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# Solvability of boundary value problems of nonlinear fractional differential equations

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Full list of author information is available at the end of the article**Abstract**

In this paper, we study the existence of multiple positive solutions for the nonlinear fractional differential equation boundary value problem  $D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0$ ,  $0 < t < 1$ ,  $u(0) = u(1) = u'(0) = 0$ , where  $2 < \alpha \leq 3$  is a real number,  $D_{0+}^{\alpha}$  is the Riemann-Liouville fractional derivative. By the properties of the Green's function, the lower and upper solution method and the Leggett-Williams fixed point theorem, some new existence criteria are established. As applications, examples are presented to illustrate the main results.

**MSC:** 34A08; 34B18**Keywords:** fractional differential equation; boundary value problem; positive solution; fractional Green's function; fixed point theorem; lower and upper solution method

## 1 Introduction

Fractional differential equations have been of great interest. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self-similar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science; see [1–5]. Recently, there have appeared some papers dealing with the existence of solutions of fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, Adomian decomposition method, *etc.*); see [6–9]. Especially, boundary value problems for fractional differential equations have attracted considerable attention; see [10–31]. As is well known, the aim of finding solutions to boundary value problems is of main importance in various fields of applied mathematics. Recently, there seems to be a new interest in the study of the boundary value problems for fractional differential equations.

Bai and Lü [28] studied the following two-point boundary value problem of fractional differential equations:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0,$$

where  $1 < \alpha \leq 2$  is a real number and  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville fractional derivative. They obtained the existence of positive solutions by means of the Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem.

Zhang [29] considered the existence and multiplicity of positive solutions for the nonlinear fractional boundary value problem

$$\begin{aligned}
 {}^C D_{0^+}^\alpha u(t) &= f(t, u(t)), \quad 0 < t < 1, \\
 u(0) + u'(0) &= 0, u(1) + u'(1) = 0,
 \end{aligned}$$

where  $1 < \alpha \leq 2$  is a real number,  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous and  ${}^C D_{0^+}^\alpha$  is the standard Caputo fractional derivative. The author obtained the existence and multiplicity results of positive solutions by means of the Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem.

Liang and Zhang [30] investigated the following nonlinear fractional boundary value problem:

$$\begin{aligned}
 D_{0^+}^\alpha u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\
 u(0) = u'(0) = u''(0) &= u''(1) = 0,
 \end{aligned}$$

where  $3 < \alpha \leq 4$  is a real number,  $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$  and  $D_{0^+}^\alpha$  is the standard Riemann-Liouville fractional derivative. By means of the lower and upper solution method and fixed point theorems, some results on the existence of positive solutions are obtained for the above fractional boundary value problems.

Yu and Jiang [31] discussed the following two-point boundary value problem of fractional differential equations:

$$\begin{aligned}
 D_{0^+}^\alpha u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\
 u(0) = u(1) = u'(0) &= 0,
 \end{aligned}$$

where  $2 < \alpha \leq 3$  is a real number and  $D_{0^+}^\alpha$  is the standard Riemann-Liouville fractional derivative. By the properties of the Green's function, they gave some results of multiple positive solutions for singular and nonsingular boundary value problems by means of the Leray-Schauder nonlinear alternative, a fixed point theorem on cones, and a mixed monotone method.

From the above works, we can see that, although the fractional boundary value problems have been investigated by some authors, the lower and upper solution method and the fixed point theorem due to Leggett-Williams are seldom considered. In addition, in the latter work the multiplicity of the solutions was not employed. Furthermore, the solution technique of upper and lower solutions was not studied, and it was also assumed that  $2 < \alpha \leq 3$ . This paper will fill up the gap.

Motivated by all the works above, in this paper we discuss the boundary value problem

$$D_{0^+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u(1) = u'(0) = 0, \tag{1.2}$$

where  $2 < \alpha \leq 3$  is a real number and  $D_{0^+}^\alpha$  is the Riemann-Liouville fractional differentiation. Using the lower and upper solution method and the Leggett-Williams fixed point

theorem, we give some new existence criteria for the boundary value problem (1.1) and (1.2). Finally, we present an example to demonstrate our results.

The plan of the paper is as follows. In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, we establish the existence of a single positive solution for the boundary value problem (1.1) and (1.2) by the lower and upper solution method. In Section 4, we establish the existence of multiple positive solutions for the boundary value problem (1.1) and (1.2) by the Leggett-Williams fixed point theorem. Examples are presented to illustrate the main results in Section 3 and Section 4, respectively. In Section 5, we give the conclusion of the paper.

## 2 Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate the analysis of problem (1.1) and (1.2). These materials can be found in the recent literature; see [31–33].

**Definition 2.1** ([32]) The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a continuous function  $f : (0, +\infty) \rightarrow \mathbb{R}$  is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^{(n)} \int_0^t \frac{f(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of number  $\alpha$ , provided that the right side is pointwise defined on  $(0, +\infty)$ .

**Definition 2.2** ([32]) The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $f : (0, +\infty) \rightarrow \mathbb{R}$  is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds,$$

provided that the right side is pointwise defined on  $(0, +\infty)$ .

From the definition of the Riemann-Liouville derivative, we can obtain the following statement.

**Lemma 2.1** *Let  $\alpha > 0$ . If we assume  $u \in C(0, 1) \cap L^1(0, 1)$ , then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

*has  $u(t) = c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \dots + c_n t^{\alpha - n}$ ,  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ , as the unique solution, where  $n$  is the smallest integer greater than or equal to  $\alpha$ .*

**Lemma 2.2** *Assume that  $u \in C(0, 1) \cap L^1(0, 1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $C(0, 1) \cap L^1(0, 1)$ . Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \dots + c_n t^{\alpha - n}, \quad \text{for some } c_i \in \mathbb{R}, i = 1, 2, \dots, n,$$

*where  $n$  is the smallest integer greater than or equal to  $\alpha$ .*

In the following, we present the Green's function of the fractional differential equation boundary value problem.

**Lemma 2.3** ([31]) *Let  $h \in C[0, 1]$  and  $2 < \alpha \leq 3$ . The unique solution of problem*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = u'(0) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G(t, s)h(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Here  $G$  is called the Green's function of the boundary value problem (2.1) and (2.2).

The following properties of the Green's function play important roles in this paper.

**Lemma 2.4** ([31]) *The function  $G$  defined by (2.3) satisfies the following conditions:*

- (1)  $G(t, s) = G(1 - s, 1 - t)$ , for  $t, s \in (0, 1)$ ;
- (2)  $t^{\alpha-1}(1 - t)s(1 - s)^{\alpha-1} \leq \Gamma(\alpha)G(t, s) \leq (\alpha - 1)s(1 - s)^{\alpha-1}$ , for  $t, s \in (0, 1)$ ;
- (3)  $G(t, s) > 0$ , for  $t, s \in (0, 1)$ ;
- (4)  $t^{\alpha-1}(1 - t)s(1 - s)^{\alpha-1} \leq \Gamma(\alpha)G(t, s) \leq (\alpha - 1)(1 - t)t^{\alpha-1}$ , for  $t, s \in (0, 1)$ .

**Remark 2.1** Obviously, by Lemma 2.4, we have  $u(t) \geq 0$  if  $h(t) \geq 0$  on  $t \in [0, 1]$ , where  $u(t)$  and  $h(t)$  are defined as (2.1).

Now we introduce the following two definitions concerned with the upper and lower solutions of the fractional boundary value problem (1.1) and (1.2).

**Definition 2.3** A function  $\beta$  is called a lower solution of the fractional boundary value problem (1.1) and (1.2), if  $\beta \in C[0, 1]$  and  $\beta(t)$  satisfies

$$\begin{aligned} -D_{0+}^{\alpha} \beta(t) &\leq f(t, \beta(t)), \quad 0 < t < 1, 2 < \alpha \leq 3, \\ \beta(0) &\leq 0, \quad \beta(1) \leq 0, \quad \beta'(0) \leq 0, \end{aligned}$$

where  $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ .

**Definition 2.4** A function  $\gamma$  is called an upper solution of the fractional boundary value problem (1.1) and (1.2), if  $\gamma \in C[0, 1]$  and  $\gamma(t)$  satisfies

$$\begin{aligned} -D_{0+}^{\alpha} \gamma(t) &\geq f(t, \gamma(t)), \quad 0 < t < 1, 2 < \alpha \leq 3, \\ \gamma(0) &\geq 0, \quad \gamma(1) \geq 0, \quad \gamma'(0) \geq 0, \end{aligned}$$

where  $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ .

The following definition is about the nonnegative continuous concave functional.

**Definition 2.5** The map  $\theta$  is said to be a nonnegative continuous concave functional on a cone  $P$  of a real Banach space  $E$  provided that  $\theta : P \rightarrow [0, \infty)$  is continuous and

$$\theta(tx + (1 - t)y) \geq t\theta(x) + (1 - t)\theta(y),$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ .

The following lemma is fundamental in the proofs of our main results.

**Lemma 2.5** ([33]) *Let  $P$  be a cone in a real Banach space  $E$ ,  $P_c = \{x \in P \mid \|x\| \leq c\}$ ,  $\theta$  a nonnegative continuous concave functional on  $P$  such that  $\theta(x) \leq \|x\|$ , for all  $x \in \overline{P}_c$ , and  $P(\theta, b, d) = \{x \in P \mid b \leq \theta(x), \|x\| \leq d\}$ . Suppose  $A : \overline{P}_c \rightarrow \overline{P}_c$  is completely continuous and there exist constants  $0 < a < b < d \leq c$  such that*

(C1)  $\{x \in P(\theta, b, d) \mid \theta(x) > b\} \neq \emptyset$  and  $\theta(Ax) > b$  for  $x \in P(\theta, b, d)$ ;

(C2)  $\|Ax\| < a$  for  $x \leq a$ ;

(C3)  $\theta(Ax) > b$  for  $x \in P(\theta, b, c)$  with  $\|Ax\| > d$ .

*Then  $A$  has at least three fixed points  $x_1, x_2, x_3$  with*

$$\|x_1\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \text{and} \quad \theta(x_3) < b.$$

**Remark 2.2** If we have  $d = c$ , then condition (C1) of Lemma 2.5 implies condition (C3) of Lemma 2.5.

For convenience, we set  $q(t) = t^{\alpha-1}(1 - t)$ ,  $k(s) = s(1 - s)^{\alpha-1}$ , then

$$q(t)k(s) \leq \Gamma(\alpha)G(t, s) \leq (\alpha - 1)k(s).$$

### 3 Single positive solution

In this section, we establish the existence of single positive solution for the boundary value problem (1.1) and (1.2) by the lower and upper solution method. In this section, we set  $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ . As an application, an example is given to illustrate the main results.

**Lemma 3.1** *If  $u$  is a positive solution of (1.1) and (1.2), then there exist two constants  $r$  and  $R$  such that  $r\rho(t) \leq u(t) \leq R\rho(t)$ , where  $\rho(t) = \int_0^1 G(t, s) ds$ .*

*Proof* Since  $u \in C[0, 1]$ , there exists  $M' > 0$  so that  $|u(t)| \leq M'$  for  $t \in [0, 1]$ . Taking

$$r = \min_{(t,u) \in [0,1] \times [0,M']} f(t, u(t)), \quad R = \max_{(t,u) \in [0,1] \times [0,M']} f(t, u(t)).$$

In view of Lemma 2.3, we have

$$r \int_0^1 G(t, s) ds \leq u(t) = \int_0^1 G(t, s)f(s, u(s)) ds \leq R \int_0^1 G(t, s) ds.$$

By direct computation, we have

$$\rho(t) = \int_0^1 G(t,s) ds.$$

Thus we finish the proof of Lemma 3.1. □

**Theorem 3.1** *The fractional boundary value problem (1.1) and (1.2) has a positive solution  $u$  if the following conditions are satisfied:*

(H<sub>f</sub>)  $f(t, u) \in C([0, 1] \times [0, +\infty), \mathbb{R}^+)$  is nondecreasing relative to  $u$ ,  $f(t, \rho(t)) \not\equiv 0$  for  $t \in (0, 1)$  and there exists a positive constant  $\mu < 1$  such that

$$k^\mu f(t, u) \leq f(t, ku), \quad \forall 0 \leq k \leq 1.$$

*Proof* At first, we will prove that the functions  $\beta(t) = k_1 g(t)$ ,  $\gamma(t) = k_2 g(t)$  are lower and upper solutions of (1.1) and (1.2), respectively, where  $0 < k_1 \leq \min\{\frac{1}{a_2}, (a_1)^{\frac{\mu}{1-\mu}}\}$ ,  $k_2 \geq \max\{\frac{1}{a_1}, (a_2)^{\frac{\mu}{1-\mu}}\}$ ,

$$a_1 = \min\left\{1, \inf_{t \in [0,1]} f(t, \rho(t))\right\} > 0,$$

$$a_2 = \max\left\{1, \sup_{t \in [0,1]} f(t, \rho(t))\right\}$$

and

$$g(t) = \int_0^1 G(t,s) f(s, \rho(s)) ds.$$

In view of Lemma 2.3 and Remark 2.1, we know that  $g(t)$  is a positive solution of the following problem:

$$\begin{aligned} -D_{0^+}^\alpha g(t) &= f(t, \rho(t)), \quad 0 < t < 1, 2 < \alpha \leq 3, \\ g(0) &= g(1) = g'(0) = 0. \end{aligned}$$

From the conclusion of Lemma 3.1, we know that

$$a_1 \rho(t) \leq g(t) \leq a_2 \rho(t), \quad t \in [0, 1].$$

Thus, by virtue of the assumptions of Theorem 3.1, one shows that

$$\begin{aligned} k_1 a_1 &\leq \frac{\beta(t)}{\rho(t)} \leq k_1 a_2 \leq 1, \\ \frac{1}{k_2 a_2} &\leq \frac{\rho(t)}{\gamma(t)} \leq \frac{1}{k_2 a_1} \leq 1, \\ (k_1 a_1)^\mu &\geq k_1, \quad (k_2 a_2)^\mu \leq k_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 f(t, \beta(t)) &= f\left(t, \frac{\beta(t)}{\rho(t)}\rho(t)\right) \geq \left(\frac{\beta(t)}{\rho(t)}\right)^\mu f(t, \rho(t)) \\
 &\geq (k_1 a_1)^\mu f(t, \rho(t)) \geq k_1 f(t, \rho(t)), \\
 k_2 f(t, \rho(t)) &= k_2 f\left(t, \frac{\rho(t)}{\gamma(t)}\gamma(t)\right) \geq k_2 \left(\frac{\rho(t)}{\gamma(t)}\right)^\mu f(t, \gamma(t)) \\
 &\geq k_2 (k_2 a_2)^{-\mu} f(t, \gamma(t)) \geq f(t, \gamma(t)).
 \end{aligned}$$

It implies that

$$\begin{aligned}
 -D_{0^+}^\alpha \beta(t) &= k_1 f(t, \rho(t)) \leq f(t, \beta(t)), \quad 0 < t < 1, 2 < \alpha \leq 3, \\
 -D_{0^+}^\alpha \gamma(t) &= k_2 f(t, \rho(t)) \geq f(t, \gamma(t)), \quad 0 < t < 1, 2 < \alpha \leq 3.
 \end{aligned}$$

Obviously,  $\beta(t) = k_1 g(t)$ ,  $\gamma(t) = k_2 g(t)$  satisfy the boundary conditions (1.2). So,  $\beta(t) = k_1 g(t)$ ,  $\gamma(t) = k_2 g(t)$  are lower and upper solutions of (1.1) and (1.2), respectively.

Next, we will prove the fractional boundary value problem

$$\begin{aligned}
 -D_{0^+}^\alpha u(t) &= g(t, u(t)), \quad 0 < t < 1, 2 < \alpha \leq 3, \\
 u(0) = u(1) &= u'(0) = 0,
 \end{aligned} \tag{3.1}$$

has a solution, where

$$g(t, u(t)) = \begin{cases} f(t, \beta(t)), & \text{if } u(t) \leq \beta(t), \\ f(t, u(t)), & \text{if } \beta(t) \leq u(t) \leq \gamma(t), \\ f(t, \gamma(t)), & \text{if } \gamma(t) \leq u(t). \end{cases}$$

Thus, we consider that the operator  $T : C[0, 1] \rightarrow C[0, 1]$  is defined as follows:

$$Tu(t) = \int_0^1 G(t, s)g(s, u(s)) ds,$$

where  $G(t, s)$  is defined as (2.3). It is clear that  $T$  is continuous in  $C[0, 1]$ . Since the function  $f(t, u)$  is nondecreasing in  $u$ , this shows that, for any  $u \in C([0, 1], [0, +\infty))$ ,

$$f(t, \beta(t)) \leq g(t, u(t)) \leq f(t, \gamma(t)), \quad \text{for } t \in [0, 1].$$

The operator  $T : C[0, 1] \rightarrow C[0, 1]$  is continuous in view of continuity of  $G(t, s)$  and  $g(t, u(t))$ . By means of the Arzela-Ascoli theorem,  $T$  is a compact operator. Therefore, from the Leray-Schauder fixed point theorem, the operator  $T$  has a fixed point, *i.e.*, the fractional boundary value problem (3.1) has a solution.

Finally, we will prove that the fractional boundary value problem (1.1) and (1.2) has a positive solution.

Suppose that  $u^*(t)$  is a solution of the fractional boundary value problem (3.1). Since the function  $f(t, u)$  is nondecreasing in  $u$ , we know that

$$f(t, \beta(t)) \leq g(t, u^*(t)) \leq f(t, \gamma(t)), \quad \text{for } t \in [0, 1].$$

Thus,

$$\begin{aligned}
 -D_{0+}^\alpha z(t) &\geq f(t, \gamma(t)) - g(t, u^*(t)) \geq 0, \\
 z(0) = z(1) = z'(0) &= 0,
 \end{aligned}$$

where  $z(t) = \gamma(t) - u^*(t)$ . By Remark 2.1,  $z(t) \geq 0$ , i.e.,  $u^*(t) \leq \gamma(t)$  for  $t \in [0, 1]$ . Similarly,  $\beta(t) \leq u^*(t)$  for  $t \in [0, 1]$ . Therefore,  $u^*(t)$  is a positive solution of the fractional boundary value problem (1.1) and (1.2). We have finished the proof of Theorem 3.1.  $\square$

In the following, we present a simple example to explain our results.

**Example 3.1** Consider the boundary value problem

$$\begin{aligned}
 D_{0+}^{\frac{5}{2}} u(t) + f(t, u) &= 0, \quad 0 < t < 1, \\
 u(0) = u(1) = u'(0) &= 0
 \end{aligned} \tag{3.2}$$

and

$$f(t, u) = t + u^\mu, \quad 0 < \mu < 1.$$

*Proof* Since  $k^\mu \leq 1$  for  $0 < \mu < 1$  and  $0 \leq k \leq 1$ . It is easy to verify that

$$k^\mu f(t, u) = k^\mu t + k^\mu u^\mu \leq t + (ku)^\mu = f(t, ku).$$

Thus, by Theorem 3.1, we know that the boundary value problem (3.2) has a positive solution  $u$ .  $\square$

#### 4 Multiple positive solutions

In this section, we establish the existence of multiple positive solutions for the boundary value problem (1.1) and (1.2) by the Legget-Williams fixed point theorem. In this section, we set  $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ . As an application, an example is given to illustrate the main results.

Let the Banach space  $E = C[0, 1]$  be endowed with the norm  $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ . Define the cone  $P \subset E$  by

$$P = \left\{ u \in E : u(t) \geq \frac{q(t)}{\alpha - 1} \|u\|, t \in [0, 1] \right\}.$$

Let the nonnegative continuous concave functional  $\theta$  on the cone  $P$  be defined by

$$\theta(u) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u(t)|.$$

Suppose that  $u$  is a solution of the boundary value problem (1.1) and (1.2). Then

$$u(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad t \in [0, 1].$$



We define an operator  $A : P \rightarrow E$  as follows:

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s)) ds, \quad t \in [0,1].$$

By Lemma 2.4, we have

$$\begin{aligned} \|Au\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)f(s,u(s)) ds, \\ (Au)(t) &\geq \frac{1}{\Gamma(\alpha)} \int_0^1 q(t)k(s)f(s,u(s)) ds \geq \frac{q(t)}{\alpha - 1} \|Au\|. \end{aligned}$$

Thus,  $A(P) \subset P$ .

Then we have the following lemma.

**Lemma 4.1**  $A : P \rightarrow P$  is completely continuous.

*Proof* The operator  $A : P \rightarrow P$  is continuous in view of the continuity of  $G(t,s)$  and  $f(t,u(t))$ . By means of the Arzela-Ascoli theorem,  $A : P \rightarrow P$  is completely continuous.  $\square$

For convenience, we denote

$$\begin{aligned} M &= \left( \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s) ds \right)^{-1}, & N &= \left( \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\alpha - 1)\sigma k(s) ds \right)^{-1}, \\ \tilde{N} &= \left( \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\alpha - 1)k(s) ds \right)^{-1}, & \sigma &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \frac{q(t)}{\alpha - 1}. \end{aligned}$$

**Theorem 4.1** Suppose  $f(t,u)$  is continuous on  $[0,1] \times [0,+\infty)$  and there exist constants  $0 < a < b < c$  such that the following assumptions hold:

- (B1)  $f(t,u) < Ma$ , for  $(t,u) \in [0,1] \times [0,a]$ ;
- (B2)  $f(t,u) \geq Nb$ , for  $(t,u) \in [1/4,3/4] \times [b,c]$ ;
- (B3)  $f(t,u) \leq Mc$ , for  $(t,u) \in [0,1] \times [0,c]$ .

Then the boundary value problem (1.1) and (1.2) has at least three positive solutions  $u_1, u_2, u_3$  with

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< a, & b &< \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c, \\ a &< \max_{0 \leq t \leq 1} |u_3(t)| \leq c, & \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| &< b. \end{aligned}$$

*Proof* We show that all the conditions of Lemma 2.5 are satisfied.

If  $u \in \bar{P}_c$ , then  $\|u\| \leq c$ . Assumption (B3) implies  $f(t,u(t)) \leq Mc$  for  $0 \leq t \leq 1$ . Consequently,

$$\begin{aligned} \|Au\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)f(s,u(s)) ds \\ &< Mc \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s) ds = c = \|u\|. \end{aligned}$$

Hence,  $A : \bar{P}_c \rightarrow \bar{P}_c$ . In the same way, if  $u \in \bar{P}_a$ , then assumption (C2) of Lemma 2.5 is satisfied.

To check condition (C1) of Lemma 2.5, we choose  $u(t) = (b + c)/2, 0 \leq t \leq 1$ . It is easy to see that  $u(t) = (b + c)/2 \in P(\theta, b, c), \theta(u) = \theta((b + c)/2) > b$ , consequently,  $\{u \in P(\theta, b, c) | \theta(u) > b\} \neq \emptyset$ . Hence, if  $u \in P(\theta, b, c)$ , then  $b \leq u(t) \leq c$  for  $1/4 \leq t \leq 3/4$ . Thus,

$$\begin{aligned} \theta(Au) &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |(Au)(t)| \geq \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} q(t)k(s)f(s, u(s)) \, ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\alpha - 1)\sigma k(s)f(s, u(s)) \, ds \\ &\geq Nb \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\alpha - 1)\sigma k(s) \, ds = b, \end{aligned}$$

i.e.,  $\theta(Au) > b$  for all  $u \in P(\theta, b, c)$ . This shows that condition (C1) of Lemma 2.5 is satisfied.

By Lemma 2.5 and Remark 2.1, the boundary value problem (1.1) and (1.2) has at least three positive solutions  $u_1, u_2$ , and  $u_3$ , satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< a, & b < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| &\leq c, \\ a < \max_{0 \leq t \leq 1} |u_3(t)| &\leq c, & \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| &< b. \end{aligned}$$

The proof is complete. □

**Corollary 4.1** *Suppose  $f(t, u)$  is continuous on  $[0, 1] \times [0, +\infty)$  and there exist constants  $0 < a_1 \leq \sigma b_1 < (\alpha - 1)\sigma b_1 < b_1 < c_1$  such that the following assumptions hold:*

- (B4)  $f(t, u) < Ma_1$ , for  $(t, u) \in [0, 1] \times [0, a_1]$ ;
- (B5)  $f(t, u) \geq \tilde{N}b_1$ , for  $(t, u) \in [1/4, 3/4] \times [\sigma b_1, c_1]$ ;
- (B6)  $f(t, u) \leq Mc_1$ , for  $(t, u) \in [0, 1] \times [0, c_1]$ .

*Then the boundary value problem (1.1) and (1.2) has at least three positive solutions  $u_1, u_2, u_3$ , satisfying*

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< a_1, & \sigma b_1 < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| &\leq c_1, \\ a_1 < \max_{0 \leq t \leq 1} |u_3(t)| &\leq c_1, & \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| &< \sigma b_1. \end{aligned}$$

*Proof* If we choose  $a = a_1, b = \sigma b_1$ , and  $c = c_1$ , then from Theorem 4.1, the conclusion holds. □

**Theorem 4.2** *If condition (B3) in Theorem 4.1 is replaced by*

$$(B3') \quad \limsup_{u \rightarrow \infty} \frac{f(t, u)}{u} < M.$$

*Then the conclusion of Theorem 4.1 also holds.*

*Proof* We only need to show there exists a number  $c'$  with  $c' > c$  and  $A : \bar{P}_{c'} \rightarrow \bar{P}_{c'}$ .

From (B3'), we know there exist  $R$  and  $\varepsilon < M$ , such that

$$\frac{f(t, u)}{u} \leq \varepsilon, \quad u \geq R. \tag{4.1}$$

Let  $L = \max_{u \in [0, R]} f(t, u), t \in [0, 1]$ .

In view of (4.1), it is easy to see that

$$f(t, u) \leq L + \varepsilon u, \quad u \geq 0. \tag{4.2}$$

Now let  $c'$  be such that

$$c' > \max \left\{ c, \frac{L}{M - \varepsilon} \right\}. \tag{4.3}$$

Then for arbitrary  $u \in \bar{P}_{c'}, t \in [0, 1]$ , from (4.2) and (4.3), we obtain

$$\begin{aligned} \|Au\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)f(s, u(s)) \, ds \\ &\leq (L + \varepsilon c') \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s) \, ds \leq c'. \end{aligned}$$

Thus,  $A : \bar{P}_{c'} \rightarrow \bar{P}_{c'}$ . □

**Theorem 4.3** *Suppose that there exist constants  $0 < a'_1 < \sigma b'_1 < (\alpha - 1)\sigma b'_1 < b'_1 < c'_1 < a'_2 < \sigma b'_2 < (\alpha - 1)\sigma b'_2 < b'_2 < c'_2 < \dots < a'_n, n \in \mathbb{N}$ , for  $i = 1, 2, \dots, n$ , such that*

(B7)  $f(t, u) < Ma'_i$ , for  $(t, u) \in [0, 1] \times [0, a'_i]$ ;

(B8)  $f(t, u) \geq \tilde{N}b'_i$ , for  $(t, u) \in [1/4, 3/4] \times [\sigma b'_i, c'_i]$ .

*Then the boundary value problem (1.1) and (1.2) has at least  $2n - 1$  positive solutions.*

*Proof* When  $n = 1$ , it is immediate from condition (B7) that  $A : \bar{P}_{a'_1} \rightarrow \bar{P}_{a'_1}$ , which means that  $A$  has at least one point  $u_1 \in \bar{P}_{a'_1}$  by the Schauder fixed point theorem.

When  $n = 2$ , it is clear that Corollary 4.1 holds (with  $c' = a'_2$ ). Then we can obtain at least three positive solutions  $u_1, u_2$ , and  $u_3$ , satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| < a'_1, \quad \sigma b'_1 < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c'_1, \\ a'_1 < \max_{0 \leq t \leq 1} |u_3(t)| \leq c'_1 \quad \text{and} \quad \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| < \sigma b'_1. \end{aligned}$$

In this way, we finish the proof by induction. The proof is complete. □

In the following, we present a simple example to illustrate our results.

**Example 4.1** Consider the boundary value problem

$$D^{\frac{5}{3}}_0+ u(t) + f(t, u) = 0, \quad 0 < t < 1, \tag{4.4}$$

$$u(0) = u(1) = u'(0) = 0, \tag{4.5}$$

where

$$f(t, u) = \begin{cases} \frac{t}{10} + 177u^2, & \text{for } u \leq 1, \\ 176 + \frac{t}{10} + u, & \text{for } u > 1. \end{cases}$$

We have

$$M = \left( \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s) ds \right)^{-1} = \frac{35\sqrt{\pi}}{8} \approx 13.7375,$$

$$N = \left( \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\alpha - 1)\sigma k(s) ds \right)^{-1} \approx 176.801.$$

Choosing  $a = \frac{1}{12}$ ,  $b = 1$ ,  $c = 14$ , we have

$$f(t, u) = \frac{t}{10} + 177u^2 \leq 1.3292 \leq Ma \approx 1.37375, \quad \text{for } (t, u) \in [0, 1] \times \left[0, \frac{1}{12}\right],$$

$$f(t, u) = 176 + \frac{t}{10} + u \geq 177.025 \geq Nb \approx 176.801, \quad \text{for } (t, u) \in [1/4, 3/4] \times [1, 14],$$

$$f(t, u) = 176 + \frac{t}{10} + u \leq 190.1 \leq Mc \approx 192.325, \quad \text{for } (t, u) \in [0, 1] \times [0, 14].$$

From Theorem 4.1, the boundary value problem (4.4) and (4.5) has at least three positive solutions  $u_1, u_2, u_3$  satisfying

$$\max_{0 \leq t \leq 1} |u_1(t)| < \frac{1}{12}, \quad 1 < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 14,$$

$$\frac{1}{12} < \max_{0 \leq t \leq 1} |u_3(t)| \leq 14, \quad \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u_3(t)| < 1.$$

## 5 Conclusions

In this paper, we have studied the existence of positive solutions for a boundary value problem of nonlinear fractional differential equations involving the Riemann-Liouville fractional derivative. The existence of a single positive solution for the given problem has been obtained by using the properties of the Green's function and the lower and upper solution method, while the existence of multiple positive solutions is based on the Leggett-Williams fixed point theorem. The main results are well illustrated with the help of examples. Our results improve the work presented in [31].

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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