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Graphs whose normalized Laplacian has three eigenvalues[☆]

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ABSTRACT

We give a combinatorial characterization of graphs whose normalized Laplacian has three distinct eigenvalues. Strongly regular graphs and complete bipartite graphs are examples of such graphs, but we also construct more exotic families of examples from conference graphs, projective planes, and certain quasi-symmetric designs.

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1. Introduction

In their pioneering monograph on spectra of graphs, Cvetković et al. [12, Sections 1.2 and 1.6] mention the spectrum of the transition matrix as one of the possible spectra to investigate graphs, and they give some properties of the coefficients of the corresponding characteristic polynomial. The spectrum of the transition matrix and the spectrum of the normalized Laplacian matrix are in (an easy) one-one correspondence, so that studying the latter is essentially the same as studying the first. The normalized Laplacian is mentioned briefly in the recent monograph by Cvetković et al. [13, Section 7.7]; however, the standard reference for it is the monograph by Chung [11], which deals almost entirely with this matrix.

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Graphs with few distinct eigenvalues have been studied for several matrices, such as the adjacency matrix [2, 6, 10, 14, 15, 19, 25], the Laplacian matrix [16, 30], the signless Laplacian matrix [1], the Seidel matrix [26], and the universal adjacency matrix [22]. One of the reasons for studying such graphs is that they have a lot of structure, and can be thought of as generalizations of strongly regular graphs (see also the manuscript by Brouwer and Haemers [3]).

Typically, graphs with few distinct eigenvalues seem to be the hardest graphs to distinguish by the spectrum. Put a bit differently, it seems that most graphs with few eigenvalues are not determined by the spectrum. Thus, the question of which graphs are determined by the spectrum (as studied in [17, 18]) is another motivation for studying graphs with few distinct eigenvalues. For the normalized Laplacian matrix, there are some recent constructions of graphs with the same spectrum by Butler and Grout [4, 5]. Some other recent work on the normalized Laplacian (energy) is done by Cavers et al. [8].

In this paper, we investigate graphs whose normalized Laplacian has three eigenvalues. The only graphs whose normalized Laplacian has one eigenvalue are empty graphs, and the (connected) ones with two eigenvalues are complete. We shall give a characterization of graphs whose normalized Laplacian has three eigenvalues. Strongly regular graphs and complete bipartite graphs are examples of such graphs, but we also construct more exotic families of examples from conference graphs, projective planes, and certain quasi-symmetric designs.

2. Basics

Throughout, Γ will denote a simple undirected graph with n vertices. The *adjacency matrix* of Γ is the $n \times n$ 01-matrix $A = [a_{uv}]$ with rows and columns indexed by the vertices, where $a_{uv} = 1$ if u is adjacent to v , and 0 otherwise. Let $D = [d_{uv}]$ be the $n \times n$ diagonal matrix where d_{uu} equals the valency d_u of vertex u . The matrix $L = D - A$ is better known as the *Laplacian matrix* of Γ . The *normalized Laplacian matrix* of Γ is the $n \times n$ matrix $\mathcal{L} = [\ell_{uv}]$ with

$$\ell_{uv} = \begin{cases} 1 & \text{if } u = v, d_u \neq 0, \\ -1/\sqrt{d_u d_v} & \text{if } u \text{ is adjacent to } v, \\ 0 & \text{otherwise.} \end{cases}$$

If Γ has no isolated vertices then $\mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$. Mohar [24] calls this matrix the transition Laplacian, but others (for example Tan [29]) use this term for the matrix $D^{-1} L$. Both matrices, and also the transition matrix $D^{-1} A$, have the same number of distinct eigenvalues, so for our purpose this makes no difference. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of \mathcal{L} or, as we shall write from now on, the \mathcal{L} -eigenvalues of Γ . The following basic results are from [11, Lemmas 1.7–8] (see also [9]).

Lemma 1. *Let $n \geq 2$. A graph Γ on n vertices has the following properties.*

- (i) $\lambda_n = 0$,
- (ii) $\sum_i \lambda_i \leq n$ with equality holding if and only if Γ has no isolated vertices,
- (iii) $\lambda_{n-1} \leq n/(n-1)$ with equality holding if and only if Γ is a complete graph on n vertices,
- (iv) $\lambda_{n-1} \leq 1$ if Γ is non-complete,
- (v) $\lambda_1 \geq n/(n-1)$ if Γ has no isolated vertices,
- (vi) $\lambda_{n-1} > 0$ if Γ is connected. If $\lambda_{n-i+1} = 0$ and $\lambda_{n-i} \neq 0$, then Γ has exactly i connected components,
- (vii) The spectrum of Γ is the union of the spectra of its connected components,
- (viii) $\lambda_i \leq 2$ for all i , with $\lambda_1 = 2$ if and only if some connected component of Γ is a non-trivial bipartite graph,
- (ix) Γ is bipartite if and only if $2 - \lambda_i$ is an eigenvalue of Γ for each i .

Because of (vii), the study of the \mathcal{L} -eigenvalues can be restricted to connected graphs without loss of generality. So from now on, Γ will be a connected graph, and the trivial \mathcal{L} -eigenvalue 0 occurs with multiplicity one.

3. Three distinct eigenvalues

In this section, we give a characterization of graphs whose normalized Laplacian has three (distinct) eigenvalues. This characterization forms the basis for the rest of the paper. Using Lemma 1, it follows that the only graphs with one \mathcal{L} -eigenvalue are the empty graphs. Using (ii) and (iii) of Lemma 1, we find that a connected graph has two \mathcal{L} -eigenvalues if and only if it is complete.

In order to describe graphs with three normalized Laplacian eigenvalues, we let $\hat{d}_u = \sum_{v \sim u} \frac{1}{d_v}$ be the *normalized valency* of u , and let $\sum_{w \sim u, v} \frac{1}{d_w}$ be the *normalized number of common neighbors* of two distinct vertices u and v . We denote this normalized number of common neighbors by $\hat{\lambda}_{uv}$ if u and v are adjacent, and by $\hat{\mu}_{uv}$ if they are not.

Theorem 1. *Let Γ be a connected graph with e edges. Then Γ has three \mathcal{L} -eigenvalues $0, \theta_1, \theta_2$ if and only if the following three properties hold.*

- (i) $\hat{d}_u = td_u^2 - (\theta_1 - 1)(\theta_2 - 1)d_u$ for all vertices u ,
 - (ii) $\hat{\lambda}_{uv} = td_u d_v + 2 - \theta_1 - \theta_2$ for adjacent vertices u and v ,
 - (iii) $\hat{\mu}_{uv} = td_u d_v$ for non-adjacent vertices u and v ,
- where $t = \frac{\theta_1 \theta_2}{2e}$.

Proof. Since \mathcal{L} is symmetric, it follows that Γ has eigenvalues $0, \theta_1$, and θ_2 if and only if $(\mathcal{L} - \theta_1 I)(\mathcal{L} - \theta_2 I)$ is a symmetric rank one matrix. If so, then its non-zero eigenvalue is $\theta_1 \theta_2$ and has eigenvector $D^{\frac{1}{2}} \mathbf{j}$ (an eigenvector of \mathcal{L} corresponding to eigenvalue 0), where \mathbf{j} is the all-ones vector. By working this out, we get the equation

$$(\mathcal{L} - \theta_1 I)(\mathcal{L} - \theta_2 I) = \frac{\theta_1 \theta_2}{2e} (D^{\frac{1}{2}} \mathbf{j})(D^{\frac{1}{2}} \mathbf{j})^T.$$

From this equation, the stated characterization follows. \square

From Theorem 1 we immediately find the below two corollaries. Recall that Γ is *strongly regular* with parameters (n, k, λ, μ) , whenever Γ is k -regular with $0 < k < n - 1$, and the number of common neighbors of any two distinct vertices equals λ if the vertices are adjacent and μ otherwise (see [3]).

Corollary 1. *A connected regular graph has three \mathcal{L} -eigenvalues if and only if it is strongly regular.*

Corollary 2. *A connected graph with three \mathcal{L} -eigenvalues has diameter two.*

Both results are not surprising, knowing that the same results hold for other matrices such as the adjacency matrix, Laplacian matrix, and signless Laplacian matrix.

4. Bipartite graphs

A complete bipartite graph is an example of a graph with three \mathcal{L} -eigenvalues; it was already observed by Chung [11, Example 1.2] that it has eigenvalues $0, 1$ (with multiplicity $n - 2$), and 2 . In this section, we give some characterizations of bipartite graphs with three \mathcal{L} -eigenvalues.

Proposition 1. *Let Γ be a connected triangle-free graph with three \mathcal{L} -eigenvalues. Then Γ is a triangle-free strongly regular graph or a complete bipartite graph.*

Proof. If Γ is regular, then it is clearly strongly-regular. So assume that Γ is non-regular. Because Γ is triangle-free, and using Theorem 1, it follows that for every pair of adjacent vertices u, v , it holds that $0 = \hat{\lambda}_{uv} = td_u d_v + 2 - \theta_1 - \theta_2$. Because G is connected and non-regular, there is a pair of adjacent vertices u, v with distinct valencies d_u and d_v . The above equation now implies that only these two

valencies occur, and that there are no odd cycles in Γ . Hence Γ is bipartite. By Corollary 2, Γ must be complete bipartite. \square

We call a graph \mathcal{L} -integral if all its \mathcal{L} -eigenvalues are integral, i.e., 0, 1, or 2. The complete bipartite graphs are such graphs; in fact, no other connected graphs are \mathcal{L} -integral.

Proposition 2. *Let Γ be connected. Then the following are equivalent.*

- (i) Γ is bipartite with three \mathcal{L} -eigenvalues,
- (ii) Γ is \mathcal{L} -integral,
- (iii) Γ is complete bipartite.

Proof. First we show that (i) implies (ii). Let Γ be bipartite with three \mathcal{L} -eigenvalues. By (i) and (ix) of Lemma 1, it clearly follows that Γ has \mathcal{L} -eigenvalues 0, 1, and 2, and hence it is integral.

Next we show that (ii) implies (iii). Let Γ be integral. By Lemma 1, the \mathcal{L} -spectrum of Γ is $\{[0]^1, [1]^{n-2}, [2]^1\}$ and hence Γ is bipartite. Because its diameter equals two, Γ is complete bipartite. It is clear that we can conclude (i) from (iii). \square

The property that complete bipartite graphs have two simple eigenvalues does not characterize them among the graphs with three \mathcal{L} -eigenvalues, as we shall see later on.

5. Biregular graphs

In this section, we shall consider biregular graphs with three distinct \mathcal{L} -eigenvalues. We call a graph with two distinct valencies k_1 and k_2 (k_1, k_2) -regular, or simply *biregular*. The complete bipartite graphs of the previous section are examples of biregular (or regular) graphs. Characterization of the biregular graphs with three distinct \mathcal{L} -eigenvalues seems to be difficult though, so we shall have a look at some special cases (also in the next section).

5.1. The valency partition

A partition $\sigma = \{V_1, \dots, V_m\}$ of the vertex set of a graph Γ is called an *equitable partition* if for all $i, j = 1, \dots, m$, the number of neighbors in V_j of $u \in V_i$ depends only on i, j , and not on u ; we denote this number by k_{ij} . We call the partition of the vertex set according to valencies the *valency partition*. The following can be obtained from Theorem 1.

Lemma 2. *Let Γ be a biregular graph with three \mathcal{L} -eigenvalues. Then the valency partition is equitable.*

Proof. Suppose Γ is (k_1, k_2) -regular, and let $V_i, i = 1, 2$, be the set of vertices of valency k_i . Fix a vertex $u \in V_i$, and let k_{ij} be the number of neighbors in V_j of $u \in V_i$. It follows that these numbers do not depend on the particular u because they are determined by the equations $k_{i1} + k_{i2} = k_i$ and $\frac{k_{i1}}{k_1} + \frac{k_{i2}}{k_2} = \hat{d}_u = tk_i^2 - (\theta_1 - 1)(\theta_2 - 1)k_i$. \square

5.2. Projective planes

To find more examples of graphs with three \mathcal{L} -eigenvalues we let Γ be such a (k_1, k_2) -regular graph, with valency partition $\{V_1, V_2\}$, and we suppose that the induced subgraph Γ_1 on V_1 is empty. By Lemma 2, $\{V_1, V_2\}$ is an equitable partition and by Theorem 1, the number of common neighbors of any two vertices in V_1 is a constant $tk_1^2k_2$ (which is k_2 times the normalized number of common neighbors). Hence we may assume that the bipartite graph between V_1 and V_2 is the incidence graph of a 2-design \mathcal{D} . Now it is convenient to switch to notation that is common in design theory. So we let $v = |V_1|, b = |V_2|, k = k_{21}, r = k_1$, and $\lambda = tk_1^2k_2$, so that \mathcal{D} is a $2-(v, k, \lambda)$ design with b blocks and replication number r . In case V_1 and V_2 have the same size, then this design is symmetric, and we obtain the following.

Proposition 3. *Let Γ be a non-bipartite biregular graph such that the valency partition has parts of equal size, and the induced graph on one of the parts is empty. Then Γ has three \mathcal{L} -eigenvalues if and only if it is obtained from the incidence graph of a projective plane by making any two vertices corresponding to the lines adjacent. If the projective plane has line size k and $v = k^2 - k + 1$ points, then the non-trivial \mathcal{L} -eigenvalues of Γ are $\frac{v}{k^2}$ and $1 + \frac{1}{k}$.*

Proof. We continue with the above notation and arguments, and assume that Γ has three \mathcal{L} -eigenvalues. The design \mathcal{D} is a symmetric $2-(v, k, \lambda)$ design, and $r = k$. Furthermore, let $a = k_{22}$. From Theorem 1, we obtain the equations

$$\frac{k}{k+a} = tk^2 - (\theta_1 - 1)(\theta_2 - 1)k, \tag{1}$$

$$\frac{\lambda}{k+a} = tk^2, \tag{2}$$

$$1 + \frac{a}{k+a} = t(k+a)^2 - (\theta_1 - 1)(\theta_2 - 1)(k+a). \tag{3}$$

By combining these three equations, we find that $t = \frac{1}{(k+a)^2}$ (note that $a \neq 0$ because Γ is not bipartite) and $\lambda = \frac{k^2}{k+a}$. The latter implies that $\lambda \neq k$, otherwise Γ would be regular and bipartite. Thus, \mathcal{D} is not a complete design. Because \mathcal{D} is also not empty, we obtain two more equations from Theorem 1:

$$\frac{\mu_{12}}{k+a} = tk(k+a), \tag{4}$$

$$\frac{\lambda_{12}}{k+a} = tk(k+a) + 2 - \theta_1 - \theta_2, \tag{5}$$

where λ_{12} and μ_{12} are the numbers of common neighbors of a vertex in V_1 and a vertex in V_2 , depending on whether they are adjacent or not, respectively. It follows that $\mu_{12} = k$, and we claim that this implies that the induced graph Γ_2 on V_2 is complete. To show this claim, consider two blocks (vertices in V_2) B_1 and B_2 . Because the design is not complete, there is a point P that is incident with B_1 , but not with B_2 . Because $\mu_{12} = k$, every neighbor of P is also a neighbor of B_2 ; in particular this holds for B_1 , which proves our claim. Thus, $a = k_{22} = v - 1$, $k_2 = k + v - 1$, and $\lambda = \frac{k^2}{k+v-1}$. When the latter is combined with the property that $\lambda(v - 1) = k(k - 1)$ (because \mathcal{D} is a symmetric design), we obtain that $v = k^2 - k + 1$ and $\lambda = 1$, i.e., \mathcal{D} is a projective plane, and Γ is as stated. Moreover, $\lambda_{12} = k - 1$, and so (4) and (5) imply that $\theta_1 + \theta_2 - 2 = \frac{1}{k+v-1} = \frac{1}{k^2}$. Together with the equation $(\theta_1 - 1)(\theta_2 - 1) = \frac{k}{(k+v-1)^2} - \frac{1}{(k+v-1)} = \frac{1}{k^3} - \frac{1}{k^2}$, which follows from (1), this determines the non-trivial \mathcal{L} -eigenvalues. On the other hand, if Γ is as stated, then the equations from Theorem 1 all hold, including a final one that was not used so far:

$$\frac{1}{k} + \frac{v-2}{k+v-1} = t(k+v-1)^2 + 2 - \theta_1 - \theta_2. \quad \square \tag{6}$$

5.3. Quasi-symmetric designs

If in the above discussion the design \mathcal{D}_Γ is not symmetric, then it seems hard to characterize Γ . In this case Γ_2 cannot be empty (unless Γ is complete bipartite) or complete. It seems natural to consider the case that Γ_2 is strongly regular, and indeed, there are such examples as we shall see. In this case, it follows from Theorem 1 that there are two block intersection sizes, depending on whether the blocks are adjacent or not. So \mathcal{D}_Γ is a quasi-symmetric design and Γ_2 is one of its (strongly regular)

block graphs. We obtain the below proposition. Here we use the notation that is common for quasi-symmetric designs (cf. [28,27]). That is, \mathcal{D}_Γ is a 2 - (v, k, λ) design with replication number $r = \lambda \frac{v-1}{k-1}$ and two block intersection sizes x and y . We do however not make the usual convention that $y > x$. The corresponding block graph Γ_2 , where two blocks are adjacent if they intersect in y points, is strongly regular with parameters (b, a, c, d) , with $b = \frac{vr}{k}$, $a = \frac{(r-1)k-x(b-1)}{y-x}$, $d = a + \rho_1\rho_2$, $c = d + \rho_1 + \rho_2$. Here $\rho_1 = \frac{r-\lambda-k+x}{y-x}$ and $\rho_2 = \frac{x-k}{y-x}$ are the (usual) non-trivial eigenvalues of Γ_2 . An important property in the following is that for a point-block pair (P, B) , the number of blocks $B' \neq B$ incident with P and intersecting B in y points equals $\frac{(\lambda-1)(k-1)-(x-1)(r-1)}{y-x}$ or $\frac{\lambda k-xr}{y-x}$, depending on whether P is incident to B or not, respectively (cf. [20, Theorem 3.2]).

Proposition 4. *Let Γ be a biregular graph with valency partition (V_1, V_2) such that Γ_1 is empty, the edges between V_1 and V_2 form the incidence relation of a quasi-symmetric design \mathcal{D}_Γ , and Γ_2 is the corresponding block graph, with notation as above. Then Γ has three \mathcal{L} -eigenvalues $0, \theta_1, \theta_2$ if and only if*

$$\begin{aligned} \frac{r}{k+a} &= tr^2 - (\theta_1 - 1)(\theta_2 - 1)r, \\ \frac{\lambda}{k+a} &= tr^2, \\ \frac{k}{r} + \frac{a}{k+a} &= t(k+a)^2 - (\theta_1 - 1)(\theta_2 - 1)(k+a), \\ \frac{x}{r} + \frac{d}{k+a} &= t(k+a)^2, \\ \frac{y}{r} + \frac{c}{k+a} &= t(k+a)^2 + 2 - \theta_1 - \theta_2, \\ \frac{\lambda k-xr}{(y-x)(k+a)} &= tr(k+a), \\ \frac{(\lambda-1)(k-1)-(x-1)(r-1)}{(y-x)(k+a)} &= tr(k+a) + 2 - \theta_1 - \theta_2, \end{aligned}$$

where $t = \frac{\theta_1\theta_2}{vr+b(k+a)}$.

Proof. This follows immediately from Theorem 1. \square

Any Steiner system, i.e., a 2 - $(v, k, 1)$ design, is a quasi-symmetric design (if $b > v$) with $y = 1, x = 0, r = \frac{v-1}{k-1}$, and block graph with parameters $(\frac{v(v-1)}{k(k-1)}, (r-1)k, r-2 + (k-1)^2, k^2)$. These parameters satisfy the above conditions and so each Steiner system gives a graph with three \mathcal{L} -eigenvalues, the non-trivial ones being $1 + \frac{1}{r}$ and $1 - \frac{1}{k} + \frac{1}{rk}$. The 2 - $(v, 2, 1)$ design of all pairs gives a graph that can also be obtained from the triangular graph $T(v+1)$ by removing all edges in a maximal clique. Note also that the graphs of Proposition 3 are degenerate cases of this construction.

Another large family of quasi-symmetric designs, the multiples of symmetric designs (i.e., each block is repeated the same number of times) do not satisfy the conditions.

Among the residuals of biplanes, only the (three) 2 - $(10, 4, 2)$ designs satisfy the above conditions, with $x = 2$ and $y = 1$, and give graphs on 25 vertices with three \mathcal{L} -eigenvalues $0, \frac{5}{6}, \frac{4}{3}$. The graph Γ_2 is the triangular graph $T(6)$.

Another example is obtained from the unique quasi-symmetric 2 - $(21, 6, 4)$ design with $b = 56, r = 16, x = 2, y = 0$. Here Γ_2 is the Gewirtz graph, and Γ has \mathcal{L} -eigenvalues $0, \frac{7}{8}, \frac{11}{8}$. Unfortunately or not, this graph on 77 vertices is strongly regular, as is well-known, cf. [20,3].

Instead of taking Γ_1 empty, we now let it be complete. Also in this case the graph between V_1 and V_2 is the incidence graph of a 2 -design \mathcal{D}_Γ , and we find some new examples by considering the case that \mathcal{D}_Γ is a quasi-symmetric design and Γ_2 is a strongly regular graph corresponding to \mathcal{D}_Γ . The following is the analogue of Proposition 4.

Proposition 5. *Let Γ be a biregular graph with valency partition (V_1, V_2) such that Γ_1 is complete, the edges between V_1 and V_2 form the incidence relation of a quasi-symmetric design \mathcal{D}_Γ , and Γ_2 is the corresponding*

block graph, with notation as above. Then Γ has three \mathcal{L} -eigenvalues $0, \theta_1, \theta_2$ if and only if

$$\begin{aligned} \frac{v-1}{v-1+r} + \frac{r}{k+a} &= t(v-1+r)^2 - (\theta_1 - 1)(\theta_2 - 1)(v-1+r), \\ \frac{v-2}{v-1+r} + \frac{\lambda}{k+a} &= t(v-1+r)^2 + 2 - \theta_1 - \theta_2, \\ \frac{k}{v-1+r} + \frac{a}{k+a} &= t(k+a)^2 - (\theta_1 - 1)(\theta_2 - 1)(k+a), \\ \frac{x}{v-1+r} + \frac{d}{k+a} &= t(k+a)^2, \\ \frac{y}{v-1+r} + \frac{c}{k+a} &= t(k+a)^2 + 2 - \theta_1 - \theta_2, \\ \frac{k}{v-1+r} + \frac{\lambda k - xr}{(y-x)(k+a)} &= t(v-1+r)(k+a), \\ \frac{k-1}{v-1+r} + \frac{(\lambda-1)(k-1) - (x-1)(r-1)}{(y-x)(k+a)} &= t(v-1+r)(k+a) + 2 - \theta_1 - \theta_2, \end{aligned}$$

where $t = \frac{\theta_1 \theta_2}{v(v-1+r) + b(k+a)}$.

A multiple of a projective plane, i.e., a design obtained from a projective plane by repeating each line λ times, is a quasi-symmetric design with parameters $2-(k^2 - k + 1, k, \lambda)$, with $r = \lambda k, x = 1, y = k, a = \lambda - 1$, and it satisfies the above conditions. Here Γ_2 is a disjoint union of cliques of size λ , and the obtained graph Γ has non-trivial \mathcal{L} -eigenvalues $\frac{v}{k^2 + k(m-1)}$ and $1 + \frac{1}{k+m-1}$. This construction is again a generalization of the construction in Proposition 3.

Other attempts to construct biregular graphs with three \mathcal{L} -eigenvalues could be inspired by the papers by Haemers and Higman on strongly regular graphs with a strongly regular decomposition [21] and by Higman on strongly regular designs [23], but we have not worked this out.

6. Cones

A cone over a graph Γ' is a graph obtained by adjoining a new vertex to all vertices of Γ' , i.e., it is a graph which has a vertex of valency $n - 1$.

Lemma 3. *Let Γ be a cone over Γ' . If Γ has three \mathcal{L} -eigenvalues then Γ' is regular or biregular, and the valency partition of Γ is equitable.*

Proof. Let v be a vertex of valency $n - 1$, and W be the set of remaining vertices, so that Γ' is the induced graph on W . From Theorem 1 we find that $\hat{d}_w = td_w^2 - (\theta_1 - 1)(\theta_2 - 1)d_w$ and $\hat{\lambda}_{vw} = td_w(n - 1) + 2 - \theta_1 - \theta_2$ for $w \in W$. Because in this case $\hat{d}_w = \hat{\lambda}_{vw} + \frac{1}{n-1}$, we obtain a quadratic equation for d_w , which shows that Γ' is regular or biregular. That the valency partition is equitable can be proven in a similar way as in Lemma 2. \square

Proposition 6. *Let Γ be a cone over a regular graph Γ' . Then Γ has three \mathcal{L} -eigenvalues if and only if Γ' is a disjoint union of (at least two) cliques of the same size d , say. In this case, the non-trivial \mathcal{L} -eigenvalues are $\frac{1}{d}$ and $1 + \frac{1}{d}$.*

Proof. As before, let v be a vertex of valency $n - 1$, and W be the set of remaining vertices, which now have constant valency d , say. If Γ has three \mathcal{L} -eigenvalues, then by Theorem 1, Γ' is a strongly regular graph with parameters $(n - 1, d - 1, \lambda, \mu)$, where $\frac{1}{n-1} + \frac{\lambda}{d} = td^2 + 2 - \theta_1 - \theta_2$ (the normalized number of common neighbors of two adjacent vertices $w, w' \neq v$) and $\frac{1}{n-1} + \frac{\mu}{d} = td^2$ (the normalized number of common neighbors of two non-adjacent vertices $w, w' \neq v$).

Moreover, by combining $\frac{n-1}{d} = t(n - 1)^2 - (\theta_1 - 1)(\theta_2 - 1)(n - 1)$ (the normalized valency of v) and $\frac{d-1}{d} + \frac{1}{n-1} = td^2 - (\theta_1 - 1)(\theta_2 - 1)d$ (the normalized valency of $w \in W$), we obtain that

$t = \frac{1}{(n-1)d^2}$, and this implies that $\mu = 0$. Thus, Γ' is a disjoint union of cliques of size d . Therefore $\lambda = d - 2$, and the above equations now show that $\{\theta_1, \theta_2\} = \{\frac{1}{d}, 1 + \frac{1}{d}\}$.

On the other hand, by checking all equations in Theorem 1, it follows that the cone over a disjoint union of d -cliques indeed has three \mathcal{L} -eigenvalues. \square

Examples of cones over biregular graphs can be constructed using certain strongly regular graphs, as we shall see next. Recall that a *conference graph* is a strongly regular graph with parameters (n, k, λ, μ) with $n = 2k + 1, k = 2\mu$, and $\lambda = \mu - 1$.

Proposition 7. *Let Γ be a graph with minimum valency one. Then Γ has three \mathcal{L} -eigenvalues if and only if it is a star graph or a cone over the disjoint union of an isolated vertex and a conference graph. The latter has non-trivial \mathcal{L} -eigenvalues $\frac{n \pm \sqrt{n-2}}{n-1}$, each with multiplicity $\frac{n-1}{2}$.*

Proof. Suppose that Γ has three \mathcal{L} -eigenvalues, and n vertices. Let u be a vertex of valency $d_u = 1$, and let v be its neighbor. Because the diameter of Γ is two, it follows that every other vertex is adjacent to v . So Γ is a cone, say over Γ' . If Γ' is regular, then Γ is a star graph by Proposition 6. So let's assume that Γ' is not regular, and hence is not empty.

Using that $\frac{1}{n-1} = \hat{\mu}_{uw} = td_w$ for all $w \neq u, v$, we obtain that $d_w = \frac{1}{t(n-1)} =: d$ is the same for all $w \neq u, v$. By combining $\frac{1}{n-1} = \hat{d}_u = t - (\theta_1 - 1)(\theta_2 - 1)$ and $\frac{1}{n-1} + \frac{d-1}{d} = \hat{d}_w = td^2 - (\theta_1 - 1)(\theta_2 - 1)d$, we find that $d = (n - 1)/2$.

It is straightforward now to show that the induced graph on the vertices except u and v is strongly regular with parameters $(n - 2, d - 1, \lambda, \mu)$, where $\frac{1}{n-1} + \frac{\lambda}{d} = td^2 + 2 - \theta_1 - \theta_2$ (the normalized number of common neighbors of two adjacent vertices $w, w' \neq u, v$), and $\frac{1}{n-1} + \frac{\mu}{d} = td^2 = \frac{1}{2}$ (the normalized number of common neighbors of two non-adjacent vertices $w, w' \neq u, v$). Using the above and the equation $0 = \hat{\lambda}_{uv} = t(n-1) + 2 - \theta_1 - \theta_2$, this implies that $\lambda = (d-3)/2$ and $\mu = (d-1)/2$. Thus, we have found that Γ' is the disjoint union of an isolated vertex and a conference graph.

On the other hand, the star graph is complete bipartite, so has three \mathcal{L} -eigenvalues. Also the cone over the disjoint union of an isolated vertex and a conference graph has three \mathcal{L} -eigenvalues; the non-trivial ones being $\frac{n \pm \sqrt{n-2}}{n-1}$ (this follows from the above equations and Theorem 1), each with multiplicity $\frac{n-1}{2}$. \square

We thus have examples where the multiplicities of the non-trivial \mathcal{L} -eigenvalues are the same. We finish this paper at the other extreme, by identifying the graphs where one non-trivial \mathcal{L} -eigenvalue is simple.

Proposition 8. *Let Γ be a graph with three \mathcal{L} -eigenvalues, of which two are simple. Then Γ is either complete bipartite or a cone over the disjoint union of two cliques of the same size.*

Proof. Let θ be the \mathcal{L} -eigenvalue with multiplicity $n - 2$. So the rank of $\mathcal{L} - \theta I$ is two. First, assume that $\theta = 1$. Using Lemma 1, it follows that the \mathcal{L} -spectrum of Γ is $\{[0]^1, [1]^{n-2}, [2]^1\}$, and hence by Proposition 2, Γ is complete bipartite.

Next, assume that $\theta \neq 1$. By considering principal submatrices of $\mathcal{L} - \theta I$ of size three, it follows that Γ has no cocliques of size three. For a vertex u , let R_u be the corresponding row in $\mathcal{L} - \theta I$. Consider now two vertices u and w that are not adjacent. Then R_u and R_w span the row space of $\mathcal{L} - \theta I$. Let v be a common neighbor of u and w . Then

$$(1 - \theta)R_v = -\frac{1}{\sqrt{d_u d_v}}R_u - \frac{1}{\sqrt{d_w d_v}}R_w.$$

This implies that if z is any fourth vertex – which is adjacent to at least one of u and w – is adjacent to v . So v is adjacent to all other vertices; $d_v = n - 1$.

We claim now that v is the only common neighbor of u and w . To show this claim, suppose that v' is another common neighbor; hence also $d_{v'} = n - 1$. Then applying the above equation to entries corresponding to v and v' shows that $(1 - \theta)^2 = -(1 - \theta) \frac{1}{n-1} = \frac{1}{d_u(n-1)} + \frac{1}{d_w(n-1)}$. This implies that $\theta = \frac{n}{n-1}$, which implies that Γ is a complete graph by (ii) and (iii) of Lemma 1; a contradiction.

Because of Proposition 7, both u and w are not vertices with valency one. Therefore there are vertices that are adjacent to one, but not the other. If z is a vertex that is adjacent to u , but not to w , then

$$(1 - \theta)R_z = -\frac{1}{\sqrt{d_u d_z}}R_u,$$

which implies that any vertex different from u and z is adjacent to u if and only if it is adjacent to z , and so $d_u = d_z$. Moreover, it tells us that $d_u = \frac{1}{\theta-1}$ (consider the entry corresponding to v in the above equation). Of course, the situation where u and w are interchanged is completely the same. It thus follows that Γ is a cone over the disjoint union of two cliques of the same size. \square

7. Concluding remarks

By computer, we checked all connected graphs with at most 10 vertices, and millions of graphs with 11 or 12 vertices and diameter two for having three normalized Laplacian eigenvalues. In this way, we obtained only a few non-regular non-bipartite examples; all of these can be constructed by the methods in this paper.

However, a classification of all graphs with three normalized Laplacian eigenvalues still seems out of reach. In this paper, we gave a combinatorial characterization that turned out to be useful in such a classification within some very special classes of graphs. In future work, it seems interesting also to consider graphs with more distinct valencies, or to find an upper bound on the number of distinct valencies in graphs with three normalized Laplacian eigenvalues.

Finally, we mention that after submitting this paper, we were informed that some of our results were obtained also by Cavers [7].

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References

- [1] F. Ayoobi, G.R. Omid, B. Tayfeh-Rezaie, A note on graphs whose signless Laplacian has three distinct eigenvalues, *Linear and Multilinear Algebra*, in press.
- [2] W.G. Bridges, R.A. Mena, Multiplicative cones – a family of three eigenvalue graphs, *Aequationes Math.* 22 (1981) 208–214.
- [3] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*. Available from: <<http://homepages.cwi.nl/~aeb/math/ipm.pdf>> (October 2010).
- [4] S. Butler, Cospectral graphs for both the adjacency and normalized Laplacian matrices, *Linear and Multilinear Algebra* 58 (2010) 387–390.
- [5] S. Butler, J. Grout, Constructing cospectral graphs for the normalized Laplacian, 2010, [arXiv:1008.3646v1](https://arxiv.org/abs/1008.3646v1). Available from: <<http://arxiv.org/abs/1008.3646>>.
- [6] D. de Caen, E.R. van Dam, E. Spence, A nonregular analogue of conference graphs, *J. Combin. Theory Ser. A* 88 (1999) 194–204.
- [7] M. Cavers, *The Normalized Laplacian Matrix and General Randić Index of Graphs*, thesis, University of Regina, 2010.
- [8] M. Cavers, S. Fallat, S. Kirkland, On the normalized Laplacian energy and general Randić index R_{-1} of graphs, *Linear Algebra Appl.* 433 (2010) 172–190.
- [9] G. Chen, G. Davis, F. Hall, Z. Li, K. Patel, M. Stewart, An interlacing result on normalized Laplacians, *SIAM J. Discrete Math.* 18 (2004) 353–361.
- [10] H. Chuang, G.R. Omid, Graphs with three distinct eigenvalues and largest eigenvalue less than 8, *Linear Algebra Appl.* 430 (2009) 2053–2062.
- [11] F.R.K. Chung, *Spectral Graph Theory*, CBMS Lecture Notes, AMS, Providence, RI, 1997.
- [12] D.M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Applications*, third ed., Johann Ambrosius Barth, Heidelberg, 1995.
- [13] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, 2010.
- [14] E.R. van Dam, Regular graphs with four eigenvalues, *Linear Algebra Appl.* 226–228 (1995) 139–162.

- [15] E.R. van Dam, Nonregular graphs with three eigenvalues, *J. Combin. Theory Ser. B* 73 (1998) 101–118.
- [16] E.R. van Dam, W.H. Haemers, Graphs with constant μ and $\bar{\mu}$, *Discrete Math.* 182 (1998) 293–307.
- [17] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum?, *Linear Algebra Appl.* 373 (2003) 241–272.
- [18] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, *Discrete Math.* 309 (2009) 576–586.
- [19] E.R. van Dam, E. Spence, Small regular graphs with four eigenvalues, *Discrete Math.* 189 (1998) 233–257.
- [20] J.M. Goethals, J.J. Seidel, Strongly regular graphs derived from combinatorial designs, *Can. J. Math.* 22 (1970) 597–614., Available from: <<http://math.ca/cjm/v22/p597>>.
- [21] W.H. Haemers, D.G. Higman, Strongly regular graphs with strongly regular decomposition, *Linear Algebra Appl.* 114–115 (1989) 379–398.
- [22] W.H. Haemers, G.R. Omid, Universal adjacency matrices with two eigenvalues, *Linear Algebra Appl.*, this issue.
- [23] D.G. Higman, Strongly regular designs and coherent configurations of type [3 2; 3], *European J. Combin.* 9 (1988) 411–422.
- [24] B. Mohar, Some applications of Laplace eigenvalues of graphs, in: G. Hahn, G. Sabidussi (Eds.), *Graph Symmetry: Algebraic Methods and Applications*, NATO ASI Ser. C, vol. 497, Kluwer, 1997, pp. 225–275.
- [25] M. Muzychuk, M. Klin, On graphs with three eigenvalues, *Discrete Math.* 189 (1998) 191–207.
- [26] J.J. Seidel, A survey of two-graphs, in: *Coll. Intern. Teorie Combinatorie*, Accademia Nazionale dei Lincei, Roma, 1976, pp. 481–511.
- [27] M.S. Shrikhande, Quasi-symmetric designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, 2007, pp. 578–582 (chapter 48).
- [28] M.S. Shrikhande, S.S. Sane, *Quasi-symmetric Designs*, Cambridge University Press, 1991.
- [29] J. Tan, On Cheeger inequalities of a graph, *Discrete Math.* 269 (2003) 315–323.
- [30] Y. Wang, Y. Fan, Y. Tan, On graphs with three distinct Laplacian eigenvalues, *Appl. Math. J. Chinese Univ. Ser. B* 22 (2007) 478–484.