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A modified Rayleigh conjecture for static problems

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Abstract

A modified Rayleigh conjecture (MRC) in scattering theory was proposed and justified by the author [A.G. Ramm, Modified Rayleigh conjecture and applications, J. Phys. A 35 (2002) L357–L361]. The MRC allows one to develop efficient numerical algorithms for solving boundary-value problems. It gives an error estimate for solutions. In this paper the MRC is formulated and proved for static problems.

MSC: 35R30

Keywords: Modified Rayleigh conjecture; Boundary-value problems

1. Introduction

Consider a bounded domain $D \subset \mathbb{R}^n$, $n = 3$, with a boundary S. The exterior domain is $D' = \mathbb{R}^3 \setminus D$. Assume that S is Lipschitz. Let S^2 denote the unit sphere in $\mathbb{R}^3$. Consider the problem

$$\nabla^2 v = 0 \text{ in } D', \quad v = f \text{ on } S, \tag{1.1}$$

$$v := O\left(\frac{1}{r}\right) \quad r := |x| \to \infty. \tag{1.2}$$

Let $\frac{x}{r} := \alpha \in S^2$. Denote by $Y_\ell(\alpha)$ the orthonormal spherical harmonics $Y_\ell = Y_{\ell m}$, $-\ell \le m \le \ell$. Let $h_\ell := \frac{Y_\ell(\alpha)}{r^{\ell+1}}$, $\ell \ge 0$, be harmonic functions in D'. Let the ball $B_R := \{x : |x| \le R\}$ contain D.

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In the region $r > R$ the solution to (1.1) and (1.2) is

$$v(x) = \sum_{\ell=0}^{\infty} c_\ell h_\ell, \qquad r > R, \tag{1.3}$$

the summation in (1.3) and below including summation with respect to m, $-\ell \le m \le \ell$, and c_ℓ being some coefficients determined by f.

The series (1.3) in general does not converge up to the boundary S. Our aim is to give a formulation of an analog of the modified Rayleigh conjecture (MRC) from [1], which can be used in numerical solution of the boundary-value problems. The author hopes that the MRC method for static problems can be used as a basis for an efficient numerical algorithm for solving boundary-value problems for Laplace equations in domains with complicated boundaries. In [4] such an algorithm was developed on the basis of the MRC for solving boundary-value problems for the Helmholtz equation. Although the boundary integral equation methods and finite element methods are widely and successfully used for solving these problems, the method based on the MRC proved to be competitive and often superior to the currently used methods.

We discuss the Dirichlet condition but a similar argument is applicable to the Neumann and Robin boundary conditions. Boundary-value problems and scattering problems in rough domains were studied in [3].

Let us present the basic results on which the MRC method is based.

Fix $\epsilon > 0$, an arbitrary small number.

Lemma 1.1. *There exist $L = L(\epsilon)$ and $c_\ell = c_\ell(\epsilon)$ such that*

$$\left\| \sum_{\ell=0}^{L(\epsilon)} c_\ell(\epsilon) h_\ell - f \right\|_{L^2(S)} \le \epsilon. \tag{1.4}$$

If (1.4) and the boundary condition (1.1) hold, then

$$\|v_\epsilon - v\|_{L^2(S)} \le \epsilon, \quad v_\epsilon := \sum_{\ell=0}^{L(\epsilon)} c_\ell(\epsilon) h_\ell. \tag{1.5}$$

Lemma 1.2. *If (1.4) holds then*

$$\|v_\epsilon - v\| = O(\epsilon) \quad \epsilon \to 0, \tag{1.6}$$

where $\|\cdot\| := \|\cdot\|_{H^m_{loc}(D')} + \|\cdot\|_{L^2(D';(1+|x|)^{-\gamma})}$, $\gamma > 1$, $m > 0$ is an arbitrary integer, and H^m is the Sobolev space.

In particular, (1.6) implies

$$\|v_\epsilon - v\|_{L^2(S_R)} = O(\epsilon) \quad \epsilon \to 0. \tag{1.7}$$

Let us formulate an analog of the modified Rayleigh conjecture (MRC):

Theorem 1 (*MRC*). *For an arbitrary small $\epsilon > 0$ there exist $L(\epsilon)$ and $c_\ell(\epsilon)$, $0 \le \ell \le L(\epsilon)$, such that (1.4) and (1.6) hold.*

Theorem 1 follows from Lemmas 1.1 and 1.2.

For the Neumann boundary condition one minimizes $\|\frac{\partial[\sum_{\ell=0}^{L} c_\ell \psi_\ell]}{\partial N} - f\|_{L^2(S)}$ with respect to c_ℓ. Analogs of Lemmas 1.1 and 1.2 are valid and their proofs are essentially the same.

If the boundary data $f \in C(S)$, then one can use the $C(S)$-norm in (1.4)–(1.7), and an analog of Theorem 1 then follows immediately from the maximum principle.

In Section 2 we discuss the usage of the MRC in solving boundary-value problems. In Section 3 proofs are given.

2. Solving boundary-value problems using the MRC

To solve problem (1.1) and (1.2) using the MRC, fix a small $\epsilon > 0$ and find $L(\epsilon)$ and $c_\ell(\epsilon)$ such that (1.4) holds. This is possible by Lemma 1.1 and can be done numerically by minimizing $\|\sum_0^L c_\ell h_\ell - f\|_{L^2(S)} := \phi(c_1, \ldots, c_L)$. If the minimum of ϕ is larger than ϵ, then increase L and repeat the minimization. Lemma 1.1 guarantees the existence of such L and such c_ℓ that the minimum is less than ϵ. Choose the smallest L for which this happens and define $v_\epsilon := \sum_{\ell=0}^{L} c_\ell h_\ell$. Then, by Lemma 1.2, v_ϵ is the approximate solution to problem (1.1) and (1.2) with the accuracy $O(\epsilon)$ in the norm $\|\cdot\|$.

3. Proofs

Proof of Lemma 1.1. We start with a claim:

Claim. *The restrictions of harmonic functions h_ℓ on S form a total set in $L^2(S)$.*

Lemma 1.1 follows from this claim. Let us prove the claim. Assume the contrary. Then there is a function $g \neq 0$ such that $\int_S g(s)h_\ell(s)\mathrm{d}s = 0$ $\forall \ell \geq 0$. This implies $V(x) := \int_S g(s)|x-s|^{-1}\mathrm{d}s = 0$ $\forall x \in D'$. Thus $V = 0$ on S, and since $\Delta V = 0$ in D, one concludes that $V = 0$ in D. Thus $g = 0$ by the jump formula for the normal derivatives of the simple layer potential V. This contradiction proves the claim. Lemma 1.1 is proved. □

Proof of Lemma 1.2. By Green's formula one has

$$w_\epsilon(x) = \int_S w_\epsilon(s) G_N(x,s)\mathrm{d}s, \quad \|w_\epsilon\|_{L^2(S)} < \epsilon, \quad w_\epsilon := v_\epsilon - v. \tag{3.1}$$

Here N is the unit normal to S, pointing into D', and G is the Dirichlet Green's function of the Laplacian in D':

$$\nabla^2 G = -\delta(x-y) \text{ in } D', \quad G = 0 \text{ on } S, \tag{3.2}$$

$$G = O\left(\frac{1}{r}\right), \quad r \to \infty. \tag{3.3}$$

From (3.1) one gets (1.7) and (1.6) with the $H^m_{loc}(D')$-norm immediately by the Cauchy inequality. Estimate (1.6) in the region $B'_R := \mathbb{R}^3 \setminus B_R$ follows from the estimate

$$|G_N(x,s)| \leq \frac{c}{1+|x|}, \quad |x| \geq R. \tag{3.4}$$

In the region $B_R \setminus D$ estimate (1.6) follows from local elliptic estimates for $w_\epsilon := v_\epsilon - v$, which imply that

$$\|w_\epsilon\|_{L^2(B_R \setminus D)} \le c\epsilon. \tag{3.5}$$

Let us recall the elliptic estimate we have used. Let $D'_R := B_R \setminus D$ and S_R be the boundary of B_R. Recall the elliptic estimate for the solution to the homogeneous Laplace equation in D'_R (see [2, p. 189]):

$$\|w_\epsilon\|_{H^{0.5}(D'_R)} \le c[\|w_\epsilon\|_{L^2(S_R)} + \|w_\epsilon\|_{L^2(S)}]. \tag{3.6}$$

The estimates $\|w_\epsilon\|_{L^2(S_R)} = O(\epsilon)$, $\|w_\epsilon\|_{L^2(S)} = O(\epsilon)$, and (3.6) yield (1.6). Lemma 1.2 is proved. □

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