Theoretical Computer Science 11 (1980) 181–205

METRIC INTERPRETATIONS OF INFINITE TREES AND SEMANTICS OF NON DETERMINISTIC RECURSIVE PROGRAMS

A. ARNOLD
Laboratoire d'Informatique, Université de Poitiers, France

M. NIVAT
Université de Paris VII et Laboratoire d'Informatique Théorique et Programmation, Paris, France

Communicated by E. Engeler
Received February 1979
Revised May 1979

Abstract. In order to define semantics of non deterministic recursive programs we are led to consider infinite computations and to replace the structure of cpo on computation domain by the structure of complete metric space. In this setting we prove the two main theorems of semantics:

(i) equivalence between operational and denotational semantics, where this last one is defined as a greatest fixed point for inclusion,

(ii) the one-many function computed by a program is the image of the set of trees computed by the scheme associated with it.

1. Introduction

In the now standard theory of computation in an ordered domain (see [5, 6, 8, 9, 13, 16]) one proves the equivalence between the definition of the computed function as the smallest fixed point of certain functional and the definition of the same function by means of terminating computation sequences of the program at a given point. This equivalence holds when the computation domain is a flat, or discrete, domain in which different defined values are incomparable: the only converging sequences are stationary sequences whose terms are all equal, for sufficiently large n to the limit of the sequence. In such a domain it is clear that any computed value is the result of some finite terminating computation sequence.

The situation is entirely different if, following Scott, one starts computing in a partially ordered domain which contains infinite ascending chains. A computed value may then be the lub of such a chain and as such can well be the result of no finite computation: a typical example is the domain of real numbers, if basic functions are the four arithmetic operations and the initial values are rational numbers after any finite amount of time one will have computed only a rational number when the result may well be irrational.

We propose in this situation to give a meaning to successful infinite computation sequences which will be said to produce a result and to define the computed function by stating that its value at a given point is the set of results of both finite terminating and infinite successful computation sequences at that point. Obviously doing so one accepts the idea that a computed function is many valued since there is absolutely no reason why all computation sequences would lead to the same result. But indeed many-valued functions were already considered as the normal output of non deterministic programs.

Our point of view thus amounts to consider deterministic programs as special cases of non deterministic programs with the advantage that our result will hold in the general case of non deterministic programs (this was in fact the original motivation of the whole study).

In order to give a meaning to successful computation sequences we found it extremely convenient to replace the order structure on the computation domain by a complete metric topology. (This is not at all to say that one cannot use the structure of a cpo to build a theory in many respects analogous to ours and indeed it has been done, see for example [14, 17].)

The results we get to are mainly conditions for the equivalence of this definition of the computed function and a mathematical definition by means of fixed point: it happens that in a very natural way one is lead to consider greatest fixed points rather than smallest. Intuitively this corresponds to the idea that, at the beginning of the computation we only know that the value of the computed function lies in a certain range, a priori the whole computation domain and in the course of the computation this range is reduced (may be to just one value but usually to a set of values). This is dual of the point of view expressed by Scott that an a priori undefined initial value gets more and more defined in the course of the computation. We have borrowed for a large part this idea of decreasing range to L. Nolin (in a uncountable number of discussions).

In the course of this study we will consider infinite trees for the following reason: algebraic infinite trees which can be generated by a recursive program scheme are at the basis of the theory called 'algebraic semantics' of recursive programs (see [5, 6, 8, 9, 11]).

The algebraic tree thus attached to a program scheme incorporates the whole semantics of the program in the sense that an interpretation being defined as a morphism, the function computed by the program resulting of the interpretation of the scheme is the morphic image of this algebraic infinite tree. Whence many results concerning classes of interpretation and families of computation domains.

Here infinite trees also play a role, in fact a crucial role. For the link between a semantics defined in an ordered structure and the semantics defined in a topological structure lies in the fact that the set of infinite trees $M^\infty(F, V)$ has both an ordered structure and a topological structure which are closely related (in fact an increasing function is order continuous iff it is continuous for the topology). The free complete

F-magma $M^\infty(F, V)$ which is studied in [4] thus appears as the mother structure in which the phenomena of computation can be better described.

2. Preliminaries

Here we recall some definitions, notations and properties about the metric space of finite and infinite trees, which are given in [4].

2.1. *The complete metric space of infinite trees*

A *graded alphabet* F is a finite set of function symbols, each $f \in F$ is given with its arity $\rho(f) \in \mathbf{N}$. We note F_i the set $\{f \in F \mid \rho(f) = i\}$. The set $X = \{x_i \mid i > 0\}$ is a set of *variables*; we note $X_0 = \emptyset$ and $X_k = \{x_1, \ldots, x_k\}$ so that $X = \bigcup_{k \geq 0} X_k$.

For any set E disjoint from F, the set $M(F, E)$ is defined inductively by

- $E \cup F_0 \subseteq M(F, E)$;
- if $f \in F_n$ with $n > 0$, if $t_1, \ldots, t_n \in M(F, E)$, then $f(t_1, \ldots, t_n) \in M(F, E)$.

It is clear that $M(F, E)$ is the free F-magma generated by E. The elements of $M(F, E)$ can be regarded as finite trees.

In the sequel of this paper we shall often use *structural induction* on the set of finite trees, like in the following definitions.

The *depth* of a finite tree t is the integer $|t|$ defined inductively by

- if $t \in E \cup F_0$, then $|t| = 1$;
- if $t = f(t_1, \ldots, t_n)$, then $|t| = 1 + \max\{|t_i| \mid 1 \leq i \leq n\}$.

We define the *truncation at depth n* of a tree t as the image of t under the mapping $\alpha_n : M(F, E) \to M(F, E \cup \{\Omega\})$, where Ω is a new symbol of arity 0, not in $F \cup E$, which indicates that a branch of a tree has been cut off in the truncation. This mapping is defined by

$$\alpha_0(t) = \Omega \quad \text{for every } t,$$

$$\alpha_{n+1}(t) = \begin{cases} t, & \text{if } t \in E \cup F_0, \\ f(\alpha_n(t_1), \ldots, \alpha_n(t_p)), & \text{if } t = f(t_1, \ldots, t_p). \end{cases}$$

If $t_1 \neq t_2$ there exists an n such that $\alpha_n(t_1) \neq \alpha_n(t_2)$; thus we note $\bar{\alpha}(t_1, t_2)$ the least integer in the non-empty set $\{n \mid \alpha_n(t_1) \neq \alpha_n(t_2)\}$ and we define the mapping $d : M(F, E) \times M(F, e) \to \mathbf{R}_+$ by

$$d(t', t'') = \begin{cases} 0, & \text{if } t' = t'', \\ 2^{-\bar{\alpha}(t', t'')}, & \text{otherwise.} \end{cases}$$

It is proved [4] that d is an *ultrametric distance* on $M(F, E)$, i.e. d fulfills the following conditions:

$$d(t', t'') = 0 \quad \text{iff} \quad t' = t'',$$

$$d(t', t'') = d(t'', t'),$$

$$d(t', t'') \leq \max(d(t, t'), d(t', t'')).$$

It is well known that the metric space $M(F, E)$ can be isometrically embedded in a *complete* metric space (i.e. where every Cauchy sequence has a limit) which is noted $M^\infty(F, E)$ and which is still an ultrametric space. It turns out that the elements of $M^\infty(F, E) - M(F, E)$ are just *infinite trees*.

The set F is always finite; if the set E is also finite, then $M^\infty(F, E)$ is a *compact space*, i.e. every sequence contains an infinite subsequence which has a limit.

Let T be an element of $M^\infty(F, E)$; we note $B^0_\varepsilon(T)$ the set $\{t \in M(F, E) \mid d(T, t) < \varepsilon\}$ and $B_\varepsilon(T)$ the set $\{T' \in M^\infty(F, E) \mid d(T, T') < \varepsilon\}$. It is obvious that $B^0_\varepsilon(T) = B_\varepsilon(T) \cap M(F, E)$ and it can be proved [4] that $B_\varepsilon(T) = \overline{B^0_\varepsilon(T)}$, where, for any subset P of $M^\infty(F, E)$, $\bar{P}$ is the closure of P for the topology on $M^\infty(F, E)$ induced by the distance d. Now let P be a subset of $M^\infty(F, E)$; we note $B_\varepsilon(P) = \{T' \in M^\infty(F, E) \mid \exists T \in P \text{ s.t. } d(T, T') < \varepsilon\}$ and $B^0_\varepsilon(P) = B_\varepsilon(P) \cap M(F, E)$; we have $B^0_\varepsilon(P) = B^0_\varepsilon(\bar{P}) = \bigcup_{T \in P} B^0_\varepsilon(T)$ and $B_\varepsilon(P) = \overline{B^0_\varepsilon(P)} = \bigcup_{T \in P} B_\varepsilon(T)$.

2.2. *The composition product*

Here we give in a simplified way some definitions and properties about the composition product. The full study of this product is done in [4], upon a more formal framework which will not be given here.

Let $t' \in M(F, E \cup X_n)$ and $t_1, \ldots, t_n \in M(F, E \cup X_q)$. We note $\vec{t}$ the vector $\langle t_1, \ldots, t_n \rangle$; then $t' \cdot \vec{t}$ is the element of $M(F, E \cup X_q)$ defined by induction on t' by

- if $t' \in E \cup F_0$, then $t' \cdot \vec{t} = t'$,
- if $t' = x_i \in X_n$, then $t' \cdot \vec{t} = t_i$,
- if $t' = f(t'_1, \ldots, t'_p)$, then $t' \cdot \vec{t} = f(t'_1 \cdot \vec{t}, \ldots, t'_p \cdot \vec{t})$.

If $\vec{t}' = \langle t'_1, \ldots, t'_m \rangle$, where $t'_i \in M(F, E \cup X_n)$, and if $\vec{t} = \langle t_1, \ldots, t_n \rangle$, where $t_j \in M(F, E \cup X_q)$, then $\vec{t}' \cdot \vec{t} = \langle t'_1 \cdot \vec{t}, \ldots, t'_m \cdot \vec{t} \rangle$. The product defined in this way is associative. It is extended to set of trees in the following way:

Let $\vec{P} = \langle P_1, \ldots, P_m \rangle$, where $P_i \subseteq M(F, E \cup X_n)$ and let $\vec{Q} = \langle Q_1, \ldots, Q_n \rangle$, where $Q_j \subseteq M(F, E \cup X_q)$. Then $\vec{P} \cdot \vec{Q} = \langle P_1 \cdot \vec{Q}, \ldots, P_m \cdot \vec{Q} \rangle$, where $P_i \cdot \vec{Q} = \bigcup_{t \in P_i} \{t\} \cdot \vec{Q}$ and $\{t\} \cdot \vec{Q}$ is the subset of $M(F, E \cup X_q)$ defined by induction on t by

- if $t \in E \cup F_0$, then $\{t\} \cdot \vec{Q} = \{t\}$;
- if $t = x_i \in X_n$, then $\{t\} \cdot \vec{Q} = Q_i$;
- if $t = f(t_1, \ldots, t_p)$, then $\{t\} \cdot \vec{Q} = \{f(t'_1, \ldots, t'_p) \mid t'_j \in \{t_j\} \cdot \vec{Q}$.

This product is still associative.

This product is extended to sets of infinite trees as follows:

Let $\vec{P} = \langle P_1, \ldots, P_m \rangle$, where $P_i \subseteq M^\infty(F, E \cup X_n)$ and let $\vec{Q} = \langle Q_1, \ldots, Q_n \rangle$, where $Q_j \subseteq M^\infty(F, E \cup X_q)$. Then $\vec{P} \odot \vec{Q} = \langle P_1 \odot \vec{Q}, \ldots, P_m \odot \vec{Q} \rangle$, where

$$P_i \odot \vec{Q} = \bigcap_\varepsilon \overline{B^0_\varepsilon(P_i) \cdot \langle B^0_\varepsilon(Q_1), \ldots, B^0_\varepsilon(Q_n) \rangle}.$$

It is proved in [4] that this product is also associative. Moreover if each of the sets Q_j contains only one tree, say U_j, and if P_i contains only the tree T_i, then $P_i \odot \vec{Q}$ contains only one tree which will be noted $T_i \odot \langle U_1, \ldots, U_n \rangle = T_i \odot \vec{U}$. In this case if the trees T_i and U_j are finite, we have $T_i \odot \vec{U} = T_i \cdot \vec{U}$.

At last it is proved in [4] that the composition product is continuous in the following sense (see [10]):

Let $(A_i)_i$ be a sequence of subsets of $M^\infty(F, E \cup X_q)$; Let

$$\underset{i}{\mathrm{LI}}(A_i) = \{T \in M^\infty(F, E \cup X_q) \mid \forall \varepsilon > 0, \exists n \geq 0, \forall i \geq n, \exists T' \in A_i \text{ such that } d(T, T') < \varepsilon\}$$

and

$$\underset{i}{\mathrm{LS}}(A_i) = \{T \in M^\infty(F, E \cup X_q) \mid \forall \varepsilon > 0, \forall n \geq 0, \exists i \geq n, \exists T' \in A_i \text{ such that } d(T, T') < \varepsilon\}.$$

We say that the sequence $(A_i)_i$ *$\mathcal{P}$-converges* to $A = \mathrm{Lim}_i(A_i)$ iff $\mathrm{LS}_i(A_i) = \mathrm{LI}_i(A_i) = A$.

A useful property of this notion of limit is that a decreasing sequence of closed sets $\mathcal{P}$-converges to its infinite intersection.

This notion of $\mathcal{P}$-convergence is obviously extended to vector of sets.

Let now $(\vec{P}_i)_i$ and $(\vec{Q}_i)_i$ two sequences of vectors of sets which $\mathcal{P}$-converge to $\vec{P}$ and $\vec{Q}$ respectively. It is proved in [4] that if each component of each $\vec{Q}_i$ is not empty, then the sequence $(\vec{P}_i \odot \vec{Q}_i)_i$ $\mathcal{P}$-converges to $\vec{P} \odot \vec{Q}$.

2.3. *Konig's lemma*

In this paper we use several times Konig's lemma in the following form:

Let E be any set and let $(A_i)_i$ a sequence of finite non-empty subsets of E; let R be a binary relation on E such that $\forall i \geq 0, \forall y \in A_{i+1}, \exists x \in A_i$ s.t. xRy. Then there exists an infinite sequence $(a_i)_i$ of elements of E such that $\forall i \geq 0$, $a_i \in A_i$ and $a_i R a_{i+1}$.

3. Metric interpretations

Let F be a graded alphabet. A *metric interpretation* of F is a structure $I = \langle E_I, d_I, \{f_I \mid f \in F\}\rangle$ such that

- $\langle E_I, \{f_I \mid f \in F\}$ is an F-magma (or F-algebra);
- d_I is a distance on E_I and $\langle E_I, d_I\rangle$ is a complete metric space;
- each operation f_I from $E_I^{\rho(f)}$ into E_I is continuous with respect to the topology induced by the distance d_I.

For any subset A of E_I we denote by $\mathrm{Cl}(A)$ the topological closure of A in E_I.

3.1. *Continuity of an interpretation*

Let I be a metric interpretation of F. With every finite tree t in $M(F, E_I)$ we associate the element $I(t)$ of E_I defined inductively by

- if $t \in E_I$, then $I(t) = t$;
- if $t \in F_0$, then $I(t) = t_I$;
- if $t = f(t_1, \ldots, t_n)$, with $f \in F_n$, then $I(t) = f_I(I(t_1), \ldots, I(t_n))$.

Thus I can be seen as a mapping from $M(F, E_I)$ into E_I; we can partially extend this mapping to $M^\infty(F, E_I)$ as follows:

Let T be an infinite tree in $M^\infty(F, E_I)$. We say that I is *continuous at T relative to $M(F, E_I)$* [7; p. 105] (for short: *rel-continuous at T*) if

(D1) There exists $e \in E_I$ such that for any sequence $(t_i)_i$ of trees in $M(F, E_I)$ which converges to T, the sequence $(I(t_i))_i$ converges to e.

which is clearly equivalent to

(D2) There exists $e \in E_I$ such that $\forall \varepsilon > 0$, $\exists \eta > 0$, $\forall t \in M(F, E_I)$, $d(t, T) < \eta$ implies $d_I(I(t), e) < \varepsilon$.

In this case we extend I by setting $I(T) = e$.

Since, for every infinite tree T, the unit set $\{T\}$ is equal to the infinite intersection $\bigcap_\varepsilon \overline{B^0_\varepsilon(T)} = \bigcap_\varepsilon B_\varepsilon(T)$, we want to express the rel-continuity of I at T by mean of the infinite intersection $\bigcap_\varepsilon \mathrm{Cl}\, I(B^0_\varepsilon(T))$, where $I(B^0_\varepsilon(T)) = \{I(t) \mid t \in B^0_\varepsilon(t)\}$ is well defined since $B^0_\varepsilon(T)$ contains only finite trees.

Proposition 3.1. *If I is rel-continuous at T, then $\bigcap_\varepsilon \mathrm{Cl}(I(B^0_\varepsilon(T)))$ is a unit set. The converse implication holds if E_I is a compact set. In both cases $I(T)$ is equal to the unique element in the infinite intersection.*

Proof. Since E_I is a complete space, for $\bigcap_\varepsilon \mathrm{Cl}(I(B^0_\varepsilon(T)))$ to be a unit set it is sufficient that the diameter of $\mathrm{Cl}(I(B^0_\varepsilon(T)))$ goes to 0 when ε goes to 0. This is an immediate consequence of (D2).

Now let us assume that E_I is compact, that $\bigcap_\varepsilon \mathrm{Cl}(I(B^0_\varepsilon(T))) = \{e\}$ and that (D2) is false. There exists $r > 0$ and for every integer n there exists $t_n \in M(F, E_I)$ such that $d(t_n, T) < 1/n$ and $d_I(I(t_n), e) \geqslant r$. Since E_I is compact, the sequence $(I(t_n))_n$ has an accumulation point e' and clearly $d_I(e', e) \geqslant r$. On the other hand, for every integer n, $e' \in \mathrm{Cl}\{I(t_i) \mid i \geqslant n\} \subset \mathrm{Cl}(I(B^0_{1/n}(T)))$ and thus $e' = e$, a contradiction.

Let now $t \in M(F, X_k)$. We define the mapping $t_I : E_I^k \to E_I$ by: for every $\vec{e}$ in E_I^k, $t_I(\vec{e}) = I(t \cdot \vec{e})$ where $\cdot$ is the composition product of trees. Clearly t_I is continuous. In the same way, with $T \in M^\infty(F, X_k)$ we associate the partial mapping $T_I : E_I^k \to E_I$ defined by: for every $\vec{e} \in E_I^k$, T_I is defined at $\vec{e}$ iff I is rel-continuous at $T \odot \vec{e}$ and thus $T_I(\vec{e}) = I(T \odot \vec{e})$.

3.2. Extended interpretations

Since our purpose is to deal with nondeterminism, we need to define mappings from $\mathcal{P}(E_I)^k$ into $\mathcal{P}(E_I)$ associated with trees in $M^\infty(F, X_k)$. The classical way to do that is extending additively the mapping $T_I : E_I^k \to E_I$. But since we are defining multivocal mappings, we no longer need the condition that $T_I(\vec{e}) = I(T \odot \vec{e})$ must be single-valued, thus we can generalize the definition of $I(T \odot \vec{e})$ following from Proposition 3.1; this leads to an alternative definition of T_I.

Let P be a closed subset of $M^\infty(F, E_I)$. We set $\hat{I}(P) = \bigcap_\varepsilon \text{Cl}(I(B_\varepsilon^0(P)))$.

Let us notice that if P is not closed in $M^\infty(F, E_I)$ we can define $\hat{I}(P)$ in the same way and we get $\hat{I}(P) = \hat{I}(\bar{P})$. Let us notice also that if $P = \{t\}$ with $t \in M(F, E_I)$, then $\hat{I}(P) = \{I(t)\}$. Thus $\hat{I} : \mathcal{P}(M^\infty(F, E_I)) \to \mathcal{P}(E_I)$ is an extension of $I : M(F, E_I) \to E_I$ and also, from Proposition 3.1, of the partial mapping: $M^\infty(F, E_I) \to E_I$.

Thus with every infinite tree T in $M^\infty(F, X_k)$ we associate the mapping, noted still T_I, from $\mathcal{P}(E_I)^k$ into $\mathcal{P}(E_I)$ defined by: for every $\vec{A} \in \mathcal{P}(E_I)^k$, $T_I(\vec{A}) = \hat{I}(\{T\} \odot \vec{A})$. The restriction of T_I to E_I^k is an extension of the partial function $T_I : E_I^k \to E_I$ previously defined; it is the reason for keeping the same notation.

In the same way we can associate multivocal functions with sets of trees as follows: if P is a closed subset of $M^\infty(F, X_k)$, P_I is the function from $\mathcal{P}(E_I)^k$ into $\mathcal{P}(E_I)$ defined by: for every $\vec{A}$ in $\mathcal{P}(E_I)^k$, $P_I(\vec{A}) = \hat{I}(P \odot \vec{A})$.

The following lemma shows that functions interpreting sets of trees are additive extensions of functions interpreting trees:

Lemma 3.2. *Let P a closed subset of $M^\infty(F, X_k)$. For every $\vec{A}$ in $\mathcal{P}(E_I)^k$, $P_I(\vec{A}) = \bigcup_{T \in P} T_I(\vec{A})$.*

Proof. First we can assume, from [4], Lemmas 26 and 27 that for every T in P, $T \odot \vec{A}$ is not empty.

It is clear that for each $T \in P$, $T_I(\vec{A}) \subset P_I(\vec{A})$.

Let $e \in P_I(\vec{A})$ and let ε and $\varepsilon' > 0$. From the definition of $P_I(\vec{A}) = \hat{I}(P \odot \vec{A})$, there exist $T_{\varepsilon,\varepsilon'} \in P$, $T' \in \{T_{\varepsilon,\varepsilon'}\} \odot \vec{A}$ and $t \in B_\varepsilon^0(T')$ such that $d_I(e, I(t)) < \varepsilon'$. Since P is compact, there exists $T^* \in P$ an accumulation point of $\{T_{\varepsilon,\varepsilon'} | \varepsilon, \varepsilon' > 0\}$, such that $\forall \varepsilon, \forall \varepsilon', \forall \varepsilon'', \exists T \in P, T' \in \{T\} \odot \vec{A}, t \in B_\varepsilon^0(T')$ such that $d_I(e, I(t)) < \varepsilon'$ and $d(T, T^*) < \varepsilon''$. But then, setting $\varepsilon'' = \varepsilon$, we get $t \in B_\varepsilon^0(B_\varepsilon(T^*) \odot \vec{A})$. Since

$$B_\varepsilon(T^*) \odot \vec{A} \subset \overline{B_\varepsilon^0(T^*) \cdot B_\varepsilon^0(\vec{A})},$$

which is included, from [4, Proposition 21] in $B_\varepsilon(T^* \odot \vec{A})$, we get $t \in B_\varepsilon^0(T^* \odot \vec{A})$ and $I(t) \in I(B_\varepsilon^0(T^* \odot \vec{A}))$ with $d(e, I(t)) < \varepsilon'$. As this is true for every ε', $e \in \text{Cl}(I(B_\varepsilon^0(T^* \odot \vec{A})))$ and therefore

$$e \in \bigcap_\varepsilon \text{Cl}(I(B_\varepsilon^0(T^* \odot \vec{A}))) = \hat{I}(T^* \odot \vec{A}) = T_I^*(\vec{A}).$$

3.3. *Uniform rel-continuity of extended interpretations*

The final result of this section is that the function interpreting the composition product of two sets of trees is equal to the product of the two functions interpreting these sets. In order to prove it, we need some comditions on I. We split the proof of this result in several lemmas, each one given with minimal hypothesis.

Lemma 3.3. *Let $T \in M^\infty(F, X_k)$ and $Q \in \mathcal{P}(M^\infty(F, E_I))^k$. Then $\hat{I}(\{T\} \odot \hat{I}(Q)) \subset \hat{I}(\{T\} \odot Q)$.*

Proof. We assume that $\{T\} \odot Q$ is not empty. Clearly $\{T\} \odot \hat{I}(Q)$ is included in $B_\varepsilon^0(T) \cdot B_\varepsilon^0(\hat{I}(Q)) = B_\varepsilon^0(T) \cdot \hat{I}(Q)$ for a given $\varepsilon > 0$; Thus

$$B_\varepsilon^0(\{T\} \odot \hat{I}(Q)) \subset B_\varepsilon^0(B_\varepsilon^0(T) \cdot \hat{I}(Q)) = \bigcup_{t \in B_\varepsilon^0(T)} B_\varepsilon^0(\{t\} \cdot \hat{I}(Q)).$$

But we can easily prove by induction on $t \in B_\varepsilon^0(T)$ that

$$B_\varepsilon^0(\{t\} \cdot \hat{I}(Q)) \subset B_\varepsilon^0(T) \cdot \hat{I}(Q) \tag{1}$$

hence

$$B_\varepsilon^0(\{T\} \odot \hat{I}(Q)) \subset B_\varepsilon^0(T) \cdot \hat{I}(Q). \tag{2}$$

By definition of $\hat{I}$, we get

$$\hat{I}(Q) \subset \mathrm{Cl}(I(B_\varepsilon^0(Q))) \tag{3}$$

and (2) becomes

$$B_\varepsilon^0(\{T\} \odot \hat{I}(Q)) \subset B_\varepsilon^0(T) \cdot \mathrm{Cl}(I(B_\varepsilon^0(Q))) \tag{4}$$

and

$$I(B_\varepsilon^0(\{T\} \odot \hat{I}(Q))) \subset I(B_\varepsilon^0(T) \cdot \mathrm{Cl}(I(B_\varepsilon^0(Q)))). \tag{5}$$

Since for every t in $B_\varepsilon^0(T)$, t_I is continuous,

$$\begin{aligned} I(t \cdot \mathrm{Cl}(I(B_\varepsilon^0(Q)))) &= t_I(\mathrm{Cl}(I(B_\varepsilon^0(Q)))) \subset \mathrm{Cl}(t_I(I(B_\varepsilon^0(Q)))) \\ &= \mathrm{Cl}(I(t \cdot I(B_\varepsilon^0(Q)))) \end{aligned} \tag{6}$$

and since one can easily prove by induction on t that

$$I(t \cdot I(B_\varepsilon^0(Q))) = I(t \cdot B_\varepsilon^0(Q)) \tag{7}$$

we get

$$I(t \cdot \mathrm{Cl}(I(B_\varepsilon^0(Q)))) \subset \mathrm{Cl}(I(t \cdot B_\varepsilon^0(Q)) \tag{8}$$

hence

$$\begin{aligned} I(B_\varepsilon^0(T) \cdot \mathrm{Cl}(I(B_\varepsilon^0(Q)))) &\subset \bigcup_{t \in B_\varepsilon^0(T)} \mathrm{Cl}(I(t \cdot B_\varepsilon^0(Q))) \\ &\subset \mathrm{Cl}(I(B_\varepsilon^0(T) \cdot B_\varepsilon^0(Q))). \end{aligned} \tag{9}$$

From [4, Proposition 21]

$$B_\varepsilon^0(T) \cdot B_\varepsilon^0(Q) \subset B_\varepsilon^0(\{T\} \odot Q) \tag{10}$$

thus, from (9) and (5)

$$I(B_\varepsilon^0(\{T\} \odot \hat{I}(Q))) \subset \mathrm{Cl}(I(B_\varepsilon^0(\{T\} \odot Q))) \tag{11}$$

and

$$\mathrm{Cl}(I(B_\varepsilon^0(\{T\} \odot \hat{I}(Q)))) \subset \mathrm{Cl}(I(B_\varepsilon^0(\{T\} \odot Q))). \tag{12}$$

This being true for every ε we get

$$\hat{I}(\{T\} \odot \hat{I}(Q)) \subset \hat{I}(\{T\} \odot Q).$$

Lemma 3.4. *Let $t(x_1, \ldots, x_p) \in M(F, X_p)$ and $W_1, \ldots, W_p \in M^\infty(F, E_I)$. If I is rel-continuous at each W_i, then I is rel-continuous at $t(W_1, \ldots, W_p)$ and $\hat{I}(t(W_1, \ldots, W_p)) = I(t(\hat{I}(W_1), \ldots, \hat{I}(W_p))$.*

Proof. Let $e_i = \hat{I}(W_i)$ and $W = t(W_1, \ldots, W_p) = \{t\} \odot \langle W_1, \ldots, W_p\rangle$. Let $e = t_I(e_I(e_1, \ldots, e_p) = I(t(\hat{I}(W_1), \ldots, \hat{I}(W_p)))$. Let $(w_n)_n$ a sequence of trees in $M(F, E_I)$ which converges to W. For n large enough $w_n = t(w_1^{(n)}, \ldots, w_p^{(n)})$ and the sequences $(w_i^{(n)})_n$ converge to W_i; then, because of the rel-continuity of I at W_i, $(I(w_i^{(n)}))_n$ converges to $\hat{I}(W_i) = e_i$. Hence $I(w_n) = I(t(w_1^{(n)}, \ldots, w_p^{(n)})) = t_I(I(w_1^{(n)}), \ldots, I(w_p^{(n)}))$ and since t_I is continuous, the sequence $I(w_n)$ converges to $t_I(e_1, \ldots, e_p) = e$.

Let $P \subset M^\infty(F, E_I)$. We say that I is *uniformly rel-continuous* on P if

(D3) $\forall \varepsilon > 0, \exists \eta > 0, \forall T \in P, \forall t, t' \in M(F, E_I)$,
$d(t, T) < \eta$ and $d(t', T) < \eta$ implies $d_I(I(t), I(t')) < \varepsilon$.

It is just an exercise in topology to prove that if I is uniformly rel-continuous on P, then it is rel-continuous at every point P. Hence for every $T \in P$, $\hat{I}(T)$ can be identified to an element of E_I and we get

(D4) $\forall \varepsilon > 0, \exists \eta > 0, \forall T \in P, \forall t \in M(F, E_I), d(t, T) < \eta$
implies $d_I(I(t), \hat{I}(T)) < \varepsilon$.

Lemma 3.5. *Let $T \in M^\infty(F, X_k)$ and $Q = \langle Q_1, \ldots, Q_k\rangle \in \mathcal{P}(M^\infty(F, E_I))^k$. If I is rel-continuous at every point of each Q_i and uniformly rel-continuous on $\{T\} \odot Q$, then $\hat{I}(\{T\} \odot Q) \subset \hat{I}(\{T\} \odot \hat{I}(Q))$.*

Proof. We can assume that $\{T\} \odot Q \neq \emptyset$. For any $\varepsilon > 0$, let $t_\varepsilon \in B_\varepsilon^0(T)$. Clearly $\lim_{\varepsilon \to 0}(t_\varepsilon) = T$, and, from the $\mathcal{P}$-continuity of the composition product [4], $\operatorname{Lim}_{\varepsilon \to 0}(\{t_\varepsilon\} \odot Q) = \{T\} \odot Q$. Hence, from the definition of Lim [4],

$$\forall V \in \{T\} \odot Q, \forall \varepsilon_1, \forall \varepsilon_2, \exists U \in \{t_{\varepsilon_1}\} \odot Q \text{ such that } d(V, U) < \varepsilon_2. \tag{13}$$

Now let $e \in \hat{I}(\{T\} \odot Q)$ and let $\varepsilon, \varepsilon', \varepsilon'' > 0$. From the definition of $\hat{I}$ we get

$$\exists V \in \{T\} \odot Q, v \in M(F, E_I) \text{ such that } d(V, v) < \varepsilon' \text{ and } d_I(e, I(v)) < \varepsilon. \tag{14}$$

Applying (13) with $\varepsilon_2 = \varepsilon'$ and $\varepsilon_1 = \varepsilon''$ we get

$$\exists U \in \{t_{\varepsilon''}\} \odot Q \text{ such that } d(V, U) < \varepsilon'. \tag{15}$$

Let us write $t_{\varepsilon''} = t(x_{i_1}, \ldots, x_{i_p})$; then $U = t(W_1, \ldots, W_p)$ with $W_j \in Q_{i_j}$. From Lemma 3.4 we get

$$\hat{I}(U) = I(t(\hat{I}(W_1), \ldots, \hat{I}(W_p))) \in I(t_{\varepsilon''} \cdot \hat{I}(Q)) \subset I(B^0_{\varepsilon''}(T) \cdot \hat{I}(Q)) \tag{16}$$

Moreover, as I is rel-continuous at U,

$$\exists u \in M(F, E_I) \text{ such that } d(u, U) < \varepsilon' \text{ and } d_I(I(u), \hat{I}(U)) < \varepsilon. \tag{17}$$

From (15) it comes, since d is ultrametric,

$$d(u, V) < \varepsilon'. \tag{18}$$

Since I is uniformly rel-continuous on $\{T\} \odot Q$, we can apply (D4) and from (14) and (18) it comes, with $\varepsilon' = \eta(\varepsilon)$

$$d_I(\hat{I}(V), I(v)) < \varepsilon \quad \text{and} \quad d_I(\hat{I}(V), I(u)) < \varepsilon. \tag{19}$$

Thus from (14), (17) and (19)

$$d_I(e, \hat{I}(U)) < 4\varepsilon. \tag{20}$$

Since this is true for every ε and since, from (16), $\hat{I}(U) \in I(B^0_{\varepsilon''}(T) \cdot \hat{I}(Q))$ we get

$$e \in \mathrm{Cl}(I(B^0_{\varepsilon''}(T) \cdot \hat{I}(Q))). \tag{21}$$

From [4, Proposition 21] it comes

$$B^0_{\varepsilon''}(T) \cdot \hat{I}(Q) \subset B^0_{\varepsilon''}(\{T\} \odot \hat{I}(Q)) \tag{22}$$

hence

$$e \in \mathrm{Cl}(I(B^0_{\varepsilon''}(\{T\} \odot \hat{I}(Q)))) \subset \mathrm{Cl}(I(B^0_{\varepsilon_1}(\{T\} \odot \hat{I}(Q)))). \tag{23}$$

This being true for every ε_1 we get $e \in \hat{I}(\{T\} \odot \hat{I}(Q))$.

From the previous lemmas it comes.

Proposition 3.6. *Let P a closed subset of $M^\infty(F, X_p)$, $Q = \langle Q_1, \ldots, Q_p \rangle$ a closed element of $\mathcal{P}(M^\infty(F, X_q))^p$ and $\vec{A} = \langle A_1, \ldots, A_1 \rangle$ an element of $\mathcal{P}(M^\infty(F, E_I))^q$. If I is rel-continuous at every point of each $Q_i \odot \vec{A}$ and if for every T in P, I is uniformly rel-continuous on $\{T\} \odot Q \odot \vec{A}$, then $P_I(Q_I(\vec{A})) = (P \odot Q)_I(\vec{A})$.*

Proof. Since $P \odot Q = \bigcup_{T \in P} \{T\} \odot Q$ we get, from Lemma 3.2 $(P \odot Q)_I(\vec{A}) = \bigcup_{T \in P} (\{T\} \odot Q)_I(\vec{A})$. Also from Lemma 3.2, $P_I(Q_I(\vec{A})) = \bigcup_{T \in P} T_I(Q_i(\vec{A}))$. Thus it is sufficient to prove the result when P is equal to $\{T\}$.

But $T_I(Q_I(\vec{A})) = \hat{I}(\{T\} \odot \hat{I}(Q \odot \vec{A}))$ and $(\{T\} \odot Q)_I(\vec{A}) = \hat{I}(\{T\} \odot Q \odot \vec{A}) = \hat{I}(\{T\} \odot (Q \odot \vec{A}))$. The wanted equality is then a direct consequence of Lemmas 3.5 and 3.3.

We say that an interpretation $I = \langle E_I, d_I, \{f_I, f \in F\} \rangle$ is *strongly contractive* if

- the diameter of E_I is bounded;

- each f_I is a contracting mapping, i.e. there exists $c_f < 1$ such that for $e_1, e'_1, \ldots, e_n, e'_n$ in E_I, $d_I(f_I(e_1, \ldots, e_n), f_I(e'_1, \ldots, e'_n)) \leq c_f \times \max\{d_I(e_i, e'_i) \mid 1 \leq i \leq n\}$.

These conditions are used in [3]. We prove here that they are sufficient in order to prove Proposition 3.6.

Proposition 3.7. *If I is strongly contractive, then it is uniformly rel-continuous on all* $M^\infty(F, E_I)$.

Proof. Let b a real number bounding the diameter on E_I and let $c = \max(c_f \mid f \in F\}$. Since F is finite, $c < 1$. Thus (D3) is a consequence of the following property:

$$\forall t, t' \in M(F, E_I),\ d(t, t') < 2^{-n} \text{ implies } d_I(I(t), I(t')) \leq b \times c^n.$$

which is easily proved by induction on n

- if $n = 0$, then $d_I(I(t), I(t')) \leq b = b \times c^0$;
- if $d(t, t') < 2^{-(n+1)}$,
  - if $t \in t_I \cup F_0$, then $t' = t$ and $d_I(I(t), I(t')) = 0 \leq b = c^{n+1}$,
  - if $t = f(t_1, \ldots, t_p)$, then $t' = f(t'_1, \ldots, t'_p)$ with $d(t_i, t'_i) < 2^{-n}$.

By induction hypothesis $d_I(I(t_i), I(t'_i)) \leq b \times c^n$ and

$$d_I(I(t), I(t')) \leq c_f \times b \times c^n \leq b \times c^{n+1}.$$

4. Non deterministic recursive programs and their computations

4.1. Definitions

As in [11] we consider that a (non deterministic) recursive program P is a pair $\langle \Sigma, I \rangle$, where Σ is a (non deterministic) recursive program scheme on a graded alphabet F and I an interpretation of F. Hereafter this interpretation will be a metric one.

A *non deterministic recursive program scheme* (ndrps) on a graded alphabet F is a set of equations

$$\Sigma = \begin{cases} \phi_i(x_1, \ldots, x_{n_i}) = \tau_i \\ i = 1, \ldots, k, \end{cases}$$

where $\Phi = \{\phi_1, \ldots, \phi_k\}$ is a set of unknown function symbols disjoint from F, with $\rho(\phi_i) = \eta_i$, or is a binary symbol not in $F \cup \Phi$ and $\tau_i \in M(F \cup \Phi \cup \{or\}, X_{n_i})$.

In the sequel we shall note F_+ the set $F \cup \{or\}$.

An elementary step of computation with P has one of the three following forms:

- making a non deterministic choice;
- replacing an unknown function symbol by its definition (copy rule);
- executing an operation f_I.

Grouping together the first two cases we get the formal definition:

Let $t, t' \in M(F_+ \cup \Phi, E_I \cup X_n)$; we define the two relations $t \to_\Sigma t'$ and $t \to_I t'$ by

(1) $t \to_\Sigma t'$ iff

(1.1) $t = or(t_1, t_2)$ and ($t' = t_1$ or $t' = t_2$),

(1.2) $t = \phi_i(t_1, \ldots, t_{n_i})$ and $t' = \tau_i \cdot \langle t_1, \ldots, t_{n_i}\rangle$,

(1.3) $t = Y(t_1, \ldots, t_m)$ with $Y \in F_+ \cup \Phi$ and there exists t_i' such that $t_i \to_\Sigma t_i'$ and $t' = Y(t_1, \ldots, t_i', \ldots, t_m)$;

(2) $t \to_I t'$ iff

(2.1) $t = f(e_1, \ldots, e_n)$ with $f \in F$, $e_1, \ldots, e_n \in E_I$ and $t' = f_I(e_1, \ldots, e_n) \in E_I$,

(2.2) $t = Y(t_1, \ldots, t_m)$ with $Y \in F_+ \cup \Phi$ and there exists t_i' such that $t_i \to_I t_i'$ and $t' = Y(t_1, \ldots, t_i', \ldots, t_m)$.

Then we call a *computation from* $t \in M(F_+ \cup \Phi, E_I)$ *with* P, any finite or infinite sequence $t_0, t_1, t_2, \ldots$ of elements of $M(F_+ \cup \Phi, E_I)$ such that $t_0 = t$ and for every $i \geq 0$, $t_i \to_\Sigma t_{i+1}$ or $t_i \to_I t_{i+1}$.

4.2. Result of a computation: intuitive approach

Now we have to define the result of a computation. The usual way for doing it is to assume that the computation domain E_I is a cpo; then by replacing each unknown function symbol, and also the symbol or, by the least value $\perp$ of E_I, one gets an increasing sequence, of which the l.u.b. is, by definition, the result of the computation (see, for example, [11, 12, 15]).

In [3, 4] we introduced a new definition of the result of a computation in which we need not an order on the computation domain. This definition is based upon the following remark: Let $t = \phi(t_1, \ldots, t_n)$ a term, where ϕ is an unknown function symbol. Since we do not yet know this function ϕ we cannot assign to t a definite value; but we do not express this fact by assigning to t a special value $\perp$ intended to mean 'undefined'; we prefer to say that the possible values of t, if any, are in the computation domain, that we express by replacing t by E_I. In this way we associate with a computation a decreasing sequence of sets and the computation is said successful if the infinite intersection of these sets contains only one element of E_I, which is the result of the computation.

Let us give an example, where we use this way of defining the result of a computation.

Let Σ be the scheme

$$\phi(x) = f(x, \phi(x))$$

and let I be its interpretation which has as domain the interval [0, 1] of the real line with the usual distance and where $f_I(x, y) = 1/(1 + xy)$. Thus the program $P = \langle \Sigma, I\rangle$ can be written

$$\phi(x) = \frac{1}{1 + x\phi(x)}.$$

The only computation from $\phi(1)$ with P is

$$\phi(1) \underset{\Sigma}{\rightarrow} f(1, \phi(1)) \underset{\Sigma}{\rightarrow} f(1, f(1, \phi(1)))$$

$$\underset{\Sigma}{\rightarrow} f(1, f(1, f(1, \phi(1)))) \underset{\Sigma}{\rightarrow} \cdots$$

which can also be written as

$$\phi(1) \underset{\Sigma}{\rightarrow} \frac{1}{1+\phi(1)} \underset{\Sigma}{\rightarrow} \frac{1}{1+\frac{1}{1+\phi(1)}} \underset{\Sigma}{\rightarrow} \frac{1}{1+\frac{1}{1+\frac{1}{1+\phi(1)}}} \underset{\Sigma}{\rightarrow} \cdots$$

or

$$t_1 \underset{\Sigma}{\rightarrow} t_2 \underset{\Sigma}{\rightarrow} t_3 \cdots,$$

where $t_1 = \phi(1)$ and $t_{i+1} = f(1, t_i) = 1/(1+t_i)$.

To t_1 we assign the set $[0, 1]$, thus to t_2 we assign the set $1/(1+[0, 1]) = 1/[1, 2] = [\frac{1}{2}, 1]$ and we can prove easily by induction that the interval associated with t_n is defined by

$$u_{2n} = \left[\frac{\text{fib}(2n-1)}{\text{fib}(2n)}, \frac{\text{fib}(2n)}{\text{fib}(2n+1)}\right], \qquad u_{2n+1} = \left[\frac{\text{fib}(2n+1)}{\text{fib}(2n+2)}, \frac{\text{fib}(2n)}{\text{fib}(2n+1)}\right],$$

where fib(n) is the nth element of the Fibonacci sequence defined by

$$\text{fib}(0) = 0, \text{fib}(1) = 1, \text{fib}(n+2) = \text{fib}(n+1) + \text{fib}(n).$$

Then we can prove that the length of the interval u_{n+1} is less than the half of the length of u_n; hence the infinite intersection $\bigcap_n u_n$ of intervals contains a single point which is

$$\lim_{n\to\infty} \frac{\text{fib}(n)}{\text{fib}(n+1)} = \frac{-1+\sqrt{5}}{2}.$$

4.3. Result of a computation: formal definition

Let us define the mapping π_I from $M(F_+ \cup \Phi, E_I)$ into $\mathcal{P}(E_I)$ by induction on t
- if $t \in E_I$, then $\pi_I(t) = \{t\}$;
- if $t \in F_0$, then $\pi_I(t) = \{t_I\}$;
- if $t = f(t_1, \ldots, t_m)$, then $\pi_I(t) = \{f_I(e_1, \ldots, e_m\} \mid e_i \in \pi_I(t_i)\}$;
- if $t = or(t', t'')$, then $\pi_I(t) = \pi_I(t') \cup \pi_I(t'')$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\pi_I(t) = E_I$.

It should be noticed that if $t \in M(F, E_I) \subseteq M(F_+ \cup \Phi, E_I)$, then $\pi_I(t) = \{I(t)\}$.

Lemma 4.1. *Let $t, t' \in M(F_+ \cup \Phi, E_I)$. If $t \rightarrow_\Sigma t'$, then $\pi_I(t') \subseteq \pi_I(t)$ and if $t \rightarrow_I t'$, then $\pi_I(t) = \pi_I(t')$.*

Proof. We prove this lemma by induction on the definition of $\to_\Sigma$ and $\to_I$:
- if $t = or(t_1, t_2)$ and $t' = t_1$ or $t' = t_2$, then $\pi_I(t') \subseteq \pi_I(t_1) \cup \pi_I(t_2) = \pi_I(t)$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then for any $t' \pi_I(t') \subseteq \pi_I(t) = E_I$;
- if $t = Y(t_1, \ldots, t_m)$ and $t' = Y(T_1, \ldots, t_i', \ldots, t_m)$ with $t_i \to_\Sigma t_i'$ and thus $\pi_I(t_i') \subset \pi_I(t)$, then

$$\text{if } Y \in \Phi, \quad \pi_I(t) = \pi_I(t') = E_I,$$

$$\text{if } Y = or, \quad \pi_I(t') \subset \pi_I(t),$$

$$\begin{aligned}\text{if } Y = f, \quad \pi_I(t') &= \{f_I(e_1, \ldots, e_m) \mid e_1 \in \pi_I(t_1), \ldots, \\ &\qquad e_i \in \pi_I(t_i'), \ldots, e_m \in \pi_I(t_m)\} \\ &\subset \{f_I(e_1, \ldots, e_m) \mid e_1 \in \pi_I(t_1), \ldots, \\ &\qquad e_i \in \pi_I(t_i), \ldots, e_m \in \pi_I(t_m)\} = \pi_I(t);\end{aligned}$$

- if $t = f(e_1, \ldots, e_m)$ and $t' = f_I(e_1, \ldots, e_m)$, then $\pi_I(t) = \{t'\} = \pi_I(t')$;
- if $t = Y(t_1, \ldots, t_m)$ and $t' = Y(t_1, \ldots, t_i', \ldots, t_m)$ with $t_i \to_I t_i'$ and thus $\pi_I(t_i) = \pi_I(t_i')$ the proof is like above but replacing $\subseteq$ by $=$.

Thus for any computation $t_0, t_1, \ldots, t_n, \ldots$ the sequence $(\pi_I(t_n))_n$ is decreasing for inclusion. We say that this computation is *successful* if $\lim_{n\to\infty} \delta(\pi_I(t_n)) = 0$, where $\delta(\pi_I(t_n))$ is the diameter of $\pi_I(t_n)$ (i.e. $\delta(\pi_I(t_n)) = \max\{d_I(e, e') \mid e, e' \in \pi_I(t_n)\}$). In this case $(\pi_I(t_n))_n$ is a Cauchy filter base and since E_I is a complete space the infinite intersection $\bigcap_n \mathrm{Cl}(\pi_I(t_n))$ contains one and only one element of E_I which is the limit of the Cauchy fiter base [7]. This element is the *result* of the successful computation $c = t_0, t_1, \ldots, t_n, \ldots$ and is noted $\mathrm{Res}(c)$.

Let us notice that this definition is compatible with the classical definition of the result of a finite computation. Let $t_0, t_1, \ldots, t_n$ be a finite computation with $t_n \in E_I$ which is of course the result of this computation; we get

$$\bigcap_{1\leq i\leq n} \mathrm{Cl}\,(\pi_I(t_i) = \mathrm{Cl}(\pi_I(t_n)) = \{t_n\}.$$

We have seen that $t_0, t_1, \ldots, t_n, \ldots$ is a successful computation implies that $\bigcap_n \mathrm{Cl}(\pi_I(t_n))$ is a unit set. The converse implication holds whenever E_I is compact. The proof is exactly like in Proposition 3.1.

Finally, for every t in $M(F_+ \cup \Phi, E_I)$ we note $\mathrm{Val}_p(t)$ the set $\{\mathrm{Res}(c) \mid c$ is a successful computation from t with $P\}$.

5. Formal and effective computations

We want now to prove that the result of a successful computation with a program $\langle \Sigma, I \rangle$ is the interpretation of the result of a formal computation, done in the space of trees using only the scheme Σ, like in the case of deterministic programs [11]

5.1. Formal computations

Let t be a tree in $M(F_+ \cup \Phi, X_m)$. A *formal computation from t with the scheme Σ* is a finite or infinite sequence $t_0, t_1, \ldots, t_n, \ldots$ of elements of $M(F_+ \cup \Phi, X_m)$ such that $t_0 = t$ and for every $i \geq 0$, $t_i \to_\Sigma t_{i+1}$.

Let us define the mapping Π from $M(F_+ \cup \Phi, X_m)$ into $\mathcal{P}(M^\infty(F, X_m))$ by induction on t

- if $t \in X_m$, then $\Pi(t) = \{t\}$;
- if $t \in F_0$, then $\Pi(t) = \{t\}$;
- if $t = or(t', t'')$, then $\Pi(t) = \Pi(t') \cup \Pi(t'')$;
- if $t = f(t_1, \ldots, t_n)$, then $\Pi(t) = \{f\} \odot \langle \Pi(t_1), \ldots, \Pi(t_n) \rangle$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\Pi(t) = M^\infty(F, X_m)$.

It is clear that, by definition, $\Pi(t)$ is a closed subset of $M^\infty(F, X_m)$ for every t in $M(F_+ \cup \Phi, X_m)$.

Moreover it is easy to prove, like in Lemma 4.1, that if $t \to_\Sigma t'$, then $\Pi(t') \subseteq \Pi(t)$.

We say that a formal computation $c = t_0, t_1, \ldots, t_n, \ldots$ *is successful* if $\lim_{n\to\infty} \delta(\Pi(t_n)) = 0$. Then, since $M^\infty(F, X_m)$ is complete, $\bigcap_n \Pi(t_n)$ contains one and only one element which is the result, noted $\mathrm{Res}(c)$, of the successful computation c.

Unfortunately this definition of successful computations is not exactly the same as the one used in [1, 2]: it is straightforward, taking into account some results of [4] comparing the set of infinite trees as a cpo and as a complete metric space, that the definition of successful computation s given in [1, 2] that we shall call here *successful' computations* is equivalent to the following definition:

Let Π' be the mapping from $M(F_+ \cup \Phi, X_m)$ into $M^\infty(F, X_m)$ which is defined exactly like Π except that $\Pi'(or(t', t'')) = M^\infty(F, X_m)$. We say that the formal computation $t_0, t_1, \ldots, t_n, \ldots$ *is successful'* iff $\lim_{n\to\infty} \delta(\Pi'(t_n)) = 0$ and its result, $\mathrm{Res}'(c)$, is the unique element in $\bigcap_n \Pi'(t_n)$.

The difference between Π and Π' comes from the fact that, in order to define the result of an infinite computation we associated, in [1, 2], with every tree t with non terminal symbols occurring in a computation sequence the tree obtained by substituting the 'bottom element' to any non terminal symbol and, in particular, to or. As mentioned in the introduction we present in this paper another point of view: with such a tree t is associated the set of values that this tree can have and thus or is interpreted as the set-theoretical union.

Since the main result of this paper (Theorem 6.7) relies upon Theorem 6.3 which was proved in [1, 2] using successful' computations, and upon Theorem 5.9 which is proved in this paper using successful computations, we have to prove the equivalence of these two notions.

From the previous definitions of Π and Π' it follows that for every t, $\Pi'(t)$ is a closed set containing $\Pi(t)$. And we have still, by the same proof as in Lemma 4.1, $t \to_\Sigma t'$ implies $\Pi'(t') \subseteq \Pi'(t)$.

Since $\Pi(t_n) \subset \Pi'(t_n)$, if the formal computation $c = t_0, t_1, \ldots, t_n, \ldots$ is successful', it is also successful and $\mathrm{Res}(c) = \mathrm{Res}'(c)$.

Proposition 5.1. *If the formal computation c from t_0 is successful, there exists a successful′ formal computation c' from t_0 such that* $\mathrm{Res}(c') = \mathrm{Res}(c)$.

Proof. With every $t \in M(F_+ \cup \Phi, X_m)$ we associate the finite set $S(t) \subseteq M(F_+ \cup \Phi, X_m)$ defined inductively by

- if $t \in X_m$, then $S(t) = \{t\}$;
- if $t \in F_0$, then $S(t) = \{t\}$;
- if $t = or(t', t'')$, then $S(t) = S(t') \cup S(t'')$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $S(t) = \{t\}$;
- if $t = f(t_1, \ldots, t_n)$, then $S(t) = \{f(t'_1, \ldots, t'_n) | t'_i \in S(t_i)\}$.

It can be easily proved by induction that for every $t' \in S(t)$ we have $t \to^*_\Sigma t'$ and $\Pi(t') = \Pi'(t')$ and that $\Pi(t) = \bigcup_{t' \in S(t)} \Pi(t')$.

We prove also by induction the following property:

If $t_1 \to_\Sigma t_2$, for every $t'_2 \in S(t_2)$ there exists $t'_1 \in S(t_1)$ such that $t'_1 \to^*_\Sigma t'_2$:

- if $t_1 = \Phi_i(u_1, \ldots, u_{n_i})$, then $S(t_1) = \{t_1\}$, hence $t'_1 = t_1 \to_\Sigma t_2 \to^*_\Sigma t'_2$;
- if $t_1 = or(u_1, u_2)$, then
  - either $t_2 = u_j$ with $j \in \{1, 2\}$, hence $S(t_2) \subseteq S(t_1)$ and we take $t'_1 = t'_2$,
  - or there exist $j \in \{1, 2\}$, u'_1, u'_2 such that $u_j \to_\Sigma u'_j$, $u_{3-j} = u'_{3-j}$, $t_2 = or(u'_1, u'_2)$; since $S(t_2) = S(u'_1) \cup S(u'_2)$, there eixts $i \in \{1, 2\}$ such that $t'_2 \in S(u'_i)$; if $i = 3 - j$, $S(u'_i) = S(u_{3-j}) \subseteq S(t_1)$ and we take $t'_1 = t'_2$; if $i = j$, by induction hypothesis there exists $t'_1 \in S(u_j) \subseteq S(t_1)$ such that $t'_1 \to^*_\Sigma t'_2$;
- if $t_1 = f(u_1, \ldots, u_n)$ there exist i and u'_i such that $u_i \to_\Sigma u'_i$ and $t_2 = f(u_1, \ldots, u'_i, \ldots, u_n)$; but $t'_2 = f(v_1, \ldots, v_n)$ with $v_1 \in S(u_1), \ldots, v_i \in S(u'_i), \ldots, v_n \in S(u_n)$; by induction hypothesis there exists $v'_i \in S(u_i)$ such that $v'_i \to^*_\Sigma v_i$; hence $t'_1 = f(v_1, \ldots, v'_i, \ldots, v_n) \in S(t_1)$ and $t'_1 \to^*_\Sigma t'_2$.

Let now $c = t_0, t_1, \ldots, t_n, \ldots$ be a successful formal computation and let $\{T\} = \bigcap_n \Pi(t_n)$. Let $S'(t_n)$ be the finite set $\{t' \in S(t_n) | T \in \Pi(t')\}$. Since $\Pi(t_n) = \bigcup_{t' \in S(t_n)} \Pi(t')$ there exists $t' \in S(t_n)$ such that $T \in \Pi(t')$, hence $S'(t_n)$ is not empty. Moreover for any t' in $S'(t_{n+1}) \subset S(t_{n+1})$ there exists $t'' \in S(t_n)$ such that $t'' \to^*_\Sigma t'$, hence $\Pi(t') \subseteq \Pi(t'')$, $T \in \Pi(t'')$ and $t'' \in S'(t_n)$.

We can apply König's lemma:

There exists a sequence $t'_0, \ldots, t'_n, \ldots$ such that $t'_n \in S'(t_n)$, $t'_n \to^*_\Sigma t'_{n+1}$ and $t_0 \to^*_\Sigma t'_0$. From this sequence we extract a formal computation $c' = t_0 = t''_0, \ldots, t''_m, \ldots$ (not necessarily infinite). We have

$$\lim_{n \to \infty} \delta(\Pi'(t''_n)) = \lim_{n \to \infty} \delta(\Pi'(t'_n)) = \lim_{n \to \infty} \delta(\Pi(t'_n)).$$

Since $\Pi(t'_n) \subseteq \Pi(t_n)$, $\delta(\Pi(t'_n)) \leq \delta(\Pi(t_n))$ and since c is successful, c' is successful′. Moreover

$$\bigcap_n \Pi'(t''_n) = \bigcap_n \Pi'(t'_n) = \bigcap_n \Pi(t'_n) \quad \text{and} \quad T \in \bigcap_n \Pi(t'_n) \subseteq \bigcap_n \Pi(t_n) = \{T\},$$

thus $\mathrm{Res}(c') = \mathrm{Res}(c) = T$.

It follows that the set $\mathrm{Val}_\Sigma(t)$ equal to $\{\mathrm{Res}(c) \mid c$ is a successful computation from t with $\Sigma\}$ is also equal to $\{\mathrm{Res}'(c) \mid c'$ is a successful' computation from t with $\Sigma\}$ which is considered in [1, 2].

As in [1, 2] we note $L_i^\infty(\Sigma)$ the set $\mathrm{Val}_\Sigma(\phi_i(x_1, \ldots, x_{n_i})) \subseteq M^\infty(F, X_{n_i})$ and we note $\overrightarrow{L^\infty(\Sigma)}$ the k-uple $\langle L_1^\infty(\Sigma), \ldots, L_k^\infty(\Sigma)\rangle$.

5.2. Interpretation of formal results

Let us define the mapping $\hat{\Pi}_I$ from $M(F_+ \cup \Phi, E_I)$ into $\mathcal{P}(M(F, E_I))$ by induction on t

- if $t \in F_0 \cup E_I$, then $\hat{\Pi}_I(t) = \{t\}$;
- if $t = or(t', t'')$, then $\hat{\Pi}_I(t) = \hat{\Pi}_I(t') \cup \hat{\Pi}_I(t'')$;
- if $t = f(t_1, \ldots, t_n)$, then $\hat{\Pi}_I(t) = \{f(u_1, \ldots, u_n) \mid u_i \in \hat{\Pi}_I(t_i)\}$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\hat{\Pi}_I(t) = E_I$.

It is immediate, from the definitions, that $\pi_I(t) = I(\hat{\Pi}_I(t))$.

Lemma 5.2. *Let $t \in M(F_+ \cup \Phi, X_m)$ and $\vec{e} \in E_I^m$. The diameter of $\hat{\Pi}_I(t \cdot \vec{e})$ is less than or equal to the diameter of $\Pi(t)$.*

Proof. We prove by induction on t that for every u in $\hat{\Pi}_I(t \cdot \vec{e})$ there exist v, v' in $\Pi(t)$ such that $d(u, v \cdot \vec{e}) \leq d(v, v')$:

- if $t = x_i \in X_m$, then $\hat{\Pi}_I(t \cdot \vec{e}) = \{e_i\}$ and we take $v = v' = x_i$;
- if $t \in F_0$, then $\hat{\Pi}_I(t) = \Pi(t) = \{t\}$ and we take $v = v' = t$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\hat{\Pi}_I(t \cdot \vec{e}) = E_I$ and $\Pi(t) = M^\infty(F, X_m)$; we take v, v' in $M(F, X_m)$ with different roots and we have $d(u, v \cdot \vec{e}) \leq d(v, v') = \frac{1}{2}$.
- if $t = f(t_1, \ldots, t_n)$, then $u = f(u_1, \ldots, u_n)$ with $u_i \in \hat{\Pi}_I(t_i \cdot \vec{e})$; by induction hypothesis there exist v_i, $v_i' \in \Pi(t_i)$ such that $d(u_i, v_i \cdot \vec{e}) \leq d(v_i, v_i')$; hence $v = f(v_1, \ldots, v_n)$ and $v' = f(v_1', \ldots, v_n')$ belong to $\Pi(t)$, $v \cdot \vec{e} = f(v_1 \cdot \vec{e}, \ldots, v_n \cdot \vec{e})$ and

$$\begin{aligned} d(u, v \cdot \vec{e}) &= \tfrac{1}{2} \max\{d(u_i, v_i \cdot \vec{e}) \mid i = 1, \ldots, n\} \\ &\leq \tfrac{1}{2} \max\{d(v_i, v_i') \mid i = 1, \ldots, n\} = d(v, v'). \end{aligned}$$

- if $t = or(t_1, t_2)$, then there exists $i \in \{1, 2\}$ such that $u \in \hat{\Pi}_I(t_i \cdot \vec{e})$; by induction hypothesis, there exist $v, v' \in \Pi(t_i) \subseteq \Pi(t)$ such that $d(u, v \cdot \vec{e}) \leq d(v, v')$.

Let now $u_1, u_2 \in \hat{\Pi}_I(t \cdot \vec{e})$. We have $d(u_1, u_2) \leq \max(d(u_1, v_1, \cdot \vec{e}), d(v_1 \cdot \vec{e}, v_2 \cdot \vec{e}), d(u_2, v_2 \cdot \vec{e}))$. Since $d(v_1 \cdot \vec{e}, v_2 \cdot \vec{e}) \leq d(v_1, v_2)$ and from the previous result, $d(u_1, u_2) \leq \max(d(v_1', v_1), d(v_1, v_2), d(v_2, v_2')) \leq \delta(\Pi(t))$, hence the result.

Proposition 5.3. *Let $T \in M^\infty(F, X_m)$ be the result of a formal successful computation from $t \in M(F_+ \cup \Phi, X_m)$ and let $\vec{e} \in E_I^m$. If I is rel-continuous at $T \odot \vec{e}$, then there exists a successful computation from $t \cdot \vec{e}$ with $\langle \Sigma, I\rangle$, which has $I(T \odot \vec{e})$ as result.*

Proof. Let $t_0, t_1, \ldots, t_n, \ldots$ be a formal computation such that $t = t_0$ and $\{T\} = \bigcap_n \Pi(t_n)$. The sequence $c = t_0 \cdot \vec{e}, \ldots, t_n \cdot \vec{e}$ is a computation from $t \cdot \vec{e} = t_0 \cdot \vec{e}$ with $\langle \Sigma, I\rangle$.

Since $\lim_{n\to\infty}\delta(\Pi(t_n))=0$ we get, from Proposition 5.3, $\lim_{n\to\infty}\delta(\hat{\Pi}_I(t_n\cdot\vec{e}))=0$, and since $\lim_{n\to\infty} t_n\cdot\vec{e}=T\odot\vec{e}$, for every $\varepsilon>0$ there exists n_ε such that $\hat{\Pi}_I(t_{n_\varepsilon}\cdot\vec{e})\subseteq B^0_\varepsilon(T\odot\vec{e})$; thus $\pi_I(t_{n_\varepsilon}\cdot\vec{e})\subseteq I(B^0_\varepsilon(T\odot\vec{e}))$. Since I is rel-continuous at $T\odot\vec{e}$ it follows that $\lim_{n\to\infty}\delta(\pi_I(t_n\cdot\vec{e}))=0$. Thus the computation c is successful and its result is

$$\bigcap_n \mathrm{Cl}(\pi_I(t_n\cdot\vec{e}))\subseteq\bigcap_\varepsilon \mathrm{Cl}(I(B^0_\varepsilon(T\odot\vec{e})))=I(T\odot\vec{e}).$$

Before proving the converse of Proposition 5.3 we need some lemmas.

Lemma 5.4. *Let $t_0, t_1,\ldots,t_n,\ldots$ be a computation from $t=t_0\in M(F_+\cup\Phi, E_I)$ with $\langle\Sigma, I\rangle$. There exists a sequence $t'_0,\ldots,t'_n,\ldots$, such that for every $n\geqslant 0$ $t'_n\to^*_I t_n$ and $t'_n\to_\Sigma t'_{n+1}$ or $t'_n=t'_{n+1}$.*

Proof. We construct the sequence $(t'_n)_n$ by induction. First we set $t'_0=t_0$.

Let us assume that we have constructed t'_n. We have $t'_n\to^*_I t_n$. If $t_n\to_I t_{n+1}$, then we set $t'_{n+1}=t'_n$; if $t_n\to_\Sigma t_{n+1}$, it is proved in [11] that there exists t'_{n+1} such that $t'_n\to_\Sigma t'_{n+1}\to^*_I t_{n+1}$.

We say that a scheme Σ is *reduced* if for every $i\in\{1,\ldots,k\}$ the set $L^\infty_i(\Sigma)$ is not empty.

Lemma 5.5. *If Σ is reduced, then for any t in $M(F_+\cup\Phi, X_m)$,* $\mathrm{Val}_\Sigma(t)$ *is not empty.*

Proof. Let t be in $M(F_+\cup\Phi, X_m)$ and let t' be the tree obtained by deriving each occurrence of a symbol Ω in t so that t' does not contain Ω, and $t\to^*_\Sigma t'$.

Since Σ is reduced, for each $i\in\{1,\ldots,k\}$ there exists a successful formal computation $t^{(i)}_0,\ldots,t^{(i)}_n,\ldots$ with $t^{(i)}_0=\phi_i(x_1,\ldots,x_{n_i})$; we have, by definition, $\lim_{n\to\infty}\delta(\Pi(t^{(i)}_n))=0$.

Let $t_0=t$ and for every $n>0$ let t_n be the tree obtained by replacing in t' every occurrence of each ϕ_i by $t^{(i)}_n$. Clearly $t_n\to^*_\Sigma t_{n+1}$. Moreover it can be easily proved by induction on t' that

$$\delta(\Pi(t_n))\leqslant\max\{\delta(\Pi(t^{(i)}_n))\mid 1\leqslant i\leqslant k\},$$

hence $\lim_{n\to\infty}\delta(\Pi(t_n))=0$.

It follows that there exists a successful formal computation from t, hence $\mathrm{Val}_\Sigma(t)$ is not empty.

Lemma 5.6. *Let $t_0, t_1,\ldots,t_n,\ldots$ be a formal computation from $t_0\in M(F_+\cup\Phi, X_m)$, not necessarily successful. If Σ is reduced, there exists $T\in\overline{\mathrm{Val}_\Sigma(t_0)}$ such that $T\in\bigcap_n\Pi(t_n)$.*

Proof. From Lemma 5.5 each $\mathrm{Val}_\Sigma(t_n)$ is not empty. Since $t_n\to_\Sigma t_{n+1}$, clearly

$\mathrm{Val}_\Sigma(t_{n+1}) \subseteq \mathrm{Val}_\Sigma(t_n)$. Hence $\overline{(\mathrm{Val}_\Sigma(t_n))_n}$ is a decreasing sequence of nonempty closed subsets of the compact space $M^\infty(F, X_m)$. Therefore $A = \bigcap_n \overline{\mathrm{Val}_\Sigma(t_n)}$ is not empty. Let T be in A; we have $T \in \overline{\mathrm{Val}_\Sigma(t_0)}$; moreover, since, by definition every element of $\mathrm{Val}_\Sigma(t_n)$ belongs to $\Pi(t_n)$ we have $\overline{\mathrm{Val}_\Sigma(t_n)} \subseteq \overline{\Pi(t_n)} = \Pi(t_n)$, hence $T \in \bigcap_n \Pi(t_n)$.

Lemma 5.7. *Let $t \in M(F_+ \cup \Phi, X_m)$ and $\vec{e} \in E_I^m$; let $\varepsilon < 2^{-|t|}$ and $T \in \Pi(t)$. For every $u \in M(F, E_I)$, $d(u, T \odot \vec{e}) < \varepsilon$ implies $I(u) \in \pi_I(t \cdot \vec{e})$.*

Proof. This result is proved by induction on t:

- if $t = x_i \in X_m$, then $T = x_i$, $t \cdot \vec{e} = T \odot \vec{e} = e_i$ and $|t| = 1$; hence $u = e_i$, and $I(u) = e_i \in \pi_I(t \cdot \vec{e}) = \{e_i\}$;
- if $t = a \in F_0$, then $T = t = t \cdot \vec{e} = T \odot \vec{e} = a$ and $|t| = 1$; hence $u = a$ and $I(u) = a_I \in \pi_I(t \cdot \vec{e}) = \{a_I\}$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $I(u) \in \pi_I(t \cdot \vec{e}) = E_I$ for every $u \in M(F, E_I)$;
- if $t = or(t_1, t_2)$, then $\exists j \in \{1, 2\}$ such that $T \in \Pi(t_j)$ and $|t_j| \leq |t| - 1$, hence $d(u, T \odot \vec{e}) < 2^{-|t|} < 2^{-|t_j|}$ and by induction hypothesis $I(u) \in \pi_I(t_j \cdot \vec{e}) \subseteq \pi_I(t \cdot \vec{e})$;
- if $t = f(t_1, \ldots, t_n)$, then $T = f(T_1, \ldots, T_n)$ with $T_i \in \Pi(t_i)$ and $u = f(u_1, \ldots, u_n)$ with $d(u_i, T_i \odot \vec{e}) < 2\varepsilon$. But $|t_i| \leq |t| - 1$, hence $2\varepsilon < 2^{-|t|+1} \leq 2^{-|t_i|}$, and, by induction hypothesis, $I(u_i) \in \pi_I(t_i \cdot \vec{e})$; thus $I(u) = f_I(I(u_1), \ldots, I(u_n)) \in \pi_I(f(t_1 \cdot \vec{e}, \ldots, t_n \cdot \vec{e})) = \pi_I(t \cdot \vec{e})$.

From these lemmas we deduce

Proposition 5.8. *Let $t \in M(F_+ \cup \Phi, X_m)$ and $\vec{e} \in E_I^m$. If Σ is a reduced scheme, then for any successful computation c from $t \cdot \vec{e}$ with $\langle \Sigma, I \rangle$, there exists $T \in \overline{\mathrm{Val}_\Sigma(t)}$ such that I is rel-continuous at $T \odot \vec{e}$ and* $\mathrm{Res}(c) = I(T \odot \vec{e})$.

Proof. Let $c = t_0, t_1, \ldots, t_n, \ldots$ a computation with $\langle \Sigma, I \rangle$ such that $t_0 = t \cdot \vec{e}$ and $\lim_{n\to\infty} \pi_I(t_n) = 0$.

From Lemma 5.4, there exists a sequence $t'_0, t'_1, \ldots, t'_n, \ldots$ such that $t'_0 = t_0 = t \cdot \vec{e}$, $t'_n \to_\Sigma t'_{n+1}$ and $\lim_{n\to\infty} \pi_I(t'_n) = \lim_{n\to\infty} \pi_I(t_n) = 0$.

It can be easily proved that there exists a sequence $(t''_n)_n$ of elements of $M(F_+ \cup \Phi, X_m)$ such that $t''_0 = t$ and for every n $t'_n = t''_n \cdot \vec{e}$ and $t''_n \to_\Sigma t''_{n+1}$.

From Lemma 5.6 there exists $T \in \overline{\mathrm{Val}_\Sigma(t)}$ such that $T \in \bigcap_n \Pi(t''_n)$.

From Lemma 5.7 we get $\forall n$, $\exists \varepsilon > 0$ such that

$$I(B_\varepsilon^0(T \odot \vec{e})) \subseteq \pi_I(t_n) = \pi_I(t''_n \cdot \vec{e}),$$

hence, since $\lim_{n\to\infty} \delta(\pi_I(t_n)) = 0$, $\forall \eta > 0$, $\exists \varepsilon > 0$ such that $\delta(I(B_\varepsilon^0(T \odot \vec{e}))) < \eta$ which implies that I is rel-continuous at $T \odot \vec{e}$.

Moreover

$$I(T \odot \vec{e}) = \bigcap_\varepsilon \mathrm{Cl}(I(B_\varepsilon^0(T \odot \vec{e}))) \subseteq \bigcap_n \mathrm{Cl}(\pi_I(t_n)),$$

hence $I(T \odot \vec{e}) = \mathrm{Res}(c)$.

From Propositions 5.3 *and* 5.8 *we get*

Theorem 5.9. *Let* $t \in M(F_+ \cup \Phi, X_m)$ *and* $\vec{e} \in E_I^m$. *If* Σ *is a reduced scheme and if* $\mathrm{Val}_\Sigma(t)$ *is closed, then*

$$\mathrm{Val}_{\langle \Sigma, I\rangle}(t \cdot \vec{e}) = \{I(T \odot \vec{e}) \mid T \in \mathrm{Val}_\Sigma(t) \text{ and } I \text{ is rel-continuous at } T \odot \vec{e}\}.$$

Let us remark that it follows from [1, 2, 4] that $\mathrm{Val}_\Sigma(t)$ is closed whenever Σ is a Greibach scheme.

6. Greatest fixed points

Given a program $\langle \Sigma, I\rangle$ we can associate with each unknown function symbol ϕ_i the function g_i from $E_I^{n_i}$ into $\mathscr{P}(E_I)$ defined by $g_i(\vec{e}) = \mathrm{Val}_{\langle \Sigma, I\rangle}(\phi_i(\vec{e}))$ which can be viewed as the meaning assigned to ϕ_i by the program.

In this section we prove that under some natural conditions the functions g_i can be defined as fixed points of a functional mapping attached to the program.

6.1. *Programs and program schemes as functional mappings*

For every integer n let $\mathscr{F}_n(E_I)$ be the set of mappings from $\mathscr{P}(E_I)^n$ into $\mathscr{P}(E_I)$ ordered by the following order:

$$g \subseteq g' \quad \text{iff} \quad \forall \vec{A} = \langle A_1, \ldots, A_n\rangle \in \mathscr{P}(E_I)^n,\ g(\vec{A}) \subseteq g'(\vec{A}).$$

The maximal element of $\mathscr{F}_n(E_I)$ is the function G_n defined by $\forall \vec{A} \in \mathscr{P}(E_I)^n$, $G_n(\vec{A}) = E_I$.

Let now $\mathscr{F}(E_I) = \mathscr{F}_{n_1}(E_I) \times \cdots \times \mathscr{F}_{n_k}(E_I)$ be the set of k-uples of functions ordered componentwise by inclusion; its maximal element is $\vec{G} = \langle G_{n_1}, \ldots, G_{n_k}\rangle$.

With every $g = \langle g_1, \ldots, g_k\rangle \in \mathscr{F}(E_I)$ and with every integer m we associate the mapping $\sigma_g^{(m)}$ from $M(F_+ \cup \Phi, X_m)$ into $\mathscr{F}_m(E_I)$ defined inductively, with $\sigma_g^{(m)}$ abbreviated by σ, as follows; for every $\vec{A} = \langle A_1, \ldots, A_n\rangle \in \mathscr{P}(E_I)^n$

- if $t = x_i \in X_m$, then $\sigma(t)(\vec{A}) = \mathrm{Cl}(A_i)$;
- if $t = or(t_1, t_2)$, then $\sigma(t)(\vec{A}) = \sigma(t_1)(\vec{A}) \cup \sigma(t_2)(\vec{A})$;
- if $t = f(t_1, \ldots, t_p)$, then $\sigma(t)(\vec{A}) = \mathrm{Cl}\{f_I(e_1, \ldots, e_p) \mid e_i \in \sigma(t_i)(\vec{A})\}$;
- if $t = a \in F_0$, then $\sigma(t)(\vec{A}) = \{a_I\}$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\sigma(t)(\vec{A}) = g_i(\sigma(t_1)(\vec{A}), \ldots, \sigma(t_{n_i})(\vec{A}))$.

Let us remark that for every t, $\sigma(t)(\vec{A})$ is always a closed subset of E_I. It should be noted also that if $\vec{g} \subseteq \vec{g}'$, then $\sigma_{\vec{g}}^{(m)}(t)(\vec{A}) \subseteq \sigma_{\vec{g}'}^{(m)}(t)(\vec{A})$.

Then with the program $\langle \Sigma, I\rangle$, where

$$\Sigma = \begin{cases} \phi_i(x_1, \ldots, x_{n_i}) = \tau_i \\ i = 1, \ldots, k, \end{cases}$$

we associate the mapping $\hat{\Sigma}_I$ from $\mathcal{F}(E_I)$ into itself defined by

$$\hat{\Sigma}_I(\vec{g}) = \langle \sigma_{\vec{g}}^{(n_1)}(\tau_1), \ldots, \sigma_{\vec{g}}^{(n_k)}(\tau_k) \rangle.$$

Clearly, if $\vec{g} \subseteq \vec{g}'$ we have $\hat{\Sigma}_I(\vec{g}) \subseteq \hat{\Sigma}_I(\vec{g}')$. Hence since $\vec{G}$ is the maximal element of $\mathcal{F}(E_I)$, the sequence $\hat{\Sigma}_I^n(\vec{G})$ is decreasing and we define $\nu(\hat{\Sigma}_I) \in \mathcal{F}(E_I)$ by $\nu(\hat{\Sigma}_I)(\vec{A}) = \bigcap_n \hat{\Sigma}_I^n(\vec{G})(\vec{A})$, which is a closed set.

In a similar way we define the functional $\hat{\Sigma}$ associated with the scheme Σ (see [1, 2, 4]).

Let $\mathcal{D}_\Sigma$ be the set

$$\mathcal{P}(M^\infty(F, X_{n_1})) \times \mathcal{P}(M^\infty(F, X_{n_1})) \times \cdots \times \mathcal{P}(M^\infty(F, X_{n_k}))$$

ordered by inclusion; its maximal element is $\vec{D} = \langle M^\infty(F, X_{n_1}), \ldots, M^\infty(F, X_{n_k}) \rangle$.

With every $\vec{P} = \langle P_1, \ldots, P_k \rangle \in \mathcal{D}_\Sigma$ we associate the mapping $\sigma_{\vec{P}}^{(m)}$ from $M(F_+ \cup \Phi, X_m)$ into $\mathcal{P}(M^\infty(F, X_m))$ defined inductively with $\sigma_{\vec{P}}^{(m)}$ abbreviated by σ, as follows:

- if $t = x_i \in X_m$, then $\sigma(t) = \{x_i\}$;
- if $t = or(t_1, t_2)$, then $\sigma(t) \cup \sigma(t_2)$;
- if $t \in F_0$, then $\sigma(t) = \{t\}$;
- if $t = f(t_1, \ldots, t_p)$, then $\sigma(t) = \{f\} \odot \langle \sigma(t_1), \ldots, \sigma(t_p) \rangle$;
- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\sigma(t) = \bar{P}_i \odot \langle \sigma(t_1), \ldots, \sigma(t_{n_i}) \rangle$.

Thus, by definition, $\sigma(t)$ is closed and if $\vec{P} \subseteq \vec{Q}$, then $\sigma_{\vec{P}}^{(m)}(t) \subseteq \sigma_{\vec{Q}}^{(m)}(t)$.

Then with the scheme Σ is associated the mapping $\hat{\Sigma}$ from $\mathcal{D}_\Sigma$ into itself defined by

$$\hat{\Sigma}(\vec{P}) = \langle \sigma_{\vec{P}}^{(n_1)}(\tau_1), \ldots, \sigma_{\vec{P}}^{(n_k)}(\tau_k) \rangle.$$

Clearly $\vec{P} \subseteq \vec{Q}$ implies $\hat{\Sigma}(\vec{P}) \subseteq \hat{\Sigma}(\vec{Q})$, hence $\hat{\Sigma}^n(\vec{D})$ is a decreasing sequence of closed sets and we note $\nu(\hat{\Sigma})$ the infinite intersection $\bigcap_n \hat{\Sigma}^n(\vec{D})$.

The following result was proved in [4]:

Proposition 6.1. *$\nu(\hat{\Sigma})$ is the greatest fixed point of $\hat{\Sigma}$.*

Now and further on we assume that the following property holds: for every n, for every T in $M^\infty(F, X_n)$, I is uniformly rel-continuous on $\{T\} \odot \langle E_I, \ldots, E_I \rangle$ that we will express by saying: I is *uniformly rel-continuous*

We can then apply Proposition 3.6 and we get

Proposition 6.2. *If I is uniformly rel-continuous, then for every $\vec{P}$ in $\mathcal{D}_\Sigma$, $\hat{\Sigma}(\vec{P})_I = \hat{\Sigma}_I(\vec{P}_I)$.*

Proof. It is sufficient to prove that for every $t \in M(F_+ \cup \Phi, X_m)$ and for every $\vec{A} \in \mathcal{P}(E_I)^n$ we have

$$(\sigma_{\vec{P}}^{(m)}(t))_I(\vec{A}) = \sigma_{\vec{P}_I}^{(m)}(t)(\vec{A}).$$

This proof is done by induction on t, setting $\sigma = \sigma_{\vec{P}}^{(m)}$ and $\sigma_I = \sigma_{\vec{P}_I}^{(m)}$:

- if $t = x_i \in X_m$, then $\sigma_I(t)(\vec{A}) = \mathrm{Cl}(A_i)$ and $\sigma(t) = \{x_i\}$, hence

$$\sigma(t)_I(\vec{A}) = \bigcap_{\varepsilon} \mathrm{Cl}(I(B_\varepsilon^0(\{x_i\} \odot \vec{A}))) = \bigcap_{\varepsilon} \mathrm{Cl}(I(B_\varepsilon^0(A_i))) = \mathrm{Cl}(I(A_i)) = \mathrm{Cl}(A_i);$$

- if $t = a \in F_0$, then $\sigma_I(t)(\vec{A}) = \{a_I\}$ and $\sigma(t) = \{a\}$, hence $(\sigma(t))_I = \{a_I\}$;
- if $t = or(t', t'')$, then $\sigma_I(t)(\vec{A}) = \sigma_I(t')(A) \cup \sigma_I(t'')(\vec{A})$ and $\sigma(t) = \sigma(t') \cup \sigma(t'')$.
  By induction hypothesis $\sigma_I(t')(\vec{A}) = \sigma(t')_I(\vec{A})$ and $\sigma_I(t'')(\vec{A}) = \sigma(t'')_I(\vec{A})$; from Lemma 3.2, $\sigma(t')$ and $\sigma(t'')$ being closed, $\sigma(t)_I(\vec{A}) = \sigma(t')\vec{A} \cup \sigma(t'')_I(\vec{A})$; hence the result;
- if $t = f(t_1, \ldots, t_p)$, then

$$\begin{aligned}\sigma_I(t)(\vec{A}) &= \mathrm{Cl}\{f_I(e_1, \ldots, e_p) \mid e_i \in \sigma_I(t_i)(\vec{A})\}\\ &= \mathrm{Cl}(I(\{f(e_1, \ldots, e_p) \mid e_i \in \sigma_I(t_i)(\vec{A})\}))\\ &= (f(x_1, \ldots, x_p))_I(\langle \sigma_I(t_1)(\vec{A}), \ldots, \sigma_I(t_p)(\vec{A})\rangle);\end{aligned}$$

  from induction hypothesis this is equal to

$$\begin{aligned}&(f(x_1, \ldots, x_p))_I(\langle \sigma(t_1)_I(\vec{A}), \ldots, \sigma(t_p)_I(\vec{A})\rangle) =\\ &\quad = (f(x_1, \ldots, x_p))_I(\langle \sigma(t_1)_I, \ldots, \sigma(t_p)_I\rangle(\vec{A}))\end{aligned}$$

and from Proposition 3.6 to

$$\begin{aligned}&(\{f(x_1, \ldots, x_p)\} \odot \langle \sigma(t_1), \ldots, \sigma(t_p)\rangle)_I(\vec{A}) =\\ &\quad = \sigma(f(t_1, \ldots, t_p))_I(\vec{A}) = \sigma(t)_I(\vec{A});\end{aligned}$$

- if $t = \phi_i(t_1, \ldots, t_{n_i})$, then $\sigma_I(t)(\vec{A}) = (P_i)_I(\sigma_I(t_1)\vec{A}), \ldots, \sigma_I(t_{n_i})(\vec{A}))$; but $(P_i)_I = (\bar{P}_i)_I$ and by induction hypothesis $\sigma_I(t_j)(\vec{A}) = \sigma(t_j)_I(\vec{A})$, hence

$$\sigma_I(t)(A) = (\bar{P}_i)_I(\langle \sigma(t_1)_I, \ldots, \sigma(t_{n_i})_I\rangle(\vec{A}))$$

  and, from Proposition 3.6,

$$\sigma_I(t)(\vec{A}) = (\bar{P}_i \odot \langle \sigma(t_1), \ldots, \sigma(t_{n_i})\rangle)_I(\vec{A})$$

which is equal to $\sigma(t)_I(\vec{A})$.

6.2. Greibach schemes

We say that a scheme

$$\Sigma = \begin{cases} \phi_i(x_1, \ldots, x_{n_i}) = \tau_i \\ i = 1, \ldots, k \end{cases}$$

is a Greibach scheme if each term τ_i is contracting, knowing that the set of contracting terms of $M(F_+ \cup \Phi, X_m)$ is defined inductively by

- if $t = f(t_1, \ldots, t_p)$ with $f \in F$, then t is contracting;
- if t_1 and t_2 are contracting, then $or(t_1, t_2)$ is contracting.

The main result of [1, 2] (see also [4]) is

Theorem 6.3. *If Σ is a Greibach scheme, then $\overrightarrow{L^\infty(\Sigma)}$ is the greatest fixed point of $\hat{\Sigma}$.*

We shall prove that under the hypothesis of uniform rel-continuity of I, $\overrightarrow{(L^\infty(\Sigma))_I}$ is the greatest fixed point of $\hat{\Sigma}_I$, which is also equal in this case to $\nu(\hat{\Sigma}_I)$.

As an immediate consequence of Proposition 6.2 we get

Proposition 6.4. *$\overrightarrow{(L^\infty(\Sigma))_I}$ is a fixed point of $\hat{\Sigma}_I$ if I is uniformly rel-continuous.*

From the monotonicity of $\hat{\Sigma}_I$, and from this proposition it follows that $\overrightarrow{(L^\infty(\Sigma))_I} \subseteq \nu(\hat{\Sigma}_I)$. Thus we have just to prove the reverse inclusion; this proof needs one preliminary lemma.

Lemma 6.5. *Let $\vec{g} = \langle g_1, \ldots, g_k \rangle \in \mathcal{F}(E_I)$, $\vec{P} = \langle P_1, \ldots, P_k \rangle \in \mathcal{D}_\Sigma$ such that every P_i is a finite set of finite trees and let $\varepsilon > 0$ such that $\forall i \leqslant k$, $\forall \vec{A} \in \mathcal{P}(E_I)^{n_i}$, $g_i(\vec{A}) \subseteq \mathrm{Cl}(I(B^0_\varepsilon(P_i) \cdot \vec{A}))$. Then for any integer m, for any $u \in M(F_+ \cup \Phi, X_m)$ and for any $\vec{A} \in \mathcal{P}(E_I)^m$, $\sigma^{(m)}_{\vec{g}}(u)(\vec{A}) \subseteq \mathrm{Cl}(I(B^0_{\varepsilon'}(\sigma^{(m)}_{\vec{P}}(u)) \cdot \vec{A}))$, where $\varepsilon' = \frac{1}{2}\varepsilon$ if u is contracting, ε otherwise.*

Proof. This lemma is proved by induction on u.

There is no difficulty in the case where $u \in F_0 \cup X_m$, $u = or(u_1, u_2)$ and $u = f(u_1, \ldots, u_p)$.

Let us note $\sigma'' = \sigma^{(m)}_{\vec{g}}$, $\sigma = \sigma^{(m)}_{\vec{P}}$ and let us assume that $u = \phi_i(u_1, \ldots, u_{n_i})$. Then $\sigma'(u)(\vec{A}) = g_i(\sigma'(u_1)(\vec{A}), \ldots, \sigma'(u_{n_i})(\vec{A}))$ and by hypothesis $\sigma'(u)(\vec{A}) \subseteq \mathrm{Cl}(I(B^0_\varepsilon(P_i) \cdot \langle \sigma'(u_1(\vec{A}), \ldots, \sigma'(u_{n_i})(\vec{A}) \rangle))$, which is included, from induction hypothesis, in

$$B = \mathrm{Cl}(I(B^0_\varepsilon(P_i) \cdot \langle \mathrm{Cl}(I(B^0_\varepsilon(\sigma(u_1)) \cdot \vec{A})), \ldots, \mathrm{Cl}(I(B^0_\varepsilon(\sigma(u_{n_i})) \cdot \vec{A}))\rangle)).$$

But for every finite tree $v \in M(F, X_l)$ the function $v_I : E_I^l \to E_I$ defined by $v_I(\vec{e}) = I(v \cdot \vec{e})$ is continuous, hence

$$\begin{aligned} B &\subseteq \mathrm{Cl}(I(B^0_\varepsilon(P_i) \cdot \langle I(B^0_\varepsilon(\sigma(u_1)) \cdot \vec{A}), \ldots, I(B^0_\varepsilon(\sigma(u_{n_i})) \cdot \vec{A})\rangle)) \\ &= \mathrm{Cl}(I(B^0_\varepsilon(P_i) \cdot \langle B^0_\varepsilon(\sigma(u_1)) \cdot \vec{A}, \ldots, B^0_\varepsilon(\sigma(u_{n_i})) \cdot \vec{A}\rangle)). \end{aligned}$$

But

$$\begin{aligned} &B^0_\varepsilon(P_i) \cdot \langle B^0_\varepsilon(\sigma(u_1)), \ldots, B^0_\varepsilon(\sigma(u_{n_i}))\rangle \subseteq \\ &\quad \subseteq B^0_\varepsilon(P_i \cdot \langle \sigma(u_1), \ldots, \sigma(u_{n_i})\rangle) = B^0_\varepsilon(\sigma(u)) \end{aligned}$$

hence $B \subseteq \mathrm{Cl}(I(B^0_\varepsilon(\sigma(u)) \cdot \vec{A}))$.

Proposition 6.6. *If Σ is a Greibach scheme, then $\nu(\hat{\Sigma}_I) \subseteq \overrightarrow{(L^\infty(\Sigma))_I}$.*

Proof. Let us write $\hat{\Sigma}^n_I(\vec{G}) = \langle g^{(n)}_1, \ldots, g^{(n)}_k \rangle$. Let $t = \langle t_1, \ldots, t_k \rangle$ a fixed element of $M(F, X_{n_1}) \times \cdots \times M(F, X_{n_k})$ and let us write $\hat{\Sigma}^n(t) = \langle P^{(n)}_1, \ldots, P^{(n)}_k \rangle$ which is included in $\hat{\Sigma}^n(\vec{D})$. For any $\vec{A} \in \mathscr{P}(E)^{n_i}$ we have $B_1^0(t_i) \cdot \vec{A} = E_I$, hence $g_i^0(\vec{A}) = G_{n_i}(\vec{A}) = E_I \subseteq B_1^0(t_i) \cdot \vec{A} \subseteq \mathrm{Cl}(I(B_1^0(t_i) \cdot \vec{A})) = \mathrm{Cl}(I(B_1^0(P_i^{(0)}) \cdot \vec{A}))$. Since Σ is a Greibach scheme we can prove by induction, using Lemma 6.5 that, for every $n \geq 0$, for every $i \leq k$ and for every $\vec{A} \in \mathscr{P}(E_I)^{n_i}$, it is true that $g_i^{(n)}(A) \subseteq \mathrm{Cl}(I(B_{2^{-n}}^0(P_i^{(n)}) \cdot \vec{A}))$. Let now $a \in \bigcap_n g_i^{(n)}(\vec{A})$. For any n, $a \in \mathrm{Cl}(I(B_{2^{-n}}^0(P_i^{(n)}) \cdot \vec{A}))$ and since $P_i^{(n)}$ is a finite set, there exists $t_n \in P_i^{(n)}$ such that $a \in \mathrm{Cl}(I(B_{2^{-n}}^0(t_n) \cdot \vec{A}))$. But the set $\{t_n \mid n \geq 0\}$, included in the compact set $M^\infty(F, X_{n_i})$ has an accumulation point T; thus for every $n \geq 0$ there exists $n' \geq n$ such that $d(t_{n'}, T) < 2^{-n}$, hence $B_{2^{-n'}}^0(t_n) \subseteq B_{2^{-n}}^0(t_{n'}) = B_{2^{-n}}^0(T)$. Since $B_{2^{-n}}^0(T) \cdot \vec{A} \subseteq B_{2^{-n}}^0(\{T\} \odot \vec{A})$ we have

$$a \in \bigcap_n \mathrm{Cl}(I(B_{2^{-n}}^0(\{T\} \odot \vec{A})) = \hat{I}(T \odot \vec{A}) = T_I(\vec{A}).$$

Let now $\hat{\Sigma}^n(\vec{D}) = \langle Q_1^{(n)}, \ldots, Q_k^{(n)} \rangle$. Since $P_i^{(n)} \subseteq Q_i^{(n)}$, $t_n \in Q_i^{(n)}$, and since $(Q_i^{(n)})_n$ is a decreasing sequence, the accumulation point T belongs to $\bigcap_n Q_i^{(n)}$ which is the ith component of $\nu(\hat{\Sigma}) = \overrightarrow{L^\infty(\Sigma)}$. It follows that $a \in T_I(\vec{A}) \subseteq (L_i^\infty(\Sigma))_I(\vec{A})$; therefore $\bigcap_n g_i^{(n)}(\vec{A}) \subseteq (L_i^\infty(\Sigma))_I(\vec{A})$; hence $\nu(\hat{\Sigma}_I) \subseteq \overrightarrow{(L^\infty(\Sigma))_I}$.

We can now establish the equivalence between operational and denotational semantics of non deterministic programs.

Theorem 6.7. *If Σ is a Greibach scheme and if I is uniformly rel-continuous, then*
(*i*) *the greatest fixed point of $\hat{\Sigma}_I$ is $\nu(\hat{\Sigma}_I)$ which is equal to $\overrightarrow{(L^\infty(\Sigma))_I}$,*
(*ii*) *For every $i \leq k$ and $\vec{e} \in E_I^{n_i}$* $\mathrm{Val}_I(\phi(\vec{e})) = \nu(\hat{\Sigma}_I)_i(\vec{e})$.

Proof. Point (i) is a direct consequence of Propositions 6.4 and 6.6.

Point (ii) is a direct consequence of Theorem 5.9, since if I is uniformly rel-continuous it is continuous at any point, and since $\mathrm{Val}_\Sigma(\phi_i) = L_i^\infty(\Sigma)$.

Acknowledgment

We gratefully acknowledge the help of Luc Boasson and Bruno Courcelle who contributed many ideas and constructive criticisms at various stages of the elaboration of this work.

The second author was given the opportunity of exposing the ideas which lead to the present paper in a number of circumstances:
- Séminaire avancé de Sémantique d'Antibes, Septembre 77;
- Semester on Universal Algebra of the Stefan Banach Mathematic Center, Varsovie, May 78;
- Third Course on Mathematics of Computation, Mathematisch Centrum Amsterdam, August 78.

Let the organizers of these schools and seminars be heartily thanked.

References

[1] A. Arnold and M. Nivat, Non deterministic recursive program schemes, in: *Fundamentals of Computation Theory*, Lecture Notes in Computer Science **56** (Springer, Berlin, 1977) 12–21.

[2] A. Arnold and M. Nivat, Algebraic semantics of non deterministic recursive program schemes, *Math. Systems Theory*, to appear.

[3] A. Arnold and M. Nivat, Interpretations métriques de schémas de programme, *Proc. 1er Colloque AFCET-SMF de Mathématiques appliquées*, Ecole Polytechnique Vol. **1** (1978), 191–208.

[4] A. Arnold and M. Nivat, The metric space of infinite trees. Algebraic and topological properties, Rapport IRIA-Laboria No. 323 (1978).

[5] B. Courcelle and M. Nivat, Algebraic families of interpretations, *Proc. 17th Symposium on Foundations of Computer Science*, Houston (1976) 137–146.

[6] B. Courcelle and M. Nivat, The algebraic semantics of recursive program schemes, *Proc. 7th Symposium on Mathematical Foundations of Computer Science*, Lecture Notes in Computer Science **64** (Springer, Berlin, 1978) 16–30.

[7] J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

[8] J.A. Goguen and J.W. Thatcher, Initial algebra semantics, *Proc. 15th Symposium on Switching and Automata Theory*, New Orleans (1974) 63–77.

[9] J.A. Goguen, J.W. Thatcher, E. Wagner and J.B. Wright, Initial algebra semantics and continuous algebras. *J. ACM* **24** (1977) 68–95.

[10] K. Kuratowski, *Topology* (Academic Press, New York, 1966).

[11] M. Nivat, On the interpretation of polyadic recursive program schemes, *Symposia Mathematica* **15** (1975) 255–281.

[12] M. Nivat, Interprétation universelle d'un schéma de programme récursif, *Riv. Informatica* **7** (1977) 9–16.

[13] D. Park, Fixpoint induction and proof of program properties, *Machine Intelligence* **5** (1969) 59–78.

[14] G. Plotkin, A powerdomain construction, *SIAM J. Comput* **5** (1976) 452–487.

[15] D. Scott, The lattice of flow diagrams, *Proc. Symposium on Semantics of Algorithm Languages*, Lecture Notes in Mathematics **182** (Springer, Berlin, 1971) 311–366.

[16] D. Scott, Outline of a theory of computation, Oxford Programming Research Group, Memo No. 2 (1972).

[17] M.B. Smyth, Power domains, *J. Comput. System Sci.* **16** (1978) 23–36.