Meet continuity properties of posets

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\textbf{A B S T R A C T} \\
In this paper we consider a number of properties of posets that are not directed complete: in particular, meet continuous posets, locally meet continuous posets and PI-meet continuous posets are introduced. Characterizations of (locally) meet continuous posets are presented. The main results are: (1) A poset is meet continuous iff its lattice of Scott closed subsets is a complete Heyting algebra; (2) A poset is a meet continuous poset with a lower hereditary Scott topology iff its supertopology is contained in its local Scott topology and the lattice of all local Scott closed sets is a complete Heyting algebra; and (3) A poset with a lower hereditary Scott topology is meet continuous iff it is locally meet continuous, iff it is PI-meet continuous.

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1. Introduction

A meet continuous lattice is a complete lattice in which binary meets distribute over directed suprema (see [2]). This algebraic notion has a purely topological characterization that can be generalized to the setting of directed complete partial orders (dcpo) in [2,3]: A dcpo $P$ is meet continuous if for any $x \in P$ and any directed subset $D$ with $x \leq \sup D$, one has $x \in \text{cl}_\sigma (\downarrow D \cap \downarrow x)$, where $\text{cl}_\sigma (\downarrow D \cap \downarrow x)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$. Our goal is to further generalize the concept of meet continuity to the setting of posets that are not directed complete; these posets recently have received increasing attention (see [5–13]). The concept of a meet continuous dcpo can be extended to the setting of posets using slight modifications of the definition where it relies on the Scott topology. Using a newly-defined intrinsic topology — the local Scott topology, the concept of locally meet continuous posets is also introduced. We show that the local Scott topology has its own distinguishing properties for general posets. Characterizations and properties of (locally) meet continuous posets are obtained in terms of the lattice properties of the Scott topology and of the local Scott topology. Posets for which the Scott topology coincides with the local Scott topology are meet continuous iff they are locally meet continuous, iff they are PI-meet continuous, while the three kinds of meet continuities do not imply each other generally. Comprehensive comparisons of the three kinds of meet continuities are given and some subtle (counter) examples are presented in the last section. The results obtained in this paper reveal some subtle properties of posets that show that topology plays an important role in their study.

2. Preliminaries

The following are some basic notions of domain theory which can be found in [1,2,4].

In a poset $P$, a principal ideal (principal filter) is a set of the form $\downarrow x = \{ y \in P : y \leq x \}$ ($\uparrow x = \{ y \in P : x \leq y \}$). A closed interval is a set of the form $\uparrow x \cap \downarrow y$ for $x \leq y$. Note that closed intervals are always non-empty. A subset $A$ of $P$ is called

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order convex if \( x, z \in A \) and \( z \leq y \leq x \) implies \( y \in A \). The notation \( \sup_x A \) denotes the supremum of the subset \( A \subseteq \downarrow x \) in the principal ideal \( \downarrow x \).

Recall that for a poset \( P \) and all \( x, y \in P \), we say that \( x \) approximates \( y \), written \( x \ll y \) if whenever \( D \) is a directed set for which \( \sup D \) exists, if \( y \leq \sup D \), then \( x \leq d \) for some \( d \in D \). The poset \( P \) is said to be continuous if every element is the directed supremum of elements that approximate it.

A subset \( A \) of a poset \( P \) is Scott closed if \( \langle A \rangle = A \) and for any directed set \( D \subseteq A \), \( \sup D \in A \) whenever \( \sup D \) exists. The complement of a Scott closed set is a Scott open set, and the family of these sets forms a topology, called the Scott topology, denoted \( \sigma(P) \). The topology generated by the complements of all principal filters \( \uparrow x \) (resp., principal ideals \( \downarrow x \)) is called the lower topology (resp., upper topology) and denoted \( \omega(P) \) (resp., \( \nu(P) \)). The join of the lower and upper topologies is called the interval topology. The common refinement \( \sigma(P) \vee \omega(P) \) of the Scott and the lower topology is called the Lawson topology, denoted \( \lambda(P) \).

A map \( f : P \to Q \) between posets is Scott continuous if it is continuous with respect to the Scott topologies on \( P \) and \( Q \). It is easy to show that \( f \) is Scott continuous iff it is monotone and preserves all existing directed suprema.

The following results will be used in what follows: the point to note is that they refer to sets that may not be directed complete. The proof of each is straightforward.

**Lemma 2.1.** Let \( P \) be a poset and \( U \subseteq P \) a subset. Then for all \( \emptyset \neq A \subseteq U \), \( \sup A = \sup_U A \) whenever one of them exists, where \( \sup_U A \) denotes the supremum of \( A \) in \( U \) (with the inherited order).

**Lemma 2.2.** Let \( P \) be a poset and \( U \) an upper set of \( P \). Let \( \sigma(P)|_U := \{W \cap U : W \in \sigma(P)\} \). If \( U \in \sigma(P) \), then \( \sigma(P)|_U = \sigma(U) \).

**Lemma 2.3.** Let \( P \) be a poset and \( A \subseteq P \). If \( s, t \) are two upper bounds of \( A \) with \( s \leq t \) and if \( A \) has a supremum in \( \downarrow t \), denoted \( \sup A \), then \( A \) has a supremum in \( \downarrow s \), in which case \( \sup A = \sup_U A \).

**Lemma 2.4.** Let \( P \) be a poset and \( M \subseteq P \) an order convex subset. Then for any non-empty subset \( A \) of \( M \), if \( \sup_M A := t \) exists, then \( \sup_M A = \sup_U A \).

### 3. Meet continuous posets

**Definition 3.1.** Let \( P \) be a poset. Then \( P \) is called a meet continuous poset (an MC-poset, in short) if for any \( x \in P \) and any directed subset \( D \), if \( \sup D \) exists and \( x \leq \sup D \), then \( x \in \mathcal{C}_x (\downarrow \sup D \downarrow x) \), where \( \mathcal{C}_x (\downarrow \sup D \downarrow x) \) is the Scott closure of the set \( \downarrow \sup D \downarrow x \).

**Remark 3.2.** For dcpo \( \sigma(P) \) the preceding definition of meet continuity is equivalent to the standard one ([2, Definition III-2.1]).

**Example 3.3.** Let \( P = \{0, 1, 2, \ldots\} \cup \{\infty, a\} \). Define a partial order “\( \leq \)” on \( P \) such that \( 0 < 1 < 2 < \cdots < \infty \) and \( 0 < a < \infty \). Then for the directed subset \( D = \{0, 1, 2, \ldots\} \) and \( x = a \), we have that \( \downarrow \sup D \downarrow x = \{0\} \) and \( a = x \notin \mathcal{C}_x (\downarrow \sup D \downarrow x) \). This shows that \( P \) is not meet continuous.

The proofs for Theorem 3.4 to Proposition 3.7 are similar to those of the analogous results for dcpo in [2].

**Theorem 3.4.** A poset \( P \) is meet continuous iff for all \( U \in \sigma(P) \) and all \( x \in P \), one has \( \uparrow(U \cap \downarrow x) \in \sigma(P) \).

**Proof.** \( \Rightarrow \): Let \( x \in P \) and \( U \in \sigma(P) \). Suppose that \( D \) is a directed subset for which \( \sup D \) exists and satisfies \( \sup D \in \uparrow(U \cap \downarrow \downarrow x) \). Then there is \( y \in U \cap \downarrow \downarrow x \) such that \( y \leq \sup D \). By the meet continuity of \( P \), we have \( \downarrow \downarrow \downarrow y \cap \downarrow U \neq \emptyset \). So, \( \downarrow \downarrow \downarrow y \cap \downarrow U \cap \downarrow \downarrow \downarrow x = \emptyset \). This shows that \( \uparrow(U \cap \downarrow x) \) is Scott open.

\( \Leftarrow \): Let \( x \in P \) and \( D \) a directed subset for which \( \sup D \) exists with \( x \leq \sup D \). If \( x \) is not in \( \mathcal{C}_x (\downarrow \sup D \downarrow x) \), then there is \( U \in \sigma(P) \) such that \( x \in U \) and \( \downarrow U \cap \downarrow \sup D \cap \downarrow \downarrow x = \emptyset \). This implies \( \uparrow(U \cap \downarrow x) \cap D = \emptyset \). It is clear that \( \sup D \in \uparrow(U \cap \downarrow x) \). Then by the assumption that \( \uparrow(U \cap \downarrow x) \) is Scott open, there is \( d \in \uparrow(U \cap \downarrow x) \cap D \), a contradiction. This shows that \( x \in \mathcal{C}_x (\downarrow \sup D \downarrow x) \) and hence \( P \) is meet continuous. \( \Box \)

By Theorem 3.4, we immediately have

**Corollary 3.5.** A poset \( P \) is an MC-poset iff for any \( U \in \sigma(P) \) and any lower subset \( A \) of \( P \), one has

\( \uparrow(U \cap A) = \bigcup_{x \in A} \uparrow(U \cap \downarrow x) \in \sigma(P) \).

**Proposition 3.6.** Let \( P \) be an MC-poset.
(i) If \( U \in \lambda(P) \), then \( \uparrow U \in \sigma(P) \);
(ii) If \( X \) is an upper set, then \( \text{int}_x X = \text{int}_x X \);
(iii) If \( X \) is a lower set, then \( \mathcal{C}_x X = \mathcal{C}_x X \).

**Proof.** Straightforward. \( \Box \)

**Proposition 3.7.** If \( P \) is a continuous poset, then \( P \) is an MC-poset.

**Proof.** Let \( x \in P \) and \( D \) a directed set with existing \( \sup D \geq x \). Then \( \downarrow x \subseteq \downarrow D \) and \( \downarrow x \subseteq \downarrow D \cap \downarrow x \). So, by the Scott closedness of \( \mathcal{C}_x (\downarrow \sup D \downarrow x) \) and the continuity of \( P \), we have \( x = \sup \downarrow x \in \mathcal{C}_x (\downarrow \sup D \downarrow x) \), as desired. \( \Box \)
We arrive at a characterization of MC-posets via the lattice of Scott closed subsets.

**Theorem 3.8.** Let $P$ be a poset. Then the following conditions are equivalent:

(1) $P$ is an MC-poset;

(2) $\sigma(P)^{op}$ is a complete Heyting algebra (cha, in short).

**Proof.** (1) $\Rightarrow$ (2): It is clear that $\sigma(P)^{op} \cong \sigma^*(P)$ (the lattice of all Scott closed sets of $P$). It suffices to show that the frame distributive law

$$F \land \left( \bigvee_{i \in I} F_i \right) = \bigvee_{i \in I} (F \land F_i)$$

holds for $\sigma^*(P)$, where $F, F_i \in \sigma^*(P)$ ($i \in I$). It is clear that $F \land \left( \bigvee_{i \in I} F_i \right) \geq \bigvee_{i \in I} (F \land F_i)$. To show $F \land \left( \bigvee_{i \in I} F_i \right) \leq \bigvee_{i \in I} (F \land F_i)$, let $x \in F \land \left( \bigvee_{i \in I} F_i \right) = F \land \bigvee_{i \in I} F_i$. Then for all $U \in \sigma(P)$ with $x \in U$, by Corollary 3.5, we have $x \in U \land F$ and $x \in \uparrow (U \land F) \in \sigma(P)$. And then there is $i_0 \in I$ such that $\uparrow (U \land F) \cap F_{i_0} \neq \emptyset$. This implies that $(U \land F) \cap F_{i_0} = U \land (F \land F_{i_0}) \neq \emptyset$. By the arbitrariness of $U \in \sigma(P)$, we have $x \in \bigcap_{i \in I} \sigma(P)(U \land F_i) = \bigvee_{i \in I} (F \land F_i)$. This finishes the proof of the frame distributivity of $\sigma^*(P)$. So, $\sigma^*(P)$ is a complete Heyting algebra.

(2) $\Rightarrow$ (1): Let $x \in P$ and $D$ a directed subset with existing $\text{sup } D = x$. Then $\{ \downarrow d : d \in D \}$ is a directed subset of $\sigma^*(P)$. So, $\text{sup } D \in \sigma_\sigma(\downarrow D) = \bigvee_{d \in D} \downarrow d$ and thus $x \leq \bigvee_{d \in D} \downarrow d$. By (2),

$$x \in \downarrow x \subseteq x \land \left( \bigvee_{d \in D} d \right) = \bigvee_{d \in D} (\downarrow d \land \downarrow x) = \sigma_\sigma \left( \bigcup_{d \in D} (\downarrow d \land \downarrow x) \right) = \sigma_\sigma(\downarrow D \land \downarrow x).$$

Thus $P$ is an MC-poset. □

We now introduce a new construction of posets that gives rise to an MC-poset from two given MC-posets.

**Definition 3.9.** Let $P$ and $Q$ be posets and $m$ a maximal element of $P$. Then the vertical sum w.r.t. $m$ of $P$ and $Q$, denoted $P \vee_m Q$, is the set $P \cup Q$ with a partial order defined by

$$x \leq y \text{ iff } \begin{cases} x \leq_P y, & x, y \in P; \\ x \leq_Q y, & x, y \in Q; \\ x \leq_P m, & y \in Q. \end{cases}$$

Since dcpos always have maximal elements, the vertical sum w.r.t. $m$ for dcpos can always be constructed.

**Lemma 3.10.** Let $P$ and $Q$ be posets and $m$ a maximal element of $P$. And let $D$ be a non-empty subset of $P \vee_m Q$.

(1) If $D \subseteq P$, then $\text{sup } D = \text{sup}_P D$ whenever one of them exists;

(2) If $D \subseteq Q$, then $\text{sup } D = \text{sup}_Q D$ whenever one of them exists;

(3) If $D$ is a directed set for which $\text{sup } D$ exists and $D \cap Q \neq \emptyset$, then $D' = D \cap Q \neq \emptyset$ is directed and $\text{sup } D = \text{sup}_Q D'$.

**Proof.** Straightforward. □

**Proposition 3.11.** Let $P$ and $Q$ be posets and $m$ a maximal element of $P$. Then $\sigma(P) = \sigma(P \vee_m Q)|_P$ and $\sigma(Q) = \sigma(P \vee_m Q)|_Q$.

**Proof.** Straightforward. □

**Theorem 3.12.** Let $P$ and $Q$ be MC-posets and $m$ a maximal element of $P$. Then $P \vee_m Q$ is also an MC-poset.

**Proof.** Let $x \in P \vee_m Q$ and let $D$ be a directed set for which $\text{sup } D$ exists and suppose $x \leq \text{sup } D$ in $P \vee_m Q$. We divide the proof into three cases.

Case 1: $\text{sup } D \in P$. This implies $D \subseteq P$ and $x \in P$. It follows from Lemma 3.10(1) that $\text{sup}_P D = \text{sup } D \geq x$. By the meet continuity of $P$ and Proposition 3.11, $x \in \sigma_\sigma(\downarrow D \cap \downarrow x) = \sigma_\sigma(P \vee_m Q)(\downarrow D \cap \downarrow x)$.

Case 2: $\text{sup } D \in Q$ and $x \in P$. Trivially, $x \leq m \leq \text{sup } D$. Since $Q$ is Scott open in $P \vee_m Q$, we have $D \cap Q \neq \emptyset$ and $m \leq \downarrow D$. So, $x \leq \downarrow x \leftarrow \downarrow m \subseteq \downarrow x \cap \downarrow D$ and $x \in \sigma_\sigma(P \vee_m Q)(\downarrow D \cap \downarrow x)$.

Case 3: $\text{sup } D \in Q$ and $x \in Q$. Since $Q$ is Scott open in $P \vee_m Q$, we have $D \cap Q \neq \emptyset$. By Lemma 3.10(3), $D' = D \cap Q$ is directed, and $x \leq \text{sup } D = \text{sup}_Q D'$. Since $D'$ is directed and $Q$ is an MC-poset, $x \in \sigma_\sigma(Q)(\downarrow D' \cap \downarrow x)$, where $\downarrow x = \downarrow x \cap \downarrow Q$. It follows from Proposition 3.11 that $x \in \sigma_\sigma(Q)(\downarrow Q \cap \downarrow x) = \sigma_\sigma(P \vee_m Q)(\downarrow D \cap \downarrow x)$.

To sum up, in all cases, $x \in \sigma_\sigma(P \vee_m Q)(\downarrow D \cap \downarrow x)$ and $P \vee_m Q$ is an MC-poset. □

### 4. Posets with lower hereditary Scott topologies

It is known that in a dcpo for any Scott closed set, the relative Scott topology agrees with the Scott topology on that sub-dcpo (see, e.g. Exercise 1-1.26 of [2]). We consider related questions for general posets.

**Definition 4.1.** (See [5]). The Scott topology on a poset $P$ is called lower hereditary if for every Scott closed subset $A$, the relative Scott topology on $A$ agrees with the Scott topology of the poset $A$.
Example 4.2. If \( \mathbb{N} \) with its usual order is augmented with two incomparable upper bounds \( a \) and \( b \), then in the resulting poset \( P = \mathbb{N} \cup \{ a, b \} \), the singleton \( \{ a \} \) is Scott open, while \( \{ a \} \) is not Scott open in the Scott closed subset \( \downarrow a \). So, the Scott topology of \( P \) is not lower hereditary.

Lemma 4.3 (See [5]). Let \( P \) be a poset. The following statements are equivalent:

1. The Scott topology on \( P \) is lower hereditary;
2. For any \( x \in P \), the inclusion map from the poset \( \downarrow x \) into \( P \) is Scott-continuous;
3. Any minimal upper bound of any directed set in \( P \) is \( (\text{the}) \) least upper bound for that directed set;
4. For any \( x \in P \) and any directed subset \( D \), \( x = \sup_{D} \) implies \( x = \sup_{P} D \).

Proof. (1) \( \iff \) (2) \( \iff \) (3): See Lemma 3.2 in [5]. We complete the proof for the lemma by proving (3) \( \iff \) (4).

(3) \( \implies \) (4): Suppose that \( D \) is a directed subset with \( x = \sup_{D} \). Then \( x \) is a minimal upper bound of \( D \). By (3), we have \( \sup_{D} = x \).

(4) \( \implies \) (3): Let \( y \) be a minimal upper bound of a directed set \( D \). Then \( y = \sup_{D} D \). By (4), we have \( y = \sup_{P} D \). \( \Box \)

Applying Lemma 4.3(3), we immediately obtain the following result.

Corollary 4.4. Every dcpo has a lower hereditary Scott topology.

Proposition 4.5. Let \( P \) be a poset. If the interval topology on \( P \) is compact, then \( P \) is a dcpo, hence has a lower hereditary Scott topology. In particular, every Lawson compact poset is a dcpo and has a lower hereditary Scott topology.

Proof. Let \( D \subseteq P \) be a directed set. Since the interval topology on \( P \) is compact, there is an element \( x \in P \) which is a cluster point of \( D \), regarded as a net. We show that \( x \) is an upper bound of \( D \) by contradiction. Suppose that there is \( d_{0} \in D \) with \( x \not\in \uparrow d_{0} \). Then \( U = P \backslash \uparrow d_{0} \) is an open neighborhood of \( x \). However, \( d \not\in U \) whenever \( d \geq d_{0} \). This means that \( D \) is eventually not in \( U \), a contradiction. So, \( x \) is also an upper bound of \( D \). To show that \( x \) is also the least one, let \( t \) be any upper bound of \( D \). Then \( \downarrow t \) is a closed set and the cluster point \( x \in \downarrow t \) and \( x \leq t \). So, \( \sup_{D} = x \), showing that \( P \) is a dcpo. By Corollary 4.4, \( P \) has a lower hereditary Scott topology. \( \Box \)

Proposition 4.6. Let \( P \) and \( Q \) be posets with lower hereditary Scott topologies. Then

1. Every convex subset \( A \) of \( P \) in the inherited order has also a lower hereditary Scott topology. In particular, Scott closed sets of \( P \) have lower hereditary Scott topologies;
2. The product \( P \times Q \) has also a lower hereditary Scott topology.

Proof. (1) Let \( D \subseteq A \) be a directed set with a minimal upper bound \( z \in A \). Then \( z \) is also a minimal upper bound of \( D \) in \( P \) because of the convexity of \( A \) and hence the supremum of \( D \) in \( P \) by Lemma 4.3(3).

(2) Let \( D \subseteq P \times Q \) be a directed set with a minimal upper bound \( z := (u, v) \). Then it is easy to see that \( u \) is a minimal upper bound of \( p(D) \) and \( v \) is a minimal upper bound of \( q(D) \), where \( p : P \times Q \rightarrow P \) and \( q : P \times Q \rightarrow Q \) are the projection maps. Since \( P \) and \( Q \) have lower hereditary Scott topologies, we have \( \sup p(D) = u \) and \( \sup q(D) = v \). Thus \( \sup D = (\sup p(D), \sup q(D)) = (u, v) = z \). By Lemma 4.3(3), \( P \times Q \) has a lower hereditary Scott topology. \( \Box \)

Lemma 4.7. Let \( P \) be a poset with a lower hereditary Scott topology and \( A \) a non-empty Scott closed set. If \( x \in A \) and \( D \subseteq A \), then we have \( \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) = \text{cl}_{\sigma}(A)(\downarrow D \cap \downarrow x) \).

Proof. Straightforward. \( \Box \)

Proposition 4.8. Let \( P \) be an MC-poset with a lower hereditary Scott topology and \( A \) a non-empty Scott closed set of \( P \). Then \( A \) in the inherited order is also an MC-poset. In particular, every principal ideal of \( P \) is an MC-poset.

Proof. Let \( D \) be a directed set in \( A \) with existing \( \sup_{P} D \geq x \in A \). Then \( \sup_{A} D = \sup_{P} D \) and \( x \in \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) = \text{cl}_{\sigma}(A)(\downarrow D \cap \downarrow x) \) by Lemmas 2.4, 4.3(3) and 4.7 and the meet continuity of \( P \). So, \( A \) in the inherited order is an MC-poset by Definition 3.1. \( \Box \)

Theorem 4.9. Let \( P \) be a poset with a lower hereditary Scott topology. Then \( P \) is an MC-poset if and only if every principal ideal is an MC-poset.

Proof. \( \Rightarrow \): Follows from Proposition 4.8.

\( \Leftarrow \): Assume each principal ideal of \( P \) is an MC-poset. Let \( D \) be a directed set in \( P \) with existing \( \sup_{D} D \geq x \). Let \( h = \sup_{D} D \). Then \( \downarrow h \) is an MC-poset and \( x \in \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) \), where \( \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) \) is the Scott closure of \( \downarrow D \cap \downarrow x \) in \( \downarrow h \). Since \( \downarrow h \) is Scott closed in \( P \), by Lemma 4.7, we have \( x \in \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) = \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) \). Then by Definition 3.1, \( P \) is an MC-poset. \( \Box \)

From Theorem 4.9 and Corollary 4.4, we immediately have the following.

Corollary 4.10. A dcpo is a meet continuous dcpo if and only if every principal ideal is a meet continuous dcpo.

Proposition 4.11. Let \( P \) be a poset with a lower hereditary Scott topology. Then each closed interval is an MC-poset iff each principal filter is an MC-poset.
Theorem 4.6. Let \( \sigma \) be the poset. Then
\[
\text{Proposition 5.1 (a). If } P \text{ is a poset and } U \subseteq P, \text{ the set } U \text{ is called a local Scott open set if the following two conditions hold:}
\]
(1) \( U \) is an upper set;
(2) For any directed set \( D \) and local supremum \( z \) of \( D \), \( D \cap U \neq \emptyset \) whenever \( z \in U \).

It is easy to verify that the family of the local Scott open subsets of \( P \) forms a topology, called the local Scott topology of \( P \) and denoted \( \sigma(P) \).

Example 5.2. Let \( P \) be the poset defined in Example 4.2. If \( U \in \sigma(P) \) is non-empty, then \( \{a, b\} \subseteq U \). This shows that \( \sigma(P) \) is not a \( T_0 \)-topology. Thus, \( \sigma(P) \neq \sigma(P) \). Noticing that \( \{b\} \) is not \( \sigma(P) \) -open, we have that \( \uparrow a \) is not \( \sigma(P) \) -closed.

Proposition 5.3. For any poset \( P \), we have \( \sigma(P) \subseteq \sigma(P) \).

Example 5.4. Let \( P \) be a poset and \( U \) an upper set of \( P \). Let \( \sigma(P) \) := \( \{W \cap U : W \in \sigma(P)\} \). If \( U \in \sigma(P) \), then \( \sigma(P) \) \( \subseteq \) \( \sigma(P) \).

Proof. Straightforward.

Proposition 5.5. Let \( P \) be a poset. \( P \) is called a locally meet continuous poset (an LMC-poset, in short) if for any \( x \in P \) and any directed subset \( D \) with a local supremum \( z = \sup D \geq x \), one has \( x \in cl_\sigma(\downarrow D \cap \downarrow x) \), where \( cl_\sigma(\downarrow D \cap \downarrow x) \) is the \( \sigma(P) \) -closure of the set \( \downarrow D \cap \downarrow x \).

Theorem 5.6. A poset \( P \) is an LMC-poset if and only if for all \( U \in \sigma(P) \) and \( x \in P \), one has \( \uparrow (U \cap \downarrow x) \subseteq \sigma(P) \).

Proof. \( \Rightarrow \) Let \( x \in P \) and \( U \in \sigma(P) \). Suppose \( z = \sup D \in \uparrow (U \cap \downarrow x) \) for some directed set \( D \) with a local supremum \( z \). Then there is \( y \in U \cap \downarrow x \) such that \( y \leq z \). By the locally meet continuity of \( P \), we have \( \downarrow D \cap \downarrow y \cap U \neq \emptyset \). This implies that \( D' \subseteq \uparrow (U \cap \downarrow y) \neq \emptyset \), showing \( \uparrow (U \cap \downarrow x) \subseteq \sigma(P) \).

\( \Leftarrow \) Let \( D \) be a directed set with a local supremum \( z = \sup D \geq x \). Suppose that \( x \not\in cl_\sigma(\downarrow D \cap \downarrow x) \). Then there is \( U \in \sigma(P) \) such that \( x \in U \) and \( U \cap \downarrow D \cap \downarrow x = \emptyset \). Then by the assumption that \( \uparrow (U \cap \downarrow x) \) is local Scott open, there is \( d \in \uparrow (U \cap \downarrow x) \) and \( D \cap \downarrow x \neq \emptyset \), a contradiction. This shows that \( x \in cl_\sigma(\downarrow D \cap \downarrow x) \) and \( P \) is an LMC-poset.

From Theorem 5.6, we immediately have the following corollary.

Corollary 5.7. A poset \( P \) is an LMC-poset if and only if for any \( U \in \sigma(P) \) and any lower subset \( A \) of \( P \), one has
\[
\uparrow (U \cap A) = \bigcup_{x \in A} \uparrow (U \cap \downarrow x) \subseteq \sigma(P).
\]

Definition 5.8. For a poset \( P \), the common refinement \( \sigma(P) \vee \omega(P) \) of the local Scott topology and the lower topology is called the local Lawson topology and denoted \( \lambda_\sigma(P) \).

Proposition 5.9. Let \( P \) be an LMC-poset. Then
(i) If \( U \in \lambda_\sigma(P) \), then \( \uparrow U \subseteq \sigma(P) \);
(ii) If \( X \) is an upper set, then \( int_\sigma X = int_\sigma X \);
(iii) If \( X \) is a lower set, then \( cl_\sigma X = cl_\sigma X \).

Proof. (i) Let \( D \) be a directed set with a local supremum \( z = \sup D \in \uparrow U \). Then there is \( x \in U \) such that \( x \leq z \). Since \( U \in \lambda_\sigma(P) \), there are \( V \in \sigma(P) \) and \( F \subseteq P \) finite such that \( x \in V \) \( \cap \uparrow F \subseteq U \). By the locally meet continuity of \( P \), we have \( x \in cl_\sigma(\downarrow D \cap \downarrow x) \) and \( \uparrow (V \cap \downarrow D \cap \downarrow x) \neq \emptyset \). Since \( x \not\in \uparrow F \), it is easy to see that \( \uparrow (V \cap \downarrow D \cap \downarrow x) \cap F \neq \emptyset \), which implies that \( \uparrow (V \cap \downarrow D \cap \downarrow x) \cap F = \emptyset \) and \( U \cap \downarrow D \neq \emptyset \). This implies that \( D \subseteq \uparrow U \neq \emptyset \). So, \( \uparrow U \subseteq \sigma(P) \).
For any poset $P$ and $x$, we have $\text{int}_{\sigma}(x) \subseteq \lambda_{I}(P)$, we have $\text{int}_{\sigma}X \subseteq \text{int}_{\lambda}(X)$. By (i), $\uparrow \text{int}_{\lambda}(X) \in \sigma(P)$ and $\text{int}_{\lambda}(X) \subseteq \uparrow \text{int}_{\lambda}(X) \subseteq \text{int}_{\sigma}(X)$. So, $\text{int}_{\sigma}X = \text{int}_{\lambda}(X).$

The equivalence of (ii) and (iii) is straightforward. □

**Proposition 5.10.** If $P$ is an LMC-poset, Then $\sigma(P)^{\emptyset}$ is a complete Heyting algebra.

**Proof.** Similar to the proof of (1) ⇒ (2) of Theorem 3.8 and is omitted. □

**Problem.** For a poset $P$, if $\sigma(P)^{\emptyset}$ is a cHa, prove or disprove that $P$ is an LMC-poset.

For the construction of vertical sum w.r.t. $m$, in terms of Lemma 2.3 and Proposition 5.4, similar arguments to those used to prove Proposition 3.11 and Theorem 3.12 can prove the following proposition and theorem.

**Proposition 5.11.** Let $P$ and $Q$ be posets and $m$ a maximal element of $P$. Then $\sigma(P) = \sigma(P \cup m Q)|_{P}$ and $\sigma(Q) = \sigma(P \cup m Q)|_{Q}$.

**Theorem 5.12.** Let $P$ and $Q$ be LMC-posets and $m$ a maximal element of $P$. Then $P \cup m Q$ is also an LMC-poset.

6. Comparisons of meet continuities on posets

We have had two notions of meet continuity on posets, namely, MC-posets and LMC-posets. We now define another meet continuity notion as follows.

**Definition 6.1.** Let $P$ be a poset. If every principal ideal of $P$ is an MC-poset, then $P$ is called a PI-meet continuous poset (a PIMC-poset, in short).

The following lemma will be helpful to compare our notions of meet continuity.

**Lemma 6.2.** For any poset $P$ and $x \in P$, we have $\sigma(P)|_{\downarrow x} := \{U \cap \downarrow x : U \in \sigma(P)\} \subseteq \sigma(\downarrow x)$.

**Proof.** Let $U \in \sigma(P)$. It is clear that $U \cap \downarrow x$ is an upper set in $\downarrow x$. Suppose $D \subseteq \downarrow x$ is a directed set with existing supremum $\sup_{D} \in U \cap \downarrow x$. Since $U$ is local Scott open, $U \cap D \neq \emptyset$ and $U \cap \downarrow x \cap D \neq \emptyset$. This shows that $U \cap \downarrow x \in \sigma(\downarrow x)$, as desired. □

**Proposition 6.3.** Let $P$ be a poset, then $P$ is an LMC-poset.

**Proof.** Let $D$ be a directed set with a local supremum $z = \sup_{D} x$. By the assumption, $\downarrow z$ is an MC-poset and $x \in \text{cl}(\downarrow D \cap \downarrow x)$. It follows from Lemma 6.2 that $x \in \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) \subseteq \text{cl}_{\sigma}(\downarrow D \cap \downarrow x) \subseteq \text{cl}_{\sigma}(\downarrow D \cap \downarrow x)$, where $\text{cl}_{\sigma}(\downarrow D \cap \downarrow x)$ is the closure of $\downarrow D \cap \downarrow x$ in $\downarrow z$ with respect to the relative local Scott topology on $\downarrow z$. This shows that $P$ is an LMC-poset. □

**Proposition 6.4.** Let $P$ be a poset satisfying for all $x \in P$, $\sigma(P)|_{\downarrow x} = \sigma(\downarrow x)$. If $P$ is an LMC-poset, then $P$ is a PIMC-poset.

**Proof.** Let $x \in P$ and $y \in \downarrow x$. Let $D \subseteq \downarrow x$ be a directed set with existing supremum $\sup_{D} z \geq y$. The locally meet continuity of $P$ implies $y \in \text{cl}_{\sigma}(\downarrow D \cap \downarrow y) \cap \downarrow x = \text{cl}_{\sigma}(\downarrow D \cap \downarrow y) \cap \downarrow x = \text{cl}_{\sigma}(\downarrow D \cap \downarrow y) = \text{cl}_{\sigma}(\downarrow D \cap \downarrow y)$. Since $\sigma(P)|_{\downarrow x} = \sigma(\downarrow x)$, we have that $\text{cl}_{\sigma}(\downarrow D \cap \downarrow y) = \text{cl}_{\sigma}(\downarrow D \cap \downarrow y)$. Thus $\downarrow x$ is an MC-poset, as desired. □

**Theorem 6.5.** Let $P$ be a poset satisfying for all $x \in P$, $\sigma(P)|_{\downarrow x} = \sigma(\downarrow x)$. Then $P$ is a PIMC-poset if and only if $P$ is an LMC-poset.

**Proof.** Apply Propositions 6.3 and 6.4. □

The following lemma characterizes when the local Scott topology and the Scott topology are equal.

**Lemma 6.6.** Let $P$ be a poset. The following statements are equivalent:

1. $P$ has a lower hereditary Scott topology;
2. $\sigma(P) = \sigma(P);$
3. $\forall P \subseteq \sigma(P)$.

**Proof.** (1) ⇒ (2): It follows from Proposition 5.3 that $\sigma(P) \subseteq \sigma(P)$. To show $\sigma(P) \subseteq \sigma(P)$, let $U \in \sigma(P)$, then for any directed subset $D$ with a local supremum $z = \sup_{D} \in U$, by (1) and Lemma 4.3(3), $z = \sup_{D} \in U$. The Scott openness of $U$ implies that $U \cap D \neq \emptyset$. This means that $U$ is local Scott open and $P \subseteq \sigma(P)$.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $D$ be a directed set with a local supremum $z = \sup_{D}$. By Lemma 4.3(4), it suffices to show that $z$ is the supremum of $D$. It is clear that $z$ is an upper bound of $D$. Let $s$ be any upper bound of $D$. Then $\downarrow z = \sigma(P)$-closed by (3). So $\sup_{D} D = z \in s$ and $z \leq s$. This shows that $z = \sup_{D}$, as desired. □

**Theorem 7.** Let $P$ be a poset with a lower hereditary Scott topology. Then $P$ is a PIMC-poset if and only if $P$ is an LMC-poset.

**Proof.** Apply Lemma 6.6(2) and Theorem 6.5. □
**Theorem 6.8.** A poset $P$ is an MC-poset with a lower hereditary Scott topology iff $\sigma_{l}(P)$ is equal to $\sigma(P)$ and $\sigma_{l}(P)^{\text{op}}$ is a cHa, iff $\nu(P) \subseteq \sigma_{l}(P)$ and $\sigma_{l}(P)^{\text{op}}$ is a cHa.

**Proof.** Apply Theorem 3.8 and Lemma 6.6. □

**Corollary 6.9.** Let $P$ be a poset with a lower hereditary Scott topology. Then $P$ is an MC-poset iff $P$ is a PIMC-poset, iff $P$ is an LMC-poset.

**Proof.** Apply Theorems 4.9 and 6.7. □

**Example 6.10.** The poset in Example 4.2 is of an MC-poset, a PIMC-poset and an LMC-poset with principal ideals being meet continuous dcpo. But its Scott topology is not lower hereditary, showing that the condition in Corollary 6.9 is not necessary.

We present some further examples to show the differences between the three notions of meet continuity in general.

**Example 6.11.** If the poset consisting of two parallel copies of $\mathbb{N}$ is augmented with two incomparable upper bounds, then the resulting poset $P$ is meet continuous but not each principal ideal. And $P$ is not locally meet continuous. This example shows that an MC-poset need not be a PIMC-poset or an LMC-poset.

**Example 6.12.** Let $Y = \{1, 2\}$. Let $I = [0, 1]$ be the unit interval. Construct a poset $P$ with the product poset $Y \times I$ in pointwise order by eliminating the element $(1, 1)$ and adding two incomparable $a$ and $b$ which are upper bounds of $\{(1) \times [0, 1]\}$ and are only below the element $(2, 1)$ in $P$. Then it is straightforward to show that $P$ is an LMC-poset. But, $P$ is not an MC-poset nor a PIMC-poset.

The following example shows that a PIMC-poset with each principal ideal being a dcpo need not be an MC-poset.

**Example 6.13.** Let $P = ((\{0, 2\} \times I) \setminus \{(0, 1)\}) \cup \{(1, 1), (0, 2)\}$ be ordered by the inherited order of $\mathbb{R} \times \mathbb{R}$. Then $P$ has two maximal elements $(0, 2)$ and $(2, 1)$. It is easy to see that $P$ is a PIMC-poset. But, $P$ is not an MC-poset.

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**References**


