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On the Associated Graded Modules of Canonical Modules

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INTRODUCTION

The *canonical modules* of Cohen–Macaulay rings give a useful technical tool in commutative algebra, algebraic geometry, and singularity theory via various duality theorems. But not much is known about the canonical modules themselves. In this paper, we investigate the structure of the canonical modules, expecially their associated graded modules. We ask the following two questions. Let K be a canonical module over a Cohen–Macaulay local ring R. Then

(1) When is the associated graded module $G(K)$ of K a Cohen–Macaulay (or more specially a canonical) $G(R)$-module?

(2) What can be said about the Hilbert–Samuel function of K?

Section 1 and Section 2 are preliminaries. In Section 1 we introduce certain numerical invariants (*genera* and *reduction exponent*) for maximal Cohen–Macaulay modules and study their properties. In Section 2 we collect some basic facts concerning canonical modules which we shall use later.

In Section 3 we give necessary and sufficient conditions for the associated graded module of a canonical module to be canonical.

In Section 4 we examine the genera and the reduction exponents of canonical modules.

In Section 5 we consider the case of dimension one in detail and give some examples.

Notation and Terminology. All rings in this paper are commutative noetherian rings with unit. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. We denote by $l(M)$, $\mu(M)$, and $e(M)$ the

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length, the minimal number of generators, and the multiplicity of M, respectively. Put $r(M) = \dim_k \operatorname{Ext}_R^d(k, M)$, where $d = \dim(M)$, and $G(M) = \bigoplus_{n \geqslant 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$. A graded ring $A = \bigoplus_{n \geqslant 0} A_n$ with $A_0 = k$ a field is said to be a *homogeneous* k-algebra if it is generated by A_1 over k. Let $M = \bigoplus_{n \in \mathbf{Z}} M_n$ be a finitely generated graded A-module. Then $H(M, n)$ and $F(M, t)$ stand for the Hilbert function $\dim_k(M_n)$ and the Hilbert series $\sum_{n \in \mathbf{Z}} H(M, n)t^n$, respectively. The (Castelnuovo's) *regularity* $\operatorname{reg}(M)$ of M is the least integer m such that M is *m-regular*, i.e., $[H^i_P(M)]_j = 0$ for all i, j such that $i + j > m$, where $P = A_+ = \bigoplus_{n > 0} A_n$. We put $a(M) = \operatorname{reg}(M) - \dim(M)$. If M is Cohen–Macaulay, then $a(M) = \deg F(M, t)$ (cf. [7]).

1. Hilbert Coefficients and Reduction Exponents

In [8, 9], we introduced the notions of *Hilbert coefficients* and various *genera* for noetherian local rings and homogeneous algebras by using their Hilbert functions. Here we extend these invariants to finitely generated modules and prove some of their properties which we shall use later. Especially, we examine the relationship between the Hilbert coefficients and the *reduction exponents* of modules. For more details concerning the theory of reductions and the theory of genera used in this paper, see [6, 10, 8, 9].

Throughout this section, $(R, \mathfrak{m}, k)$ is a d-dimensional Cohen–Macaulay local ring with infinite residue field and M is a Cohen–Macaulay R-module with $\dim(M) = \dim(R)$. A parameter ideal I for R is a *minimal M-reduction* of $\mathfrak{m}$ if $I\mathfrak{m}^n M = \mathfrak{m}^{n+1}M$ for some n. The *reduction exponent* $\delta(M)$ of M is the least integer n such that $I\mathfrak{m}^n M = \mathfrak{m}^{n+1}M$ for some minimal M-reduction I of $\mathfrak{m}$. If $\delta(M) = 0$, then M is called a *Ulrich* R-module.

Proposition 1.1 (cf. [1]). *In general* $\mu(M) \leqslant e(M)$ *and* $r(M) \leqslant e(M)$, *and the following conditions are equivalent*:

(1) *M is a Ulrich R-module.*

(2) $\mu(M) = e(M)$.

(3) $r(M) = e(M)$.

(4) $\operatorname{reg} G(M) = 0$.

(5) $l(M/\mathfrak{m}^{n+1}M) = \mu(M)\binom{n+d}{d}$ *for all* $n \geqslant 0$.

Moreover, if R is a regular local ring, these conditions are also equivalent to the following condition:

(6) *$G(M)$ has a linear resolution as a graded $G(R)$-module.*

Proof. Let I be a minimal M-reduction of $\mathfrak{m}$. Then $e(M) = l(M/IM)$ and $r(M) = l(IM : \mathfrak{m}/IM)$. Hence $e(M) - \mu(M) = l(\mathfrak{m}M/IM)$ and $e(M) -$

$r(M) = l(M/IM : \mathfrak{m})$. Since I is generated by an M-regular sequence, $\bigoplus_{n \geq 0} I^n M/I^{n+1}M$ is isomorphic to $M/IM[X_1, ..., X_d]$. Hence if $\mathfrak{m}M = IM$, then $l(M/\mathfrak{m}^{n+1}M) = \mu(M)\binom{n+d}{d}$ for all $n \geq 0$. From these facts, the conditions (1), (2), (3), and (5) are equivalent each other. By [10, Theorem 5.1], $\delta(M) \leq \operatorname{reg} G(M)$ and the equality holds if $G(M)$ is Cohen–Macaulay. Hence (1) and (4) are equivalent (cf. [1, Corollary 1.6]). Finally, the equivalence of (4) and (6) follows from [7, Theorem 8].

It is well known that there are (uniquely determined) integers $e_i(M) = e_i$, $0 \leq i \leq d$, such that

$$l(M/\mathfrak{m}^{n+1}M) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-d)^d e_d, \qquad n \gg 0.$$

We call $e_i(M)$ the ith *Hilbert coefficient* of M. We define the *sectional genus* $g_s(M)$ and the *Δ-genus* $g_\Delta(M)$ of M by

$$g_s(M) = e_1(M) - e(M) + \mu(M)$$

and

$$g_\Delta(M) = e(M) + (d-1)\mu(M) - l(\mathfrak{m}M/\mathfrak{m}^2M),$$

respectively (cf. [9]). The following proposition can be proved similarly as in [8, Theorem 4.3; 9, Theorem 3.3] (which treat the case $M = R$). So we omit the proof. Note that for any minimal M-reduction I of $\mathfrak{m}$, $g_\Delta(M) = l(\mathfrak{m}^2M/I\mathfrak{m}M)$ and $G(M)$ is Cohen–Macaulay if and only if $IM \cap \mathfrak{m}^{n+1}M = I\mathfrak{m}^nM$ for all n(cf. [14]).

PROPOSITION 1.2. *The genera $g_s(M)$ and $g_\Delta(M)$ are non-negative integers, and the following conditions are equivalent*:

(1) $g_s(M) = 0$.

(2) $g_\Delta(M) = 0$.

(3) $\delta(M) \leq 1$.

(4) $\operatorname{reg} G(M) \leq 1$.

(5) $l(M/\mathfrak{m}^{n+1}M) = e(M)\binom{n+d-1}{d} + \mu(M)\binom{n+d-1}{d-1}$ *for all* $n \geq 0$.

Moreover if these conditions are satisfied, then $G(M)$ is Cohen–Macaulay.

COROLLARY 1.3. (1) $e_1(M) \geq 0$, *and* $e_1(M) = 0$ *if and only if M is a Ulrich R-module.*

(2) $e_1(M) = 1$ *if and only if* $l(M/\mathfrak{m}^{n+1}M) = \mu(M)\binom{n+d}{d} + \binom{n+d-1}{d}$ *for all* $n \geq 1$.

LEMMA 1.4. (1) *$r(R) \leqslant e(R)$ and the equality holds if and only if R is a regular local ring.*

(2) *$r(R) = e(R) - 1$ if and only if $\delta(R) = 1$.*

(3) *$\delta(R) = 2$, then $r(R) \geqslant e(R) + d - 1 - \mathrm{emb}(R)$.*

Proof. Let I be a minimal reduction of $\mathfrak{m}$. Then $e(R) - r(R) = l(R/I : \mathfrak{m})$ and $I \cap \mathfrak{m}^2 = I\mathfrak{m}$.

(1) This follows from Proposition 1.2.

(2) $e(R) - r(R) = 1 \Leftrightarrow I : \mathfrak{m} = \mathfrak{m}$ (i.e., $\mathfrak{m}^2 \subset I \neq \mathfrak{m}) \Leftrightarrow \mathfrak{m}^2 = I\mathfrak{m}$ and $I \neq \mathfrak{m}$, i.e., $\delta(R) = 1$.

(3) Assume that $I\mathfrak{m}^2 = \mathfrak{m}^3$. Then $(I : \mathfrak{m}) \supset I + \mathfrak{m}^2$ and $e(R) - r(R) \leqslant l(R/I + \mathfrak{m}^2) = l(R/\mathfrak{m}^2) - l(I + \mathfrak{m}^2/\mathfrak{m}^2) = l(R/\mathfrak{m}^2) - l(I/I\mathfrak{m}) = 1 + \mathrm{emb}(R) - d$.

EXAMPLE 1.5. Assume that $d = 1$. Then $M = \mathfrak{m}^i$ is a Cohen–Macaulay R-module with $\delta(M) = \max\{\delta(R) - i, 0\}$. Hence $\mathfrak{m}^i$ is a Ulrich R-module if and only if $i \geqslant \delta(R)$ (these modules are all isomorphic to each other). If $G(R)$ is Cohen–Macaulay, then $G(\mathfrak{m}^i)$ is Cohen–Macaulay for all i. Conversely, if $G(\mathfrak{m})$ is Cohen–Macaulay, then $G(R)$ is Cohen–Macaulay. If $R = k[[t^4, t^5, t^{11}]]$ and $M = \mathfrak{m}$, then $\delta(R) = 3$, $\delta(M) = 2$, and $G(R)$ and $G(M)$ are not Cohen–Macaulay. Therefore even if $\delta(M) = 2$, $G(M)$ is not necessarily a Cohen–Macaulay $G(R)$-module.

2. CANONICAL MODULES OF COHEN–MACAULAY RINGS

In this section, we collect some (more or less well-known) facts on the canonical modules of Cohen–Macaulay local rings and Cohen–Macaulay homogeneous algebras. For details, see [4, 2, 3].

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and put $E = E_R(k)$, the injective envelope of k. A finitely generated R-module K is called a *canonical R-module* if it satisfies the following equivalent conditions:

(1) R is Cohen–Macaulay and $K \otimes_R \hat{R}$ is isomorphic to $\mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), E)$ as an $\hat{R}$-module, where $d = \dim(R)$.

(2) K is a Gorenstein R-module of rank one, i.e., $id_R(K) = \mathrm{depth}_R(K)$ and $\mathrm{End}_R(K) \cong R$.

(3) The trivial extension $R \ltimes K$ of R by K is a Gorenstein local ring.

(Usually, canonical modules are defined by the condition (1) without the Cohen–Macaulay condition on R.) If a canonical R-module exists, it is unique up to isomorphisms, and any Gorenstein R-module is a direct sum of canonical modules. If K is a canonical R-module, then K is a Cohen–Macaulay R-module with $\mu(K)=r(R)$, and K/xK is a canonical R/xR-modules for any non-zero divisor x of R.

LEMMA 2.1. *Let K be a canonical R-module and I a parameter ideal of R. Then $l(K/IK)=l(R/I)$ and $(IK:K)=I$. In particular, for any ideal J of R, $JK\subset IK$ if and only if $J\subset I$.*

Proof. By the above remark, K/IK is a canonical module over an artinian local ring R/I. Hence it is isomorphic to $E_{R/I}(k)$ and is a faithful R/I-module. Therefore we have

$$l(K/IK)=l(E_{R/I}(k))=l(R/I)$$

and

$$(IK:K)/I=\operatorname{ann}_{R/I}(K/IK)=0.$$

COROLLARY 2.2. *If K is a canonical R-module, then $e(K)=e(R)$.*

Proof. We may assume that k is an infinite field. Take I in Lemma 2.1 to be a minimal reduction of $\mathfrak{m}$. Then

$$e(K)=l(K/IK)=l(R/I)=e(R).$$

Henceforth let A be a d-dimensional Cohen–Macaulay homogeneous algebra over a field k, and put $P=A_+$. A finitely generated graded A-module $K=\bigoplus_{n\in\mathbf{Z}}K_n$ is said to be a *canonical graded A-module* if K_P is a canonical A_P-module. Put $K_A=\underline{\operatorname{Hom}}_k(H_P^d(A),k)$. Then K_A is a canonical graded A-module, and any canonical graded A-module is isomorphic to some $K_A(n)$, $n\in\mathbf{Z}$. We call K_A *the canonical module of A*. A finitely generated graded A-module is a Gorenstein module if and only if it is isomorphic to some $K_A(a_1)\oplus\cdots\oplus K_A(a_r)$, $a_i\in\mathbf{Z}$. If x is a homogeneous A-regular element of degree r, then $K_{A/xA}\cong(K_A/xK_A)(r)$.

LEMMA 2.3 (cf. [13, Theorem 4.4]). *We have*

$$F(K_A,t)=(-1)^dF(A,t^{-1}).$$

Proof. We may assume that k is an infinite field. If $d=0$, then $K_A=\underline{\operatorname{Hom}}_k(A,k)$ and

$$F(K_A,t)=\sum_{n\in\mathbf{Z}}H(K_A,n)t^n=\sum_{n\in\mathbf{Z}}H(A,-n)t^n=F(A,t^{-1}).$$

If x is a homogeneous A-regular element of degree one, then $K_{A/xA} \cong (K_A/xK_A)(1)$ and

$$(1-t)F(K_A, t) = F(K_A/xK_A, t) = tF(K_{A/xA}, t).$$

Hence assuming that k is an infinite field and by induction on d, we have

$$\begin{aligned} F(K_A, t^{-1}) &= (t-1)^{-1}F(K_{A/xA}, t^{-1}) \\ &= (t-1)^{-1}(-1)^{d-1}F(A/xA, t) = (-1)^d F(A, t). \end{aligned}$$

(The difference between our formula and that in [13] stems from the fact that Stanley takes $K_A(-a(A))$ as the definition of the canonical module of A.)

Let $M = \bigoplus_{n \in \mathbf{Z}} M_n$ be a finitely generated graded A-module and let r be an integer. We say that M is *homogeneous in degree* r if M is generated by its elements of degree r, in other words, $M_n = 0$ for any $n < r$ and $A_1 M_n = M_{n+1}$ for any $n \geqslant r$. By the classical Castelnuovo's lemma, if M is positively graded (i.e., $M_n = 0$ for any $n < 0$) and is 0-regular, then M is homogeneous in degree zero (cf. [7, Theorem 2]). Let $B = A \ltimes M$ be the trivial extension of A by M. Then B is a noetherian graded ring with $B_n = A_n \times M_n$, and B is a homogeneous k-algebra if and only if M is homogeneous in degree one.

Put $(1-t)^d F(A, t) = a_0 + a_1 t + \cdots + a_m t^m$ with $a_m \neq 0$. We say that A is a *level ring* if $a_m = r(A)$.

LEMMA 2.4 (cf. [5]). *The following conditions are equivalent*:

(1) *A is a level ring.*

(2) *K_A is homogeneous in degree $-a(A)$.*

(3) *$A \ltimes K_A(-a(A)-1)$ is a (Gorenstein) homogeneous k-algebra.*

3. ASSOCIATED GRADED MODULES OF CANONICAL MODULES

In this section, we give conditions for the associated graded module of a canonical module to be canonical. Throughout this section $(R, \mathfrak{m}, k)$ stands for a d-dimensional Cohen–Macaulay local ring and K is a canonical R-module. If $G(R)$ is Cohen–Macaulay, we denote the canonical module of $G(R)$ by $K_{G(R)}$.

LEMMA 3.1. *Let M be a finitely generated R-module. Then $G(M)$ is a canonical $G(R)$-module if and only if M is a canonical R-module and $G(M)$ is a Gorenstein $G(R)$-module.*

Proof. Under each condition, $G(R)$ is Cohen–Macaulay.

(If) By the assumption, $G(M) \cong K_{G(R)}{}^r$ for some r. Then by Corollary 2.2,

$$e(R) = e(M) = e(G(M)) = e(K_{G(R)})r = e(R)r.$$

Hence $r = 1$ and $G(M)$ is a canonical $G(R)$-module.

(Only if) By Foxby–Reiten's theorem, $G(R \ltimes M) \cong G(R) \ltimes G(M)(-1)$ is a Gorenstein ring. This implies that $R \ltimes M$ is a Gorenstein ring. Hence M is a canonical R-module again by Foxby–Reiten's theorem.

LEMMA 3.2. *Let R be an artinian local ring such that $\delta(R) = r$, and put $E = E_R(k)$. Then $l(\mathfrak{m}^r E) = 1$.*

Proof. Since $\mathfrak{m}^r \neq 0$ and $\mathfrak{m}^{r+1} = 0$, we have $\mathfrak{m} \subset \operatorname{ann}(\mathfrak{m}^r) \neq R$. Therefore $\mathfrak{m} = \operatorname{ann}(\mathfrak{m}^r)$. From the isomorphisms

$$\begin{aligned} E/\mathfrak{m}^r E &\cong \operatorname{Hom}(\operatorname{Hom}(R/\mathfrak{m}^r, R), E) \\ &\cong \operatorname{Hom}(\operatorname{ann}(\mathfrak{m}^r), E) = \operatorname{Hom}(\mathfrak{m}, E), \end{aligned}$$

we get $l(\mathfrak{m}^r E) = l(E) - l(E/\mathfrak{m}^r E) = l(R) - l(\mathfrak{m}) = 1$.

LEMMA 3.3. *Put $(1-t)^d F(G(K), t) = a_0 + a_1 t + \cdots + a_m t^m$ with $a_m \neq 0$. Then $a_0 = r(R)$, and if $G(K)$ is Cohen–Macaulay, then $a_m = 1$.*

Proof. The first assertion is trivial. For the second assertion, we may assume that k is an infinite field. Take a minimal reduction $I = (x_1, ..., x_d)$ of $\mathfrak{m}$ and put $S = R/I$ and $L = K/IK$. Then L is a canonical S-module and is isomorphic to $E_S(k)$. Let x_i^* be the initial form of x_i in $G(R)$. Then by the assumption, $x_1^*, ..., x_d^*$ is a $G(K)$-regular sequence of degree one and $G(K)/(x_1^*, ..., x_d^*)\, G(K) \cong G(L)$ (cf. [14]). Hence $(1-t)^d F(G(K), t) = F(G(L), t)$ and our assertion follows from Lemma 3.2.

Let A be a d-dimensional homogeneous k-algebra, and put $(1-t)^d F(A, t) = a_0 + a_1 t + \cdots + a_m t^m$ with $a_m \neq 0$. We say that A is *symmetric* if $a_i = a_{m-i}$, $0 \leqslant i \leqslant m$ (or equivalently, $F(A, t) = (-1)^d t^a F(A, t^{-1})$, where $a = \deg F(A, t)$). If A is Gorenstein, then A is symmetric, but the converse does not hold in general (cf. [13]).

THEOREM 3.4 (cf. [15]). *Let R be a Gorenstein local ring. Then $G(R)$ is Gorenstein if and only if $G(R)$ is Cohen–Macaulay and symmetric.*

Proof. We may assume that k is an infinite field and $G(R)$ is Cohen–Macaulay. Take a minimal reduction $I = (x_1, ..., x_d)$ of $\mathfrak{m}$. Then as in

Lemma 3.3, $x_1^*, \ldots, x_d^*$ is a $G(R)$-regular sequence of degree one, $G(R)/(x_1^*, \ldots, x_d^*) \cong G(R/I)$ and $(1-t)^d F(G(R), t) = F(G(R/I), t)$. Hence we may assume that R is artinian. Then the assertion is proved in [15].

THEOREM 3.5. *The following conditions are equivalent:*

(1) *$G(K)$ is a canonical $G(R)$-module.*

(2) *For any (or some) Gorenstein R-module M, $G(M)$ is a Gorenstein $G(R)$-module.*

(3) *$G(R)$ and $G(K)$ are Cohen–Macaulay and they satisfy the equation $F(G(K), t) = (-1)^d t^a F(G(R), t^{-1})$, where $a = a(G(R))$.*

Proof. We may assume that R is complete.

$(1) \Rightarrow (2)$. Since $M \cong K^r$ for some r and $G(K)$ is a Gorenstein $G(R)$-module, $G(M) \cong G(K)^r$ is a Gorenstein $G(R)$-module.

$(2) \Rightarrow (1)$. Assume that $G(M)$ is a Gorenstein $G(R)$-module for some Gorenstein R-module M. Suppose that $M \cong K^r$. Since $G(M) \cong G(K)^r$ is Gorenstein, $G(K)$ is a Gorenstein $G(R)$-module. Hence $G(K)$ is a canonical $G(R)$-module by Lemma 3.1.

$(1) \Rightarrow (3)$. By the assumption, we have $G(K) \cong K_{G(R)}(-a)$. Hence by Lemma 2.3,

$$F(G(K), t) = t^a F(K_{G(R)}, t) = t^a (-1)^d F(G(R), t^{-1}).$$

$(3) \Rightarrow (1)$. We have only to show that $G(R \ltimes K) \cong G(R) \ltimes G(K)(-1)$ is a Gorenstein ring. By Theorem 3.4, this is equivalent to showing that $G(R \ltimes K)$ is Cohen–Macaulay and symmetric. Since $G(R)$ and $G(K)$ are Cohen–Macaulay, the ring $G(R \ltimes K)$ is Cohen–Macaulay. On the other hand, by the assumption,

$$a(G(K)) = \deg F(G(K), t) = \deg F(G(R), t) = a(G(R)).$$

Hence $a(G(R \ltimes K)) = a(G(R)) + 1$ and we have

$$\begin{aligned} F(G(R \ltimes K), t) &= F(G(R), t) + tF(G(K), t) \\ &= (-1)^d t^a F(G(K), t^{-1}) + (-1)^d t^{a+1} F(G(R), t^{-1}) \\ &= (-1)^d t^{a+1} \{F(G(R), t^{-1}) + t^{-1} F(G(K), t^{-1})\} \\ &= (-1)^d t^{a+1} F(G(R \ltimes K), t^{-1}). \end{aligned}$$

Therefore $G(R \ltimes K)$ is symmetric.

COROLLARY 3.6. *If $G(K)$ is a canonical $G(R)$-module, then $G(R)$ is a level ring and $r(G(R))=r(R)$.*

Proof. Since $K_{G(R)} \cong G(K)(a(G(R)))$ is homogeneous in degree $-a(G(R))$, $G(R)$ is a level ring by Lemma 2.4, and $r(G(R))=\mu(G(K))=\mu(K)=r(R)$.

4. GENERA AND REDUCTION EXPONENTS OF CANONICAL MODULES

As in the previous section, $(R, \mathfrak{m}, k)$ is a d-dimensional Cohen–Macaulay local ring with infinite residue field and K is a canonical R-module. We examine the conditions on genera and reduction exponents under which the associated graded modules of canonical modules are Cohen–Macaulay or canonical.

PROPOSITION 4.1. (1) *$\delta(K)=0$ if and only if R is a regular local ring.*

(2) *$\delta(K) \leqslant 1$ if and only if $\mathrm{emb}(R)=e(R)+\dim(R)-1$.*

(3) *If $G(R)$ is Cohen–Macaulay, then $\delta(K)=\delta(R)$.*

Proof. Let I be a minimal reduction of $\mathfrak{m}$. Assume that $\delta(K)=0$. Then $\mathfrak{m}K=IK$ and we have $\mathfrak{m}=I$ by Lemma 2.1. Therefore R is a regular local ring. Assume that $\delta(K)\leqslant 1$. Then $\mathfrak{m}^2K=I\mathfrak{m}K\subset IK$. Hence $\mathfrak{m}^2\subset I$ by Lemma 2.1 and $\mathfrak{m}^2=\mathfrak{m}^2\cap I=I\mathfrak{m}$. Therefore $\delta(R)\leqslant 1$, i.e., $\mathrm{emb}(R)=e(R)+d-1$. Assume that $G(R)$ is Cohen–Macaulay and $\delta(K)\leqslant n$. Take I such that $\mathfrak{m}^{n+1}K=I\mathfrak{m}^nK$. Then $\mathfrak{m}^{n+1}K\subset IK$ and we have $\mathfrak{m}^{n+1}\subset I$ by Lemma 2.1. Hence $\mathfrak{m}^{n+1}=\mathfrak{m}^{n+1}\cap I=I\mathfrak{m}^n$ since $G(R)$ is Cohen–Macaulay (cf. [14]). The converse assertions are clear.

By Proposition 4.1(3), if $\delta(R)=2$, then $\delta(K)=2$. The converse does not hold in general (cf. Example 5.11). If $\delta(R)\leqslant 2$, then $G(R)$ is Cohen–Macaulay (cf. [12, Theorem 2.1]).

PROPOSITION 4.2. (1) *$e_1(K)=0$ if and only if R is a regular local ring.*

(2) *$e_1(K)=1$ if and only if $\mathrm{emb}(R)=e(R)+\dim(R)-1$ and $e(R)\geqslant 2$.*

(3) *$e_1(K)=2$ never occurs.*

(4) *$e_1(K)=3$ if and only if $g_s(K)=1$ and $r(R)=e(R)-2$.*

Proof. (1) This follows from Corollary 1.3 and Proposition 4.1.

(2) If $e_1(K)=1$, then by Proposition 1.2 and Lemma 1.4,

$$1=e_1(K)\geqslant e(K)-\mu(K)=e(R)-r(R)\geqslant 1.$$

Hence $e(R)-r(R)=1$ and $\delta(R)=1$ by Lemma 1.4. The converse follows from Proposition 4.1, Proposition 1.2, and Lemma 1.4.

(3) Assume that $e_1(K)=2$. Then by Proposition 1.2 and Lemma 1.4, we have

$$2=e_1(K)>e(R)-r(R)\geqslant 2,$$

which is a contradiction.

(4) Assume that $e_1(K)=3$. Then, as in (3),

$$3=e_1(K)>e(R)-r(R)\geqslant 2.$$

Hence $e(R)-r(R)=2$ and $g_s(K)=e_1(K)-e(R)+r(R)=1$. The converse is clear.

PROPOSITION 4.3. (1) *Assume that* $\mathrm{emb}(R)=e(R)+\dim(R)-1$. *Then* $G(K)$ *is canonical* $G(R)$*-module, and if* R *is not regular,* $l(K/\mathfrak{m}^{n+1}K)=e(R)\binom{n+d}{d}-\binom{n+d-1}{d-1}$ *for all* $n\geqslant 0$.

(2) *Assume that* $\mathrm{emb}(R)=e(R)+\dim(R)-2$ *and* R *is not Gorenstein. Then* $G(K)$ *is not a canonical* $G(R)$*-module.*

Proof. (1) Put $S=R\ltimes K$. Then S is a Gorenstein local ring with $g_\Delta(S)=g_\Delta(R)+e(R)-r(R)=1$. Hence $G(S)\cong G(R)\ltimes G(K)(-1)$ is Gorenstein by [12]. Therefore $G(K)$ is a canonical $G(R)$-module. On the other hand, since $(1-t)^dF(G(R),t)=1+(e(R)-1)t$, we have $(1-t)^dF(G(K),t)=(e(R)-1)+t$ if R is not regular. This implies our second assertion.

(2) Suppose that $G(K)$ is a canonical $G(R)$-module. Then $G(R)$ is Cohen–Macaulay and by the assumption, we have

$$(1-t)^dF(G(R),t)=1+(\mathrm{emb}(R)-\dim(R))t+t^2.$$

Since $G(R)$ is a level ring by Corollary 3.6, we have $r(R)=1$, which contradicts our assumption.

PROPOSITION 4.4. *The following conditions are equivalent*:

(1) $\delta(K)=2$ *and* $G(K)$ *is Cohen–Macaulay.*

(2) $\delta(K)=2$ *and* $g_\Delta(K)=1$.

Proof. (1) $\Rightarrow$ (2). Since $\mathrm{reg}\, G(K)=\delta(K)=2$, by Lemma 3.3, we have

$$(1-t)^dF(G(K),t)=r(R)+(l(\mathfrak{m}K/\mathfrak{m}^2K)-dr(R))t+t^2.$$

Hence $g_\Delta(K)=e(K)+(d-1)r(R)-l(\mathfrak{m}K/\mathfrak{m}K^2)=1$.

(2) $\Rightarrow$ (1). Let I be a minimal reduction of $\mathfrak{m}$. Then

$$1=g_\Delta(K)=l(\mathfrak{m}^2K/I\mathfrak{m}K)\geqslant l(\mathfrak{m}^2K\cap IK/I\mathfrak{m}K).$$

Assume that $\mathfrak{m}^2K \cap IK \neq I\mathfrak{m}K$. Then $\mathfrak{m}^2K \cap IK = \mathfrak{m}^2K$, i.e., $\mathfrak{m}^2K \subset IK$. Hence $\mathfrak{m}^2 \subset I$ by Lemma 2.1 and $\mathfrak{m}^2 = \mathfrak{m}^2 \cap I = I\mathfrak{m}$. Thus $\delta(K) \leqslant \delta(R) \leqslant 1$, which is a contradiction. Hence $\mathfrak{m}^2K \cap IK = I\mathfrak{m}K$. By the assumption, we may assume that $I\mathfrak{m}^2K = \mathfrak{m}^3K$. Then $IK \cap \mathfrak{m}^{n+1}K = I\mathfrak{m}^nK$ for all n. Therefore $G(K)$ is Cohen–Macaulay $G(R)$-module (cf. [14, 16]).

THEOREM 4.5. *If $\delta(R) = 2$ and $g_\Delta(K) = 1$, then $G(K)$ is Cohen–Macaulay and the following conditions are equivalent:*

(1) *$G(K)$ is a canonical $G(R)$-module.*

(2) *$G(R)$ is a level ring and $r(G(R)) = r(R)$.*

(3) $\operatorname{emb}(R) = e(R) + \dim(R) - 1 - r(R)$.

Proof. Since $\delta(R) = 2$, $G(R)$ is Cohen–Macaulay. The first assertion follows from Proposition 4.4. Since $\operatorname{reg} G(R) = \operatorname{reg} G(K) = \delta(K) = 2$, we have

$$(1-t)^d F(G(R), t) = 1 + (v-d)t + (e+d-v-1)t^2$$

and

$$(1-t)^d F(G(K), t) = r + (e-r-1)t + t^2,$$

where $v = \operatorname{emb}(R)$, $e = e(R)$, and $r = r(R)$ (cf. Lemma 3.3). Therefore the equivalence of (1) and (3) follows from Theorem 3.5. By Corollary 3.6, the condition (1) implies the condition (2). Finally, if the condition (2) is satisfied, then $r(R) = r(G(R)) = e + d - 1 - v$. Thus the condition (3) is satisfied. This completes the proof.

5. THE CASE OF DIMENSION ONE

In this section, $(R, \mathfrak{m}, k)$ stands for a one-dimensional Cohen–Macaulay local ring with infinite residue field. We denote by $Q = Q(R)$ and $\bar{R}$ the total quotient ring and the integral closure of R, respectively. Put $E = E_R(k)$ and fix a minimal reduction xR of $\mathfrak{m}$. Let M be a finitely generated torsion-free R-module.

LEMMA 5.1. *For any $\mathfrak{m}$-primary ideal I and an M-regular element y in $\mathfrak{m}$, $l(IM/yIM) = l(M/yM)$.*

Proof. $l(IM/yIM) = l(M/yIM) - l(M/IM) = l(M/yIM) - l(yM/yIM) = l(M/yM)$.

Put $Q(M) = M \otimes_R Q$, $B(M) = \bigcup_{n=0}^{\infty} (\mathfrak{m}^n M : \mathfrak{m}^n)_{Q(M)}$, and $g(M) = e_1(M)$. Then as in [8, Theorem 5.1],

$$l(M/\mathfrak{m}^n M) = e(M)n - g(M) + l(\mathfrak{m}^n B(M)/\mathfrak{m}^n M)$$

for all $n \geqslant 0$, and we have $\mathfrak{m}B(M) = xB(M)$, $e(M) = l(B(M)/\mathfrak{m}B(M))$, $g(M) = l(B(M)/M)$, $g_s(M) = l(\mathfrak{m}B(M)/\mathfrak{m}M)$, and $\delta(M) = \operatorname{reg} G(M) = \min\{n \mid B(M) = (\mathfrak{m}^n M : \mathfrak{m}^n)\}$.

PROPOSITION 5.2. (1) *For any* $n \geqslant 0$, $l(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M) \leqslant e(M)$, *and the equality holds if and only if* $\delta(M) \leqslant n$.

(2) $g_s(M) \geqslant g_\Delta(M)$, *and the equality holds if and only if* $\delta(M) \leqslant 2$.

Proof. (1) By Lemma 5.1, $e(M) = l(M/xM) = l(\mathfrak{m}^n M/x\mathfrak{m}^n M)$. Hence $e(M) - l(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M) = l(\mathfrak{m}^{n+1} M/x\mathfrak{m}^n M)$.

(2) Since $g_s(M) = l(\mathfrak{m}^2 B(M)/x\mathfrak{m}M)$ and $g_\Delta(M) = l(\mathfrak{m}^2 M/x\mathfrak{m}M)$, we have $g_s(M) - g_\Delta(M) = l(\mathfrak{m}^2 B(M)/\mathfrak{m}^2 M)$.

Henceforth we assume that R has a canonical module K. The following lemma and its corollary may be well known (cf. [4]).

LEMMA 5.3. $H^1_\mathfrak{m}(R) \cong Q/R$ *and* $l(K/IK) = l(R : I/R)$ *for any* $\mathfrak{m}$*-primary ideal* I.

Proof. Since $\operatorname{Ext}^1_R(R/\mathfrak{m}^n, R) \cong \operatorname{Hom}_R(\mathfrak{m}^n, R)/R \cong (R : \mathfrak{m}^n)/R$, $H^1_\mathfrak{m}(R) \cong \bigcup_{n=0}^{\infty} (R : \mathfrak{m}^n)/R = Q/R$. For the second assertion we may assume that R is complete. Then

$$K/IK \cong R/I \otimes \operatorname{Hom}(H^1_\mathfrak{m}(R), E) \cong \operatorname{Hom}(\operatorname{Hom}(R/I, H^1_\mathfrak{m}(R)), E)$$

and

$$\operatorname{Hom}(R/I, H^1_\mathfrak{m}(R)) \cong \operatorname{Hom}(R/I, Q/R) \cong (R : I)/R.$$

This implies our assertion.

COROLLARY 5.4. *Let* I *and* J *be* $\mathfrak{m}$*-primary ideals. Then* $IK \subset JK$ *if and only if* $(R : I) \supset (R : J)$. *In particular,* $\delta(K) \leqslant n$ *if and only if* $(R : x\mathfrak{m}^n) = (R : \mathfrak{m}^{n+1})$.

Proof. Put $L = I + J$. Then by Lemma 5.3, $l(LK/JK) = l(R : J/R : L)$. Hence $IK \subset JK \Leftrightarrow JK = LK \Leftrightarrow R : J = R : L$ $(= (R : I) \cap (R : J)) \Leftrightarrow (R : I) \supset (R : J)$.

PROPOSITION 5.5. (1) *$g(K)=l(R/x^nR:\mathfrak{m}^n)$ and $g_s(K)=l(R:x^{n-1}\mathfrak{m}/R:\mathfrak{m}^n)$ for all $n \geqslant \delta(K)$.*

(2) $g_\Delta(K) = e(R) + r(R) - l(R:\mathfrak{m}^2/R) = l(R:x\mathfrak{m}/R:\mathfrak{m}^2)$.

Proof. For any $n \geqslant 0$, we have $l(K/\mathfrak{m}^nK) = l(R:\mathfrak{m}^n/R) = l(x^nR:\mathfrak{m}^n/x^nR) = l(R/x^nR) - l(R/x^nR:\mathfrak{m}^n) = e(K)n - l(R/x^nR:\mathfrak{m}^n)$. Hence our assertions follow from Lemma 5.1 and Lemma 5.3.

PROPOSITION 5.6. *$g_s(K)=1$ if and only if $\delta(K)=2$ and $G(K)$ is Cohen–Macaulay.*

Proof. Assume that $g_s(K)=1$. Then $1 \leqslant g_\Delta(K) \leqslant g_s(K)=1$ and $\delta(K) \geqslant 2$ by Proposition 1.2 and Proposition 5.2. Hence $g_s(K)=g_\Delta(K)$ and $\delta(K)=2$ by Proposition 5.2, and $G(K)$ is Cohen–Macaulay by Proposition 4.4. Conversely, if $\delta(K)=2$ and $G(K)$ is Cohen–Macaulay, then $g_s(K)=g_\Delta(K)=1$ by Proposition 5.2 and Proposition 4.4.

PROPOSITION 5.7. *If $g(K) \leqslant 3$, then $G(K)$ is Cohen–Macaulay. Moreover, $g(K)=3$ and $G(K)$ is a canonical $G(R)$-module if and only if R is a cubic plane curve singularity (i.e., $\mathrm{emb}(R)=2$ and $e(R)=3$).*

Proof. This follows from Proposition 1.2, Proposition 4.2, Proposition 5.6, and Theorem 3.5.

From now on, we assume that R is analytically unramified (hence $\bar{R}$ is a finitely generated R-module), and put $C=R:\bar{R}$ and $c(R)=l(\bar{R}/C)$. We say that an ideal I is *normal* if I^n is integrally closed for all n.

LEMMA 5.8. *The following conditions are equivalent:*

(1) $C=\mathfrak{m}^2$.

(2) $\mathfrak{m}^2\bar{R}=\mathfrak{m}^2$ *and* $c(R)=2e(R)$.

(3) $\delta(R)=2$, $c(R)=2e(R)$, *and* $\mathfrak{m}$ *is normal.*

Moreover, if R is Gorenstein, these conditions are also equivalent to each of the following conditions:

(4) $l(\bar{R}/R)=e(R) \geqslant 3$.

(5) $\mathrm{emb}(R)=e(R)-1$ *and* $\mathfrak{m}$ *is normal.*

Proof. (1)$\Rightarrow$(2). $c(R) = l(\bar{R}/\mathfrak{m}^2\bar{R}) = 2l(\bar{R}/\mathfrak{m}\bar{R}) = 2e(R)$.

(2)$\Rightarrow$(1). $C \supset \mathfrak{m}^2\bar{R}$ and $l(C/\mathfrak{m}^2\bar{R}) = l(\bar{R}/\mathfrak{m}^2\bar{R}) - l(R/C) = 2e(R) - c(R)=0$. Hence $C=\mathfrak{m}^2\bar{R}=\mathfrak{m}^2$.

(2)$\Leftrightarrow$(3). We may assume that $c(R)=2e(R)$. Then $\mathfrak{m}^2\bar{R}=\mathfrak{m}^2 \Leftrightarrow \bar{R}=(\mathfrak{m}^2:\mathfrak{m}^2) \Leftrightarrow \bar{R}=B(R)$ and $B(R)=(\mathfrak{m}^2:\mathfrak{m}^2) \Leftrightarrow \mathfrak{m}^n$ is integrally closed for all $n \geqslant \delta(R)$ and $\delta(R) \leqslant 2$ (cf. [11, Lemma 5.1]) $\Leftrightarrow \delta(R) \leqslant 2$ and $\mathfrak{m}$ is normal. If $\delta(R) \leqslant 1$, then $\mathfrak{m}\bar{R} = \mathfrak{m}$, i.e., $C = \mathfrak{m}$, which is a contradiction. Hence $\delta(R) = 2$.

For the conditions (4) and (5), see [11, Theorem 4.1 and Proposition 5.2].

PROPOSITION 5.9. *If* $C = \mathfrak{m}^2$ *and* $r(R) = e(R) - \mathrm{emb}(R)$, *then* $G(K)$ *is a canonical* $G(R)$*-module.*

Proof. By Lemma 5.8, we have $\delta(R) = 2$. Hence $r(R) = e(R) - \mathrm{emb}(R) = g(R) - e(R) + 1$ by Proposition 5.2, and $l(R : \mathfrak{m}^2/R) = l(R : C/R) = l(\bar{R}/R) = g(R) = e(R) + r(R) - 1$. Therefore our assertion follows from Theorem 4.5 and Proposition 5.5.

EXAMPLE 5.10 (cf. [9, Example 3.5, (3)]). Put $R = k[[t^e, t^{e+1}, ..., t^{2e-r-1}]]$ with $1 \leqslant r \leqslant (e-1)/2$. Then $(1-t)F(G(R), t) = 1 + (e-r-1)t + rt^2$ and R satisfies the conditions in Proposition 5.9. Therefore $G(K)$ is a canonical $G(R)$-module.

EXAMPLE 5.11. Put $R = k[[t^4, t^5, t^{11}]]$. Then we have $(1-t)F(G(R), t) = 1 + 2t + t^3$ and $(1-t)F(G(K), t) = 2 + t + t^2$. Hence $G(R)$ is not Cohen–Macaulay, $\delta(R) = 3$, $g(K) = 3$, and $\delta(K) = 2$. Therefore, by Proposition 5.7, $G(K)$ is Cohen–Macaulay but is not a canonical $G(R)$-module.

REFERENCES

1. J. BRENNAN, J. HERZOG, AND B. ULRICH, Maximally generated Cohen–Macaulay modules, *Math. Scand.* **61** (1987), 181–203.
2. R. M. FOSSUM, H.-B. FOXBY, P. GRIFFITH, AND I. REITEN, Minimal injective resolutions with applications to dualizing modules and Gorenstein modules, *Publ. Math. I.H.E.S.* **45** (1975), 193–215.
3. S. GOTO AND K. WATANABE, On graded rings, I, *J. Math. Soc. Japan* **30** (1978), 179–213.
4. J. HERZOG AND E. KUNZ, Der kanonische Modul eines Cohen–Macaulay-Rings, *in* "Lecture Notes in Math., "Vol. 238, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
5. T. HIBI, Level rings and algebras with straightening laws, *J. Algebra* **117** (1988), 343–362.
6. D. G. NORTHCOTT AND D. REES, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.
7. A. OOISHI, Castelnuovo's regularity of graded rings and modules, *Hiroshima Math. J.* **12** (1982), 127–144.
8. A. OOISHI, Genera and arithmetic genera of commutative rings, *Hiroshima Math. J.* **17** (1987), 47–66.
9. A. OOISHI, Δ-genera and sectional genera of commutative rings, *Hiroshima Math. J.* **17** (1987), 361–372.

10. A. OOISHI, Reductions of graded rings and pseudo-flat graded modules, *Hiroshima Math. J.* **18** (1988), 463–477.
11. A. OOISHI, Genera of curve singularities, to appear.
12. J. D. SALLY, Tangent cones at Gorenstein singularities, *Compositio Math.* **40** (1980), 167–175.
13. R. P. STANLEY, Hilbert functions of graded algebras, *Adv. in Math.* **28** (1978), 57–83.
14. P. VALABREGA AND G. VALLA, Form rings and regular sequences, *Nagoya Math. J.* **72** (1978), 93–101.
15. J. WATANABE, The Dilworth number of Artin Gorenstein rings, peprint, 1986.
16. J. D. SALLY, Super-regular sequences, *Pacific J. Math.* **84** (1979), 465–481.