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Positivity Conditions for Quadratic Forms and Applications

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1. INTRODUCTION

In this work we are concerned with the quadratic form

$$J(q, V, D) = \int_D [|\overrightarrow{\nabla V}|^2 + q(\boldsymbol{x})V^2]\, d\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x}$ is, in general, and n-dimensional variable and $D \subset \mathfrak{R}^n$ is bounded with a piecewise smooth boundary.

We will give various conditions on the functions $q(\boldsymbol{x})$, $V(\boldsymbol{x})$, and on the domain D that will imply positivity of the quadratic form. Of course, if $q(\boldsymbol{x})$ is positive for all $\boldsymbol{x}$, then it is clear that $J(q, V, D)$ will also be positive. However, the results given below will investigate the extent to which $q(\boldsymbol{x})$ may be negative and still maintain positivity of $J(q, V, D)$. We will provide large classes of easily identified functions $q(\boldsymbol{x})$, for which $J(q, V, D) \geq 0$. In the case where $\boldsymbol{x}$ is a one dimensional variable, such a study has

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been performed by Barnes [6]. Similar problems have also been studied by Nehari [15].

Let D be any three dimensional domain so that $\boldsymbol{x} = (x, y, z)$. We will first give an example of the type of result we have in mind. Suppose that D has volume equal to that of a sphere of radius 4 (approximately 268.0825...), that $|q(\boldsymbol{x})| \leq 1$, and that

$$\int_D q(\boldsymbol{x})\, d\boldsymbol{x} \geq 152.849\ldots \tag{2}$$

After a short computation, it follows from Theorem 3 below that, for all such functions $q(\boldsymbol{x})$ and for all V in the Sobolev space $W_0^{1,2}(D)$, the quadratic form $J(q, V, D)$ is positive:

$$J(q, V, D) = \int_D [|\overrightarrow{\nabla V}|^2 + q(\boldsymbol{x})V^2]\, d\boldsymbol{x} \geq 0. \tag{3}$$

The inequality is sharp and equality holds when D is a sphere and $q(\boldsymbol{x})$ is a certain step function taking on only values of ± 1. Thus, $q(\boldsymbol{x})$ may be negative for a sizable portion of its domain. However, according to (2), its positive values must dominate overall. For example, if $q(\boldsymbol{x}) = +1$ over more than 79% of the domain and $q(\boldsymbol{x}) = -1$ elsewhere, then it is easy to verify (2) so (3) holds.

A similar result holds in two dimensions. Theorem 1 shows that, if D is a two dimensional domain (so that $\boldsymbol{x} = (x, y)$) having area 4π ($\approx 12.566\ldots$) and if

$$|q(\boldsymbol{x})| \leq 2 \qquad \text{and} \qquad \int_D q\, d\boldsymbol{x} \geq 8.42467\cdots,$$

then for all functions V in the Sobolev space $W_0^{1,2}(D)$ it follows that

$$J(q, V, D) = \int_D [|\overrightarrow{\nabla V}|^2 + q(\boldsymbol{x})V^2]\, d\boldsymbol{x} \geq 0. \tag{4}$$

Another short calculation shows that if $q(\boldsymbol{x}) = +1$ over more than 84% of the domain and $q(\boldsymbol{x}) = -1$ elsewhere, then (3) holds.

These results can be thought of as higher dimensional relatives of the classical one dimensional Wirtinger inequality [12]:

$$\text{if } \int_0^{2\pi} y(x)\, dx = 0 \qquad \text{then } \int_0^{2\pi} [y'^2 - y^2]\, dx > 0 \tag{5}$$

unless $y = A \cos x + B \sin x$.

We will also apply our results to establish the existence of a solution of Dirichlet's problem for elliptic partial differential equations of the form $\nabla^2 V + q(\mathbf{x})V = 0$ where the function $q(\mathbf{x})$ may change its sign. Such problems are called indeterminate.

2. Positivity Conditions for Two-Dimensional Quadratic Forms

Suppose, first of all, that we consider the two dimensional case where $\mathbf{x} = (x, y)$. The partial differential equation for a two dimensional eigenvalue problem

$$V_{xx} + V_{yy} + [\lambda - q(x, y)]V = 0 \qquad \text{with } V(x, y) = 0 \text{ on } \partial D \tag{6}$$

is closely associated with the quadratic form (1). It is well known that the first eigenvalue $\lambda_1(q)$ of (6) is characterized as the minimum of

$$\lambda_1(q) = \min_{V \in W_0^{1,2}(D)} J(q, V, D) = \min_{\substack{V \in W_0^{1,2}(D) \\ |V|_2 = 1}} \int_D [V_x^2 + V_y^2 + q(x, y)V^2]\, dx\, dy$$

where the minimum is taken over all normalized functions $V(x, y)$ in the Sobolev space $W_0^{1,2}(D)$ of functions that vanish on ∂D. In view of this fact, it follows that the quadratic form $J(q, V, D)$ is positive definite if and only if $\lambda_1(q) > 0$.

We will first consider a special case of (6) where the domain is a circle D^* having the same area as D, a radius R, and the coefficient function a constant, say $Q_{R,h}(x, y) \equiv -h^2$. It is easy to see that, if h^2 is large enough, then $\lambda_1(Q_{R,h}) < 0$. It follows from a monotonicity argument that $\lambda_1(Q_{R,h})$ is a decreasing function of h. Since $\lambda_1(Q_{R,0}) > 0$, it follows that there is a unique value of h, call it h_0, satisfying $\lambda_1(Q_{R,h_0}) = 0$. A direct solution of (6) shows that $h_0 = j_0/R$ where j_0 is the smallest root of the Bessel function $J_0(r)$. This yields the first and most elementary positivity condition; if $q(x, y)$ is any function defined on D^* that satisfies $q(x, y) \geq -(j_0/R)^2$, then $J(q, V, D^*) \geq 0$ for all functions $V \in W_0^{1,2}(D)$. But we will show that this result may be considerably improved.

Generalizing the function $Q_{R,h}(x, y)$, suppose that h, H are given constants and consider the eigenvalue problem (6) for the step function $q_s(x, y)$ defined on D^* by

$$q_s(x, y) = \begin{cases} -h^2 & \text{if } x^2 + y^2 \leq s^2 \\ H^2 & \text{if } s^2 < x^2 + y^2 \leq R^2. \end{cases} \tag{7}$$

Here, we require that $0 \leq s \leq R$ and $\pi R^2 = A$, the area of D^*. Note that the choice $s = R$ yields the previous function $Q_{R,h}(x, y)$.

Suppose that $h > j_0/R$ so that it is possible to have $\lambda_1(q_s) < 0$. By monotonicity $\lambda_1(q_s)$ is a decreasing function of s. However, $\lambda_1(q_0) > 0$ and $\lambda_1(q_R) < 0$. Thus, there is a unique value of $s \in [0, R]$, call it s_0, such that $\lambda_1(q_{s_0}) = 0$. A direct solution of (8) (using the Bessel functions $I_n(z)$, $J_n(z)$, $K_n(z)$) and taking account of the jump discontinuity in $q_s(x, y)$ show that $s_0 = s_0(h, H, R)$ is the largest root of the equation

$$\begin{aligned} &H J_0(hs)[I_1(HS)K_0(HR) + I_0(HR)K_1(HS)] \\ &\quad + hJ_1(HS)[I_0(HS)K_0(HR) - I_0(HR)I_0(HR)K_0(HS)] = 0 \end{aligned} \tag{8}$$

satisfying $0 \leq s_0 \leq R$. This yields a second positivity condition; if $q(x, y)$ is any function defined on D^* that satisfies $q(x, y) \geq q_{s_0}(x, y)$, then $J(q, V, D^*) \geq 0$ for all functions $V \in W_0^{1,2}(D)$. However, we will show that this little result may still be vastly improved by using rearrangement inequalities in the style of [18, 13, 4].

To introduce the ideas, suppose that h, H, and m are given constants satisfying $-h^2 \leq m \leq H^2$. Let D be a bounded domain in the x, y-plane and let $C(h, H, m, D)$ be the set of all functions $q(x, y)$ defined on D satisfying

$$-h^2 \leq q(x, y) \leq H^2 \qquad \text{and} \qquad \int_D q(x, y)\, dx\, dy = m.$$

Define the function $F(h, H, R, s)$ by

$$F(h, H, R, s) = \int_{D^*} q_s(x, y)\, dx\, dy = \pi[H^2R^2 - (H^2 + h^2)s^2].$$

Given a function $f(x, y)$ defined on D, the rearrangements of it into symmetrically increasing and decreasing order will be denoted by $\hat{f}^+(x, y)$ and $\hat{f}^-(x, y)$, respectively. They are defined on the circular domain D^* which has the same area as D and will satisfy the following conditions.

1. Both functions $\hat{f}^\pm(x, y)$ have circular symmetry,

$$\hat{f}^\pm(x, y) = \hat{f}^\pm(r), \qquad \text{for } 0 \leq r \leq R \text{ where } r = \sqrt{x^2 + y^2}.$$

2. $\hat{f}^+(r)$ is a nondecreasing and $\hat{f}^-(r)$ a nonincreasing function of r.

3. Both functions are equimeasurable to $f(x, y)$. That is, denoting by $A(z)$ the area of the subset of D for which $f(x, y) \geq z$ and similarly denoting by $A^+(z)$ and $A^-(z)$ the area of the set in D^* for which $\hat{f}^+(x,$

$y) \geq z$ and $\hat{f}^-(x, y) \geq z$, respectively, then

$$A(z) = A^+(z) = A^-(z) \qquad \text{for all } z.$$

4. The two rearrangements are connected by the formula

$$\hat{f}^-(r) = \hat{f}^+(\sqrt{R^2 - r^2}), \qquad 0 \leq r \leq R.$$

5. Almgren and Lieb [2] show that symmetrization is a continuous operation when applied to smooth functions. This means that the decreasing rearrangement of an eigenfunction of (6), say $\hat{V}_1^-$, will also vanish on the boundary of D^* and it will belong to $W_0^{1,2}(D)$. Furthermore, the following inequalities are well known [18, 4]:

$$\begin{aligned} \int_D |\vec{\nabla} f|^2\, dx\, dy &\geq \int_{D^*} |\vec{\nabla} \hat{f}^-|^2\, dx\, dy, \\ \int_D f^2 g\, dx\, dy &\geq \int_{D^*} (\hat{f}^-)^2 \hat{g}^+\, dx\, dy. \end{aligned} \tag{9}$$

We are now in a position to state a theorem.

THEOREM 1. *Suppose that D is any two dimensional domain having area πR^2, that $V(x, y) \in W_0^{1,2}(D)$, that $q(x, y) \in C(h, H, m, D)$, and that s_0 is determined by* (8).
Then if $0 \leq h < j_0/R$, it follows that

$$J(q, V, D) = \int_D [V_x^2 + V_y^2 + q(x, y)V^2]\, dx\, dy > 0. \tag{10}$$

If, however, $h \geq j_0/R$ and in addition

$$\int_D q(x, y)\, dx\, dy \geq F(h, H, R, s_0), \tag{11}$$

then (10) *still holds for all such functions $q(x, y)$ and V. Furthermore, the inequality is sharp, and equality is attained when D is the circle D^* and $q_{s_0}(x, y)$ is defined by* (7).

Proof. We will apply symmetrization to the first eigenfunction $V_1(x, y)$. Since $V_1 \in W_0^{1,2}(D)$, is follows [2] that $\hat{V}_1^- \in W_0^{1,2}(D)$ and

$$J(q, V, D) \geq \lambda_1(q, D) = J(q, V_1, D) \geq J(\hat{q}^+, \hat{V}_1^-, D^*) \geq \lambda_1(\hat{q}^+, D^*).$$

Next, select $s \in [0, R]$ so that

$$\int_{D^*} q_s(x, y)\, dx\, dy = \int_{D^*} \hat{q}^+(x, y)\, dx\, dy.$$

This condition and the fact that $-h^2 \leq \hat{q}^+(r) \leq H^2$ implies that

$$\int_0^r q_s(t)t\,dt \leq \int_0^r \hat{q}^+(t)t\,dt \qquad \forall r \in [0, R]. \tag{12}$$

To prove this inequality, we define a function $G(x)$ by

$$G(x) = \int_0^x [q_s(t) - \hat{q}^+(t)]\,t\,dt$$

and note that $G(0) = G(R) = 0$. Furthermore, the bounds on $q(x, y)$ show that $G'(r) \leq 0$ for $0 \leq r \leq s$ while $G'(r) \geq 0$ for $s \leq r \leq R$. This implies that $G(x) \leq 0$ for all x, proving (12).

Integrating by parts, we see that

$$\int_0^R [\hat{q}^+(r) - q_s(r)][\hat{V}_1^-(r)]^2 r\,dr$$
$$= -\int_0^R \frac{d}{dr}[\hat{V}_1^-(r)]^2 \int_0^r [\hat{q}^+(t) - q_s(t)]t\,dt\,dr.$$

Since $\hat{V}_1^-$ is decreasing, the right side of this equation is nonnegative. This, the minimum principle, and (9) give

$$\lambda_1(q) = J(q, V_1) \geq J(\hat{q}^+, \hat{V}_1^-) \geq J(q_s, \hat{V}_1^-) \geq \lambda_1(q_s).$$

The first part of the theorem, where $h < j_0/R$, from this inequality and from the comments made on page 3 above. If, however, $h \geq j_0/R$, then it is possible for $\lambda_1(q_s)$ to be negative for some choices of s. Now $q_s(x, y)$ is a strictly decreasing function of s, and Eq. (11) implies that $s \geq s_0$. Therefore, there is a unique value of s_0 for which $\lambda_1(q_{s_0}) = 0$ so that $\lambda_1(q) \geq \lambda_1(q_s) \geq \lambda_1(q_{s_0}) = 0$. ∎

A second proof of this theorem can be given, using ideas given by Banks [5]. Although his theorem was stated only for the vibrating membrane problem, he observed that it can be easily generalized to a large class of other problems. The following theorem is one such generalization.

THEOREM 2 (Banks). *Let $p(x, y)$ and $q(x, y)$ be nonnegative real continuous functions defined in a domain D with a piecewise smooth boundary C such that*

$$\int_D p(x, y)\,dx\,dy = \int_D q(x, y)\,dx\,dy.$$

Consider the eigenvalue problems

$$\nabla^2 u + (\lambda - p(x, y))u = 0 \qquad u \equiv 0 \text{ on } C. \tag{13}$$

$$\nabla^2 v + (\mu - q(x, y))v = 0 \qquad v \equiv 0 \text{ on } C. \tag{14}$$

Let $v_1(x, y)$ denote the eigenfunction corresponding to the lowest eigenvalue $\mu_1(q)$ of (14) and define

$$A(z) = \{(x, y) \mid v_1^2(x, y) \geq z\}.$$

If $\int_{A(z)} (p - q)\, dx\, dy \geq 0$ for all $z \geq 0$, then

$$\lambda_1 \leq \mu_1$$

where $\lambda_1(p)$ is the lowest eigenvalue of (13).

It is clear that, for any given domain D and function $q(x, y) \in C(h, H, m, D)$, there exists a function, say $q^*(x, y)$, that takes on only values of $-h^2$ or H^2 and satisfies the inequality $\int_{A(z)} (q^* - q)\, dx\, dy \geq 0$ for all $z \geq 0$. Thus, Theorem 2 shows that $\lambda_1(q) \geq \lambda_1(q^*)$. However, the increasing rearrangement of any such function q^* is simply a function $q_s(x, y)$ of the form of Eq. (7) giving the second proof. ▌

Next, we will use some Bessel functions to give explicit solution formulae for the problem. Using radial symmetry, the two dimensional problem reduces to a one dimensional problem for $V = V(r)$ (where $r^2 = x^2 + y^2$) of the form

$$V_{rr} + \frac{1}{r}V_r + (\lambda - \hat{q}_s)V = 0 \qquad \text{with } V'(0) = V(R) = 0. \tag{15}$$

We now solve the differential equation (15) in each of the intervals $[0, s_0]$ and $[s_0, R]$, and use the condition that the eigenfunction has a continuous derivative at $r = s_0$. This process yields the equation (8).

Consider now the claims made in (4) above. Setting $h = H = 1$, $R = 2$, and making a short computer calculation using (8) shows that $s_0 \approx 0.81189 \cdots$, so that $F(1, 1, 2, 0.81189 \cdots) \approx 8.42467 \cdots$ This computation justifies Eq. (4).

This result has also been verified independently by using the software package *Sleign*. It is a well respected set of routines for computing eigenvalues of second order Sturm–Liouville equations [3]. It even handles singular problems such as the one considered here. A subroutine based on *Sleign* was written that computes $\lambda_1(q_s)$ as a function of s and then solves the equation $\lambda_1(q_s) = 0$ for s_0 using bisection. The numerical results were identical to those obtained by solving (8) directly.

2.1. The Three-Dimensional Case

Consider the three dimensional eigenvalue problem

$$V_{xx} + V_{yy} + V_{zz} + [\lambda - q(x, y, z)]V = 0$$
$$\text{where } \mathbf{x} = (x, y, z) \in D \subset \mathfrak{R}^3.$$

Let D^* be a sphere having the same volume as D and define a function $q_s(x, y, z)$, analogous to (7), on D^* by

$$q_s(x, y, z) = \begin{cases} -h^2, & \text{if } 0 \leq x^2 + y^2 + z^2 \leq s^2, \\ H^2, & \text{if } s^2 < x^2 + y^2 + z^2 \leq R^2. \end{cases} \tag{16}$$

Here, s is selected so that

$$\int_D q_s(x, y, z)\, dx\, dy\, dz = \int_{D^*} q(x, y, z)\, dx\, dy\, dz.$$

Let R be the radius of D^*. The three dimensional equations analogous to (8) are given by

$$G(h, H, s, R) = \int_D q_s(x, y, z)\, dx\, dy\, dz = \frac{4\pi}{3}[(h^2 + H^2)s^3 - h^2R^3].$$

It is easy to see that, if $h < \pi/R$, then $\lambda_1(q) > 0$. However, if $h \geq \pi/R$, then there will exist a unique value of s_0 that makes $\lambda_1(q_s) = 0$. Since the root must correspond to the first eigenvalue, it must be the largest root in the interval $[0, R]$ of the equation

$$\begin{aligned} \sin(hs_0)\{\sinh[H(R - s_0)] + s_0 H \cosh[H(R - s_0)]\} \\ = \sinh[H(R - s_0)]\{\sin(hs_0) - s_0 h \cos(hs_0)\}. \end{aligned} \tag{17}$$

THEOREM 3. *Suppose that the volume of D is $\frac{4}{3}\pi R^3$, that $V(x, y, z) \in W_0^{1,2}(D)$, that $q(x, y, z) \in C(h, H, m, D)$, that s_0 is determined by* (17), *and that $0 \leq h \leq \pi/R$. Then*

$$J(q, V, D) = \int_D [V_x^2 + V_y^2 + V_z^2 + q(x, y, z)V^2]\, dx\, dy\, dz > 0. \tag{18}$$

However, if $h \geq \pi/R$ and in addition

$$\int_D q(x, y, z)\, dx\, dy\, dz \geq G(h, H, R, s_0),$$

then (18) *still holds for all such functions* $q(x, y, z)$ *and* V. *Furthermore, the inequality is sharp, and equality is attained when* D *is the sphere* D^* *and* $q_{s_0}(x, y, z)$ *is defined by* (16).

Consider now the claims made in (3) above. Setting $h = H = 2$, and $R = 4$ and solving (17) shows that $s_0 \approx 2.396 \cdots$, so that $G(2, 2, 4, 2.396) \approx 152.849 \cdots$ This computation justifies the remarks made in Eq. (3) above.

As in the two dimensional case, this result has also been verified independently by using the software package *Sleign*. The numerical results were identical to those obtained by solving (17) directly.

2.2. Remarks on the Second Eigenvalue.

We will confine our remarks to the two dimensional case; however, the methods could easily be used to deal with any number of dimensions. The basic idea has its roots in the work of Pólya [17]. We will denote the second eigenvalue of (6) by $\lambda_2(D, q)$. The corresponding eigenfunction, $V_2(x, y)$, will have exactly two nodal domains, say $D = D_1 \cup D_2$. We now consider the first eigenvalue of the problem (6) when the domain is restricted to D_1 with Dirichlet boundary conditions and with coefficient function $q(x, y)$. It follows that the first eigenvalue of D_1 is equal to the second eigenvalue computed over all of D and that the first eigenfunction of D_1 is nothing more than the function V_2 restricted to D_1, and similar remarks hold for D_2. We will use this observation to obtain positivity conditions for the second eigenvalue. Such a construction will not work for eigenvalues of order greater than two since, in these cases, the exact number of nodal domains is unknown.

First, we note that the previous results for the first eigenvalue imply that

$$\lambda_2(D, q) = \lambda_1(D_1, q) = \lambda_1(D_2, q) \geq \max\{\lambda_1(D_1^*, q_1), \lambda_1(D_2^*, q_2)\}.$$

Here, D_1^* and D_2^* are two circular domains obtained using the rearrangement theorems given in the previous section and q_i are the radially symmetric piecewise constant functions analogous to (7). The second eigenvalue of the disconnected domain $D_1^* \cup D_2^*$ is $\max\{\lambda_1(D_1^*, q_1), \lambda_1(D_2^*, q_2)\}$.

The smallest possible value for this second eigenvalue will occur when D_1^* and D_2^* are congruent and the two symmetric functions q_i defined on each of the two domains D_1 and D_2 are translates of each other. We use the notation $D_{1/2}^*$ to denote a circular domain having an area $\frac{1}{2}$ that of D and $q_{1/2}$ to mean the corresponding radially symmetric function defined on $D_{1/2}^*$ analogous to Eq. (7) having

$$\int_{D_{1/2}^*} q_{1/2}(x, y)\, dx\, dy = \frac{1}{2} \int_D q(x, y)\, dx\, dy.$$

Thus we have shown the following.

THEOREM 4. *Let $\lambda_2(D, q)$ denote the second eigenvalue of* (6) *on D. Then*

$$\lambda_2(D, q) \geq \lambda_1(D_{1/2}^*, q_{1/2}).$$

Therefore, a positivity condition for λ_2 can be obtained from a corresponding condition for λ_1.

The second eigenvalue satisfies the variational condition

$$\lambda_2(q) = \min J(V, q, D) = \min \int_D [V_x^2 + V_y^2 + q(x, y)V^2]\, dx\, dy.$$

Here, the minimum is over all $V \in W_0^{1,2}(D)$ that satisfy $(V, V) = 1$ and $(V, V_1) = 0$ where V_1 is the first eigenfunction. Thus, it follows that if $\lambda_1(D_{1/2}^*, q_{1/2}) \geq 0$ and if $V = 0$ on ∂D then

$$\int_D [V_x^2 + V_y^2 + q(x, y)V^2]\, dx\, dy > 0 \qquad \text{for all } V \text{ satisfying } (V, V_1) = 0.$$

This shows a more direct connection to the one dimensional Wirtinger inequality than the results given in Theorems 1 and 3 above.

3. Dirichlet's Problem with Indeterminate Coefficient Functions

As an application of Theorem 3 we will establish the existence of a solution of Dirichlet's problem for a linear elliptic partial differential equation of the second order in three dimensions:

$$U_{xx} + U_{yy} + U_{zz} - q(x, y, z)U = 0. \tag{19}$$

Here we allow the coefficient q to have any sign. Such problems are called indeterminate.

Theorem 3 gives conditions that make the first eigenvalue $\lambda_1(q)$ of (19) positive, and it turns out that this is all that is needed to prove the existence theorem.

THEOREM 5. *Suppose that the $\lambda_1(q)$ is positive. Then the Dirichlet problem*

$$U_{xx} + U_{yy} + U_{zz} - q(x, y, z)U = 0$$
$$\text{with } U(x, y, z) = f(x, y, z) \text{ on } \partial D$$

has a unique solution $U(x, y, z)$ in D.

The proof of this theorem can be obtained simply by direct imitation of that given by Garabedian [11] where it was assumed that $q > 0$. However, the only use made of this assumption was to show that, if the inner product and norm are defined by

$$(u, v) = \int_D [u_x v_x + u_y v_y + u_z v_z + quv]\, dx\, dy\, dz, \qquad \|u\| = \sqrt{(u, u)}, \quad (20)$$

then one does obtain a valid inner product and norm. However, even if we allow q to be indeterminate but with $\lambda_1 > 0$, then it is easy to see that (20) defines a valid norm and inner product and that the classical proof can still be used. The only condition that requires much checking is that if $\|u\| = 0$, then $u = 0$. This result follows from the minimum property and the positivity of λ_1.

4. POSITIVITY OF THE n-TH EIGENVALUE

Consider for a moment the one dimensional equations

$$u'' + (\lambda - q(x))u = 0 \quad (21)$$

and

$$y'' - q(x)y = 0. \quad (22)$$

Theorems guaranteeing the positivity of the first eigenvalue of (21) for various boundary conditions have been given [6]. Such studies were first initiated by Nehari [15]. However, he considered the equation $y'' + \mu p(x)y = 0$ and obtained some results connecting oscillation theory for the equation $z'' + p(x)z = 0$ for positive $p(x)$ with the assumption that the first eigenvalue satisfied $\mu_1 \geq 1$. In this work, we consider the equation (21) instead since, in this case, there is no requirement that the coefficient $q(x)$ be positive.

It appears that such positivity conditions for the higher eigenvalues of

(21) have not been considered. The methods used here are quite different from those used for the first eigenvalue. We will give the details only for Dirichlet boundary conditions.

To begin the study, let us define the quadratic form associated with (21)

$$J(q, u) = \int_a^b [(u')^2 + q(x)u^2]\, dx, \text{ and suppose } \int_a^b u^2\, dx = 1.$$

If we impose the orthogonality condition $(u, u_1) = 0$, then the minimum of J is λ_2. The successive minimum problems, $J(q, u)$ = minimum subject to the auxiliary conditions

$$(u, u_1) = 0, (u, u_2) = 0, \cdots, (u, u_{n-1}) = 0,$$

define the eigenfunctions and eigenvalues of (21) with the natural boundary condition $u'(x) = 0$ at $x = a$ and b. If we give the boundary conditions $u(a) = u(b) = 0$ to the equation (21), then the minimums of J are computed over only those functions satisfying these conditions and it yields the corresponding eigenvalue and eigenfunction of (21). Thus, a positivity condition for the nth eigenvalue will yield positivity of the associated quadratic form when subjected to an orthogonality condition.

The following result deals with such positivity conditions for the nth eigenvalue.

THEOREM 6. *Suppose that y_0 is a solution of* (22) *and that u_n is the nth eigenfunction of* (21) *with boundary conditions $u_n(a) = u_n(b) = 0$. Then the following relation holds*:

$$\lambda_n \int_a^b [y_0(x)u_n(x)]^2\, dx = \int_a^b [u_n'^2(x)y_0^2(x) - y_0'^2(x)u_n^2(x)]\, dx. \tag{23}$$

Thus, the nth eigenvalue λ_n is positive if and only if there exists a nontrivial solution $y_0(x)$ of (22) *satisfying the integral inequality*

$$\int_a^b [y_0'^2(x)u_n^2(x) - u_n'^2(x)y_0^2(x)]\, dx \le 0. \tag{24}$$

Proof. Substitute $\lambda_n u_n = qu_n - u_n''$ into the left side of (23) to obtain

$$\begin{aligned} \lambda_n \int_a^b [y_0(x)u_n(x)]^2\, dx &= \int_a^b q y_0^2 u_n^2\, dx - \int_a^b u_n u_n'' y_0^2\, dx \\ &= \int_a^b y_0 y_0'' u_n^2\, dx - \int_a^b u_n u_n'' y_0^2\, dx. \end{aligned} \tag{25}$$

When f, g satisfy appropriate boundary conditions, the one dimensional "Green's formula" $\int fg'' - gf''\,dx = 0$ holds. Use the equations (21), (22) and apply Green's theorem to each of the two integrals in (25). This computation gives the following results:

$$\begin{aligned}\lambda_n \int_a^b [y_0 u_n]^2\,dx &= \int_a^b [y_0 u_n^2 y_0'' - y_0^2 u_n u_n'']\,dx \\ &= \int_a^b y_0 (y_0 u_n^2]''\,dx - \int_a^b u_n [y_0^2 u_n]''\,dx \\ &= 2\int_a^b y_0^2 u_n'^2 - u_n^2 y_0'^2\,dx + \int_a^b y_0^2 u_n [u_n'' - q(x) u_n]\,dx \\ &= 2\int_a^b y_0^2 u_n'^2 - u_n^2 y_0'^2\,dx - \lambda_n \int_a^b y_0^2 u_n^2\,dx.\end{aligned}$$

Finally, solving this equation for the eigenvalue λ_n yields (23). ∎

Now the nth eigenvalue is either positive or else it is not, leading to the following corollary.

COROLLARY 7. *If there exists one nontrivial solution, say* y_0, *of Eq.* (22) *that satisfies* (24), *then* $\lambda_n(q) \geq 0$ *so that every solution* y *of* (22) *satisfies* (24).

It is clear that other kinds of boundary conditions could be used in these theorems with the same result. The only thing necessary is that there exists a nontrivial solution of (22) satisfying (24) as well as the boundary conditions

$$y^2(x) u_n(x) u_n'(x)]_a^b = 0 \qquad \text{and} \qquad y(x) y'(x) u_n^2(x)]_a^b = 0.$$

In particular, various combinations of Dirichlet and Neumann conditions might be used. Now, any condition which implies positivity of an eigenvalue will generate Wirtinger-type inequalities using the Max–Min criteria discussed above.

We will now consider conditions that will insure positivity of the nth eigenvalue for (21). In a classic work, Krein [14] obtained upper and lower bounds for the equation $y'' + \lambda p(x) y = 0$, assuming that constants a, b are given and that $q(x)$ is bounded $a \leq q(x) \leq b$, with Dirichlet boundary conditions given to y. His work was, in a way, a one dimensional version of the ideas given by Pólya and Szegö [18]. These various methods have been extended in many different ways. See, for example, [16, 4] and the references given there. One such extension was given in [8] where the general problem of minimizing the nth eigenvalue of some very general differential operators was considered. It is not difficult to adapt those methods to show that if $-h^2 \leq q(x) \leq H^2$ then $\lambda_n(q) \geq \lambda_n(q_n)$ where the function $q_n(x)$ is defined by the following conditions:

1. The function $q_n(x)$ is periodic in $[0, 1]$ with period $1/n$ so that

$$q_n(x) = q_n(x + 1/n) \qquad \text{for } x, x + 1/n \in [0, 1].$$

2. It is symmetric in each interval $[(i - 1)/n, i/n]$ so that

$$q_n((2i - 1)/2n + t) = q_n((2i - 1)/2n - t)$$

whenever $(2i - 1)/2n + t, (2i - 1)/2n - t \in [(i - 1)/n, i/n]$.

3. For each x, either $q_n(x) = -h^2$ or else $q_n(x) = H^2$.
4. $\int_0^2 q(x)\,dx = \int_0^1 q_n(x)\,dx$.

Since $q_n(x)$ is periodic, it is easy to see that the nth eigenvalue $\lambda_n(q_n)$ of (21) is equal to the first eigenvalue of the equation $w'' + (\tau - q_n(x))w = 0$ when restricted to the subinterval $[(i - 1)/n, i/n]$ and given the boundary conditions $w((i - 1)/n) = w(i/n) = 0$. This makes it easy to obtain a positivity condition for $\lambda_n(q)$ using the corresponding theorems for τ_1.

5. The n-th Eigenvalue for Partial Differential Equations

Analogues of Theorems 6 and 7 hold for partial differential equations, and the proofs of the corresponding theorems are similar. Let D be a two dimensional domain and consider the equations

$$\nabla^2 U + (\lambda - q(x, y))U = 0 \qquad \text{with } U = 0 \text{ on } \partial D \tag{26}$$

and

$$\nabla^2 V - q(x, y)V = 0. \tag{27}$$

THEOREM 8. *Suppose that V is a solution of* (27) *and that U_n is the nth eigenfunction of* (26) *with the boundary condition $U = 0$ on ∂D. Then the following relation holds*:

$$\lambda_n \int_D [V(x, y)U_n(x, y)]^2\,dx$$
$$= \int_D [|\overrightarrow{\nabla V}|^2(x, y)U_n^2(x, y) - [|\overrightarrow{\nabla U_n}|^2(x, y)V^2(x, y)]\,dx.$$

Thus, the nth eigenvalue λ_n is positive if and only if there exists a nontrivial solution V of (27) *for which the following integral inequality holds*:

$$\int_a^b [[|\overrightarrow{\nabla V}|^2(x, y)U_n^2(x, y) - [|\overrightarrow{\nabla U_n}|^2(x, y)] V^2(x, y)] \, dx \leq 0. \tag{28}$$

Furthermore, if there exists one nontrivial solution, say V_0, of Eq. (27) *for which* (28) *holds, then $\lambda_n(q) \geq 0$. Thus,* (28) *must hold for every solution V of* (27).

Theorem 4 above gives a positivity condition for the case $n = 2$. It seems to be much more difficult to obtain positivity conditions for λ_n with $n \geq 3$ since, for partial differential equations, the number of nodal domains may be much smaller than the index n [9].

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