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Generalised Einstein mass-variation formulae: II Superluminal relative frame velocities

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ABSTRACT

In part I of this paper we have deduced generalised Einstein mass variation formulae assuming relative frame velocities $v < c$. Here we present corresponding new expressions for superluminal relative frame velocities $v > c$. We again use the notion of the residual mass $m_0(v)$ which for $v > c$ is defined by the equation $m(v) = m_0(v)[(v/c)^2 - 1]^{-1/2}$ for the actual mass $m(v)$. The residual mass is essentially the actual mass with the Einstein factor removed, and we emphasise that we make no restrictions on $m_0(v)$. Using this formal device we deduce corresponding new mass variation formulae applicable to superluminal relative frame velocities, assuming only the extended Lorentz transformations and their consequences, and two invariants that are known to apply in special relativity. The present authors have previously speculated a dual framework such that both the rest mass m_0^* and the residual mass at infinite velocity m_∞^* (by which we mean p_∞^*/c, assuming finite momentum at infinity) are equally important parameters in the specification of mass as a function of its velocity, and the two arbitrary constants can be so determined. The new formulae involving two arbitrary constants may also be exploited so that the mass remains finite at the speed of light, and two distinct mass profiles are determined as functions of their velocity with the rest mass assumed to be alternatively prescribed at the origin of either frame. The two profiles so obtained $(M(U), m(u))$ and $(M^*(U), m^*(u))$ although distinct have a common ratio $M(U)/M^*(U) = m(u)/m^*(u)$ that is a function of $v > c$, indicating that observable mass depends upon the frame in which the rest mass is prescribed.

Introduction

In part I of this paper [4] we have deduced new Einstein mass variation expressions for relative frame velocities $v < c$. In this paper we present corresponding new formulae for superluminal relative frame velocities $v > c$. In two recent papers [2,3] the present authors have proposed extensions of the Lorentz transformations for superluminal motion, and we have described a duality involving both sub and super luminal velocities, such that any observer in either set of frames (subluminal or superluminal) would be unable to distinguish whether they belong to the sub or super set of frames. In other words, the present authors have proposed in [3] a dual world view, and one immediate consequence of this is that if the rest mass m_0^* is perceived to be a defining parameter of some importance, then pursuing this duality, so also is the limiting value $m_\infty^* = p_\infty^*/c$ where p_∞^* is defined to be the limiting value of the momentum as the velocity becomes infinite. For superluminal relative frame velocities, one observer perceives a subluminal particle velocity, while the other observer necessarily perceives a superluminal particle velocity. In this paper we produce new special relativistic mass variation formulae involving both sub and super luminal motions, which we exploit in the context of superluminal relative frame velocity to produce explicit new formulae corresponding to $\mathscr{E} = mc^2$, and involving both the rest mass m_0^* and the residual mass (as defined by (2.6)) at infinite relative velocity m_∞^*.

In this paper, assuming only the extended Lorentz transformations (2.1) and their consequences, we seek to develop the formalism of special relativity without making any assumptions on the variation of mass with velocity. To facilitate the analysis, we introduce the concept of the residual mass $m_0(v)$ as being defined by the equation $m(v) = m_0(v)[(v/c)^2 - 1]^{-1/2}$ for $v > c$, namely the residual mass is the actual mass with the Einstein factor removed. Again we emphasise that initially we make no restrictions on $m_0(v)$, and that this formal device merely facilitates the analysis.

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Of course for zero velocity, both the actual mass and the residual mass happen to coincide and are both equal to the rest mass. This seemingly trivial remark nevertheless implies that both the rest mass m_0^* and the limiting residual mass m_∞^*, in fact have the same status as being the values of the residual mass at zero and infinite velocities respectively.

Based on two invariances which are known to apply in special relativity, we deduce new expressions for the variation of mass with velocity involving two arbitrary constants. The two assumed invariances, are the usual force invariance in the direction of relative motion for two non-accelerating frames and a second invariance involving mass which is not so well known, but nevertheless applies in special relativity. In this way the new expressions have a corresponding status to the Einstein formula but involve an additional arbitrary constant. As an example, the additional degree of freedom can be exploited to incorporate both the rest mass m_0^* and the value of the residual mass m_∞^* at infinite relative velocity. Alternatively, the additional degree of freedom can be exploited to ensure that the actual mass remains finite at $v = c$, namely the arbitrary constants can be chosen to satisfy $m_0(c) = 0$.

In the following section we present a brief summary of the extended Lorentz transformations and their consequences. In the subsequent section we show how corresponding mass-momentum relations might be deduced without any assumptions on the variation of mass with velocity. Using these relations together with the force invariance and another invariance involving mass (see Eq. (2.5)) we deduce in the section thereafter the governing ordinary differential equation restricting the variation of mass with velocity (namely Eq. (3.9)). On solving this equation we eventually deduce new mass variation formulae, which include the Einstein expression as a special case. In two subsequent sections we examine two possible applications of the new formulae, and some brief concluding remarks are made in the final section of the paper.

Extended Lorentz transformations of special relativity

We consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 1. We note that the coordinate notation adopted here is slightly different to that normally used in special relativity involving primed and unprimed variables. We do this purposely because it is convenient to view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose the notation employed in nonlinear continuum mechanics is preferable. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

For $c < v < \infty$, the two extended Lorentz transformations derived in [2] can be expressed as

$$x = \frac{\varepsilon(X - vT)}{\left[(v/c)^2 - 1\right]^{1/2}}, \quad t = \frac{\varepsilon(T - vX/c^2)}{\left[(v/c)^2 - 1\right]^{1/2}}, \tag{2.1}$$

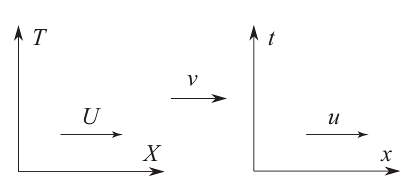

Fig. 1. Two inertial frames moving along x-axis with relative velocity v.

where $\varepsilon = \pm 1$ corresponding to the two distinct cases which arise from the two distinct sets of constraints,

$$x = -\varepsilon cT, \quad t = -\varepsilon X/c, \quad v = \infty.$$

Subsequent to the publication of [2], the work of Vieira [8] appearing at about the same time, was brought to our attention, who proposed two alternative derivations of the same extended Lorentz transformations. One algebraic in character and the other geometric. The almost simultaneous appearance of the same extended Lorentz transformations for superluminal motion is an entirely positive outcome, in that there is now some considerable commonality of agreement in the basic equations underlying superluminal motion. After the publication of [2], Andréka et al. [1] confirmed this giving yet another derivation of the above extended transformations in one spatial and one time dimensions. However, motivated by [2], Jin and Lazar [5] point out that these extended Lorentz transformations are not entirely new and have a prior history and we refer the reader to this paper for details of these earlier contributions. In some of these earlier contributions the transformations for $v > c$ are presented but not formally derived as such, and the main original contribution of [2] is the mode of their derivation as arising from the tangent vector for special relativity, combined with an initial condition for infinite relative velocity. The particular derivation of [2] is entirely mathematical in nature, not dependent on any ad-hoc physical reasoning, and quite distinct from all previous derivations including those cited by Jin and Lazar [5]. Quite recently Mantegna [6] makes use of an idea from [2] to formulate a possible solution to the problem of spinless tachyon localisation.

Irrespective of sub or super luminal relative frame motion, with velocities $U = dX/dT$ and $u = dx/dt$, (2.1) yields the well known Einstein addition of velocity law

$$u = \frac{U - v}{(1 - Uv/c^2)}, \tag{2.2}$$

and as an immediate consequence of (2.2) is the identity

$$[1 - (u/c)^2](1 - Uv/c^2)^2 = [1 - (v/c)^2][1 - (U/c)^2], \tag{2.3}$$

which is not so well-known, but is nevertheless fundamental to the development of the formulation of special relativity. Another important formula arising from (2.2) is

$$\left(\frac{1 + U/c}{1 - U/c}\right) = \left(\frac{1 + u/c}{1 - u/c}\right)\left(\frac{1 + v/c}{1 - v/c}\right), \tag{2.4}$$

and both (2.3) and (2.4) apply for both sub and super luminal motions. These two formulae reveal that at least one of the velocities u, v or U must not exceed the speed of light, and clearly both formulae need re-arrangement depending upon the particular values of the three velocities. In this paper we need to take the square root of (2.3) and the logarithm of (2.4).

As described in part I of this paper [4], in the following development of special relativity we adopt the two invariances, namely

$$\frac{dp}{dt} = \frac{dP}{dT}, \quad \frac{dm}{dx} = \frac{dM}{dX}, \tag{2.5}$$

which are known to apply in special relativity, and in this sense we claim that the resulting new mass variation formulae carry a corresponding status as the Einstein expression. The new formulae involve two arbitrary constants, while the Einstein expression involves only the rest mass as a single arbitrary constant. We have previously noted in part I that together the above two invariances imply that the energy-mass rates are the same in both frames, namely $d\mathscr{E}/dm = dE/dM$, which is clearly the case in conventional special relativity when both energy-mass rates are then equal to c^2.

In the next section for relative frame velocities $v > c$, we suppose that $U < c$, while $u > c$, and we extend the new mass variation expressions given in [4] assuming only the validity of the extended Lorentz transformations (2.1) and their consequences (2.2)–(2.4), but not making any assumptions of the mass variation with velocity, other than adopting the structure,

$$m(u) = \frac{m_0(u)}{[(u/c)^2 - 1]^{1/2}}, \quad M(U) = \frac{M_0(U)}{[1 - (U/c)^2]^{1/2}}, \tag{2.6}$$

where $m_0(u)$ and $M_0(U)$ are referred to as the residual masses, and denote arbitrary functions to be determined subsequently. The particular assumed structure (2.6) is not restrictive in any sense and merely facilitates the mathematical analysis. Together with the formal identities (2.3) and (2.5) it is sufficient to duplicate the essential structure of the Lorentz invariant mass-momentum relations.

In particular, as fully described in [4] we may again introduce the variable parameter $\lambda(u, U) = m_0(u)/M_0(U)$ to extend the Lorentz invariant mass-momentum relations and to derive new mass variation formulae based on maintaining the two invariances (2.5), and noting that the Einstein mass variation arises from $\lambda \equiv 1$. For superluminal relative frame motion with $u, v > c$ and $U < c$, we define the variables (ξ, η, γ) by

$$\xi = \log\left(\frac{1 + U/c}{1 - U/c}\right), \quad \eta = \log\left(\frac{u/c + 1}{u/c - 1}\right), \quad \gamma = \log\left(\frac{v/c + 1}{v/c - 1}\right), \tag{2.7}$$

for which again $\xi = \eta + \gamma$ while the inverses are given by

$$U = c\tanh(\xi/2), \quad u = c\coth(\eta/2), \quad v = c\coth(\gamma/2). \tag{2.8}$$

In the following section the above relations are exploited to deduce new mass variation formulae for superluminal relative frame velocities v.

Generalised Lorentz invariant mass-momentum relations for $v > c$

As fully described in part I of this paper [4] we determine new mass variation formulae assuming that the Lorentz invariant mass-momentum relations of special relativity may be extended, and in this section we present the analytical details for superluminal motion with $v > c$, assuming that $U < c$ and $u > c$. Accordingly, one observer registers a subluminal velocity while the other identifies the motion as superluminal. The residual masses $m_0(u)$ and $M_0(U)$ are defined through the relations (2.6) and that the appropriate square root relation arising from (2.3) becomes

$$\left[(u/c)^2 - 1\right]^{1/2}(Uv/c^2 - 1) = \left[1 - (U/c)^2\right]^{1/2}\left[(v/c)^2 - 1\right], \tag{3.1}$$

noting that for $U < c$ and $u, v > 0$, $Uv/c^2 > 1$ from (2.2). On multiplication of (2.2) by $m(u)$, we have

$$p = \frac{m_0(u)(v - U)}{\left[(u/c)^2 - 1\right]^{1/2}(Uv/c^2 - 1)} = \frac{\lambda(Mv - P)}{\left[(v/c)^2 - 1\right]^{1/2}}, \tag{3.2}$$

on using (3.1), $P = MU$ and λ is as previously defined, namely $\lambda = m_0(u)/M_0(U)$. Further, from $(2.6)_2$ and (3.1) we have

$$m = \frac{m_0(u)}{M_0(U)} \frac{M_0(U)(Uv/c^2 - 1)}{\left[1 - (U/c)^2\right]^{1/2}\left[(v/c)^2 - 1\right]^{1/2}} = \lambda\frac{(Pv/c^2 - M)}{\left[(v/c)^2 - 1\right]^{1/2}}, \tag{3.3}$$

and (3.2) and (3.3) constitute the appropriate generalisation of the Lorentz invariant mass-momentum relations, with inverse relations characterised by $1/\lambda$, thus

$$P = \frac{(mv + p)}{\lambda\left[(v/c)^2 - 1\right]^{1/2}}, \quad M = \frac{(pv/c^2 + m)}{\lambda\left[(v/c)^2 - 1\right]^{1/2}}.$$

The above equations constitute the formal generalisation of the equations of special relativity, without making any assumptions on the variation of mass with velocity. In the subsequent analysis, we supplement these equations with the two invariances (2.5), the first representing force invariance, while the second is an assumed invariance that is known to hold in special relativity.

On differentiating (3.2) and (3.3), making use of the superluminal Lorentz transformations (2.1), we may deduce from the two invariances (2.5), the two differential relations

$$\begin{aligned} &\lambda(v\,dM - dP) + d\lambda(Mv - P) = \varepsilon\,dP(1 - Uv/c^2), \\ &\lambda(v\,dP/c^2 - dM) + d\lambda(Pv/c^2 - M) = \varepsilon\,dM(1 - v/U), \end{aligned} \tag{3.4}$$

which we may view as two equations for the two unknowns $Q = dP/dM$ and $\mu = d\lambda/dM$. On solving (3.4) we find the intermediate relation

$$Q = \frac{c^2}{\lambda v}\left\{\mu\left(M - \frac{Pv}{c^2}\right) + \lambda + \varepsilon\left(1 - \frac{v}{U}\right)\right\},$$

and eventually after a long calculation we have

$$\begin{aligned} Q &= \frac{\left\{(1-\beta)\varepsilon Mc^2/U + P[\beta(\lambda + \varepsilon) - \varepsilon v/U]\right\}}{[(\beta\lambda + \varepsilon)M - \varepsilon Pv/c^2]}, \\ \mu &= -(\lambda + \varepsilon)\frac{[\beta\lambda + \varepsilon(1 - v/U)]}{[(\beta\lambda + \varepsilon)M - \varepsilon Pv/c^2]}, \end{aligned} \tag{3.5}$$

where it is convenient to introduce β defined by

$$\beta = \frac{(v/c)^2 - 1}{Uv/c^2 - 1}, \tag{3.6}$$

noting that $\beta - 1 = uv/c^2$. Now from $P = MU$, we may deduce from (3.5)

$$\begin{aligned} M\frac{dU}{dM} &= -\frac{\varepsilon\left[1 - \left(\frac{U}{c}\right)^2\right]\left[v + (\beta - 1)\frac{c^2}{U}\right]}{[\beta\lambda + \varepsilon(1 - Uv/c^2)]}, \\ M\frac{d\lambda}{dM} &= -\frac{(\lambda + \varepsilon)\left[\beta\lambda + \varepsilon\left(1 - \frac{v}{U}\right)\right]}{[\beta\lambda + \varepsilon(1 - Uv/c^2)]}, \end{aligned} \tag{3.7}$$

and division of these two equations provides the basic determining equation for λ, thus

$$\frac{dU}{d\lambda} = \frac{\varepsilon\left[1 - \left(\frac{U}{c}\right)^2\right]\left[v + (\beta - 1)\frac{c^2}{U}\right]}{(\lambda + \varepsilon)[\beta\lambda + \varepsilon(1 - v/U)]}. \tag{3.8}$$

On making use of the relations (2.8), and after much simplification, Eq. (3.8) eventually reduces to the standard first order ordinary differential equation

$$\frac{d\sigma}{d\xi} - \frac{\sigma}{2}\frac{\sinh(\xi - \gamma/2)}{\cosh(\xi - \gamma/2)} = \frac{-\varepsilon\sinh\xi}{4\cosh(\gamma/2)\cosh(\xi - \gamma/2)}, \tag{3.9}$$

where $\sigma = 1/(\lambda + \varepsilon)$ and $\varepsilon = \pm 1$. For the two cases $\varepsilon \pm 1$, Eq. (3.9) can be formally written as

$$\frac{d}{d\xi}\left\{\frac{\sigma}{[\cosh(\xi - \gamma/2)]^{1/2}}\right\} = \frac{-\varepsilon\sinh\xi}{4\cosh(\gamma/2)[\cosh(\xi - \gamma/2)]}, \tag{3.10}$$

and on writing $\sinh\xi$ as $\sinh(\xi - \gamma/2 + \gamma/2)$ and expanding can be integrated to yield

$$\sigma = \frac{\varepsilon}{2} - \frac{\varepsilon}{4}\tanh\left(\frac{\gamma}{2}\right)\left[\cosh\left(\xi - \frac{\gamma}{2}\right)\right]^{1/2}\int_0^{\xi - \frac{\gamma}{2}}\frac{d\rho}{(\cosh\rho)^{1/2}} + C_1\left[\cosh\left(\xi - \frac{\gamma}{2}\right)\right]^{1/2}, \tag{3.11}$$

where C_1 denotes on arbitrary constant of integration.

From (3.11) and $\sigma = (\lambda+\varepsilon)^{-1}$ we may deduce

$$\lambda = m_0(u)/M_0(U) = \frac{\frac{1}{2}+\frac{\tanh(\gamma/2)}{4}[\cosh(\xi-\gamma/2)]^{1/2}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}-\varepsilon C_1[\cosh(\xi-\gamma/2)]^{1/2}}{\frac{\varepsilon}{2}-\frac{\varepsilon\tanh(\gamma/2)}{4}[\cosh(\xi-\gamma/2)]^{1/2}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+C_1[\cosh(\xi-\gamma/2)]^{1/2}}, \quad (3.12)$$

and we comment that all the integrals given in this paper, can if necessary, be expressed in terms of elliptic integrals. Conceptually and in terms of understanding the structure of the new solutions, it is perhaps easier to leave them in the above form.

With λ so determined from the above, we now proceed to integrate $(3.7)_1$ as follows

$$\begin{aligned}\frac{dM}{M} &= \frac{\varepsilon\, dU}{\left[1-(U/c)^2\right]}\left\{\varepsilon\frac{U}{c^2}-\frac{\beta U(\lambda+\varepsilon)}{c^2(\beta-1+Uv/c^2)}\right\}\\ &= \frac{U\,dU}{c^2\left[1-(U/c)^2\right]}\left\{1-\frac{\beta\varepsilon}{\sigma(\beta-1+Uv/c^2)}\right\}\\ &= \frac{1}{2}\tanh(\xi/2)\,d\xi-\frac{\varepsilon\sinh\xi\,d\xi}{4\sigma\cosh(\gamma/2)\cosh(\xi-\gamma/2)},\end{aligned} \quad (3.13)$$

where we have exploited (3.6) and the transformations (2.7) and their inverses (2.8). From (3.10) it is clear that (3.13) can be reformulated

$$\frac{dM}{M} = \frac{1}{2}\tanh(\xi/2)\,d\xi+\frac{d\left(\sigma/[\cosh(\xi-\gamma/2)]^{1/2}\right)}{\sigma/[\cosh(\xi-\gamma/2)]^{1/2}},$$

for both $\varepsilon = \pm 1$, so that in both cases

$$M(U) = \frac{C_2\cosh(\xi/2)\sigma}{[\cosh(\xi-\gamma/2)]^{1/2}},$$

where C_2 denotes a further arbitrary constant. From this equation, $(2.6)_2$ and $\sigma = (\lambda+\varepsilon)^{-1}$ we may deduce

$$m_0(u)+\varepsilon M_0(U) = \frac{C_2}{[\cosh(\xi-\gamma/2)]^{1/2}}, \quad (3.14)$$

which along with (3.12) provides two equations for the determination of the two unknown functions $m_0(u)$ and $M_0(U)$. From (3.12) and (3.14) we may deduce

$$\begin{aligned}M_0(U) &= C_2\left\{C_1-\frac{\varepsilon}{4}\tanh(\gamma/2)\int_0^{\xi-\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{\varepsilon}{2[\cosh(\xi-\gamma/2)]^{1/2}}\right\},\\ m_0(u) &= C_2\left\{-\varepsilon C_1+\frac{1}{4}\tanh(\gamma/2)\int_0^{\xi-\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{1}{2[\cosh(\xi-\gamma/2)]^{1/2}}\right\},\end{aligned} \quad (3.15)$$

and on adopting the notation $M_0(U) = N_0(\xi)$ and $m_0(u) = n_0(\eta)$ we have

$$\begin{aligned}N_0(\xi) &= C_2\left\{C_1-\frac{\varepsilon}{4}\tanh(\gamma/2)\int_0^{\xi-\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{\varepsilon}{2[\cosh(\xi-\gamma/2)]^{1/2}}\right\},\\ n_0(\eta) &= C_2\left\{-\varepsilon C_1+\frac{1}{4}\tanh(\gamma/2)\int_0^{\eta+\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{1}{2[\cosh(\eta+\gamma/2)]^{1/2}}\right\}.\end{aligned} \quad (3.16)$$

From these expressions we may deduce

$$\begin{aligned}\frac{dN_0(\xi)}{d\xi} &= -\frac{\varepsilon C_2\sinh\xi}{4\cosh(\gamma/2)[\cosh(\xi-\gamma/2)]^{3/2}},\\ \frac{dn_0(\eta)}{d\eta} &= -\frac{C_2\sinh\eta}{4\cosh(\gamma/2)[\cosh(\eta+\gamma/2)]^{3/2}},\end{aligned} \quad (3.17)$$

which are required in the corresponding energy formulae

$$\begin{aligned}E &= M(U)c^2-c^2\int_0^{\xi}\operatorname{sech}(\zeta/2)\frac{dN_0(\zeta)}{d\zeta}\,d\zeta,\\ \mathscr{E} &= m(u)c^2+c^2\int_0^{\eta}\operatorname{cosech}(\psi/2)\frac{dn_0(\psi)}{d\psi}\,d\psi,\end{aligned} \quad (3.18)$$

noting that from $(3.17)_2$ the integral in $(3.18)_2$ is overall well-behaved at the origin, and that both expressions in (3.18) are assumed to adopt their mass energies as the datum energy.

Application of formulae using rest mass and residual mass at infinite relative velocity

Pursuing the duality proposed by the authors in [2,3], if the rest mass m_0^*, which happens to coincide with the residual mass at zero velocity, is taken to be a defining parameter of some importance, then so also is the value of the residual mass m_∞^* at infinite velocity. These considerations lead us to propose the following boundary conditions on (3.15) for the determination of the two arbitrary constants C_1 and C_2; thus

$$\begin{aligned}&U=0,\quad \xi=0,\quad \eta=-\gamma,\quad M_0(0)=m_0^*,\\ &u=\infty,\quad \xi=\gamma,\quad \eta=0,\quad m_0(\infty)=m_\infty^*,\end{aligned} \quad (4.1)$$

and from (3.15) and (4.1) we obtain

$$\begin{aligned}m_0^* &= C_2\left\{C_1+\frac{\varepsilon}{4}\tanh\left(\frac{\gamma}{2}\right)\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{\varepsilon}{2[\cosh(\gamma/2)]^{1/2}}\right\},\\ m_\infty^* &= C_2\left\{-\varepsilon C_1+\frac{1}{4}\tanh\left(\frac{\gamma}{2}\right)\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{1}{2[\cosh(\gamma/2)]^{1/2}}\right\}.\end{aligned} \quad (4.2)$$

On solving (4.2) we may deduce

$$C_1 = \left(\frac{\varepsilon m_0^*-m_\infty^*}{m_0^*+\varepsilon m_\infty^*}\right)\left(\frac{\tanh(\gamma/2)}{4}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{1}{2[\cosh(\gamma/2)]^{1/2}}\right),$$

$$C_2 = \frac{\varepsilon m_0^*+m_\infty^*}{2}\left(\frac{\tanh(\gamma/2)}{4}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}+\frac{1}{2[\cosh(\gamma/2)]^{1/2}}\right)^{-1},$$

assuming that $m_0^*+\varepsilon m_\infty^*$ is non-zero. From the particular values $\eta = -\gamma$ and $\xi = \gamma$ we may deduce respectively

$$\begin{aligned}m_0(-v) = n_0(-\gamma) &= \left(\frac{m_\infty^*-\frac{\varepsilon m_0^*\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}}{1+\frac{\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}}\right),\\ M_0(c^2/v) = N_0(-\gamma) &= \left(\frac{m_0^*-\frac{\varepsilon m_\infty^*\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}}{1+\frac{\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}}\int_0^{\gamma/2}\frac{d\rho}{(\cosh\rho)^{1/2}}}\right),\end{aligned} \quad (4.3)$$

noting the apparent symmetries, both with regard to being even functions of γ, and having the same and equal dependency on m_0^* and m_∞^*. We emphasize that m_∞^* is the limiting value of the residual mass $m_0(u)$ in the limit of infinite velocity u, or alternatively $m_\infty^* = p_\infty^*/c$ where p_∞^* denotes the limiting momentum. It is clear from $(4.3)_1$ that the given expressions represent a weighted average of the values m_0^* and m_∞^*. Fig. 2 shows the variation of $m_0(v)/m_0^*$ for v/c in the interval $(1,\infty)$ as given by $(4.3)_1$, again noting that $m_0(v)$ is an even function and that the actual mass is given by $m(v) = \sinh(\gamma/2)m_0(v)$. The corresponding values for γ lie in the interval $(0,\infty)$. In Fig. 3 we let $V = c^2/v$ and the figure shows the variation of $M_0(V)/m_0^*$ as arising from $(4.3)_2$ for $V/c = c/v$ in the interval $(0,1)$ noting that the actual mass is determined from $M(V) = \cosh(\gamma/2)M_0(V)$. In both of these figures, the value $m_\infty^* = m_0^*/2$ is arbitrarily adopted, and the dashed curve corresponds to $\varepsilon = 1$ while the solid curve corresponds to $\varepsilon = -1$. In

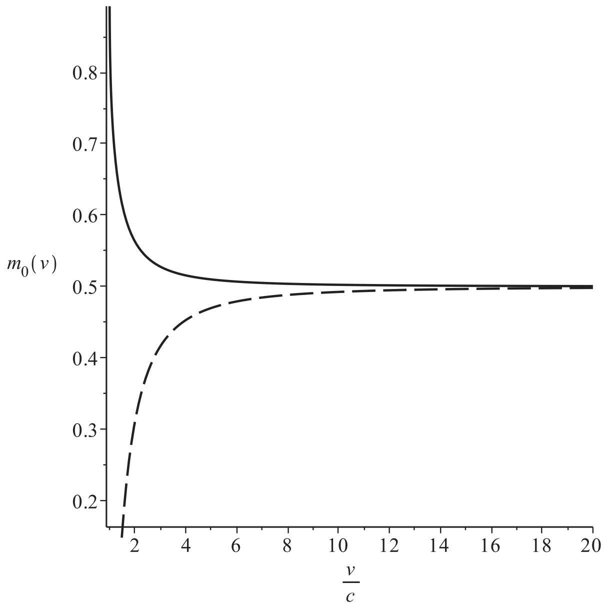

Fig. 2. Variation of residual mass $m_0(v)/m_0^*$ from (4.3)$_1$ for $\varepsilon = 1$ (dashed) and $\varepsilon = -1$ (solid).

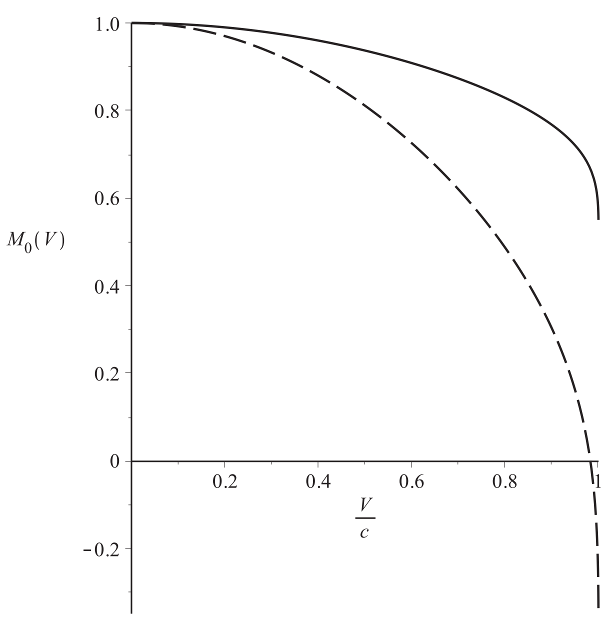

Fig. 3. Variation of residual mass $M_0(v)/m_0^*$ from (4.3)$_2$ for $\varepsilon = 1$ (dashed) and $\varepsilon = -1$ (solid).

the figures, the Einstein expression corresponds to the lines $m_0(v)/m_0^* = 1/2$ in Fig. 2 and $M_0(V)/m_0^* = 1$ in Fig. 3.

Further from (3.17) and (3.18) we find

$$E = M(U)c^2 - \frac{C_2 c^2 \varepsilon}{\cosh(\gamma/2)} \left\{ \frac{\cosh(\xi - \gamma)/2}{[\cosh(\xi - \gamma/2)]^{1/2}} - [\cosh(\gamma/2)]^{1/2} \right\},$$

$$\mathscr{E} = m(u)c^2 - \frac{C_2 c^2}{\cosh(\gamma/2)} \left\{ \frac{\sinh(\eta + \gamma/2)/2}{[\cosh(\eta + \gamma/2)]^{1/2}} - \frac{\sinh(\gamma/2)}{[\cosh(\gamma/2)]^{1/2}} \right\},$$

and by making use of Eq. (3.14), these expressions can be simplified to yield

$$\begin{aligned} E &= \left\{ M(U)U - \varepsilon m(u)u \left[\left(\frac{v}{c}\right)^2 - 1 \right]^{1/2} \right\} \frac{c^2}{v} + E_0, \\ \mathscr{E} &= \left\{ m(u)u - \varepsilon M(U)U \left[\left(\frac{v}{c}\right)^2 - 1 \right]^{1/2} \right\} \frac{c^2}{v} + \mathscr{E}_0, \end{aligned} \tag{4.4}$$

where E_0 and $\mathscr{E}_0$ are defined by

$$E_0 = [m_0^* + \varepsilon m_0(-v)]c^2,$$

$$\mathscr{E}_0 = [m_\infty^* + \varepsilon M_0(c^2/v)](c^3/v),$$

and these relations can be shown from (4.3) and (4.4) to become

$$E_0 = (m_\infty^* + \varepsilon m_0^*)c^2 \left(1 + \frac{\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}} \int_0^{\gamma/2} \frac{d\rho}{[\cosh\rho]^{1/2}} \right)^{-1},$$

$$\mathscr{E}_0 = (m_0^* + \varepsilon m_\infty^*) \frac{c^3}{v} \left(1 + \frac{\sinh(\gamma/2)}{2[\cosh(\gamma/2)]^{1/2}} \int_0^{\gamma/2} \frac{d\rho}{[\cosh\rho]^{1/2}} \right)^{-1}.$$

We may provide an independent check on the relations (4.4) by confirming that they satisfy the following Lorentz invariant energy-momentum relations

$$\mathscr{E} = \frac{\varepsilon(E - Pv)}{\left[(v/c)^2 - 1\right]^{1/2}} + \text{const}, \quad E = -\frac{\varepsilon(\mathscr{E} + pv)}{\left[(v/c)^2 - 1\right]^{1/2}} + \text{const}, \tag{4.5}$$

where the arbitrary constants are determined by assumed initial data. We may readily verify that the derived expressions (4.4) indeed satisfy (4.5).

The Lorentz invariant energy-momentum relations (4.5) are most easily verified as follows. From the energy and momentum equations in both frames, and noting that $f = dp/dt$ and $F = dP/dT$ are assumed to coincide, we have using the extended Lorentz transformation (2.1)$_2$

$$\begin{aligned} \frac{d}{dT}\left\{ \mathscr{E} - \frac{\varepsilon(E - Pv)}{\left[(v/c)^2 - 1\right]^{1/2}} \right\} &= \frac{fu\varepsilon(1 - Uv/c^2)}{\left[(v/c)^2 - 1\right]^{1/2}} - \frac{\varepsilon(fU - fv)}{\left[(v/c)^2 - 1\right]^{1/2}} \\ &= \frac{f\varepsilon}{\left[(v/c)^2 - 1\right]^{1/2}} \{(U - v) - (U - v)\} \\ &= 0, \end{aligned}$$

and similarly

$$\begin{aligned} &\frac{d}{dT}\left\{ E + \frac{\varepsilon(\mathscr{E} - pv)}{\left[(v/c)^2 - 1\right]^{1/2}} \right\} \\ &= fU + \frac{\varepsilon}{\left[(v/c)^2 - 1\right]^{1/2}} \{fu + fv\} \frac{\varepsilon(1 - Uv/c^2)}{\left[(v/c)^2 - 1\right]^{1/2}} \\ &= \frac{f}{\left[(v/c)^2 - 1\right]} \left\{ U\left[\left(\frac{v}{c}\right)^2 - 1 \right] + (U - v) + v\left(1 - \frac{Uv}{c^2}\right) \right\} = 0, \end{aligned}$$

and therefore we may deduce the energy-momentum relations (4.5). We emphasise that the results (4.5) must hold irrespective of any assumed mass variation, since they apply assuming only the energy and momentum equations $dE/dT = FU$, $F = dP/dT$ and force invariance $F = f$.

On re-writing (4.4) as

$$E = \left(P - \varepsilon p\left[(v/c)^2 - 1\right]^{1/2}\right)(c^2/v) + E_0,$$
$$\mathscr{E} = -\left(p + \varepsilon P\left[(v/c)^2 - 1\right]^{1/2}\right)(c^2/v) + \mathscr{E}_0, \tag{4.6}$$

we see that (4.6) admits the following interesting identity

$$(\mathscr{E} - \mathscr{E}_0)^2 + (E - E_0)^2 = (p^2 + P^2)c^2.$$

Although the relations (4.6) have been derived within the context of a particular mass variation (namely (3.15) or (3.16)) the relations (4.6) can be confirmed directly for any mass variation since

$$\frac{d}{dT}\left\{E - \frac{c^2}{v}\left(P - \varepsilon p\left[(v/c)^2 - 1\right]^{1/2}\right)\right\} = f\left\{U - \frac{c^2}{v} + \frac{c^2}{v}\left(1 - \frac{Uv}{c^2}\right)\right\}$$
$$= 0,$$

and

$$\frac{d}{dT}\left\{\mathscr{E} + \frac{c^2}{v}\left(p + \varepsilon P\left[(v/c)^2 - 1\right]^{1/2}\right)\right\}$$
$$= \frac{f\varepsilon}{\left[(v/c)^2 - 1\right]^{1/2}}\left\{(U - v) + \frac{c^2}{v}\left(1 - \frac{Uv}{c^2}\right) + \frac{c^2}{v}\left(\left[\frac{v}{c}\right]^2 - 1\right)\right\}$$
$$= 0,$$

using only the extended Lorentz transformation $(2.1)_2$, the energy and momentum Eqs. (2.5) and force invariance. Together with (4.5) and (4.6) may be combined to give

$$\mathscr{E} - \mathscr{E}_0 = \frac{\varepsilon[(E - E_0) - Pv]}{\left[(v/c)^2 - 1\right]^{1/2}}, \quad p = \frac{\varepsilon\left[P - (E - E_0)v/c^2\right]}{\left[(v/c)^2 - 1\right]^{1/2}}, \tag{4.7}$$

and the inverse relations

$$E - E_0 = -\frac{\varepsilon[(\mathscr{E} - \mathscr{E}_0) + pv]}{\left[(v/c)^2 - 1\right]^{1/2}}, \quad p = \frac{\varepsilon\left[p + (E - E_0)v/c^2\right]}{\left[(v/c)^2 - 1\right]^{1/2}}, \tag{4.8}$$

again emphasising that (4.7) and (4.8) apply independently of any assumed mass variation.

Finally in this section, we comment that directly from the transformations (2.7) and (2.8) and the integral (3.14) we may deduce

$$m_0(u) + \varepsilon M_0(U) = \frac{C_2}{[\cosh(\xi - \gamma/2)]^{1/2}}$$
$$= \frac{C_2\left[1 - (U/c)^2\right]^{1/2}\left[(v/c)^2 - 1\right]^{1/4}}{\left(\left[1 + (U/c)^2\right](v/c) - 2(U/c)\right)^{1/2}},$$

and therefore from the two sets of conditions (4.1) we have respectively

$$m_0(-v) + \varepsilon m_0^* = \frac{C_2}{[\cosh(\gamma/2)]^{1/2}} = C_2\left(\frac{c}{v}\right)^{1/2}\left[(v/c)^2 - 1\right]^{1/4},$$
$$m_\infty^* + \varepsilon M_0(c^2/v) = \frac{C_2}{[\cosh(\gamma/2)]^{1/2}} = C_2\left(\frac{c}{v}\right)^{1/2}\left[(v/c)^2 - 1\right]^{1/4}. \tag{4.9}$$

We may confirm the relations (4.9), since from (4.2) we have

$$m_\infty^* + \varepsilon m_0^* = \frac{C_2}{\left[\cosh\left(\frac{\gamma}{2}\right)\right]^{1/2}}\left(1 + \frac{\sinh\left(\frac{\gamma}{2}\right)}{2\left[\cosh\left(\frac{\gamma}{2}\right)\right]^{1/2}}\int_0^{\frac{\gamma}{2}}\frac{d\rho}{(\cosh\rho)^{1/2}}\right), \tag{4.10}$$

while from the expressions (4.3) we obtain

$$m_0(-v) + \varepsilon m_0^* = m_\infty^* + \varepsilon M_0\left(\frac{c^2}{v}\right) = \frac{(m_\infty^* + \varepsilon m_0^*)}{\left(1 + \frac{\sinh\left(\frac{\gamma}{2}\right)}{2\left[\cosh\left(\frac{\gamma}{2}\right)\right]^{1/2}}\int_0^{\frac{\gamma}{2}}\frac{d\rho}{(\cosh\rho)^{1/2}}\right)}, \tag{4.11}$$

and (4.9) follows directly from (4.10) and (4.11). In the following section, we consider another application of the formulae displaying finite mass at the speed of light.

Application with finite mass at the speed of light

Finally, we present an application of the two parameter family of mass variation (3.15) and the integral relation (3.14) for which the arbitrary constant C_1 is determined such that the mass remains finite. It eventuates that with an appropriate choice of the constant C_1 we may arrange for both $m(u)$ and $M(U)$ to be finite in the limit $u, U \to c$. From (3.15), it is clear that if the constant C_1 is such that

$$C_1 = \frac{\varepsilon}{4}\tanh(\gamma/2)\int_0^\infty \frac{d\rho}{(\cosh\rho)^{1/2}} = \frac{\varepsilon}{4\sqrt{2}}\tanh(\gamma/2)B\left(\frac{1}{4}, \frac{1}{2}\right),$$

where $B(1/4, 1/2) = 5.244115\ldots$ denotes the usual beta function, and then both $m_0(u)$ and $M_0(U)$ vanish in the limit $u, U \to c$. From L'Hôpital's rule we have

$$M_c = \lim_{U\to c}\frac{M_0(U)}{\left[1 - (U/c)^2\right]^{1/2}} = \lim_{\xi\to\infty}\frac{N_0(\xi)}{\operatorname{sech}(\xi/2)} = \lim_{\xi\to\infty}\frac{dN_0(\xi)/d\xi}{-\frac{1}{2}\frac{\tanh(\xi/2)}{\cosh(\xi/2)}},$$
$$m_c = \lim_{u\to c}\frac{m_0(u)}{\left[(u/c)^2 - 1\right]^{1/2}} = \lim_{\eta\to\infty}\frac{n_0(\eta)}{\operatorname{cosech}(\eta/2)} = \lim_{\eta\to\infty}\frac{dn_0(\eta)/d\eta}{-\frac{1}{2}\frac{\coth(\eta/2)}{\sinh(\eta/2)}}, \tag{5.1}$$

and from (3.17) and (5.1) we may eventually deduce the limiting values as follows

$$M_c = \frac{\varepsilon C_2 e^{5\gamma/4}}{\sqrt{2}(e^\gamma + 1)} = \frac{\varepsilon C_2 (v/c + 1)^{5/4}}{2\sqrt{2}(v/c)(v/c - 1)^{1/4}},$$

$$m_c = \frac{C_2 e^{-\gamma/4}}{\sqrt{2}(e^\gamma + 1)} = \frac{C_2 (v/c - 1)^{5/4}}{2\sqrt{2}(v/c)(v/c + 1)^{1/4}},$$

$$m_c M_c = \frac{\varepsilon C_2^2}{8}\operatorname{sech}^2(\gamma/2), \quad m_c = \varepsilon M_c e^{-3\gamma/2},$$

$$M_c^2 = \frac{C_2^2}{8}e^{3\gamma/2}\operatorname{sech}^2(\gamma/2), \quad M_c = \frac{C_2}{2\sqrt{2}}\frac{e^{3\gamma/4}}{\cosh(\gamma/2)},$$

and from which we may deduce the interesting relations

$$m_c M_c = \frac{\varepsilon C_2^2\left[(v/c)^2 - 1\right]}{8(v/c)^2}, \quad \frac{m_c}{M_c} = \varepsilon\left(\frac{(v/c) - 1}{(v/c) + 1}\right)^{3/2},$$

noting however that in the case $\varepsilon = -1$, both of the latter relations imply that one of m_c or M_c is necessarily negative. Although the notion of negative mass is sometimes entertained as a physical concept, this particular application may only be physically sensible in the case $\varepsilon = 1$.

Assuming that C_1 is determined such that the masses remain finite at $u = U = c$, then for $u, v > c$ and $U < c$ we may deduce the following expressions

$$M_0(U) = \varepsilon C_2\left\{\frac{1}{2[\cosh(\xi-\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{\xi-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\},$$
$$m_0(u) = C_2\left\{\frac{1}{2[\cosh(\eta+\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\eta+\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}, \tag{5.2}$$

with variables (ξ, η, γ) defined by (2.7) and (2.8). Now for $U, v > c$ and $u < c$ and adopting the convention that the residual masses $M_0(U)$ and $m_0(u)$ for $U > c$ and $u < c$ are defined respectively by

$$M(U) = \frac{M_0(U)}{\left[(U/c)^2 - 1\right]^{1/2}}, \quad m(u) = \frac{m_0(u)}{\left[1 - (u/c)^2\right]^{1/2}}, \tag{5.3}$$

we may show that the above Eqs. (5.2) still apply except that now the variables (ξ, η, γ) are defined by

$$\xi = \log\left(\frac{U/c+1}{U/c-1}\right), \quad \eta = \log\left(\frac{1+u/c}{1-u/c}\right), \quad \gamma = \log\left(\frac{v/c+1}{v/c-1}\right), \tag{5.4}$$

for which as usual $\xi = \eta + \gamma$ while the inverses are given by

$$U = c\coth(\xi/2), \quad u = c\tanh(\eta/2), \quad v = c\coth(\gamma/2).$$

With mass solutions so defined, we may plot mass variation profiles for both sub and superluminal velocities. We might for example position the particle at the origin of the (X, T) frame and impose the conditions

$$U = 0, \quad u = -v, \quad \xi = 0, \quad \eta = -\gamma, \quad M(0) = m_0^*, \tag{5.5}$$

where m_0^* denotes the assumed known rest mass, and in this case C_2 is determined from the equation

$$m_0^* = \varepsilon C_2\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\},$$

and (5.2) becomes

$$M_0(U) = N_0(\xi) = m_0^*\frac{\left\{\frac{1}{2[\cosh(\xi-\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{\xi-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}},$$
$$m_0(u) = n_0(\eta) = \varepsilon m_0^*\frac{\left\{\frac{1}{2[\cosh(\eta+\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\eta+\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}. \tag{5.6}$$

Figs. 4 and 5 show the mass variations $M(U)/m_0^*$ and $m(u)/m_0^*$ for both sub and superluminal velocities as determined respectively from $(5.6)_1$ and $(5.6)_2$.

Alternatively, we might fix C_2 by positioning the particle at the origin of the (x, t) frame and imposing the conditions

$$U = v, \quad u = 0, \quad \xi = \gamma, \quad \eta = 0, \quad m(0) = m_0^*, \tag{5.7}$$

where m_0^* is the assumed known rest mass such that

$$m_0^* = C_2\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\},$$

and (5.2) becomes

$$M_0^*(U) = N_0^*(\xi) = \varepsilon m_0^*\frac{\left\{\frac{1}{2[\cosh(\xi-\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4}\int_{\xi-\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}},$$
$$m_0^*(u) = n_0^*(\eta) = m_0^*\frac{\left\{\frac{1}{2[\cosh(\eta+\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\eta+\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}, \tag{5.8}$$

noting that we have appended an asterisk to designate that the solution (5.8) is an alternative solution to (5.6). Generally, the two alternative conditions (5.5) and (5.7) generate distinct mass trajectories as functions of velocity, namely (5.6) and (5.8) respectively. Notice however, the necessary correspondence that (5.8) is obtained from (5.6) with the interchanges γ to $-\gamma$, m_0 to M_0 and η to ξ. Figs. 6 and 7 show the mass variations $M^*(U)/m_0^*$ and $m^*(u)/m_0^*$ for both sub and superluminal velocities as determined respectively from $(5.8)_1$ and $(5.8)_2$.

The figures are obtained in the following manner. For example, for the conditions (5.5) ξ lies in the interval $(0, \infty)$ while the variable η lies in the interval $(-\gamma, \infty)$. Now it is important to emphasise that with the mass solutions defined by either (5.6) and (5.8), the same expressions apply for both sub and superluminal velocities except that in the two regions ξ and η are defined differently; namely (2.7) for $U < c$ and $u > c$, while (5.4) applies for $U > c$ and $u < c$. As described in part I, we might visualise the solution

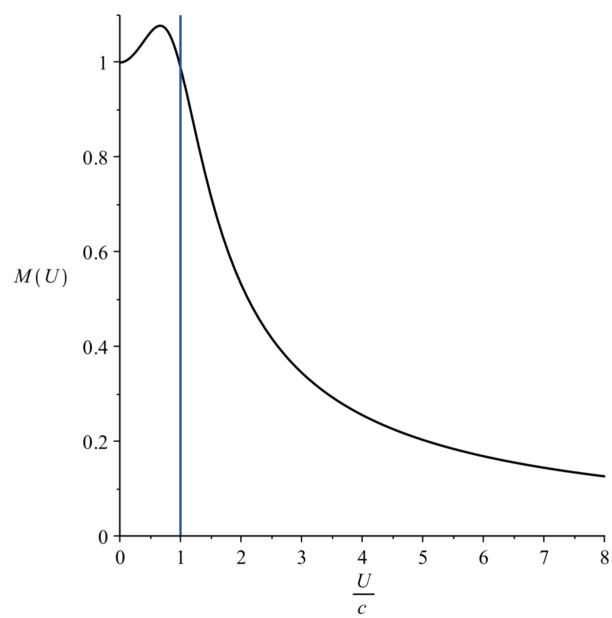

Fig. 4. Variation of mass $M(U)/m_0^*$ from $(5.6)_1$, $(2.6)_2$ and $(5.3)_1$ for $\varepsilon = 1$.

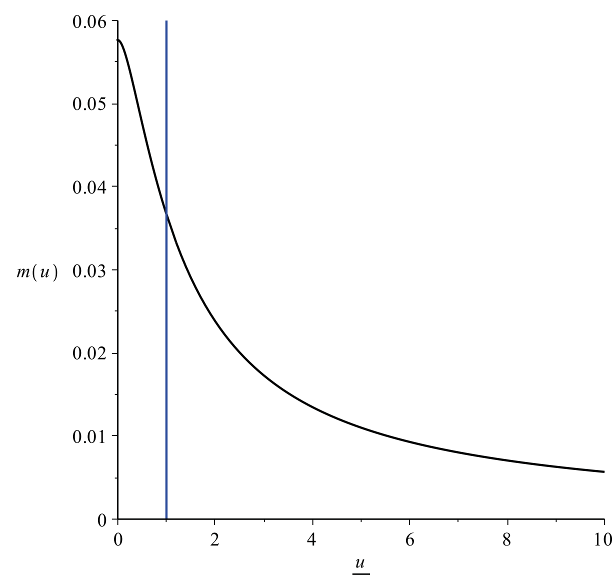

Fig. 5. Variation of mass $m(u)/m_0^*$ from $(5.6)_2$, $(2.6)_1$ and $(5.3)_2$ for $\varepsilon = 1$.

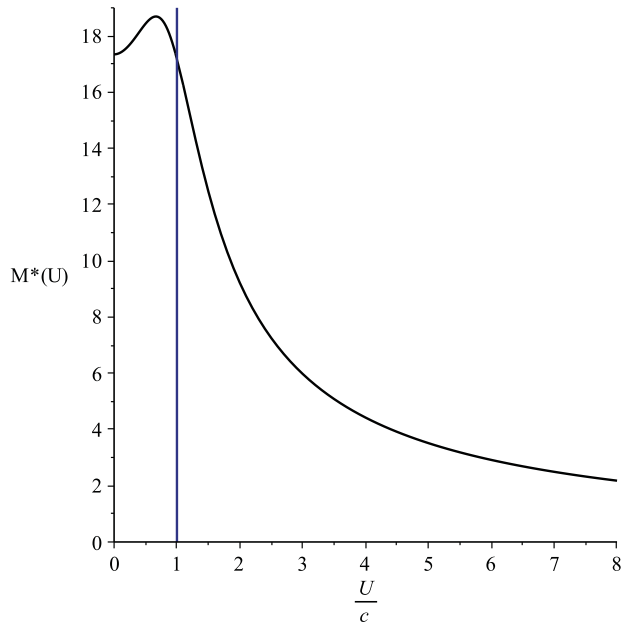

Fig. 6. Variation of mass $M^*(U)/m_0^*$ from (5.8)$_1$, (2.6)$_2$ and (5.3)$_1$ for $\varepsilon = 1$.

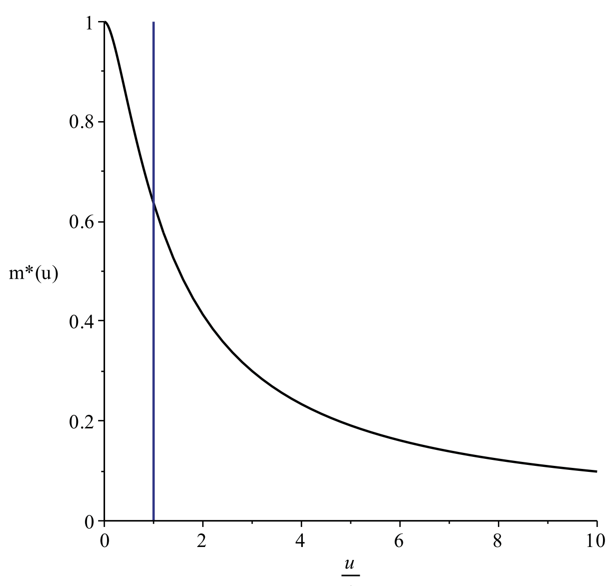

Fig. 7. Variation of mass $m^*(u)/m_0^*$ from (5.8)$_2$, (2.6)$_1$ and (5.3)$_2$ for $\varepsilon = 1$.

as a symmetrically folded sheet with the fold corresponding to $\xi = \eta = \infty$ and any prescribed data on the residual mass at one edge of the sheet is automatically inherited at the other edge of the sheet. Thus for example, for $u, v > c$ and $U < c$ and the conditions (5.5), the assumed initial condition $M_0(0) = m_0^*$ implies for $U, v > c$ and $u < c$ that $M_0(\infty) = m_0^*$. There is a corresponding relation arising from the value of $m_0(-v)$ for $u, v > c$ and $U < c$, which generates the same value for $m_0(-c^2/v)$ for $U, v > c$ and $u < c$. This is because as functions of ξ and η the curves above and below the speed of light are the same curve, and it is only the re-interpretation to the velocity that changes for below and above c. This means that to determine the actual masses $M(U)$ and $m(u)$, we multiply $M_0(U)$ by $\cosh(\xi/2)$ for $U < c$ and $\sinh(\xi/2)$ for $U > c$ for $M(U)$; and $\cosh(\eta/2)$ for $u < c$ and $\sinh(\eta/2)$ for $u > c$ for $m(u)$. Thus, with these definitions of ξ and η, any prescribed values

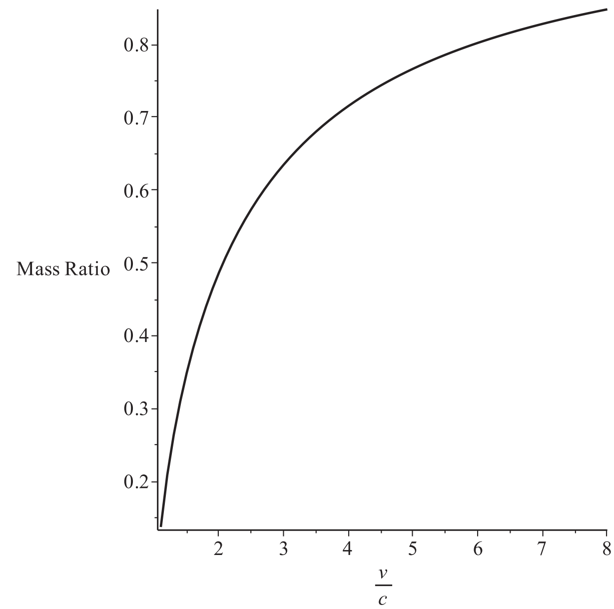

Fig. 8. Variation of mass ratio $M(U)/M^*(U) = m(u)/m^*(u)$ from (5.9), for $v > c$ and $\varepsilon = 1$.

for $U = 0$, namely at $\xi = 0$ are inherited at $U = \infty$ since the value $\xi = 0$ corresponds to both $U = 0$ and $U = \infty$, so that for the conditions (5.5) we have that $M_0(U) \sim m_0^*$ at both $U = 0$ and $U = \infty$. This means that $M(U) \sim m_0^*$ at $U = 0$, while at $U = \infty, M(U) \sim cm_0^*/U$. Similar comments apply to the conditions (5.7) and the variable η. In this case the variable ξ lies in the interval (γ, ∞) while the variable η lies in the interval $(0, \infty)$. We note that from (5.6) and (5.8) the ratios $M_0(U)/M_0^*(U)$ and $m_0(u)/m_0^*(u)$ coincide and are both equal to

$$\frac{M(U)}{M^*(U)} = \frac{m(u)}{m^*(u)} = \varepsilon \frac{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} - \frac{\tanh(\gamma/2)}{4} \int_{\gamma/2}^{\infty} \frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}{\left\{\frac{1}{2[\cosh(\gamma/2)]^{1/2}} + \frac{\tanh(\gamma/2)}{4} \int_{-\gamma/2}^{\infty} \frac{d\rho}{(\cosh\rho)^{1/2}}\right\}}, \tag{5.9}$$

which for a prescribed relative frame velocity $v > c$ provides the mass scaling arising from the two alternative prescriptions of the rest mass. Fig. 8 shows the variation of the mass ratio $M(U)/M^*(U) = m(u)/m^*(u)$ as determined from (5.9), for $v > c$ and $\varepsilon = 1$.

Conclusions

The unresolved issues associated with dark energy and dark matter imply that our understanding of mass may not be quite adequate, see for example [7]. In special relativity, mass as a function of its velocity is prescribed by a single arbitrary constant, termed its rest mass. In parts I and II of this paper we have posed the question as to whether there might exist other mass variations, and we have determined new mass variations involving two arbitrary constants.

As usual in special relativity we have considered two moving frames such that the (x, t) frame is moving with constant velocity v with respect to the (X, T) frame. We consider a moving particle having velocity u with respect to the (x, t) frame and U with respect to the (X, T) frame. If $v < c$ the standard Lorentz transformations apply, while if $v > c$, then the so-called extended Lorentz transformations (2.1) have been proposed by the authors [2,3], noting that there are two possible versions corresponding to $\varepsilon = \pm 1$. At the outset we have assumed that the Einstein variation of mass formula $m(v) = m_0^*\left[1 - (v/c)^2\right]^{-1/2}$ does not apply, and we have proposed the question of determining other mass variation formulae that preserve the structure of the Lorentz invariant mass-momentum relations as well as force invariance in the direction of relative motion and another invariance (see (2.5)) that is known to apply in special relativity. Based only on these assumptions, we have determined new mass variation expressions such as (3.15) involving two arbitrary constants C_1 and C_2, and $m_0(u)$ and $M_0(U)$ for $u > c$ and $U < c$ are defined by

$$m(u) = \frac{m_0(u)}{\left[(u/c)^2 - 1\right]^{1/2}}, \quad M(U) = \frac{M_0(U)}{\left[1 - (U/c)^2\right]^{1/2}}, \tag{6.1}$$

where $m(u)$ and $M(U)$ denote the perceived masses from the two frames, and we have termed $m_0(u)$ and $M_0(U)$ as the perceived residual masses, being the actual mass with the Einstein factor removed. The two arbitrary constants C_1 and C_2 can be determined in terms of $M_0(0) = m_0^*$ and $m_0(\infty) = m_\infty^*$, noting that m_0^* coincides with the accepted notion of rest-mass ($M(0)$) while $m_\infty^* = p_\infty/c$, where p_∞ denotes the finite momentum at infinite velocity. We note that $m_\infty^* = p_\infty/c$ is immediately evident from (6.1), in the limit of infinite velocity u. Further, it is clear from either (4.4) or (4.6) that finite momentum for infinite velocity ensures that the resulting energies remain finite. We note however, that from $(6.1)_1$, $m(u) \simeq c m_\infty^*/u$ as $u \to \infty$, so the actual mass is necessarily zero for infinite velocity.

For subluminal relative frame motion $v < c$, the new formulae involve the integral $\int^x d\rho/(\sinh\rho)^{1/2}$, while for superluminal relative frame motion $v > c$, the new formulae are in terms of the integral $\int^x d\rho/(\cosh\rho)^{1/2}$. Both of these integrals can, if necessary, be expressed in terms of standard elliptical functions, but the resulting expressions are not particularly helpful in terms of generating insight. It is clear that the integral involving sinh generates imaginary numbers for negative values of the integration variable and as detailed in part I of this paper [4], care must be exercised in proposing specific initial data for a boundary value problem to have real outcomes. For subluminal v there are two branches for the new solutions depending on the sign of $\xi - \gamma/2$, giving rise to allowable (U, u) regions. This apparent limitation for $v < c$ tends to reinforce the robustness of the Einstein formula $m(v) = m_0^*[1 - (v/c)^2]^{-1/2}$ for subluminal relative frame velocities. For $v > c$ the integral involving cosh does not exhibit this difficulty, there are no corresponding restrictions on allowable initial data, and two illustrative boundary value problems are formulated. In Section "Application of formulae using rest mass and residual mass at infinite relative velocity" for $U < c$ and $u, v > c$, new formulae are presented for the relativistic mass variation assuming that both the rest mass m_0^* and the residual mass $m_\infty^* = p_\infty/c$ are known values. In [2,3] a dual set of frames (i.e. sub and super) are proposed, so that if this approach has any veracity, it means that both m_0^* and m_∞^* are defining parameters of equal importance. Further, for both $v < c$ and $v > c$, the new mass variations also admit finite mass solutions at the speed of light, and for which the residual mass vanishes at $v = c$, namely $m_0(c) = 0$. The existence of finite mass solutions at the speed of light essentially means that there is more mass as compared to that predicted by Einstein's formula for mass, since the singularity at the speed of light becomes steeper.

Finally, mass profiles having finite mass at the speed of light are exploited to determine two mass profiles as functions of velocity assuming that the rest mass is alternatively prescribed at the origin of either frame. The two profiles so obtained $(M(U), m(u))$ and $(M^*(U), m^*(u))$ are distinct but have common ratios $M(U)/M^*(U) = m(u)/m^*(u)$ which are functions of $v > c$ given by (5.9), indicating that observable mass magnitudes are dependent upon the frame in which the rest mass is adopted.

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