{2n} \boldsymbol{MU}_1$.

**Proposition 2.4.** *Iterating the homology suspension homomorphism twice is the same as* $\circ$ *multiplication by* $b_1$ *in* $H_*(\boldsymbol{MU}_*; \mathbb{F}_p)$.

**Proof.** Using the fact that $\mathbb{C}P^1 \simeq S^2$ and our description of the multiplication in the proof of 2.2 we see that the $b_1 \circ$ multiplication map

$$S^2 \times \boldsymbol{MU}_n \to \boldsymbol{MU}_1 \times \boldsymbol{MU}_n \to \boldsymbol{MU}_{n+1}$$

gives precisely the defining map for the spectrum, $S^2 MU_n \to MU_{n+1}$ and the result follows.

## 3. The main relations

The formal group for $E_*$ comes from the unstable H-space map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Previous workers have applied this formal group to obtain rich information in stable homotopy theory. However, since the formal group law comes from unstable homotopy information, it should produce unstable information, and it does.

For the first part of this section we study $E_* \mathbb{C}P^\infty$ and produce a very general form of our main relations in $E_* \boldsymbol{G}_*$ needing no knowledge of $MU$ or $BP$. We then specialize to $BP_* \mathbb{C}P^\infty$ and $H_* \boldsymbol{BP}_*$ where we obtain useful explicit relations from the general theorem. In particular we rely heavily on 3.14 in the next 2 sections. The last part of this section is devoted to demonstrating how to compute with the main relations. The main results of this section are Theorems 3.4, 3.8, 3.12 and 3.14.

We do our best to follow the notation of Adams [1]. Let $E_*(-)$ and $E^*(-)$ be the unreduced homology and cohomology theories associated to a ring spectrum $E$ with coefficient rings $E_* \simeq E^{-*}$ and $\Omega$-spectrum $\boldsymbol{E}_*$. All of the theories we will consider will be equipped with a complex orientation.

**Definition 3.1.** A *complex orientation* is an element $x^E \in E^2(\mathbb{C}P^\infty)$ which restricts to an $E^*$ generator of $E^*(\mathbb{C}P^1)$ and to zero in $E^*$(point).

**Remark 3.2.** It not really necessary that $x^E \in E^2(\mathbb{C}P^\infty)$, however, it can always be so arranged and we will insist on it for the minor convenience it gives.

We have several examples of $E_*(-)$ in mind, in particular $MU_*(-)$, $BP_*(-)$, $H_*(-;R)$ ($R$ is a ring), and $K(n)_*(-)$, the Morava extraordinary $K$-theories [11].

We collect the following elementary basic facts which we need.

**Lemma 3.3.** (See [1].)

(a) $E^*(\mathbb{C}P^\infty) \simeq E^*[[x^E]]$ *the power series on* $x^E$ *over* $E^*$.

(b) $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*(\mathbb{C}P^\infty) \hat{\otimes}_{E^*} E^*(\mathbb{C}P^\infty)$.

(c) $E_*\mathbb{C}P^\infty$ *is* $E_*$ *free on* $\beta_i \in E_{2i}\mathbb{C}P^\infty$, $i \geqslant 0$, *dual to* $x^i$, *i.e.* $\langle x^i, \beta_j \rangle = \delta_{ij}$.

(d) $E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E_*\mathbb{C}P^\infty \otimes_{E_*} E_*\mathbb{C}P^\infty$.

(e) *The diagonal* $\mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ *induces a coproduct* $\psi$ *on* $E_*\mathbb{C}P^\infty$ *with* $\psi(\beta_n) = \sum_{i=0}^n \beta_i \otimes \beta_{n-i}$.

(f) *The H-space product* $m: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ *induces a coproduct* $m^*$ *on* $E^*\mathbb{C}P^\infty$ *with* $m^*(x^E) = \sum_{i,j\geqslant 0} a_{ij} x^i \otimes x^j$, *and* $a_{ij} \in E^{-2(i+j-1)} = E_{2(i+j-1)}$.

(g) $F(y,z) = y +_{F_E} z = y +_F z = \sum_{i,j\geqslant 0} a_{ij} y^i z^j$ *is a commutative associative formal group law over* $E^*$, *i.e.*

$$F(y,z) = F(z,y), \qquad F(y,0) = y$$

*and*

$$F(y, F(z,w)) = F(F(y,z), w).$$

We have now set things up so we can prove our main relations in their general form.

**Theorem 3.4.** *In the power series ring* $E_*\mathbb{C}P^\infty[[s,t]]$

$$\beta(s)\beta(t) = \beta(s +_F t)$$

*where* $\beta(r) = \sum_{i\geqslant 0} \beta_i r^i$ *and the product is that induced by the H-space structure of* $\mathbb{C}P^\infty$.

**Proof.** Let $a_{ij}^n \in E_*$ be defined by

$$\left(\sum_{i,j\geqslant 0} a_{ij} x^i \otimes x^j\right)^n = (m^*(x))^n = m^*(x^n) = \sum_{i,j\geqslant 0} a_{ij}^n x^i \otimes x^j.$$

We know $\beta_i\beta_j = \sum_{n\geqslant 0} c_n \beta_n$ for some $c_n \in E_*$. By duality

$$c_n = \left\langle x^n, \sum c_k \beta_k \right\rangle = \langle x^n, \beta_i\beta_j \rangle = \langle m^*(x^n), \beta_i \otimes \beta_j \rangle$$

$$= \left\langle \sum_{k,q\geqslant 0} a_{kq}^n x^k \otimes x^q, \beta_i \otimes \beta_j \right\rangle = a_{ij}^n.$$

So $\beta_i s^i \beta_j t^j = \sum_{n\geqslant 0} a_{ij}^n \beta_n s^i t^j$ and

$$\beta(s)\beta(t) = \sum_{i,j\geq 0} \beta_i s^i \beta_j t^j = \sum_{n\geq 0} \sum_{i,j\geq 0} a_{ij}^n \beta_n s^i t^j$$
$$= \sum_{n\geq 0} \beta_n \left( \sum_{i,j\geq 0} a_{ij} s^i t^j \right)^n = \sum_{n\geq 0} \beta_n (s +_F t)^n = \beta(s +_F t).$$

**Remark 3.5.** For $E_*(-) = H_*(-;Z)$, $s +_F t = s + t$ and $\beta(s)\beta(t) = \beta(s+t)$ just describes a divided power algebra without showing the binomial coefficients.

**Corollary 3.6.** *Define* $[1]_F(s) = s$ *and inductively* $[n]_F(s) = [n-1]_F(s) +_F s$, *then*

$$\beta(s)^n = \beta([n]_F(s)).$$

**Proof.** Just iterate 3.4.

Let $E$ and $G$ both be ring spectra with complex orientations $x^E$ and $x^G$ respectively. Let $\boldsymbol{G}_*$ be the $\Omega$-spectrum for $G$. The orientation $x^G$ can be considered as a map $x^G \in [\mathbb{C}P^\infty; \boldsymbol{G}_2] \simeq G^2\mathbb{C}P^\infty$. This induces a map $(x^G)_* : E_*\mathbb{C}P^\infty \to E_*\boldsymbol{G}_*$ and we define $b_i = (x^G)_*(\beta_i)$. As with $\beta(s)$ we have

$$(x^G)_*\beta(s) = b(s) = \sum_{n\geq 0} b_n s^n \in E_*\boldsymbol{G}_*[[s]].$$

Although $E_*\boldsymbol{G}_*$ is not necessarily a Hopf ring because it is not always a coalgebra, it does still have both products, $*$ and $\circ$.

**Definition 3.7.** In $E_*\boldsymbol{G}_*[[s,t]]$

$$b(s) +_{[F]} b(t) = b(s) +_{[F]_G} b(t) = \mathop{*}_{i,j\geq 0} [a_{ij}^G] \circ b(s)^{\circ i} \circ b(t)^{\circ j}.$$

We now prove our main general relation.

**Theorem 3.8.** *In* $E_*\boldsymbol{G}_*[[s,t]]$

(i) $b(s +_F t) = b(s) +_{[F]} b(t)$,

(ii) $b([p]_F(s)) = [p]_{[F]}(b(s))$.

**Note.** The $F$ on the left is $F_E$ with $a_{ij} = a_{ij}^E$ and the $[F]$ on the right is $[F]_G$ with $[a_{ij}] = [a_{ij}^G]$. The adornments $E$ and $G$ can safely be left out because they are the only ones which make any sense.

**Proof.** (ii) is just an iteration of (i). For (i),

| | |
|---|---|
| $b(s +_F t) = (x^G)_*(\beta(s +_F t))$ | definition of $b$'s |
| $= (x^G)_*(\beta(s)\beta(t))$ | 3.4 |
| $= (x^G)_*(m_*)(\beta(s) \otimes \beta(t))$ | definition of multiplication |

$$= (m^*x^G)_*(\beta(s)\otimes\beta(t)) \qquad \text{naturality}$$

$$= \Big(\sum_{i,j\geq 0} a_{ij}^G (x^G)^i \otimes (x^G)^j\Big)_* (\beta(s)\otimes\beta(t)) \qquad 3.3(f)$$

$$= \underset{i,j\geq 0}{*} [a_{ij}]\circ b(s)^{\circ i}\circ b(t)^{\circ j} = b(s) +_{[F]} b(t).$$

The last step follows from the definition of the $b$'s and the facts, from Section 1, that addition in $G^*(-)$ and $G_*$ translates into $*$ in $E_*G_*$ and multiplication in $G^*(-)$ and $G_*$ gives $\circ$ multiplication in $E_*G_*$.

We are interested in several different combinations of $E$ and $G$. For those which we use somewhat in this paper we make explicit here. We have displayed the $[p]$-sequence versions so blatantly because most calculations can be done using it and it is easier to handle.

**Corollary 3.9.**
(a) *Let $E = G = MU$ (or $BP$) with the canonical orientation $\mathbb{C}P^\infty \simeq MU_1 \to MU_1$.*
(i) $b(s +_F t) = b(s) +_{[F]} b(t)$.
(ii) $b([p]_F(s)) = [p]_{[F]}(b(s))$.
(b) *Let $E_*(-) = H_*(-;R)$ for a ring $R$ and $G = MU$ (or $BP$).*
(i) $b(s+t) = b(s) +_{[F]} b(t)$.
(ii) $b(ps) = [p]_{[F]}(b(s))$
(ii)′ *if* $R = Z/pZ = \mathbb{F}_p$, $b(ps) = b_0$.
(c) *Let $E = BP$ (or $MU$) and $G_* = K(Z/pZ, *)$ the* mod $p$ *Eilenberg–MacLane spectrum.*
(i) $b(s +_F t) = b(s) * b(t)$.
(ii) $b([p]_F(s)) = b(s)^{*p} = b_0$.

**Proof.** The $b_0$ in (c)(ii) follows because $K(Z/pZ, n) \xrightarrow{p} K(Z/pZ, n)$ is null homotopic. Everything else follows directly from 3.8 using the singular homology formal group law $s +_F t = s + t$.

To anyone who works with formal groups, the above rather general formulas probably do not appear very useful. The $a_{ij}$ are very difficult to handle from what is generally known about $E_*$ and $G^*$. However, for $BP$ we can extract some very explicit formulas which are useful in computing. Later we give some detailed examples of computations with 3.8.

We let $x^{BP}$ be the orientation inherited from the map $MU \to BP$. We collect some basic facts for $BP$.

**Theorem 3.10.** (See [1].)
(a) *In $MU^*[[x]]\otimes\mathbb{Q}$ we define*

$$\log x = \sum_{n>0} \frac{\mathbb{C}P^{n-1}}{n} x^n.$$

*Define* $\exp x$ *by* $\exp(\log x) = x$. *Then* $F(z, w) = \exp(\log z + \log w)$.

(b) *In* $BP^*[[x]] \otimes \mathbb{Q}$ *we define*

$$\log^{BP} x = \sum_{n\geq 0} \frac{\mathbb{C}P^{p^n-1}}{p^n} x^{p^n} = \sum_{n\geq 0} m_n x^{p^n}.$$

*Define* $\exp^{BP} x$ *by* $\exp^{BP}(\log^{BP} x) = x$. *Then* $F_{BP}(z, w) = \exp^{BP}(\log^{BP} z + \log^{BP} w)$.

Part (b) of the next theorem will be most important to us. It follows from Hazewinkel's construction of generators for $BP^*$.

**Theorem 3.11.** *Let p be the prime associated with BP.*

(a) (Hazewinkel [7, 8].)

*The generators for*

$$BP^* \simeq Z_{(p)}[v_1, v_2, \ldots] \subset BP^* \otimes \mathbb{Q}$$

*are given inductively by*

$$pm_n = v_n + \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}, \qquad m_i = \frac{\mathbb{C}P^{p^i-1}}{p^i} \in BP^* \otimes \mathbb{Q}.$$

(b) $$[p]_F(x) = \sum_{n>0}^{F_{BP}} v_n x^{p^n} \bmod (p).$$

**Proof of (b).** From (a) we obtain

$$\sum_{n>0} v_n x^{p^n} + \sum_{0<i<n} m_i v_{n-i}^{p^n} x^{p^n} = p \sum_{n>0} m_n x^{p^n}.$$

Rewritten this becomes

$$\sum_{i>0} \log^{BP} v_i x^{p^i} = p \log x - px$$

$$= p \log x - \log(\exp(px)).$$

Switch $-\log(\exp(px))$ to the other side and apply exp to both sides to obtain

$$\exp(px) +_F \sum_{i>0}^{F} v_i x^{p^i} = [p]_F(x).$$

So, if $\exp(px) = 0 \bmod (p)$ we are done. ($a_{i0} = a_{0i} = 0$, $i > 1$.)

$$BP^* \subset Z_{(p)}[m_1, m_2, \ldots] \subset BP^* \otimes \mathbb{Q} \qquad (m_0 = 1).$$

We have $\log x = \sum_{n\geq 0} m_n x^{p^n}$ (3.10(b)) for $BP$ and $\exp(\log x) = x$ defines $\exp y$. We see by construction that $\exp x = \sum_{i\geq 0} e_i x^{i+1}$ with $e_0 = 1$ and $e_i$ in degree $-2i$ of $Z_{(p)}[m_1, m_2, \ldots]$. An easy induction with 3.11 (a) shows $p^n m_n \in BP^*$. Thus it is easy to prove that for every monomial $y$ in $m$'s of degree $-2i$, $p^i y \in BP^*$, so $p^i e_i \in BP^*$ and $\exp(px) = \sum_{i\geq 0} e_i p^{i+1} x^{i+1}$ is in $pBP^*[[x]]$.

Let $QBP_*\mathbb{C}P^\infty$ denote the module of indecomposables for the ring $BP_*\mathbb{C}P^\infty$. Always $p$ denotes the prime associated with $BP$.

**Theorem 3.12.** *In* $QBP_*\mathbb{C}P^\infty \bmod(p)$,

$$\sum_{i=1}^{n} v_i^{p^{n-i}} \beta_{p^{n-i}} = 0.$$

**Proof.**

$$\begin{aligned} \beta(s)^p &= \beta([p]_F(s)) && 3.6 \\ &= \beta\Big(\sum_{n>0}^{F} v_n s^{p^n}\Big) \bmod(p) && 3.11\text{ (b)} \\ &= \prod_{n>0} \beta(v_n s^{p^n}). && 3.4 \end{aligned}$$

In $QBP_*\mathbb{C}P^\infty \bmod(p)$ this reduces to

$$0 = \sum_{n>0} \beta(v_n s^{p^n}) \text{ in positive degrees.}$$

The formula we wish to prove is precisely the coefficient of $s^{p^n}$.

**Remark 3.13.** In [33] Schochet proved $v_1^{p^n}\beta_{p^n} = (\beta_{p^n})^p$ modulo $\beta_{p^i}$, $i < n$. This motivated our conjecture for 3.12 which led to 3.4 which in turn allowed us to prove the general 3.8 which previously we could only do for $E_*(-) = H_*(-;R)$.

The next formula will be crucial to us in the next two sections. Let $\mathbb{F}_p = Z/pZ$, $Q$ be the module of indecomposables, and $I = ([p], [v_1], [v_2], \ldots)$.

**Theorem 3.14.** *In* $QH_*(\underline{BP}_1 : \mathbb{F}_p)/I^{\circ 2} \circ QH_*(\underline{BP}_1; \mathbb{F}_p)$

$$\sum_{i=1}^{n} [v_i] \circ b_{p^{n-i}}^{\circ p^i} = 0.$$

**Proof.** From 3.11 (b) and the fact that $a_{ij} \in (p, v_1, v_2, \ldots)$ if $a_{ij} \neq a_{10} = a_{01} = 1$ we have

**3.15** $$[p]_F(s) = ps + \sum_{n>0} v_n s^{p^n} \bmod(p, v_1, v_2, \ldots)^2.$$

So,
$$\begin{aligned} b_0 &= [p]_{[F]}(b(s)) && 3.9\text{ (b)(ii) and (ii)}' \\ &= [p] \circ b(s) \mathop{*}_{n>0} ([v_n] \circ b(s)^{\circ p^n}) \bmod I^{\circ 2}. && \text{by } 3.15 \end{aligned}$$

By 1.12 (c)(vii), $[p]\circ b(s)=b_0 \bmod *$ and $(p)$, so mod $*$ in positive degrees we obtain

$$0=\sum_{n>0}[v_n]\circ b(s)^{\circ p^n}$$

and the coefficient of $s^{p^n}$ gives the desired result.

**Remark.** Computations indicate that this relation is probably true in $QH_*(\boldsymbol{BP}_1;\mathbb{F}_p)$ but a proof has eluded us.

**Remark.** The shortand of Theorem 3.8, however elegant, is not in its readily computable form. Unwinding the definitions of $F$, $[F]$ and $b(r)$ we have

$$b(s+_F t)=b(s)+_{[F]}b(t)$$

becomes

$$b\Big(\sum_{i,j\geq 0}a_{ij}s^it^j\Big)=\mathop{*}_{i,j\geq 0}[a_{ij}]\circ b(s)^{\circ i}\circ b(t)^{\circ j}$$

that is,

$$\sum_{n\geq 0}b_n\Big(\sum_{i,j\geq 0}a_{ij}s^it^j\Big)^n=\mathop{*}_{i,j\geq 0}[a_{ij}]\circ\Big(\sum_{k\geq 0}b_ks^k\Big)^{\circ i}\circ\Big(\sum_{q\geq 0}b_qt^q\Big)^{\circ j}.$$

The coefficients of the $s^ut^v$ in the equation now give relations. Keep in mind that $b_0=[0_2]$, $a_{10}=a_{01}=1$ and obey the rules of Lemma 1.12 and you will find that computations are finite.

Because of the unfamiliarity of the formula in 3.8 we give the following for sake of clarity.

**Sample computation 3.16.** We restrict our attention to the $p=2$ case of 3.9 (a)(ii) for $BP$. Using Hazewinkel's generators, 3.11(a), and the definition of $[p]_F(x)$, 3.6, as $\exp(p\log x)$, 3.10 (b), we have for $p=2$,

$$[p]_F(x)=[2]_F(x)=2x-v_1x^2+2v_1^2x^3 \\ -(7v_2+8v_1^3)x^4+(30v_2v_1+26v_1^4)x^5-(111v_2v_1^2+84v_1^5)x^6+\cdots.$$

Writing down 3.9 (a)(ii) for $BP$, $p=2$, mod $s^5$ we have

$$b(2s-v_1s^2+2v_1^2s^3-(7v_2+8v_1^3)s^4) \\ =([2]\circ b(s))*([-v_1]\circ b(s)^{\circ 2})*([2v_1^2]\circ b(s)^{\circ 3}) \\ *([-7v_2-8v_1^3]\circ b(s)^{\circ 4}).$$

The reason we can ignore the rest of the terms on the right is because $b_0\circ b_i=0$, $i>0$ as $b_0=[0_2]$ and for $s^*$, $b_0\circ[a]=[0_{2-2n}]$, $a\in BP^{-2n}$ (all from Lemma 1.12). Expanding both sides further mod $s^3$ we have

$$b_0 + b_1(2s - v_1 s^2) + b_2(2s)^2 =$$
$$= ([2] \circ (b_0 + b_1 s + b_2 s^2)) * ([-v_1] \circ (b_0 + b_1 s)^{\circ 2}).$$

The coefficient of $s^0$ gives $b_0 = b_0$. For $s^1$ we get $2b_1 = [2] \circ b_1$. However, that is not new as

$$[2] \circ b_1 = ([1] * [1]) \circ b_1 = ([1] \circ b_1) * ([1] \circ b_0) + ([1] \circ b_0) * ([1] \circ b_1)$$
$$= b_1 * b_0 + b_0 * b_1 = b_1 + b_1 = 2b_1.$$

From the coefficient of $s^2$ we have $-v_1 b_1 + 4b_2 = [2] \circ b_2 + [-v_1] \circ b_1^{\circ 2}$. As above and in 1.12(c)(vii), $[2] \circ b_2 = 2b_2 + b_1^{*2}$ and our relation becomes

$$b_1^{*2} = 2b_2 - v_1 b_1 - [-v_1] \circ b_1^{\circ 2}.$$

To clean things up a bit, observe that $b_1^{\circ 2}$ is primitive, $\psi(b_1^{\circ 2}) = \psi(b_1)^{\circ 2} = (b_1 \otimes b_0 + b_0 \otimes b_1)^{\circ 2} = b_1^{\circ 2} \otimes [0_4] + 2(b_0 \circ b_1 \otimes b_0 \circ b_1)(=0) + [0_4] \otimes b_1^{\circ 2}$. So $\chi(b_1^{\circ 2}) = -b_1^{\circ 2}$ by 1.12 (b)(iv). Thus

$$-[-v_1] \circ b_1^{\circ 2} = -[v_1] \circ [-1] \circ b_1^{\circ 2} = -[v_1] \circ \chi(b_1^{\circ 2}) = -[v_1] \circ (-b_1^{\circ 2}) = [v_1] \circ b_1^{\circ 2}.$$

The final result of our labor is the formula in $BP_4\boldsymbol{BP}_1$ ($p = 2$),

$$b_1^{*2} = 2b_2 - v_1 b_1 + [v_1] \circ b_1^{\circ 2}.$$

If we reduce to $H_4(\boldsymbol{BP}_1; Z_{(2)})$ we just set $v_1 = 0$ (but not $[v_1]$). The element $[v_1] \circ b_1^{\circ 2}$ is the 4th suspension of $[v_1]$ (2.4) and so is the image of the Hurewicz homomorphism of the generator of $\pi_4\boldsymbol{BP}_1$; explicitly, $[v_1] \circ b_1^{\circ 2} = b_1^{*2} - 2b_2$.

The coefficients of $s^3$ give

$$b_1 2v_1^2 - b_2 4v_1 + b_3 8 = [2] \circ b_3 + [-v_1] \circ 2b_1 \circ b_2$$
$$+ [2v_1^2] \circ b_1^{\circ 3} + ([2] \circ b_1) * ([-v_1] \circ b_1^{\circ 2})$$

which cleans up to

$$2v_1^2 b_1 - 4v_1 b_2 + 6b_3 = 6b_1 * b_2 - 2b_1^{*3} - 2v_1 b_1^{*2}$$
$$- 2[v_1] \circ b_1 \circ b_2 + 2[v_1^2] \circ b_1^{\circ 3}.$$

As we shall see later, the group is $Z_{(2)}$ free so we can divide by 2 to get a relation. If we further divide by the unit 3 we can express $b_3$ in other terms. For our final example the coefficient of $s^4$ mod(2) gives

$$v_2 b_1 + v_1^2 b_2 = b_2^{*2} + b_1^{*4} + v_1 b_1^{*3}$$
$$+ [v_1] \circ b_2^{\circ 2} + [v_2] \circ b_1^{\circ 4}.$$

**Remark 3.17.** Recently D.C. Johnson and the second author have shown that $\operatorname{hom\,dim}_{MU_*} MU_* K(Z/p^n Z, k) = \infty$ for $n > 0$, $k > 1$ ([12]). The first counterexample to the old conjecture that $\operatorname{hom\,dim}_{MU_*} MU_* K(Z/pZ, n) = n$ was obtained by

K. Sinkinson and the second author as follows. We use 3.9 (c)(ii). From the Atiyah–Hirzebruch spectral sequence $H_*(K(Z/pZ,2); BP_*) \Longrightarrow BP_*K(Z/pZ,2)$ it is easy to see $pb_1 = 0$ and $b_1^{*p} \neq 0$ for $0 \neq b_1 \in BP_2K(Z/pZ,2)$. From 3.11 (b) $[p]_F(x) = v_n x^{p^n} \bmod x^{p^{n+1}}$ and $(p, v_1, \ldots, v_{n-1})$. From this and $b([p]_F(s)) = b_0$ the coefficient of $s^{p^n}$ tells us that $v_n b_1 = 0 \bmod (p, v_1, \ldots, v_{n-1})$. By induction we have $v_n b_1^{*n} = 0$ ($v_1 b_1 = 0$ is quite easy to check). Since we know that $b_1^{*p} \neq 0$, the annihilator ideal test of Conner and Smith [6] or the ideal annihilator test of Johnson and Wilson [10] shows that hom $\dim_{BP_*} BP_*K(Z/pZ,2) > p$ and the result follows.

## 4. The main theorem

Most of this section (the latter part) is dedicated to the computation of $H_*(\boldsymbol{MU}_*; \mathbb{F}_p)$ and giving a completely algebraic description and construction for it. The first part of the section is spent deriving corollaries of this result. These include the fact that $H_*(\boldsymbol{MU}_*; Z)$ has no torsion. Moreover we can compute $E_*\boldsymbol{MU}_*$ and give an algebraic construction for it. Of particular importance is the algebraic construction for $MU_*\boldsymbol{MU}_*$ (and $BP_*\boldsymbol{BP}_*$) because this contains all of the information for unstable complex cobordism operations.

We wish to construct Hopf rings in a purely algebraic way which give $E_*\boldsymbol{MU}_*$ and $E_*\boldsymbol{BP}_*$. We begin as at the end of Section 1 with the free Hopf ring constructed in 1.19 with $R[S] = E_*[G^*]$. We then impose the relations implied by 3.8. We denote this Hopf ring by $E_*^R\boldsymbol{G}_*$.

**Lemma 4.1.** *If $E_*\boldsymbol{G}_*$ is a Hopf ring then there is a canonical map of Hopf rings*

$$i_R : E_*^R\boldsymbol{G}_* \to E_*\boldsymbol{G}_*.$$

**Proof.** $(x^G)_* : E_*\mathbb{C}P^\infty \to E_*\boldsymbol{G}_*$ gives us the necessary map from the supplemented coalgebra of 1.19 to induce a map on the free $E_*[G^*]$ Hopf ring ring as in 1.18 ($E_*\boldsymbol{G}_*$ is an $E_*[G^*]$ Hopf ring). $E_*^R\boldsymbol{G}_*$ is a quotient of the free Hopf ring and the defining relations also hold in $E_*\boldsymbol{G}_*$ by 3.8 so the map from the free Hopf ring to $E_*\boldsymbol{G}_*$ factors through $E_*^R\boldsymbol{G}_*$.

The proof of part (a) of the following result will occupy the last half of this section. We state it now and derive its corollaries which include a computation of $E_*\boldsymbol{MU}_*$. Recall that $p$ denotes a prime and when $BP$ is present it is the prime associated with $BP$.

**Theorem 4.2.** *The following are isomorphisms of Hopf rings.*

(a) $i_R : H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p) \to H_*(\boldsymbol{MU}_*; \mathbb{F}_p)$.

(b) $i_R : H_*^R(\boldsymbol{BP}_*; \mathbb{F}_p) \to H_*(\boldsymbol{BP}_*; \mathbb{F}_p)$.

**Proof of (b).** The tensor products below are in the category of Hopf rings. By construction $H_*^R(\boldsymbol{BP}_*; \mathbb{F}_p) \simeq \mathbb{F}_p[BP^*] \otimes_{\mathbb{F}_p[MU^*]} H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p)$. The Hopf ring map $\mathbb{F}_p[MU^*] \to \mathbb{F}_p[BP^*]$ is induced by Quillen's $MU^* \to BP^*$. A similar isomorphism holds without the $R$ by Quillen's Theorem 2.1. Thus (a) implies (b).

**Corollary 4.3** [40]. *$H_*(\boldsymbol{MU}_*; Z)$ and $H_*(\boldsymbol{BP}_*; Z_{(p)})$ have no torsion.*

**Proof.** From the construction of $H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p)$ and the isomorphism 4.2 (a) we have $H_*(MU_*; \mathbb{F}_p)$ is concentrated in even degrees. (The only positive degree elements used in the construction were $b_i \in H_{2i}^R(\boldsymbol{MU}_*; \mathbb{F}_p)$.) By the Bockstein spectral sequence there can be no $p$ torsion. Similarly for $H_*(\boldsymbol{BP}_*; Z_{(p)})$.

**Remark 4.4.** Since $\boldsymbol{BP}_*$ is a $(p)$-localized space, $H_*(\boldsymbol{BP}; \mathbb{F}_q)$, for $q \neq p$, is just $\mathbb{F}_q[BP^*]$, concentrated in degree zero.

Now that $\boldsymbol{MU}_*$ and $\boldsymbol{BP}_*$ are torsion free spaces we know that $H_*(\boldsymbol{MU}_*; Z)$, $H_*(\boldsymbol{BP}_*; Z_{(p)})$, $MU_*\boldsymbol{MU}_*$ and $BP_*\boldsymbol{BP}_*$ are all Hopf rings by 1.15 (b).

**Corollary 4.5.** *The following are isomorphisms of Hopf rings.*

(a) $i_R : H_*^R(\boldsymbol{MU}_*; Z) \to H_*(MU_*; Z)$.

(b) $i_R : H_*^R(\boldsymbol{BP}_*; Z_{(p)}) \to H_*(\boldsymbol{BP}_*; Z_{(p)})$.

**Proof.** Both $H_*^R(\boldsymbol{MU}_*; Z)$ and $H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p)$ are constructed from the $b$'s and $[x]$'s, $x \in MU^*$. The defining relations for $H_*^R(\boldsymbol{MU}_*; \mathbb{F}_p)$ are just the mod$(p)$ versions of those for $H_*^R(\boldsymbol{MU}_*; Z)$ so there is a map

$$H_*^R(\boldsymbol{MU}_*; Z) \to H^R(\boldsymbol{MU}_*; \mathbb{F}_p)$$

which induces an isomorphism

$$H_*^R(\boldsymbol{MU}_*; Z) \otimes \mathbb{F}_p \xrightarrow{\simeq} H^R(\boldsymbol{MU}_*; \mathbb{F}_p).$$

Thus because $H_*(\boldsymbol{MU}_*; Z)$ has no torsion by 4.3, the map $H_*^R(\boldsymbol{MU}_*; Z) \to H_*(\boldsymbol{MU}_*; Z)$ induces isomorphisms when tensored with $\mathbb{F}_p$ for all primes. This proves (a). (b) is similar.

We consider the next corollary our most interesting. The dual of this result, an algebraic construction for $MU^*\boldsymbol{MU}_*$, is a complete description of the unstable complex cobordism operations (and unstable $BP$ operations).

**Corollary 4.6.** *The following are isomorphisms of Hopf rings.*

(a) $i_R : MU_*^R\boldsymbol{MU}_* \to MU_*\boldsymbol{MU}_*$.

(b) $i_R : BP_*^R\boldsymbol{BP}_* \to BP_*\boldsymbol{BP}_*$.

**Proof.** The Atiyah–Hirzebruch spectral sequence

$$H_*(MU_*;MU_*) \Longrightarrow MU_*\boldsymbol{MU}_*$$

is even dimensional and so collapses giving us that $MU_*\boldsymbol{MU}_*$ is $MU_*$ free. As in the proof of 4.5 we can show there is a map

$$MU^R_*\boldsymbol{MU}_* \to H^R_*(\boldsymbol{MU}_*;Z)$$

which induces an isomorphism

$$MU^R_*\boldsymbol{MU}_* \otimes_{MU_*} Z \xrightarrow{\simeq} H^R_*(\boldsymbol{MU}_*;Z).$$

Now since the map $MU^R_*\boldsymbol{MU}_* \to MU_*\boldsymbol{MU}_*$ induces an isomorphism when tensored with $Z$ (i.e. $\otimes_{MU_*} Z$) and $MU_*\boldsymbol{MU}_*$ is $MU_*$ free, we have our result. (b) is similar.

This leads us to the general computation.

**Corollary 4.7.** *$E_*\boldsymbol{MU}_*$ and $E_*\boldsymbol{BP}_*$ are Hopf rings and*

(a) $i_R : E^R_*\boldsymbol{MU}_* \to E_*\boldsymbol{MU}_*$

(b) $i_R : E^R_*\boldsymbol{BP}_* \to E_*\boldsymbol{BP}_*$

*give isomorphisms of Hopf rings for any multiplicative homology theory $E_*(-)$ with a complex orientation.*

**Proof.** Let $x^{MU}$ be the canonical complex orientation for $MU$. Let $x^E$ be the given orientation for $E$. There is a unique map of ring spectra $MU \to E$ which takes $x^{MU}$ to $x^E$ (see [1] p. 52). The map therefore takes $b_i^{MU}$ to $b_i^E$ and it induces a ring map $MU_* \to E_*$ taking $a_{ij}^{MU}$ to $a_{ij}^E$. Thus by construction it induces a map $MU^R_*\boldsymbol{MU}_* \to E^R_*\boldsymbol{MU}_*$. It also induces a map on the Atiyah–Hirzebruch spectral sequence $H_*(\boldsymbol{MU}_*;MU_*) \to H_*(\boldsymbol{MU}_*;E_*)$. $H_*(\boldsymbol{MU}_*;MU_*)$ collapses (see proof of 4.6) and the image of the above map includes an $E_*$ basis of the $E^2$ term. By naturality of the differentials $H_*(\boldsymbol{MU}_*;E_*) \Longrightarrow E_*\boldsymbol{MU}_*$ also collapses. The map of spectral sequences induces an isomorphism

$$E_* \otimes_{MU_*} H_*(\boldsymbol{MU}_*;MU_*) \to H_*(\boldsymbol{MU}_*;E_*).$$

Because the spectral sequence collapses we have an induced isomorphism on the associated graded objects for $E_* \otimes_{MU_*} MU_*\boldsymbol{MU}_*$ and $E_*\boldsymbol{MU}_*$. Both are $E_*$ free and we have an isomorphism $E_* \otimes_{MU_*} MU_*\boldsymbol{MU}_* \xrightarrow{\simeq} E_*\boldsymbol{MU}_*$. The construction of $E^R_*\boldsymbol{MU}_*$ only uses the formal group coefficients in the relations so we automatically have $E_* \otimes_{MU_*} MU^R_*\boldsymbol{MU}_* \simeq E^R_*\boldsymbol{MU}_*$. The result follows from these two isomorphisms and 4.6 (a). For (b) we first compute $MU_*\boldsymbol{BP}_*$ and proceed in a similar manner.

**Remark 4.8.** It is not always true that $E_*^R G_* \simeq E_* G_*$ for other $G_*$. Examples are easy to find, for instance $H_*(K(Z/pZ, 2*); \mathbb{F}_p)$. However, for $H_*(K(Z/pZ, *); \mathbb{F}_p)$ there is a very nice algebraic construction which we will give in [30]. For quite some time computations led the second author to conjecture an isomorphism $MU_*^R K(Z, 2*) \simeq MU_* K(Z, 2*)$. It is true for $K(Z, 2)$ and in the stable range. However, an element of order 2 in $MU_{17}K(Z, 4)$ terminated this project. This is quite far out of the stable range.

**Remark 4.9.** In the proof of 4.2 (a) we show that the mod $p$ homology of a connected component of $\boldsymbol{MU}_*$ is a polynomial algebra for all $p$. This implies the same for integer homology and because the Atiyah–Hirzebruch spectral sequence collapses, the same is true for $E$ homology. Similarly, $E_* \Omega \boldsymbol{MU}_*$ is an exterior algebra over $E_*$ on the $E$ homology suspension of the generators for $E_* \boldsymbol{MU}_{*-1}$. Similar remarks hold for $\boldsymbol{BP}_*$. In the next section we give a basis for $QH_*(\boldsymbol{BP}_*; \mathbb{F}_p)$ and we therefore have a similar basis for $QE_* \boldsymbol{BP}_*$. The same holds for $QE_* \Omega \boldsymbol{BP}_*$.

**Remark 4.10.** We can easily construct a Hopf ring which includes $E_* \Omega \boldsymbol{MU}_*$. All that is necessary is to add an element $e$ with the property $e \circ e = b_1$. An algebraic construction for $E_* \Omega \boldsymbol{MU}_*$ will follow.

If we consider $MU_* \boldsymbol{MU}_*$ as the cobordism group of all maps with even codimension, as in Section 2, we have the following geometric corollary.

**Corollary 4.11.** *Using both products, $MU_* \boldsymbol{MU}_*$ is generated by maps to a point, identity maps and linear embeddings, $\boldsymbol{b}_n : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$.*

**Proof.** $MU_* \boldsymbol{MU}_*$ is generated by the $[v]$, $v \in MU^*$, which by 2.3 (a) are just maps to a point, by $v \in MU_*$, which by 2.3 (b) are just the identity maps, and (from the proof of 4.6) any elements which cover $b$'s in homology. The $\boldsymbol{b}_n$ do this.

We can now produce, from our algebraic madness, a nontrivial geometric statement which has an analogue in the unoriented case [35].

**Corollary 4.12.** *Any map of compact stable almost complex manifolds is cobordant to one of the form $f : \amalg_i F_i \times U_i \to M$ where $f \mid F_i \times U_i$ is the composition of the projection $F_i \times U_i \to U_i$ and an embedding, $U_i \hookrightarrow M$.*

**Proof.** The description of the product (2.2) and the generators (4.11) suffices for the even codimensional result. To do the odd codimension part we need Remarks 4.9 and 4.10 and the fact that $MU_*(-)$ homology suspension is just $a \circ$ multiplication by pt. $\to S^1$.

We now begin our proof of 4.2 (a). It will occupy most of the rest of the section. We give an outline of the proof here. Fix a prime $p$ and let $H^R_*MU_*$ and $H_*MU_*$ be $H^R_*(MU_*;\mathbb{F}_p)$ and $H_*(MU_*;\mathbb{F}_p)$ respectively. First we study the size of $H^R_*MU_*$, obtaining an upper bound on the $\dim_{\mathbb{F}_p} QH^R_*MU_*$ by use of the relations computed in Section 3. When this is done we begin computing $H_*MU_*$ using the bar spectral sequence. We work by induction on degree. Assuming we know $H_{2i}MU_*$ for $i<k$ we compute the $E_\infty$ term of the spectral sequence giving us that $H_{2k-1}MU_*=0$ and a form of $H_{2k}MU_*$. We use the Hopf ring nature of the spectral sequence to show that $H^R_{2k}MU_* \to H_{2k}MU_*$ is onto. This allows us to solve the algebra extensions of the spectral sequence and show that $H_*MU_*$ is a polynomial algebra for degrees $\leqslant 2k$. We then see that the size of $H_{2k}MU_*$ is equal to the upper bound on the size of $H^R_{2k}MU_*$ and since the map $H^R_{2k}MU_* \to H_{2k}MU_*$ is onto we are done.

We begin our study of the size of $QH^R_*MU_*$ by investigating the degree zero. Recall that for a graded Hopf ring over $\mathbb{F}_p$, $H_*(*)$, the module of indecomposables, $QH_*(*)$, is defined by

$$QH_*(n)=IH_*(n)/IH_*(n)*IH_*(n)$$

where $IH_*(n)$ is the augmentation ideal, $IH_*(n)=\ker\varepsilon$, and $*$ is the additive product. By 1.12 (c)(vi), $QH_*(*)$ is a bigraded algebra over $\mathbb{F}_p$ using the $\circ$ product for multiplication and having $[1]-[0_0]$ as the unit.

Recall that $MU^*=Z[x_2,x_4,\ldots]$ with $x_{2i}$ of degree $-2i$.

**Lemma 4.13.** *As an algebra with unit,* $QH^R_0MU_* \simeq \mathbb{F}_p[[x_2]-[0_{-2}],[x_4]-[0_{-4}],\ldots]$.

**Proof.** By construction, $H^R_0MU_* \simeq \mathbb{F}_p[MU^*]$. Considering only the $*$ product structure, $\mathbb{F}_p[MU^n]\simeq \otimes \mathbb{F}_p[Z]$, one copy of $\mathbb{F}_p[Z]$ for each $Z$ free summand of $MU^n$. For $\mathbb{F}_p[Z]$, an $\mathbb{F}_p$ basis for the augmentation ideal is given by $[n]-[0]$, $0\neq n\in Z$. Now $([n]-[0])*([m]-[0])=[m+n]-[m]-[n]+[0]=0\in Q\mathbb{F}_p[Z]$. Thus $[n+m]-[0]=([m]-[0])+([n]-[0])$ in $Q\mathbb{F}_p[Z]$. In particular $n([1]-[0])=[n]-[0]$ for all $n$. Thus $Q\mathbb{F}_p[Z]=\mathbb{F}_p$ is generated by $[1]-[0]$. Since $Q(\otimes\mathbb{F}_p[Z])\simeq \oplus Q\mathbb{F}_p[Z]$, $Q\mathbb{F}_p[MU^*]$ is $\mathbb{F}_p$ free on generators $[x]-[0_{-\deg x}]$ for a $Z$ basis $\{x\}$ of $MU^*$. From $([a]-[0])\circ([b]-[0])=[ab]-[0]$ we have the desired result for $QH^R_0MU_*$.

From the construction of $H^R_*MU_*$ we now know that $QH^R_*MU_*$ is a ring with unit $[1]-[0_0]$ with generators $[x_{2i}]-[0_{-2i}]$ and $b_i$, $i>0$. We improve on this by eliminating the unnecessary $b$'s.

**Lemma 4.14.** *As an algebra with unit,* $QH^R_*MU_*$ *is generated by* $[x_{2i}]-[0_{-2i}]$, $i>0$, *and* $b_{p^i}=b_{(i)}$, $i\geqslant 0$.

**Proof.** For $m$ not a power of $p$ we write $m=m'p^i$ with $i$ maximal. Then $m'>1$ and $m'\neq 0(p)$. We will show that $b_m$ can be written in terms of lower $b$'s. We use the main relation 3.9 (b)(i), $b(s+t)=b(s)+_{[F]}b(t)$. In degree $2m$ the left hand side is $b_m(s+t)^m$. The coefficient of $s^{(m'-1)p^i}t^{p^i}$ is $m'b_m \neq 0$. The coefficient of $s^{(m'-1)p^i}t^{p^i}$ on the right hand side, $b(s)+_{[F]}b(t)$, gives the desired result.

The relations 3.14 tell us that $QH_*^R MU_*$ is not the free commutative algebra on the generators of 4.14. In fact, the relations of 3.14 are all there are. We must now describe how effectively they can cut down the upper bound on the size of $QH_*^R MU_*$. Alexander [2] has shown that Hazewinkel's generators are in the image of $MU_* \to BP_*$. Thus we can consider the $v_n$ as elements of $MU^*$ and since we are working at $p$ we can replace $[x_{2(p^n-1)}]$ with $[v_n]$. It is not necessary to use Alexander's result here. We could work with $H_*^R MU_{(p)*}$ and use Quillen's map $BP_* \to MU_{*(p)}$ and derive $H_*^R MU_*$ from $H_*^R MU_{(p)*}$. A third way is to work with $BP_*$ and prove part (a) of 4.2 from (b).

Observe that

$$([x_{2i}]-[0_{-2i}])\circ b_m = [x_{2i}]\circ b_m \quad \text{for } m>0.$$

The following will show how to add in the relations (3.14) to the algebra generated by $[x_{2i}]$ and $b_{(i)}$.

**Lemma 4.15.** *Let $I=([v_1],[v_2],\ldots)$ and $r_m = \sum_{i=1}^m [v_i]\circ b_{(m-i)}^{\circ p^i} \in QH_*^R MU_*$.*

(a) *$r_m$ is in the ideal generated by $I^{\circ 2}$ and $[x_{2i}]$, $i \neq p^n-1$.*

(b) *$(r_1, r_2, \ldots)$ is a regular ideal in the polynomial algebra $A=\mathbb{F}_p[[x_{2i}], b_{(k)}]$, $i>0$, $k\geqslant 0$, i.e. $r_n$ multiplication on $A/(r_1,r_2,\ldots,r_{n-1})$ is injective.*

**Proof.** Part (a) is immediate from 3.14 and 4.14. The relation 3.14 followed purely algebraically from the defining relation for $H_*^R MU_*$. Part (b) is more complicated. Fix $n>0$. Let $J_i=(r_n, r_{n-1},\ldots,r_{n-i+1})$, $0<i\leqslant n$. We regard $J_i$ as an ideal in various rings related to $A$. Let $A_i = A/(b_{(0)}, b_{(1)},\ldots,b_{(n-i-1)})$ and $B_i = b_{(n-i)}^{-1}A_i$. Note that $A_n = A$ and that if $J_n \subset A_n = A$ is a regular ideal for all $n$ then (b) is proven.

We will show $J_i$ is regular in $A_i$ by induction on $i$. For $i=1$, $J_1$ is a non-zero principal ideal in the integral domain $A_1$ and is therefore regular. Assume the result for $<i$. The map given by multiplication of $b_{(n-i)}$ in

$$0 \longrightarrow A_i \xrightarrow{b_{(n-i)}} A_i \longrightarrow A_{i-1} \longrightarrow 0$$

raises degree, so it is possible by induction on degree to show that $J_{i-1}$ is regular in $A_i$, assuming $J_{i-1}$ is regular in $A_{i-1}$. If we prove that $J_{i-1}\subset A_i$ is prime then $A_i/J_{i-1}$ is an integral domain and so multiplication by $r_{n-i+1}$ is injective if $0\neq r_{n-i+1}\in A_i/J_{i-1}$. However, in the degree of $r_{n-i+1}$, $A_i/J_{i-1}=A$, so $r_{n-i+1}\neq 0$.

It is prime in $B_i$ since each of its generators is a polynomial generator of the polynomial algebra $B_i$ over $\mathbb{F}_p[b_{(n-i)}, b_{(n-i)}^{-1}]$. Suppose $J_{i-1}$ is not prime in $A_i$, i.e. there exist $x,y\notin J_{i-1}\subset A_i$ with $xy\in J_{i-1}$. Since $J_{i-1}$ is prime in $B_i$, we have $x$ or $y$, say $x\in b_{(n-i)}^{-1}\circ J_{i-1}$, i.e. $b_{(n-i)}^{\circ k}\circ x\in J_{i-1}$ for some minimum $k>0$. We may assume

$k=1$ by replacing $x$ with $b_{(n-i)}^{\circ(k-1)}\circ x$. We have $b_{(n-i)}\circ x=\sum_{j=1}^{i-1}a_j\circ r_{n-j+1}$ with $a_j\in A_i$ not all divisible by $b_{(n-i)}$. We can assume that if $a_j\neq 0$ then $a_j\notin J_{j-1}$. In $A_{i-1}$ this becomes $0=\sum_{j=1}^{i-1}a_j\circ r_{n-j+1}$ with $a_j$ not all in $J_{j-1}$. This is a contradiction since $J_{i-1}$ is regular in $A_{i-1}$ by induction. Therefore $J_{i-1}$ is prime in $A_i$ and $J_i$ is regular in $A_i$. This proves the claim which completes the proof of 4.15 (b).

The polynomial algebra $\mathbb{F}_p[[x_{2i}],b_1]$, $i>0$, has a natural bigrading inherited from $QH_*^RMU_*$. Define $c_{ij}$ and $d_{ij}$ as follows.

$$c_{**}=\dim_{\mathbb{F}_p}QH_*^RMU_*$$

and

$$d_{**}=\dim_{\mathbb{F}_p}\mathbb{F}_p[[x_{2i}],b_1]_{**}.$$

We can now give our upper bound on the size of $QH_*^RMU_*$.

**Lemma 4.16.**

(a) $c_{**}\leq d_{**}$.

(b) $d_{i*}=d_{i+2,*+1}$, $i\geq 0$.

**Remark 4.17.** As we see later, $c_{ij}=d_{ij}$.

**Proof of 4.16.** By 4.14 and 4.15 (a), $c_{**}$ is less than or equal to the $\mathbb{F}_p$-dimension of $\mathbb{F}_p[[x_{2i}],b_{(j)}]_{**}$ modulo the ideal $(r_1,r_2,\dots)$. The bidegree's of $r_n$ and $b_{(n)}$ are the same, and the result follows because $(r_1,r_2,\dots)$ is a regular ideal (4.15 (b)). To prove (b) it is enough to note the bidegree of $b_1=b_{(0)}$ is $(2,1)$.

Lemma 4.16 completes our estimate of the size of $H_*^RMU_*$. We are now ready to commence our computation of $H_*MU_*$ proving our isomorphism 4.2 (a) as we go. We do this by induction on degree. By construction $H_0^RMU_*\simeq\mathbb{F}_p[MU^*]$. By 1.14 (f), $H_0MU_*\simeq\mathbb{F}_p[MU^*]$. Also $H_1^RMU_*=0$ and since $\pi_1MU_*=0$, $H_1MU_*=0$. Let $MU_*'$ be the zero component of $MU_*$, i.e. $MU_k'$ is the component of $MU_k$ which contains $[0_{2k}]$.

**Induction 4.18.** In degrees $<2k-1$,

(i) $QH_*MU_*'$ is generated by $\circ$ products of the $[x_{2i}]$ and $b_{(i)}$.

(ii) $H_*MU_*'$ is a polynomial algebra.

(iii) For $i>0$, $d_{i*}=\dim_{\mathbb{F}_p}QH_iMU_*'$.

**Proof of 4.2 (a).** Assuming 4.18 for all $k$, (i) implies $i_R$ is a surjection. 4.16 (a) and 4.18 (iii) together with the fact that $i_R$ is onto imply Remark 4.17 and we have $QH_*^RMU_*\to QH_*MU_*$ is an isomorphism. Since $H_*MU_*'$ is a free commutative algebra (by (ii)), the above isomorphism implies that the map $H_*^RMU_*\to H_*MU_*$ is really an isomorphism.

**Proof of 4.18.** For $k = 1$ there is nothing to prove. We assume 4.18 for degrees less than $2k-1$ and we wish to prove it for degrees $\leq 2k$. We know $\Omega(\Omega MU'_*) \simeq MU_{*-1}$. We use the bar spectral sequence as in [31] and [32],

$$\mathrm{Tor}^{H_*MU_*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow E_0 H_* \Omega MU'_{*+1}.$$

Our knowledge of $H_0 MU_*$ and 4.18 (ii) for degrees $< 2k-1$ imply $\mathrm{Tor}^{H_*MU_*}(\mathbb{F}_p, \mathbb{F}_p)$ is an exterior algebra on $\mathrm{Tor}_1$ up through degree $2k-1$. This is because Tor of a polynomial algebra is an exterior algebra on generators of one degree higher than those for the polynomial algebra. This is a spectral sequence of Hopf algebras and the differentials lower the homological degree. $\mathrm{Tor}_q = 0$ for $q < 1$ and all of the generators are in $\mathrm{Tor}_1$ so the differentials on the generators are zero and so the spectral sequence collapses. We have no algebra extension problems even at the prime 2 because all of the generators are in odd degree. By the homology suspension we have

**4.19** $$QH_i MU_* \simeq PH_{i+1}\Omega MU'_{*+1} \simeq QH_{i+1}\Omega MU'_{*+1}, \qquad i+1 \leq 2k-1.$$

We now use

$$\mathrm{Tor}^{H_*\Omega MU'_*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow E_0 H_* MU'_*.$$

Tor of an exterior algebra is a divided power algebra on generators of degree one higher than those of the exterior algebra. Since $H_*\Omega MU'_*$ is an exterior algebra through degree $2k-1$, $\mathrm{Tor}^{H_*\Omega MU'_*}(\mathbb{F}_p, \mathbb{F}_p)$ is a divided power Hopf algebra through degree $2k$. The spectral sequence is concentrated in even degrees through $2k$ so it collapses. It is a divided power algebra on the primitives, $= \mathrm{Tor}_1$, which are isomorphic, by the homology suspension, to $QH_*\Omega MU'_*$, so

**4.20** $$PE_0 H_{i+2} MU'_{*+1} \simeq QH_{i+1}\Omega MU'_{*+1}, \qquad i+2 \leq 2k.$$

The iterated isomorphisms of 4.19 and 4.20 are by the double suspension, which by 2.4 is just $\circ$ multiplication by $b_1 = b_{(0)}$. By induction and 4.18 (i) for lower degrees this shows that $PE_0 H_{2k} MU'_*$ is generated by the $[x_{2i}]$ and $b_{(i)}$.

A divided power Hopf algebra on $x, \Gamma(x)$, has $\mathbb{F}_p$ basis $\{\gamma_i(x)\}$ $\gamma_i(x)\gamma_j(x) = (i,j)\gamma_{i+j}(x)$. Thus the $\gamma_{p^i}(x)$ are the generators. The only primitive is $\gamma_1(x)$. We have shown that all $\gamma_1(x)$ in $E_0 H_i MU'_*$ are given by $\circ$ products of the $[x_{2j}]$ and $b_{(j)}$ for $i \leq 2k$. It is now only necessary to do the same for each $\gamma_{p^i}(x)$ for degree $= 2k$. Pick $x$ with $\gamma_{p^i}(x)$ of degree $2k$ with $i > 0$. By (i) we know that $\gamma_1(x)$ is a linear combination of elements $[y] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ i_1} \circ \ldots$, $y \in MU^*$. It is enough to prove what we want assuming $\gamma_1(x) = [y] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots$. Consider the representative in $E_0 H_* MU'_*$, say $z$, of the element $[y] \circ b_{(i)}^{\circ j_0} \circ b_{(i+1)}^{\circ j_1} \circ \ldots$. Computing the iterated coproduct of $z - \gamma_{p^i}(x)$ in $E_0 H_* MU'_*$ we see that it must lie in a lower filtration than $\gamma_{p^i}(x)$. (The coproduct of $z$ can be computed using 1.12 (c)(i).) So $z = \gamma_{p^i}(x)$ mod lower filtration and $\gamma_{p^i}(x)$ can therefore be represented in terms of $[x_{2i}]$ and $b_{(i)}$. This concludes the proof of (i) for degrees $\leq 2k$.

We will now show that $QE_0H_*MU'_*$ is so big that $H^R_*MU_*$ can map onto $H_*MU'_*$ only if it is a polynomial algebra. In the process we will show that $H_*MU'_*$ is the size of the upper bound on $H^R_*MU_*$.

**4.21** $QH_iMU'_* = PE_0H_{i+2}MU'_{*+1}$ $\quad i+2 \leq 2k$ by 4.19 and 4.20

$\dim_{\mathbb{F}_p} PE_0H_{2k}MU'_{*+1} = d_{2(k-1),*}$ $\quad$ by 4.21 and 4.18 (iii), $i = 2(k-1)$

**4.22** $= d_{2k,*+1}$ $\quad$ by 4.16 (b).

From the general properties of divided power Hopf algebras over $\mathbb{F}_p$ we have

$$e_{2k,*} = \dim_{\mathbb{F}_p} QE_0H_{2k}MU'_*$$

**4.23**
$$= \dim_{\mathbb{F}_p} PE_0H_{2k}MU'_* + \dim_{\mathbb{F}_p} QE_0H_{2k/p}MU'_*$$

$$= d_{2k,*} + e_{2k/p,*} \qquad \text{by 4.22}$$

$$(e_{2k/p,*} = 0 \text{ if } p \nmid k).$$

Because $i_R$ is onto (by (i) for degrees $\leq 2k$), we can impose the algebra structure of $H^R_*MU_*$ on $E_0H_*MU'_*$ to solve the algebra extension problems. By surjectivity and 4.16 (a) we must have

$$\dim_{\mathbb{F}_p} QH_{2k}MU'_* \leq d_{2k,*}.$$

By this and 4.23 a subspace of $QE_0H_{2k}MU'_*$ of dimension $\geq e_{2k/p,*}$ must become decomposable in $QH_{2k}MU'_*$, i.e. they must be $p$th powers. By induction there are exactly $e_{2k/p,*}$ generators in degrees of the form $(2k/p^i, *)$, $i > 0$, which can have $p$th powers so in fact they must all have nontrivial $p$th powers in degree $2k$. Thus we have a polynomial algebra, 4.18 (ii), and 4.18 (iii) holds in degree $2k$.

**Remark 4.24.** This computation could have been done directly without 4.15 using the linear algebra of the next section by giving a basis for $QE_0H_*BP'_*$ and using it to do our counting. This was how the original proof of 4.2 (b) went but we feel it is much nicer to be able to prove 4.2 (a) without resorting to massive linear algebra.

Call a Hopf algebra *bipolynomial* if it and its dual are both polynomial algebras. Because a divided power algebra is dual to a polynomial algebra our proof actually gave the following.

**Corollary 4.25** [40]. *$H_*(MU'_*; Z)$ is a bipolynomial Hopf algebra.*

**Proof.** We have just proven this for $H_*MU'_*$ for all primes. The property lifts to $Z$.

**Remark 4.26.** From [28] we know 4.24 actually determines the Hopf algebra structure of $H_*MU'_*$.

## 5. A basis for $QH_*BP_*$

Although we have given some explicit relations (3.12 and 3.14), so far most of our work has been of a very general nature, e.g. relations in $E_*G_*$ and isomorphisms $E_*^R MU_* \simeq E_* MU_*$. However, in proving these last isomorphisms we computed and could have described $E_*MU_*$ quite nicely. In this section we restrict our attention to $H_*BP_*$ and describe explicitly its generators and primitives. By Remark 4.9 this does the same for $E_*BP_*$.

Recall that by $H_*BP_*$ we mean $H_*(BP_*;\mathbb{F}_p)$. Let $QH_*BP_*$ denote the indecomposables (see 4.13). $BP'_*$ denotes the zero component of $BP_*$, i.e. $BP'_k$ is the component of $BP_k$ containing $[0_{2k}]$.

**Proposition 5.1.**
(a) *$QH_*BP_*$ is a ring under the $\circ$ product.*
(b) *$QH_0BP_* = \mathbb{F}_p[[v_i]-[0_{-2(p^i-1)}] : i>0]$ with unit $[1]-[0_0]$.*
(c) *In positive degrees, $QH_*BP_* = QH_*BP'_*$.*
(d) *$QH_*BP_*$ is generated over $QH_0BP_*$ by the $b_{(m)} = b_{p^m} \in H_{2p^m}BP_1$.*

**Proof.** Lemma 4.13 is the same as (b) and (a) follows from 1.14(f). For (c) one knows that the components of $\mathbf{BP}_*$ are all equivalent. A 'generator' in the positive dimensional homology of a nonzero component is the $*$ product of a generator of $H_*BP'_*$ and $[v]$ for some $v \in BP^*$. Hence these 'generators' actually differ from those of $H_*BP'_*$ by decomposables. (d) follows from the corresponding statement for $H_*^R BP_*$ of 4.14.

Define elements in $QH_*BP_*$,

$$v^I b^J = [v_1^{i_1} v_2^{i_2} \cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots,$$

where $I=(i_1, i_2, \ldots)$ and $J=(j_0, j_1, \ldots)$ are sequences of non-negative integers almost all zero. Let $\Delta_k$ be the sequence with 1 in the $k$th place and zeros elsewhere. Recall the notation $b_{p^i} = b_{(i)}$.

**Definition 5.2.** We call $v^I b^J$ *allowable* if

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + \cdots + p^n\Delta_{k_n} + J'$$

where $k_1 \leq k_2 \leq \cdots \leq k_n$ and $J'$ is non-negative implies $i_n = 0$.

**Theorem 5.3.**
(a) *The allowable $v^I b^J$ ($J \neq 0$) form a basis for $QH_*BP'_*$.*
(b) *The $v^I b^J \circ b_1$ with $v^I b^J$ allowable ($J$ possibly zero) form a basis for $PH_*BP'_*$.*

**Remark 5.4.** Part (b) follows immediately from part (a) and the spectral sequence computation (4.21) of the previous section.

**Remark 5.5.** From the spectral sequence computation (4.13) of the previous section, a basis for $PH_*\Omega\boldsymbol{BP}_* \simeq QH_*\Omega\boldsymbol{BP}_*$ is given by the suspension of the basis of 5.3(a) (and 5.1(b)) for $QH_{*-1}\boldsymbol{BP}_{*-1}$.

**Remark 5.6.** $QH_{2*-1}\boldsymbol{BP}'_* = 0 = PH_{2*-1}\boldsymbol{BP}'_*$.

**Remark 5.7.** A basis for $QH_{2m}\boldsymbol{BP}_n$, $m>0$, is given by all allowable $v^Ib^J$ with $m=\sum_{k\geq 0} j_kp^k$ and $n=\sum_{k\geq 0} j_k - \sum_{k>0} i_k(p^k-1)$.

**Remark 5.8.** Since $H_*(\boldsymbol{BP}'_*; Z_{(p)})$ has no torsion and is a polynomial algebra our basis lifts to it.

The rest of this section is occupied with the proof of 5.3(a). Let $I = ([v_1]-[0_{-2(p-1)}], [v_2]-[0_{-2(p^2-1)}], \ldots)$ and define

$$F_sQH_*\boldsymbol{BP}_* = I^{\circ s} \circ QH_*\boldsymbol{BP}_* \quad \text{for } s\geq 0.$$

We obtain the associated graded object

$$E_sQH_*\boldsymbol{BP}_* = F_sQH_*BP_*/F_{s-1}QH_*BP_*, \quad s\geq 0.$$

$E_*QH_*\boldsymbol{BP}_*$ is now a tri-graded ring under $\circ$ products. From 3.14 we have the relation

$$\textbf{5.9}n^* \qquad \sum_{i=1}^{n} [v_i]\circ b_{(n-i)}^{\circ p^i} = 0 \quad \text{in } E_1QH_{2p^n}\boldsymbol{BP}_1.$$

By the previous section these relations generate all relations and, in fact, provide defining relations for $E_*QH_*\boldsymbol{BP}_*$. (Remark 4.17, Lemma 4.15.) Thus a basis for $E_*QH_*\boldsymbol{BP}_*$ can be lifted to a basis for $QH_*\boldsymbol{BP}_*$.

Let $A = F_p[u_1, u_2, \ldots, b_{(0)}, b_{(1)}, \ldots]$ be triply graded so that the map given by $\lambda(u_i) = [v_i]-[0_{-2(p^i-1)}]$ and $\lambda(b_{(m)}) = b_{(m)}$, preserves the grading. Let $r_n = \sum_{i=1}^n u_ib_{(n-i)}^{p^i} \in A$ and $R=(r_1, r_2, \ldots)\subset A$. Then by the above remarks $\lambda$ induces an isomorphism

$$\textbf{5.10} \qquad \bar\lambda : A/R \xrightarrow{\simeq} E_*QH_*\boldsymbol{BP}_*,$$

so proving 5.3 amounts to showing the allowable monomials in $A$ (defined analogously to 5.2) project down to a basis of $A/R$.

We want to give an algorithm for expressing an arbitrary monomial in $E_*QH_*\boldsymbol{BP}_*$ as a linear combination of allowable monomials, which is equivalent by 5.10 to an algorithm for expressing a monomial in $A$ in a linear combination of allowable monomials and elements of the relation ideal $R$.

We need to define some specific elements in $A$. Let $N = (n_1, n_2, \dots, n_k)$ be a finite strictly increasing sequence of positive integers and let

5.11 $$s_N = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{k} b^{p^i}_{(n_{\sigma(i)}-i)}$$

where the sum is taken over all permutations $\sigma$ of $\{1, 2, \dots, k\}$ and $(-1)^{\sigma}$ is the sign of $\sigma$. Let $b_{(i)} = 0$ for $i < 0$.

Let $N_i$ denote the sequence $N$ with $n_i$ deleted and define

5.12 $$r_N = \sum_{i=1}^{k} (-1)^{i+k} s_{N_i} r_{n_i}$$

where $r_n = \sum_{j=1}^{n} u_j b^{p^j}_{(n-j)}$ as above. Then we have

**Lemma 5.13.** *Let $N = (n_1, n_2, \dots, n_k)$, $0 < n_i < n_{i+1}$,*

(i) *$r_N \in R$ for all $N$.*

(ii) *The coefficient of $u_i$ in $r_N$ is zero for $i < k$.*

(iii) *The coefficient of $u_k$ in $r_N$ is $s_N$.*

**Proof.** (i) is obvious. (ii) and (iii) are obvious when $k = 1$, so we can argue by induction on $k$. The $s_N$ satisfy

5.14 $$s_N = \sum_{i=1}^{k} (-1)^{i+k} s_{N_i} b^{p^k}_{(n_i-k)}.$$

For $i \neq j$ and $1 \leq i, j \leq k$ let $N_{ij}$ denote the sequence $N$ with $n_i$ and $n_j$ deleted. Then 5.12 and 5.14 imply

$$r_N = \sum_{i=1}^{k} (-1)^{i+k} \sum_{j \neq i} (-1)^{\varepsilon(i,j)+k-1} s_{N_{ij}} b^{p^{k-1}}_{(n_j-k+1)} r_{n_i}$$

where

$$\varepsilon(i,j) = \begin{cases} j & \text{for } j < i \\ j-1 & \text{for } j > i. \end{cases}$$

This can be rewritten as

5.15 $$r_N = \sum_{j=1}^{k} (-1)^{j+k-1} b^{p^{k-1}}_{(u_j-k+1)} \sum_{i \neq j} (-1)^{\varepsilon(j,i)+k-1} s_{N_{ij}} r_{n_i}$$

5.16 $$= \sum_{j=1}^{k} (-1)^{j+k-1} b^{p^{k-1}}_{(n_j-k+1)} r_{N_j}.$$

Therefore the inductive hypothesis implies that the coefficient of $u_i$ in $r_N$ zero for $i < k-1$, and that the coeffecient of $u_{k-1}$ is

$$\sum_{j=1}^{k}(-1)^{j+k-1}b_{(n_j-k+1)}^{p^{k-1}}s_{N_j}$$

$$=\sum_{j=1}^{k}(-1)^{j+k-1}b_{(n_j-k+1)}^{p^{k-1}}\sum_{i\neq j}(-1)^{\varepsilon(j,i)+k-1}s_{N_{ij}}b_{(n_i-k+1)}^{p^{k-1}}$$

$$=\sum_{i\neq j}(-1)^{\varepsilon(i,j)+j}b_{(n_j-k+1)}^{p^{k-1}}b_{(n_i-k+1)}^{p^{k-1}}s_{N_{ij}}$$

$$=0,\text{ since } s_{N_{ij}}=s_{N_{ji}},$$

so (ii) is proved. (iii) is an immediate consequence of 5.14 and the definition of $r_{n_i}$.

We are now ready to describe our algorithm for expressing nonallowable monomials in terms of allowable ones modulo $R$. Let

$$x_N=u_k\prod_{i=1}^{k}b_{(n_i-i)}^{p^i}.$$

By Definition 5.2 every nonallowable monomial is divisible by some $x_N$. The choice of $N$ is not unique, but that is irrelevant.

**Algorithm 5.17.** Given a monomial of the form $x_N u^I b^J$, replace it by $(x_N-r_N)u^I b^J$. (Note that the leading term of $r_N$ is $x_N$.)

**Lemma 5.18.**

(a) *The allowable $u^I b^J$ give a basis for $A/R\simeq E_*QH_*BP_*$.*

(b) *Any $u^I b^J$ can be written as a linear combination of allowable monomials in $A/R$ by iterating 5.17 a finite number of times.*

**Remark.** This lemma completes the proof of Theorem 5.3(a).

**Proof of 5.18(b).** We assign a nonnegative weight $w(x)\in\mathbb{Q}$ to each monomial $x\in A$ by the rules $w(xy)=w(x)+w(y)$, $w(u_i)=0$ and

5.19 $$w(b_{(m)})=f(m)=\frac{p^{2m+2}-2p^{2m}+1}{p^m(p^2-1)}.$$

Then we have for $m\geqslant i$,

5.20 $$p^i f(m-i)-f(m)=p^{-m}\left(\frac{p^{2i}-1}{p^2-1}\right),$$

in particular

5.21 $$pf(m-1)>f(m).$$

We want to show

**5.22** $\qquad r_N = x_N +$ terms of higher weight.

We first consider the expression $u_k s_N$. In $s_N$ the term $(-1)^\sigma \prod_{i=1}^k b^{p^i}_{(n_{\sigma(i)}-i)}$ has weight

$$\sum_{i=1}^{k} p^i f(n_{\sigma(i)} - i) = \sum_{i=1}^{k} \left( f(n_{\sigma(i)}) - \frac{p^{-n_{\sigma(i)}}}{p^2-1} \right) + \sum_{i=1}^{k} \frac{p^{2i-n_{\sigma(i)}}}{p^2-1}$$

by 5.20. The first term on the right is independent of the permutation while the second term is strictly minimized by setting $\sigma =$ identity. From 5.12 the coefficient of $u_j$, $j > k$ is given by

$$\sum_{i=1}^{k} (-1)^{i+k} s_{N_i} b^{p^j}_{(n_i-j)}.$$

Comparing this to 5.14, the coeffecient of $u_k$, we see by $j > k$ and 5.20 that the weight of each term is higher than for the corresponding term for $u_k$. This completes the proof of 5.22.

5.17 is homogeneous with respect to the triple grading on $A$ and so it stays within a certain finite dimensional vector space. Each application of it raises the weight by a positive rational number with bounded denominator, so a maximum possible weight, and therefore an allowable expression, will occur after a finite number of applications.

**Remark 5.23.** A simpler algorithm is the following. Choose $x^N$ so that each $n_i$ is minimal and replace the factor $u_k b^{p^k}_{(n_k)}$ by $u_k b^{p^k}_{(n_k)} - r_{k+n_k}$. We conjecture that this method is also effective.

We now prove 5.18 (a). We assume that we can write all $v^I b^J$ in terms of allowable $v^{I'} b^{J'}$. To prove (a) it is enough to show that the number of allowable terms is equal to the dimension of the vector space $QH_*\boldsymbol{BP}'_*$. We do several inductive steps. The main induction is on degree. For degree 2 we have $v^I b_{(0)}$ and there are no relations so induction is begun. To do our counting argument we will actually give a basis for $QE_0H_*\boldsymbol{BP}'_*$ (see Remark 4.24). In the last section we worked with $\boldsymbol{MU}'_*$ but we could have worked equally well with $\boldsymbol{BP}'_*$. We assume the reader can handle the minor changes necessary.

Define $s(J) = (0, j_0, j_1, \ldots)$ and $s^n(J) = s(s^{n-1}(J))$. If $j_0 = 0$ we can also define $s^{-1}(J) = (j_1, j_2, \ldots)$. We assume the allowable $v^I b^J$ give a basis for $QH_*\boldsymbol{BP}'_*$ for degrees $< 2k$. By the proof of the main theorem (between 4.20 and 4.21) a basis for $QE_0H_*\boldsymbol{BP}'_*$, in degrees $\leq 2k$, is given by all $v^I b^{s^n(J+\Delta_0)}$ with $v^I b^J$ allowable and $n \geq 0$. It is easy to see that this includes all $v^{I'} b^{J'}$ allowable of degree $2k$. By 4.17 we need to have exactly $d_{2k,*}$ allowable $v^I b^J$ of degree $2k$. By 4.23 it is enough to show that in degree $2k$ we have exactly $e_{2k/p,*}$ non-allowable $v^{I'} b^{J'}$ of the form $v^I b^{s^n(J+\Delta_0)}$, $n \geq 0$, $v^I b^J$ allowable. We define a map: (the Verschiebung)

$$V : QE_0H_{2k}\boldsymbol{BP}'_* \to QE_0H_{2k/p}\boldsymbol{BP}'_*$$

by

$$V(v^I b^{s^n(J+\Delta_0)}) = \begin{cases} 0 & \text{if } n = 0 \\ v^I b^{s^{n-1}(J+\Delta_0)} & \text{if } n > 0. \end{cases}$$

This gives a 1–1 correspondence between $v^I b^{s^n(J+\Delta_0)}$ with degree $2k$, $n > 0$, $v^I b^J$ allowable and the $v^{I'} b^{s^{n'}(J'+\Delta_0)}$ of degree $2k/p$, $n' \geq 0$, $v^{I'} b^{J'}$ allowable. Furthermore $V(v^I b^{s^n(J+\Delta_0)})$, $n > 0$, is allowable if and only if $v^I b^{s^n(J+\Delta_0)}$ is. By induction we see there are $e_{2k/p^2,*}$ non-allowable $v^I b^{s^n(J+\Delta_0)}$ of degree $2k$ with $n > 0$, $v^I b^J$ allowable. By 4.22 there are $d_{2k/p,*}$ $v^{I'} b^{J'+\Delta_0}$ of degree $2k/p$ with $v^{I'} b^{J'}$ allowable. If we show that these are in 1–1 correspondence with the non-allowable $v^I b^{J+\Delta_0}$ of degree $2k$ with $v^I b^J$ allowable then we will have $d_{2k/p,*} + e_{2k/p,*} = e_{2k/p,*}$ non-allowable $v^I b^{s^n(J+\Delta_0)}$, $n \geq 0$, with $v^I b^J$ allowable and our result will be proven. So, given $v^I b^{J+\Delta_0}$ of degree $2k$, not allowable, $v^I b^J$ allowable, let $n+1$ be the smallest $n+1$ such that $i_{n+1} \neq 0$. Then write

$$J = (p-1)\Delta_0 + p^2\Delta_{k_1} + \cdots + p^{n+1}\Delta_{k_n} + J''$$

with $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$, each $k_i$ minimum. Define $I' = I - \Delta_{n+1}$ and

$$J' = p\Delta_{k_1} + \cdots + p^n\Delta_{k_n} + s^{-1}(J'').$$

The following details are easily checked: $J$ can be written in this fashion, $J'$ is defined because $j''_0 = 0$, the degree of $v^{I'} b^{J'+\Delta_0}$ is $2k/p$. We need to show that $v^{I'} b^{J'}$ is allowable. We will show that if

**5.24** $$J' = p\Delta_{u_1} + p^2\Delta_{u_2} + \cdots + p^m\Delta_{u_m} + L$$

with $u_1 \leq u_2 \leq \cdots \leq u_m$, each $u_i$ minimum, $m$ maximum, then $k_i = u_i$ and $m = n$.

This will show $v^{I'} b^{J'}$ is allowable because $i'_k = 0$ if $k \leq n$. By the minimality of $u_i$ and the definition of $J'$ we have $u_i \leq k_i$. We proceed by induction. If

$$u_1 = k_1 \leq \cdots \leq u_{q-1} = k_{q-1} \leq u_q < k_q,$$

then

$$J = p\Delta_{k_1} + \cdots + p^{q-1}\Delta_{k_{q-1}} + p^q\Delta_{u_{q-1}} + p^{q+1}\Delta_{k_q} + \cdots + p^{n+1}\Delta_{k_n} + K$$

and $v^I b^J$ is not allowable ($i_{n+1} \neq 0$). This is a contradiction so $u_q = k_q$. In particular, if $m > n$, then $v^I b^J$ is not allowable.

We now have a map of $v^I b^J$ allowable to $v^{I'} b^{J'}$ allowable where degree $v^I b^{J+\Delta_0}$ is $2k$ and degree $v^{I'} b^{J'+\Delta_0}$ is $2k/p$. We need an inverse to this map. Let $v^{I'} b^{J'+\Delta_0}$ be of degree $2k/p$ with $v^{I'} b^{J'}$ allowable. Write $J'$ as in 5.24. Then define $I = I' + \Delta_{m+1}$ and

$$J = (p-1)\Delta_0 + p^2\Delta_{u_1} + \cdots + p^{m+1}\Delta_{u_m} + s(L).$$

We must show that $v^I b^J$ is allowable. Recall that from $i_q = 0$, $q \leq m$, because $v^{I'} b^{J'}$ is allowable. To prove $v^I b^J$ is allowable write

$$J = p\Delta_{v_1} + p^2\Delta_{v_2} + \cdots + p^n\Delta_{v_n} + K,$$

$v_1 \leq v_2 \leq \cdots \leq v_n$, each $v_i$ minimal, $n$ maximal, then we can show by an argument similar to the one for the other map that $v_i = u_i$ and $m = n$. Thus $v^I b^J$ is allowable and a careful check using the proofs that the maps are well defined shows that they are inverses to each other.

**Remark 5.25.** Although the proof of 5.18 (a) is motivated by the spectral sequence used to prove 4.2 (a), the argument could be rephrased so that it would be independent of Section 4. It should be regarded as a statement about $H^R_* \boldsymbol{BP}_*$.

## 6. Final remarks

We have described everything about $H_* \boldsymbol{MU}_*$ except the homology operations. The first author has done some work in this direction. As our interests are elsewhere at the present time and we may never come back to the problem we quote what is known for the benefit of others who wish to pursue the matter. We denote $[p]_F(x)/x$ as the power series $[p]_F(x)$ divided by $x$.

**Theorem 6.1.** *In* $H_*(\boldsymbol{MU}_*; \mathbb{F}_p)[[s]]$,

$$\sum_{i \geq 0} Q^i([1]) a^i s^{(p-1)i} = [p]_{[F]}(b(s))/b(s)$$

*where* $a \in \mathbb{F}_p$ *is some nonzero element.*

**Proof.** Let $q = 2p-2$ and let $L^{qi-1}$ denote the $qi-1$ dimensional lens space, i.e. the quotient of $S^{qi-1} \subset \mathbb{C}^{i(p-1)}$ by scalar multiplication by the $p$th roots of unity. Let $L^{qi} = L^{qi-1} \bigcup_f e^{qi}$ where $f: S^{qi-1} \to L^{qi-1}$ is the universal covering projection. Let $\tilde{L}^{qi}$ denote the universal cover of $L^{qi}$, i.e. $\tilde{L}^{qi}$ is $S^{qi-1}$ with $p$ copies of the disc attached.

Now we know that if $X$ is a stably complex manifold, a map $g: X \to \boldsymbol{MU}_0$ is induced by a map $f: \tilde{X} \to X$ where $\tilde{X}$ is another stably complex manifold of the same dimension. If $X$ and $\tilde{X}$ are manifolds with singularities and $f$ preserves them in an appropriate sense, then $f$ can still induce a map $g$. In particular, the covering projection $f_0: \tilde{L}^{qi} \to L^{qi}$ induces a map $g_0: L^{qi} \to \boldsymbol{MU}_0$. It follows from the definition of Dyer–Lashof operations that $g_0$ represents a nonzero scalar multiple of

$$Q^i[1] \in H^{qi}(\boldsymbol{MU}_0; \mathbb{F}_p).$$

Our program then is to show that the map $f_0$ is homologous to an appropriate map of a stably complex manifold into $\boldsymbol{MU}_0$.

Let $V_p^{qi} \subset \mathbb{C}P^{i(p-1)+1}$ denote a degree $p$ algebraic hypersurface of complex dimension $(p-1)i$ defined by the equation

$$\sum_{j=0}^{i(p-1)+1} z_i^p = 0.$$

Let $y \in \mathbb{C}P^{(p-1)i+1}$ denote the point $[0,0,\ldots,0,1]$. Then there is a linear projection

$$\pi : \mathbb{C}P^{i(p-1)+1} - \{y\} \to \mathbb{C}P^i$$

obtained by dropping the last co-ordinate. Restricting $\pi$ to $V_p^{qi}$ we get a map $f_1 : V_p^{qi} \to \mathbb{C}P^{i(p-1)}$ which induces a map $g_1 : \mathbb{C}P^{i(p-1)} \to MU_0$. The map $f_1$ can easily be seen to be a $p$-fold branched covering ramified along a degree $p$ hypersurface in $\mathbb{C}P^{i(p-1)}$.

We will show that the maps $g_0$ and $g_1$ are homologous by constructing an appropriate kind of "cobordism" between $f_0$ and $f_1$. Let $\hat{M} = \mathbb{C}P^{i(p-1)} \times I$ (where $I$ denotes the unit interval). Let $u \in \hat{M}$ denote the point $([1,0,\ldots 0],0)$. Let

$$\hat{M} \supset \mathring{D}^{qi} = \{([1, z_1, z_2 \cdots z_{i(p-1)}], 0) : \sum |z_k|^2 < \tfrac{1}{2}\},$$

i.e. $\mathring{D}^{qi}$ is an open $qi$-disc in $CP^{i(p-1)} \times \{0\}$ with center $u$. Let $U$ denote the complement of $\mathring{D}^{qi}$ in $CP^{i(p-1)} \times \{0\}$. Define an action of the group $Z/(p)$ on $U$ by

$$([z_0, z_1 \cdots z_{i(p-1)}], 0) \to ([e^{2\pi i/p} z_0, z_1 \cdots z_{i(p-1)}], 0).$$

Note that the quotient of $U$ by this action is a $D^2$ bundle over $CP^{i(p-1)-1}$ with boundary $L^{qi-1}$. Let $M$ denote the quotient of $\hat{M}$ obtained by indentifying points in $U$ conjugate under the group action. $M$ can be thought of as a manifold with singularities whose boundary is $L^{qi} \amalg \mathbb{C}P^{i(p-1)}$.

Hence $M$ is a "cobordism" between $\mathbb{C}P^{i(p-1)}$ and $L^{qi}$. We need to construct a cobordism $N$ between $V_p^{qi}$ and $\tilde{L}^{qi}$ and an appropriate map $N \to M$. Let $\hat{N} = V_p^{qi} \times I$ and consider $\hat{f} = f_1 \times \mathrm{id.} : \hat{N} \to \hat{M}$. The group action on $U$ can be lifted to one on $\hat{f}^{-1}(U)$ and we define the quotient $N$ of $\hat{N}$ in a similar way. Hence we get a map $f : N \to M$ which is a "cobordism" between $f_0$ and $f_1$, so $g_0$ and $g_1$ are homologous.

It remains then to describe the homology class represented by $g_1$ (and thereby $Q^i[1]$) in terms of familiar elements in $H_*(MU_0; \mathbb{F}_p)$. Recall that if $x \in MU^2\mathbb{C}P^{i(p-1)+1}$ is the canonical generator, then the degree $p$ hypersurface $V_p^{qi} \subset \mathbb{C}P^{i(p-1)+1}$ is dual to the cobordism class $[p]_{MU}(x)$. It follows that the map $f_1 : V_p^{qi} \to \mathbb{C}P^{i(p-1)}$ is dual to the class $[p]_{MU}(x)/x \in MU^0(CP^{i(p-1)})$. Hence the map

$$CP^{i(p-1)} \hookrightarrow \mathbb{C}P^\infty \xrightarrow{[p]_{MU}(x)/x} MU_0$$

is induced by $f_1$ and the Theorem follows.

A totally unrelated problem which we will also probably never get around to is the following. Let $m + n = k$ and let $M^n$ and $N^n$ denote weakly almost complex manifolds of dimensions $2m$ and $2n$ respectively. Let $f$ be an element of

$MU^{2k}N^n \simeq [N^n, MU_k]$. We have given an acceptable description of the homology of $MU_k$, but it would be nice to be able to describe the image of the map

$$f^* : H^*(MU_k; Z) \to H^*(N^n; Z)$$

without resorting to the space $MU_k$ (much the same as Chern classes can be handled). In particular, be duality, $MU^{2k}N^n \simeq MU_{2m}N^n$ which is represented by a bordism element

$$g : M^m \to N^n.$$

The information of $g$ is equivalent to that of $f$. It would be nice to describe the image of $f^*$ just using constructions with the map $g$. An application of this would come from the fact that elements in the ker of $H^*(MU_k; Z) \to H^*(MU_k; Z)$ give obstructions in $H^*(N^n; Z)$ to making $g$ bordant to an embedding $g_1 : M_1^m \hookrightarrow N^n$.

**Note added in proof**

Our final result allows one to compute the coaction $BP_*BP_k \to BP_*BP \otimes_{BP_*} BP_*BP_k$ in a simple way where $BP_*BP$ is the Quillen algebra $BP_*[t_1, t_2, \ldots]$ (see [1]). It is easy to compute the coproduct on both kinds of products, $*$ and $\circ$, so it is only necessary to compute

$$\mu : BP_*CP^\infty \to BP_*BP \otimes_{BP} BP_*CP^\infty.$$

Let $c$ be the canonical antiautomorphism of $BP_*BP$ and define $\beta = \sum_{i\geq 0}\beta_i$, $\beta(x) = \sum_{i\geq 0} x^i \otimes \beta_i$ and $t^F = 1 +_F t_1 +_F t_2 +_F \cdots = \sum_{i\geq 0}^F t_i$.

**Theorem.** $\mu(\beta) = \beta(c(t^F))$.

Both sides of this formula have only finite sums in each degree and by equality we mean they are degreewise the same.

**Proof.** From Adams' notes on Quillen's work [1], we combine

(11.4) (rephrased) $\mu(\beta) = \beta(b)$, (11.3)(iii) $b = c(M)$, (16.4) $M = t^F$.

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