Einstein–Kähler metrics on a class of bundles involving integral weights

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Abstract

We prove that some compact complex bundles, defined over \( \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \) (where \( \mathbb{P}_d \) is the complex projective space of complex dimension \( d \)) and depending on \( n \) integral weights \( a_1, \ldots, a_n \), have positive first Chern class if \( 1 \leq a_h \leq d_h - 1 \) for all \( h \), and carry Einstein–Kähler metrics when \( a_1 = \cdots = a_n \) and \( d_1 = \cdots = d_n \). © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

On montre que certains fibrés complexes compacts, définis au-dessus de \( \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \) (où \( \mathbb{P}_d \) désigne l’espace projectif complexe de dimension complexe \( d \)), et dépendant de \( n \) puissances entières \( a_1, \ldots, a_n \), sont à première classe de Chern positive si \( 1 \leq a_h \leq d_h - 1 \) pour tout \( h \), et admettent une métrique d’Einstein–Kähler quand \( a_1 = \cdots = a_n \) et \( d_1 = \cdots = d_n \). © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction and main results

In this article, generalizing the class of bundles that we studied in [11,12], we give examples of compact Kähler manifolds with positive first Chern class which carry Einstein–Kähler metrics. We hope these examples will allow a better understanding of the case of more general algebraic manifolds. The problem of Einstein–Kähler metrics is described and studied [1,5,17,24]; for a general survey, see [15]. As regards original papers, concerning existence theorems, we refer to [2,3,21,23,26,28], and, for obstructions, to [16,19,20].

We investigate here projective bundles over the basis

\[ B = B_{d_1}, \ldots, d_n = \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \ (n \geq 2), \]

where \( \mathbb{P}_k \) is the complex projective space of complex dimension \( k \). These manifolds, which depend on integral weights \( a_1, a_2, \ldots, a_n \), are labelled \( X_{[d], [a]} \) and defined as follows:

Let \( [d] = (d_1, \ldots, d_n) \) and \( [a] = (a_1, \ldots, a_n) \) belong to \( \mathbb{N}^n \), with \( 1 \leq a_h \leq d_h - 1 \) for all \( h = 1, \ldots, n \), and let \( b_h = d_1 + d_2 + \cdots + d_h - 1 \) and \( m = b_n \). We also set

\[ I_1 = \{0, \ldots, b_1\}, \quad I_l = \{b_{l-1} + 1, \ldots, b_l\} \quad \text{for} \ l \geq 2, \]

and, if \( Z_h = (z_k)_{k \in I_h} \in \mathbb{C}^{d_h} \),

\[ Z_h^{a_h} = (z_k^{a_h})_{k \in I_h}. \]
Using these notations, $X_{[d],[a]}$ is the manifold

$$X_{[d],[a]} = \{ p = [(Z_1), \ldots, (Z_n), [\lambda_1 Z_{a_1}^1, \ldots, \lambda_n Z_{a_n}^m]] \in \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \times \mathbb{P}_m, $$

where $\Lambda = [\lambda_1, \ldots, \lambda_n] \in \mathbb{P}_{n-1}$. \hfill (1)

$X = X_{[d],[a]}$ is a complex $m$-dimensional submanifold of $B \times \mathbb{P}_m$, and a complex projective bundle over $B$ with fibers isomorphic to $\mathbb{P}_{n-1}$.

Let us now state the main results of this paper. The proof of Theorem 1 (respectively Theorem 2) is given in Section 2.3 (respectively Section 3).

**Theorem 1.** The first Chern class $C_1(X)$ of $X$ is positive if and only if $1 \leq a_h \leq d_h - 1$ for all $h = 1, \ldots, n$.

**Theorem 2.** Suppose that $d_1 = \cdots = d_n = d$, and $a_1 = \cdots = a_n = a$ with $1 \leq a \leq d - 1$. Then $X$ admits an Einstein–Kähler metric.

Under the conditions of Theorem 2, if $G$ denotes the automorphisms group of $X$ defined in 2.2, Tian’s invariant $\alpha_G(X)$ is equal to one. When the dimensions $d_h$ are distinct, as well as the weights $a_h$, one uses a different method to compute $\alpha_G(X)$. It involves Fano manifolds which generalize bundles introduced by Calabi, and it is studied in [13].

### 2. Geometry of the bundles $X_{[d],[a]}$

#### 2.1. Description of charts and parametrizations of $X$

(a) The natural coordinates systems on the projective spaces $\mathbb{P}_{d_1-1}, \ldots, \mathbb{P}_{d_n-1}, \mathbb{P}_{n-1}$ generate an atlas of $X = X_{[d],[a]}$ with $n d_1 \ldots d_n$ charts whose domains are open dense subsets.

Let us describe one of them, labelled $(U_0, \psi_0)$. Its domain $U_0$ is the set of points

$$p = [(z_0, \ldots, z_b_1), \ldots, (z_{b_{n-1}+1}, \ldots, z_m), [\lambda_1 z_{a_1}^1, \ldots, \lambda_n z_{a_n}^m]] \in X$$

such that $z_0 \neq 0, z_{b_1+1} \neq 0, \ldots, z_{b_{n-1}+1} \neq 0, \lambda_1 \neq 0$; hence, the first components of the vectors $Z_1, \ldots, Z_n$, $A$ occurring in the description of $p$ given in Definition (1) of Section 1 are different from zero. Now, let us set

$$\begin{align*}
\zeta_1 &= \frac{z_1}{z_0} ; & \zeta_{b_1+1} &= \frac{\lambda_2 z_{b_1}^1}{\lambda_1 z_0} ; & \zeta_{b_1+2} &= \frac{z_{b_1+2}}{z_{b_1+1}} ; & \cdots ; \\
\zeta_{b_{n-1}+1} &= \frac{\lambda_n z_{a_n}^m}{\lambda_{n-1} z_{a_{n-1}+1}^1} ; & \zeta_{b_{n-1}+2} &= \frac{z_{b_{n-1}+2}}{z_{b_{n-1}+1}} ; & \cdots ; & \zeta_m &= \frac{z_m}{z_{b_{n-1}+1}} .
\end{align*}$$

Then,

$$p = [(1, \zeta_1, \ldots, \zeta_{b_1}), \ldots, (1, \zeta_{b_{n-1}+2}, \ldots, \zeta_m), [1, \zeta_{a_1}^1, \ldots, \zeta_{a_2}^1, \zeta_{b_1+1} (1, \zeta_{b_1+2}, \ldots, \zeta_{b_2}^2), \ldots, \zeta_{b_{n-1}+1} (1, \zeta_{a_{n-1}+2}, \zeta_m)]]$$

and $\psi_0$ is the one-to-one mapping from $U_0$ into $\mathbb{C}^m$ given by

$$\psi_0(p) = (\zeta_1, \ldots, \zeta_m).$$

The other charts of the atlas are obtained in an analogous way by assigning to each vector $Z_1, \ldots, Z_n$, $A$ a component different from 0 which can thus be taken equal to 1.

(b) Let $V$ be the subset of $X$ such that all the components of the vectors $Z_1, \ldots, Z_n$ and $A$ occurring in Definition (1) of Section 1 are different from zero. We now describe several parametrizations $\psi_h$ ($h = 1, \ldots, n$) of $V$ by the open set of $\mathbb{C}^m$:

$$C_m = \{ (z_k) 1 \leq k \leq m \in \mathbb{C}^m; z_k \neq 0 \text{ for all } k \}. $$

Let us choose $h \in \{1, \ldots, n\}$. For any $j = 1, \ldots, n$, we pick $Z_j = (z_k) k \in I_j \in \mathbb{C}^{d_j}$ with $z_k \neq 0$ for all $k \in I_j$, and we suppose that the first component $z_{b_{h-1}+1}$ of $Z_h$ is equal to one (we could impose this condition to any other component of $Z_h$). We identify $Z = (Z_1, \ldots, Z_n)$ with the point $(z_k) k \in I_m, k \neq b_{h-1}+1$ of $\mathbb{C}^m$ and we set

$$\psi_h(Z) = [(Z_1, \ldots, Z_n), [1^1, \ldots, 1^m]] \in V.$$
ψh is a surjective mapping from $C^m_a$ onto $V$. Suppose that $p \in V$ is such that $p = \psi_h(Z) = \psi_h(Z')$, with $Z$ and $Z' \in C^m_a$. Then, $z_k = z_k'$ when $k \in I_h$, and for any $j \neq h$, there exists $v_j$ such that
\[ z_k' = v_j z_k \quad \text{and} \quad z_k^{(d)} = (v_j z_k)^{d_j} \quad \text{for all} \ k \in I_j. \]

Consequently, $v_j^{a_j} = 1$ and $v_j$ is an $a_j$-th root of the unity in $C$. Hence, any $p \in V$ is the image by $\psi_h$ of
\[ a_h = \prod_{j=1, j \neq h}^{n} a_j \]
elements of $C^m_a$. $\psi_h$ is a parametrization of $V$ by $C^m_a$ which covers $a_h$ times $V$. To obtain a chart of $V$, we pick, for any $j \in \{1, \ldots, n\}$ different from $h$, an index $k_j \in I_j$, and we impose that the argument of the component $z_{k_j}$ of $Z_j$ belongs to some fixed interval of length $(2\pi/a_j)$.

### 2.2. Automorphisms group of $X$

**a.** To define a group $G$ of automorphisms of $X_{[d],[\alpha]}$, we use the automorphisms groups of $P_{d_1-1}, \ldots, P_{d_n-1}$ obtained by multiplication by $e^{\theta i}$ ($\theta \in \mathbb{R}$) and permutation of the homogeneous coordinates. Indeed, if $\sigma$ is such an automorphism of $P_{d_n-1}$, it induces a transformation of $X$ which maps
\[ [(Z_1), \ldots, [Z_h], \ldots, [Z_n], [\lambda_1 Z_1^{\alpha_1}, \ldots, \lambda_n Z_n^{\alpha_n}]] \in P_{d_1-1} \times \cdots \times P_{d_n-1} \times \mathbb{P}^m, \]

into
\[ ([Z_1], \ldots, \sigma(Z_h), \ldots, [Z_n], [\lambda_1 Z_1^{\alpha_1}, \ldots, \lambda_h (\sigma(Z_h))^{\alpha_h}, \ldots, \lambda_n Z_n^{\alpha_n}]). \]

If $d_h = d_k$ and $a_h = a_k = a$ (with $1 \leq h < k \leq n$), we also consider the automorphism of $X$ induced by permutation of $Z_h$ and $Z_k$; it is defined as follows:
\[ \sigma(Z_1), \ldots, [Z_k], \ldots, [Z_h], \ldots, [Z_n], [\lambda_1 Z_1^{\alpha_1}, \ldots, \lambda_k Z_k^{\alpha_k}, \ldots, \lambda_h Z_h^{\alpha_h}, \ldots, \lambda_n Z_n^{\alpha_n}]. \]

The group of automorphisms of $X$ generated by the previous ones will be denoted by $G$. Notice that the dense open subset $V$ of $X$ defined in 2.1(b) is $G$-invariant.

**b.** Now, let $\varphi \in C^\infty(X)$ be a $G$-invariant function. We want to examine the effect of this invariance on the expressions $\varphi \circ \psi_h = \psi_h$ (defined on $C^m_a$) of $\varphi$ in the parametrizations $(\psi_h)_1 \leq h \leq n$ of $V$. Suppose, to simplify the notations, that $h = 1$ and write
\[ \varphi_1 = \varphi(1, z_1, \ldots, z_n) = \varphi([Z_1], \ldots, [Z_n], [Z_1^{\alpha_1}, \ldots, Z_n^{\alpha_n}]) = \varphi(p), \]

where $Z_1 = (1, z_1, \ldots, z_n)$ considered as belonging to $c_{a+1}$ and $Z_j = (z_k)_{k \in I_j} \in C_a^d$ if $j \geq 2$.

(b.1) First, $\varphi_1$ is $t_\theta$-invariant, i.e. invariant by multiplication of the $z_k$ by any $e^{\theta i}$ ($\theta \in \mathbb{R}$). Hence, it depends only on the $x_k = |z_k|^2$, and we consider $\varphi_1$ as a function of $(x_1, \ldots, x_n) \in \mathbb{R}^m$. Then $\varphi_1$ is invariant by permutation of any $x_k, x_1$ (if $1 \leq k < l \leq n$ and $(k, l)$ belong to the same subset $I_j$).

(b.2) Now, we establish the link between $\varphi_1$ and $\varphi_h$ (when $h = 2$ for instance), using only the invariance by the automorphisms $t_{j,\theta}$. If $p \in V$, we consider the following two manners of describing $p$:
\[ p = ([Z_1] = [1, z_1, \ldots, z_{b_1}], [Z_2] = [z_{b_1+1}, \ldots, z_{b_2}], [Z_3], \ldots, [Z_n], [Z_1^{\alpha_1}, Z_2^{\alpha_2}, \ldots, Z_n^{\alpha_n}]), \]

and
\[ p = ([Z_1'] = [z_0', z_1', \ldots, z_{b_1'}], [Z_2'] = [z_{b_1'+1}, \ldots, z_{b_2'}], [Z_3'], \ldots, [Z_n'], [Z_1'^{\alpha_1}, Z_2'^{\alpha_2}, \ldots, Z_n'^{\alpha_n}]). \]

From these representations, we deduce that, from some complex numbers $v_1, \ldots, v_n$, $v \neq 0$, we have
\[ Z_k' = v_h Z_h \quad \text{and} \quad (Z_1'^{\alpha_1}, \ldots, Z_n'^{\alpha_n}) = v(Z_1^{\alpha_1}, \ldots, Z_n^{\alpha_n}). \]

Thus,
\[ z_0' = v_1, \quad 1 = v_2 z_{b_1+1}, \quad z_0'^{\alpha_1} = v, \quad 1 = v z_{b_1+1}' \]
which implies
\[ v = z_{b_1 + 1}, \quad v_1 = z_0 = v^{1/a_1} = z_{b_1 + 1}, \quad v_2 = z_{b_1 + 1} \]
and, if \( h \geq 2, v_h = z_{b_1 + 1}^{a/h} \) since, for \( k \in I_h, \)
\[ z_k' = v_h z_k, \quad z_k'' = v_h z_k = v_h z_k, \quad \text{i.e.} \quad v_h = v^{1/a_h}. \]

The relation between \( \psi_1 \) and \( \psi_2 \) follows:
\[ \psi(p) = \psi_1(1, x_1, \ldots, x_m) = \psi_2 \left( \frac{1}{x_{b_1 + 1}}, \frac{x_1}{x_{b_1 + 1}}, \ldots, \frac{x_{b_1} + 2}{x_{b_1 + 1}}, \ldots, \frac{x_{b_2} + 1}{x_{b_1 + 1}}, \ldots, \frac{x_{b_3}}{x_{b_1 + 1}}, \ldots, \frac{x_{b_{n-1} + 1}}{x_{b_1 + 1}}, \ldots, \frac{x_m}{x_{b_1 + 1}} \right). \]

(b.3) In an analogous way, let us express that \( \psi_1 \) is invariant by permutation \( \sigma_{0,1} \) of the homogeneous coordinates \( z_0 \) and \( z_1 \) of \( Z_1 \). Notice that
\[ p' = \sigma_{0,1}(p) = ([z_1, 1, z_2, \ldots, z_n], [Z_2], \ldots, [Z_n], [z_1^{a_1}, 1, z_2^{a_1}, \ldots, z_{b_1}, z_2^{a_2}, \ldots, z_n^{a_n}]). \]
To compute \( \psi_1(p') \), we must represent \( p' \) under the following form:
\[ p' = ([1, z_1', \ldots, z_{b_1}'], [Z_2'], \ldots, [Z_n'], [1, z_1', z_2', \ldots, z_{b_1}', Z_2', \ldots, Z_n']) \]
with \( Z_h' \in \mathbb{C}_{a_h} \). So we get
\[ z_1' = \frac{1}{z_1}, \quad z_2' = \frac{z_2}{z_1}, \quad \ldots, \quad z_{b_1}' = \frac{z_{b_1}}{z_1} \]
and, if \( h \geq 2, k \in I_h, \) for some \( v \neq 0, \)
\[ z_k' = v_h z_k, \quad z_k'' = \frac{z_k}{z_1}. \]
Hence, \( v_h = 1/z_1^{a_1}, \) i.e. \( v_h = z_1^{-a_1/a_h} \) for some determination of the power \( z_1^{-a_1/a_h} \) which has not to be precised since we only consider \( x_k = |z_k|^2 \), \( x_k' = |z_k'|^2 \), and \( x_k'' = x_k/x_1^{a_1/a_h}. \)

Consequently, we obtain
\[ \psi_1(1, x_1, \ldots, x_m) = \psi_1 \left( \frac{1}{x_1}, x_2/x_1, \ldots, x_{b_1}/x_1, \ldots, x_{b_2}/x_{b_1}, \ldots, x_{b_{n-1}}/x_{b_1}, \ldots, x_m/x_{b_1} \right). \]

(b.4) Finally, if \( d_1 = d_2 = d \) and \( a_1 = a_2 = a \), and if \( \sigma \) is the automorphism which permutes \([Z_1]\) and \([Z_2]\), taking into account the invariance of \( \psi \) by \( \sigma \) and the \( \tau_{j,j'} \) yields:
\[ \psi_1(1, x_1, \ldots, x_m) = \psi(p) = \psi(\sigma(p)) = \psi([Z_2], [Z_1], [Z_3], \ldots, [Z_n], [Z_2', Z_1', Z_3', \ldots, Z_n']) \]
\[ = \psi_2(x_d, \ldots, x_{2d-1}; 1, x_1, \ldots, x_{d-1}; x_{2d}, \ldots, x_m) = \psi_1 \left( \frac{1}{x_d}, \frac{x_{2d-1}}{x_d}, x_1, \ldots, \frac{x_{d-1}}{x_d}, \frac{x_{2d}}{x_d} \right) \]

2.3. **First Chern class and metric of \( X \)**

(a) If one uses the atlas defined in 2.1(a), there are two generic types of changes of coordinates.

Let us start with charts \( \alpha \) parameterizing points
\[ p = ([Z_1], \ldots, [Z_n], [z_1 Z_1^a, \ldots, z_n Z_n^a]) \in X \]
such that the first components of \( Z_1, \ldots, Z_n \) and \( \Lambda \) are equal to one; the \( (\alpha) \) coordinates of \( p \) are
\[ z_1, \ldots, z_{b_1}; \lambda_2, z_{b_1+2}, \ldots, z_{b_2}; \ldots; \lambda_n, z_{b_n-1} + 1, \ldots, z_m. \]
To change the position of the components equal to one in $Z_1, \ldots, Z_n$ and $A$, we proceed step by step, changing successively in $Z_1$, then in $Z_2, \ldots, Z_n$ and finally in $A$ (the roles of $Z_b$ being symmetric).

On the first hand, we consider chart $(\beta)$ which differs from $(\alpha)$ by the fact that the second component of $Z_1$ is now equal to one. The corresponding change of coordinates $\Gamma_{\beta,\alpha}$ from chart $(\alpha)$ to chart $(\beta)$ maps each point $(z_1, \ldots, z_m)$, with $z_1 \neq 0$, to point

\[
\left( \frac{1}{z_1}, \frac{z_2}{z_1}, \frac{z_3}{z_1}, \frac{z_{b_1}+1}{z_1}, \frac{z_{b_1}+2}{z_1}, \ldots, \frac{z_{b_2}+1}{z_1}, \frac{z_{b_2}+2}{z_1}, \ldots, \frac{z_{b_m}+1}{z_1}, \frac{z_{b_m}+2}{z_1}, z_2, \ldots, z_m \right),
\]

as is seen if we express, for instance when $E_\beta$ relations:

\[
\begin{align*}
&\left( [1, z_1, \ldots, z_1, 1, z_{b_1}+2, \ldots, z_m], [1, z_1^1, \ldots, z_1^1, z_{b_1}+1, 1, z_{b_1}+2, \ldots, z_m] \right) \\
= &\left( (u_0, 1, u_2, \ldots, u_{b_1}), [1, u_{b_1}+2, \ldots, u_m], [u_0^1, 1, u_2^1, \ldots, u_{b_1}^1, u_{b_1}+1, 1, u_{b_1}+2, \ldots, u_m] \right).
\end{align*}
\]

Let us compute the Jacobian $J_{\beta,\alpha}$ of $\Gamma_{\beta,\alpha}$. It is given by

\[
J_{\beta,\alpha} = \frac{1}{z_{b_1+1}}.
\]

On the other hand, let us denote by $(\gamma)$ the chart for which condition $\lambda_1 = 1$ of chart $(\alpha)$ becomes $\lambda_2 = 1$. The change of coordinates from chart $(\alpha)$ to chart $(\gamma)$ is given by

\[
\Gamma_{\gamma,\alpha} : (z_1, \ldots, z_{b_1}+1 \neq 0, \ldots, z_m) \to (\frac{1}{z_{b_1+1}}, \frac{z_1}{z_{b_1+1}}, \frac{z_2}{z_{b_1+1}}, \frac{z_3}{z_{b_1+1}}, \frac{z_{b_1}+1}{z_{b_1+1}}, \frac{z_{b_1}+2}{z_{b_1+1}}, \ldots, \frac{z_{b_m}+1}{z_{b_1+1}}, \frac{z_{b_m}+2}{z_{b_1+1}}, z_2, \ldots, z_m),
\]

as we obtain if we consider (for $n = 2$) the following two representations of $p \in X$, which yield the coordinates of $p$ in charts $(\alpha)$ and $(\gamma)$:

\[
p = \left( [1, z_1, \ldots, z_1, 1, z_{b_1}+2, \ldots, z_m], [1, z_1^1, \ldots, z_1^1, z_{b_1}+1, 1, z_{b_1}+2, \ldots, z_m] \right) \\
= \left( (1, z_1, \ldots, z_1, 1, z_{b_1}+2, \ldots, z_m), [z_0(1, z_1, \ldots, z_1), 1, z_{b_1}+2, \ldots, z_m] \right).
\]

The Jacobian $J_{\gamma,\alpha}$ of $\Gamma_{\gamma,\alpha}$ is equal to

\[
J_{\gamma,\alpha} = \frac{1}{z_{b_1+1}}.
\]

(b) We now look for an element $\omega \in C_1(X)$.

Let us consider a chart $(U_3, \varphi_3)$, labelled $(\delta)$ (its domain $U_3$ is an open subset of $X$ containing $V$ and $\varphi_3$ is an isomorphism between $U_3$ and $\mathbb{C}^m$), of the atlas defined in 2.1(a) for which each vector $Z_1, \ldots, Z_n$ and $A$ occurring in the description of $p = ([Z_1], \ldots, [Z_n], [\lambda_1 Z_1^1, \ldots, \lambda_n Z_n^m]) \in X$ (see Definition (1) in Section 1).

In chart $(\delta)$, we seek $\omega$ as $i \partial \overline{\partial} K_3$ where the potential $K_3$ is defined by $K_3 = \log E_3$, with

\[
E_3(p) = |Z_1|^{2r_1} \cdots |Z_n|^{2r_n} (l_1 |Z_1^1|^2 + \cdots + l_n |Z_n^m|^2)^q.
\]

Here for any $Z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, $A = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ and $a \in \mathbb{N}$, we set

\[
|Z|^2 = \sum_{j=1}^N |x_j|^2, \quad Z^a = (z_1^a, \ldots, z_N^a), \quad x_j = |z_j|^2 \text{ and } l_j = |\lambda_j|^2.
\]

This choice is natural if we consider the pull-back on $X$ of convenient multiples of the Fubini–Study metric by the natural projections of $X \subset \mathbb{P}d_1 \times \cdots \times \mathbb{P}d_n \times \mathbb{P}m$ on the factors $\mathbb{P}d_1 \times \cdots \times \mathbb{P}d_n$ and $\mathbb{P}m$.

We try to find the exponents $r_1, \ldots, r_n, q$ such that the functions $E_3$ can be viewed as the local expressions of an Hermitian metric on the determinant line bundle of $X$. Consequently the local functions $E_3$ must satisfy the following compatibility relations: $E_3 = |\Gamma_{\beta,\alpha}|^2 E_2$, if we consider for instance charts $(\alpha)$ and $(\beta)$. Under these conditions, the curvature form $\omega = i \partial \overline{\partial} E_3$ is independent of chart $(\delta)$ and represents a well defined $1$-$1$ form on $X$ which belongs to $C_1(X)$.

Let $z_1, \ldots, z_m$ be the $(\alpha)$-coordinates of $p \in U_\alpha$. If $p \in U_\alpha \cap U_\gamma$, computing the value of $E_\gamma(p)$ in terms of the coordinates of $p$ in chart $(\alpha)$, we get:
\[ E_{\gamma}(p) = (1 + x_1 + \cdots + x_{b_1})^{\ell_1} (1 + x_{b_1+2} + \cdots + x_{b_2})^{\ell_2} \cdots (1 + x_{b_{n-1}+2} + \cdots + x_m)^{\ell_n} \times \left[ \frac{1}{x_{b_1+1}} (1 + x_{b_1+2} + \cdots + x_{b_2}) + \frac{x_{b_2+1}}{x_{b_1+1}} (1 + x_{b_2+2} + \cdots + x_{b_3}) + \cdots \frac{x_{b_n+1}}{x_{b_{n-1}+1}} (1 + x_{b_n+2} + \cdots + x_m) \right]^q \]

Thus, \( E_{\gamma} = |J_{\gamma,a}|^2 E_{\alpha} \), taking into account the value of \( J_{\gamma,a} \) that we obtained in paragraph (a), we must have

\[ \frac{1}{x_{b_1+1}^q} = |J_{\gamma,a}|^2 = \frac{1}{x_{b_1+1}^q} \]

that is \( q = n \).

On the other hand, if \( p \in U_\alpha \cap U_\beta \), we see that

\[ E_{\beta}(p) = \left( 1 + x_1 + \cdots + x_{b_1} \right)^{\ell_1} \left( 1 + x_{b_1+2} + \cdots + x_{b_2} \right)^{\ell_2} \cdots \left( 1 + x_{b_{n-1}+2} + \cdots + x_m \right)^{\ell_n} \times \left[ \frac{1}{x_{b_1}} \left( 1 + x_{b_2} \right) + \frac{x_{b_3}}{x_{b_1}} \left( 1 + x_{b_4} \right) + \cdots + \frac{x_{b_n}}{x_{b_{n-1}}} \left( 1 + x_m \right) \right]^q \]

\[ = \frac{1}{x_{1^{r_1}+q_{a_1}}} E_{\alpha}(p). \]

Hence, we obtain

\[ \frac{1}{x_{1^{r_1}+q_{a_1}}} = |J_{\beta,a}|^2 = \frac{1}{x_{1^{r_1}+q_{a_1}}} \]

and, since \( q = n, r_1 + n a_1 = d_1 + (n-1) a_1 \), i.e. \( r_1 = d_1 - a_1 \). In fact, \( r_h = d_h - a_h \) for \( h = 1, \ldots, n \).

(e) And now, let us collect some properties of \( \omega \). First, thanks to the form of the potential \( K = \log E \) (where we omit any reference to some chart), \( \omega = i \partial \bar{\partial} K \) is everywhere positive definite as we shall see later, so the first Chern class of \( X \) is positive. We shall use the corresponding Kähler metric \( g \), written locally \( g_{kl} = \partial_k \bar{\partial}_l K \).

Let us check that \( \omega \) is positive definite, in chart (a) for instance. In this chart, we write \( \omega = \sum_{i=1}^{n+1} \omega_i \), with

\[ \omega_1 = i(d_1 - a_1) \partial \bar{\partial} \log(1 + x_1 + \cdots + x_{b_1}), \]

\[ \omega_h = i(d_h - a_h) \partial \bar{\partial} \log(1 + x_{b_{h-1}+2} + \cdots + x_{b_h}) \quad \text{for} \ h = 2, \ldots, n, \]

and

\[ \omega_{n+1} = i n \partial \bar{\partial} \log \left[ 1 + x_{b_1}^1 + \cdots + x_{b_1}^{a_1} + \sum_{h=2}^n x_{b_{h-1}+1} (1 + x_{b_{h-1}+2} + \cdots + x_{b_h}) \right]. \]

All these one-to-one forms are considered, at every point, as Hermitian forms on \( \mathbb{C}^m \). Clearly, \( \omega_1, \ldots, \omega_n \) are non-negative since they correspond to multiples of the Fubini–Study metric on \( \mathbb{C}^{d_1-1}, \ldots, \mathbb{C}^{d_n-1} \), respectively. \( \omega_{n+1} \) is also non-negative since it is the pull-back of the Fubini–Study metric \( i n \partial \bar{\partial} \log(1 + v_1^2 + \cdots + v_m^2) \) on \( \mathbb{C}^m \) by the following holomorphic map:

\[ (z_1, \ldots, z_m) \rightarrow (v_1, \ldots, v_m) \]

\[ = \left( z_1^{a_1}, z_{b_1+1} \bar{z}_{b_1+1}, z_{b_1+2} \bar{z}_{b_1+2}, \ldots, z_{b_{n-1}+1} \bar{z}_{b_{n-1}+1}, z_{b_{n-1}+2} \bar{z}_{b_{n-1}+2}, \ldots \right). \]

At any point \( p = (z_1, \ldots, z_m) \), we have

\[ V(p) = \sum_{h=1}^n \ker \omega_h(p) = [\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m; \ \zeta_k = 0 \ \text{for} \ k \neq b_1 + 1, \ldots, b_{n-1} + 1]. \]

To prove that \( \omega(p) \) is positive definite, one has to show that the restriction of \( \omega_{n+1}(p) \) to \( V(p) \) is itself positive definite. Suppressing the dependence in \( p \), we set:
\[ u_1 = z_{b_1+1}, \ldots, u_{n-1} = z_{b_{n-1}+1}. \]
\[ S_1 = 1 + x_1^{a_1} + \cdots + x_{b_1}^{a_1}, \ldots, S_h = 1 + x_{b_{h-1}+2}^{a_h} + \cdots + x_{b_h}^{a_h} \quad \text{for } 2 \leq h \leq n-1, \]
and
\[ S = S_1 \left[ 1 + \sum_{h=1}^{n-1} \left( u_h \sqrt{S_h / S_1} \right) \right] = S_1 T. \]

Notice that \( S_1, \ldots, S_{n-1} \) do not depend on \( u = (u_1, \ldots, u_{n-1}). \)

Now, let \( \xi = (\xi_1, \ldots, \xi_m) \in V \) and \( \xi_1^* = \zeta_{b_1+1}, \ldots, \xi_{n-1}^* = \zeta_{b_{n-1}+1}. \) We have
\[
\omega_{n+1}(\xi, \bar{\xi}) = n \sum_{\alpha, \beta = 1}^{n-1} \frac{\partial^2 \log S}{\partial u_\alpha \partial \bar{u}_\beta} \xi^*_\alpha \bar{\xi}^*_\beta = n \sum_{\alpha, \beta = 1}^{n-1} \frac{\partial^2 \log T}{\partial u_\alpha \partial \bar{u}_\beta} \xi^*_\alpha \bar{\xi}^*_\beta.
\]

Hence, by virtue of the positive definiteness of the Fubini–Study metric on \( \mathbb{C}^{n-1}, \) \( \omega_{n+1}(\xi, \bar{\xi}) = 0 \) if and only if \( \xi^*_1 = \cdots = \xi^*_n = 0, \) i.e. \( \xi = 0. \) From this, we deduce that the restriction of \( \omega_{n+1} \) to \( V \) is positive definite, as requested.

\( \mathbf{(d)} \) \( \omega \) is \( G \)-invariant. In fact, for any generator \( \tau \) of \( G \) defined in 2.2, if \( p \in X \) and \( \tau' = \tau(p) \), we can choose\( (\delta') \) whose domains also contain \( p \) and \( \tau' \) and such that \( K_{\delta'} \circ \tau = K_\delta \) in the neighborhood of \( p. \) Thus,
\[ \tau^* \omega = \delta^* (i \bar{\delta} K_{\delta'}) = i \bar{\delta} K_\delta \circ \tau = i \bar{\delta} K_\delta = \omega. \]

Finally, we shall need the local expressions \( \omega_{\eta \alpha} = \psi_\eta^*(\omega) \) of \( \omega \) (or the corresponding metric \( \psi_\eta \) of the open subset \( V \subset X \)) defined in 2.1.(b). To simplify the notations, we take \( h = 1. \) Since \( V \subset U_\alpha', \) the mapping
\[ \Gamma_{\alpha'} : \psi_\alpha' \circ \psi_1 : \mathbb{C}^m \to \mathbb{C}^m, \]
which is an \( a_1' = a_2' = \cdots = a_n' \)-fold covering, is given by
\[ \Gamma_{\alpha', 1} : Z = (\xi_1, \ldots, \xi_m) \to \left( z_1, \ldots, z_{b_1}, z_{b_1+1}, \frac{z_{b_2}}{z_{b_1+1}}, \ldots, \frac{z_{b_{n-1}+2}}{z_{b_{n-1}+1}}, \ldots, \frac{z_{b_m+2}}{z_{b_m+1}}, \ldots, \frac{z_{b_{n-1}+2}}{z_{b_{n-1}+1}}, \ldots, \frac{z_m}{z_{b_{n-1}+1}} \right), \]
as we see by writing
\[
p = [(1, z_1, \ldots, z_{b_1}), (z_{b_1+1}, \ldots, z_{b_2}), \ldots, (z_{b_{n-1}+1}, \ldots, z_m)],
\]
\[= \left( [1, u_1, \ldots, u_{b_1}], [1, u_{b_1+2}, \ldots, u_{b_2}], \ldots, [1, u_{b_{n-1}+2}, \ldots, u_{b_m}],
\right)
\]
\[1, u_{b_1+1}^1, \ldots, u_{b_1+1}^n; u_{b_1+1}^1 (1, u_{b_1+2}^2, \ldots, u_{b_2}^n); \ldots; u_{b_{n-1}+1}^1 (1, u_{b_{n-1}+2}, \ldots, u_{b_m}^n)). \]

Hence, if
\[ t_1 = 1 + \sum_{k=1}^{b_1} x_k, \quad t_h = \sum_{k=b_{h-1}+1}^{b_h} x_k \quad \text{for } h = 2, \ldots, n \quad \text{and} \quad T = 1 + \sum_{h=1}^{b_1} t_1^n + \sum_{h=2}^{b_2} t_2^n + \sum_{h=3}^{b_3} t_3^n + \cdots + \sum_{h=n}^{b_n} t_h^n, \]
we get
\[ E_\alpha (\Gamma_{\alpha', 1}(Z)) = \Gamma_{\alpha', 1}^{\alpha_1 a_1} \left( \frac{t_2}{x_{b_1+1}} \right) \cdots \left( \frac{t_n}{x_{b_{n-1}+1}} \right) \frac{d_a - a_n}{t_n} T^n. \]

Consequently, since \( \partial \bar{\eta} \log (d_{a_1} - a_2, \ldots, d_{a_n} - a_n) \) is \( \omega_1 = \Gamma_{\alpha', 1} (i \partial \bar{\eta} K_\alpha) = i \partial \bar{\eta} K_\alpha \circ \Gamma_{\alpha', 1} = i \partial \bar{\eta} K_1, \)
where the local potential \( K_1 \) is defined by \( K_1 = \log(T^n \prod_{h=1}^{a_n} t_h^{d_{a_n} - a_n}). \)
2.4. Volume element of the metric \( g \)

We want to compute the volume element of the metric \( \psi_1^2(g) \) on \( C^m_k \). For any \( Z = (z_0 = 1, z_1, \ldots, z_m) \) identified with the point \( (z_k)_{1 \leq k \leq m} \) of \( C^m_k \) (i.e. \( x_k = |z_k|^2 \neq 0 \) for all \( k \)), if \( I_h = \{d_0 + \cdots + d_{h-1}, \ldots, d_0 + \cdots + d_h - 1\} \), with \( d_0 = 0 \), \( h \in \{1, \ldots, n\} \), we set

\[
t_h = \sum_{j \in I_h} x_j, \quad T = \sum_{h=1}^n \left( \sum_{j \in I_h} x_j \right), \quad \text{where } a_j = a_h \text{ if } j \in I_h, \quad \text{and } K = \log \left( T^n \prod_{h=1}^n r_h^{d_h-a_h} \right).
\]

Let us also put \( J_1 = \{1, \ldots, d_1 - 1\} \) and \( J_h = I_h \) for \( h \geq 2 \).

**Proposition 1.** (1) Let \( g_{\lambda, \mu} = \partial_{\lambda, \mu} K \) and \( M = (g_{\lambda, \mu})_{1 \leq \lambda, \mu \leq m} \). Then

\[
det M = \prod_{h=1}^n \det B_h + \sum_{h_1 = 1}^{h-1} \det B_{h_1} \cdots \det B_{h_1-1} \Gamma(h) \det B_{h_1+1} \cdots \det B_n,
\]

where \( \det B_h \) and \( \Gamma(h) \) are defined by (A.3) and (A.7) in the subsequent proof.

(2) There is a constant \( C \) such that, at any point \( Z = (z_k)_{1 \leq k \leq m} \) satisfying \( 0 < x_k \leq 1 \) for all \( k = 1, \ldots, m \), we have

\[
det M \leq C \prod_{h=2}^n t_h^{a_h-d_h}.
\]

**Proof.** To avoid to interrupt the main stream of the article, the proof is given in Appendix A. It is much more tricky than in the case \( a_1 = \cdots = a_n = 1 \) studied in [12]. The explicit value of \( \det M \) given in part (1), in particular the cumbersome expressions of \( \det B_h \) and \( \Gamma(h) \), is essential to get the upper bound of the volume element obtained in part (2). This upper bound is itself crucial in the evaluation of Tian’s invariant \( a_G(X) \).

3. Proof of Theorem 2

3.1. Minoration of admissible functions

The functions \( \varphi \) we consider are \( g \)-admissible \( (g_{\lambda, \mu} + \partial_{\lambda, \mu} \varphi > 0) \) and \( G \)-invariant. We use the notations of 2.2, in particular as regard the expressions \( \varphi_h = \varphi \circ \psi_h \) of \( \varphi \) in parametrizations \( \psi_h \) of \( V \). Recall that we write

\[
\varphi_1(1, z_1, \ldots, z_m) = \varphi(1, x_1, \ldots, x_m)
\]

since \( \varphi_1 \) depends only on the \( x_k = |z_k|^2 \). Thus,

\[
\frac{\partial^2 \varphi}{\partial z_j \partial z_k} = \delta_{jk} \partial_j \varphi + \breve{z}_j z_k \delta_{jk} \varphi,
\]

where \( \partial_j = \partial / \partial x_j \) and \( \partial_{jk} = \partial^2 / \partial x_j \partial x_k \).

To get lower bounds of \( g \)-admissible, \( G \)-invariant functions, we proceed in a sequence of propositions. For \( p \in X \), we express that the restriction of \( K + \varphi \) to conveniently chosen holomorphic curves \( \gamma \) of \( X \) starting from \( p \) is subharmonic; the curves are such that informations concerning \( K + \varphi \) at the extremity of \( \gamma \) can be obtained by virtue of \( G \) invariance properties of \( K \) and \( \varphi \). In this way, we progressively reduce the number of variables and finally we obtain a logarithmic upper estimate of \( -(K + \varphi) \).

**Proposition 2.** Let \( \varphi \in C^\infty(X) \) be a \( g \)-admissible and \( G \)-invariant function on \( X = X_{[d], [a]} \). For \( x_1, \ldots, x_m > 0 \), if

\[
\gamma = \left( \prod_{k=1}^{d_1-1} x_k \right)^{1/(d_1-1)}
\]

we have

\[
-(K + \varphi)(1, x_1, \ldots, x_{b_1}, x_{b_1+1}, \ldots, x_m) \leq -(K + \varphi)(1, \gamma^{d_1-1}, x_{b_1+1}, \ldots, x_m),
\]
where
\[ K = \log \left( \sum_{h=1}^{n} \left( \sum_{j \in I_h} x_j^{a_h} \right)^n \prod_{h=1}^{n} \left( \sum_{j \in I_h} x_j \right)^{d_h-a_h} \right) \]
and, for any \( p \in \mathbb{N} \), \( \gamma^p = (\gamma, \ldots, \gamma) \in \mathbb{R}^p \).

**Proof. First step.** For \( 1 \leq k \leq d_1 - 1, 0 < \xi < 1 \) and \( x_{k+1}, \ldots, x_m > 0 \), let us prove the following inequality:
\[
-(K + \varphi)(1, \xi, \ldots, x_{k+1}, \ldots, x_m) \leq -(K + \varphi) \left( \frac{x_{k+1}}{\xi^{1/b}}, \ldots, \frac{x_{d_1 - d_2 - \cdots - d_{k+1} - 1}}{\xi^{a_1/a_2}}, \frac{x_m}{\xi^{a_1/a_2}} \right) + a_1 \left( \sum_{h=1}^{n} \frac{d_h}{a_h} \right) \log \frac{1}{\xi^{1/b}}.
\]
(2)

with \( b = (k + 1)/k \). For \( s > 0 \), we set
\[
\Phi(s) = s \frac{d}{ds} (K + \varphi)(\psi(s)),
\]
(3)

where
\[
\psi(s) = (s^b, s^b, s_{x_{k+1}}, \ldots, s_{x_{d_1 - d_2 - \cdots - d_{k+1} - 1}}, \ldots, s_{x_{d_1}}, s_{x_{d_2}}, \ldots, s_{x_m}).
\]

In \( \mathbb{C}^m = \{(1, \xi_1, \ldots, \xi_m) \in \mathbb{C}^{m+1}; \prod_{k=1}^{m} \xi_k \neq 0\} \), let us consider the curve defined, for any complex number \( \sigma \in \mathbb{C} \), by
\[
\gamma(\sigma) = (1, \sigma^b \sqrt{\xi}, \ldots, \sigma^b \sqrt{\xi_{d_1}}, \sigma \sqrt{x_{d_1 + 1}}, \ldots, \sigma \sqrt{x_{d_1 + 1}}, \ldots, \sigma \sqrt{x_m}),
\]
where we take the principal determinations of the powers \( \sigma^b, \sigma^b \sqrt{\xi}, \ldots, \sigma \sqrt{x_m} \). Since \( K + \varphi \) is strictly plurisubharmonic, hence its Laplacian is positive. On the other hand, it depends only on \( s = \sigma \tilde{\sigma} \) and we write it \( \psi = \psi(s) \), with \( \psi \) defined above. But,
\[
\frac{\partial^2 \psi}{\partial \sigma \partial \tilde{\sigma}}(\sigma, \tilde{\sigma}) = \psi'(\sigma, \tilde{\sigma}) + \sigma \tilde{\sigma} \psi''(\sigma, \tilde{\sigma}) = \left( s \frac{d}{ds} \psi(s) \right)' \geq 0.
\]
Thus, \( \Phi'(s) \geq 0 \) and \( \Phi \) is an increasing function, with \( \Phi(s) \) explicitly given by
\[
\Phi(s) = b \sigma^b \zeta \left( \partial_1 (K + \varphi) + \cdots + \partial_k (K + \varphi) + \sum_{p=k+1}^{d_1-1} x_p \partial_p (K + \varphi) + \sum_{h=2}^{n} \sum_{p \in I_h} \frac{a_h}{a_p} x_p \partial_p (K + \varphi) \right),
\]
where the derivatives are taken at \( \psi(s) \).

Setting \( s_0 = \zeta^{-1/b} \geq 1 \), we compute all derivatives of \( K + \varphi \) at
\[
P_0 = \psi(s_0) = \{(1^{k+1}, y_{k+1}, \ldots, y_m) \}
\]
with \( y_p = s_0 x_p \) for \( p = k + 1, \ldots, d_1 - 1 \) and \( y_p = s_{a_1/a_2} x_p \) if \( p \in I_h \) and \( h \geq 2 \).

Consequently, for \( 1 \leq s \leq s_0 \),
\[
\Phi(1) \leq \Phi(s) \leq \Phi(s_0) = b \sum_{p=1}^{k} \partial_p (K + \varphi) + \sum_{p=k+1}^{d_1-1} x_p \partial_p (K + \varphi) + \sum_{h=2}^{n} \sum_{p \in I_h} \frac{a_h}{a_p} x_p \partial_p (K + \varphi).
\]
(4)

Let us evaluate \( \Phi(s_0) \).

By definition of \( K \), we write
for any \( \eta > 0 \). Thus, for any \( \eta > 0 \) and any \( i \in \{1, \ldots, k\} \), the following relation is satisfied:

\[
\varphi(1, \eta^k, \ldots, \eta, y_k, \eta y_{k+1}, \ldots, y_m) = \left(1, \eta^{k-1}, \eta, \ldots, \eta y_{k+1}, \eta y_{k+1} \eta^{a_1/a_2}, \ldots, \eta y_{k+1} \eta^{a_1/a_2} \eta^{a_2/a_3}, \ldots, \eta y_{n+1} \eta^{a_1/a_2} \ldots \eta^{a_{n-1}/a_n} \right),
\]

for any \( a_1, \ldots, a_m > 0 \). Then, taking into account (4) and the Definition (3) of \( \Phi(x) \), we obtain for \( 1 \leq s \leq \eta \),

\[
\Phi(0) = \eta \sum_{i=1}^{k} \frac{d_i}{a_i} + \sum_{p=k+1}^{d_1-1} a_i \varphi - \sum_{h=2}^{n} \sum_{p \in I_h} \frac{d_h}{a_h} y_p \partial_p \varphi.
\]

By summation on \( i = 1, \ldots, k \), we deduce that

\[
(k + 1) \sum_{j=1}^{k} \partial_j \varphi + \sum_{p=k+1}^{d_1-1} y_p \partial_p \varphi + \sum_{h=2}^{n} \sum_{p \in I_h} \frac{d_h}{a_h} y_p \partial_p \varphi = 0,
\]

which is equality (6) since \( b = (k + 1) / k \).
\[
\frac{d}{ds} \left[ (K + \psi)(\psi(s)) \right] \leq a_1 \sum_{h=1}^{n} \frac{d_h}{a_h} \tag{8}
\]

Finally, integrating this inequality between 1 and \( s_0 = \zeta^{-b} \) yields (2).

**Second step.** Let \( 1 \leq k \leq d_1 - 1 \). We want to prove by induction on \( k \) that, for \( x_1, \ldots, x_k > 0 \) and \( \gamma_k = (\prod_{j=1}^{k} x_j)^{1/k} \), we have

\[-(K + \psi)(1, x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) \leq -(K + \psi)(1, \gamma_{k-1}, x_{k+1}, \ldots, x_m). \tag{L_k} \]

So let us assume \((L_{k-1})\) is valid for some \( k \in [2, \ldots, d_1 - 1] \) (assertion clearly true when \( k = 2 \)); we have to show \((L_k)\).

First, the definition of \( K \) yields

\[
K \left( 1, \frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{b_1}}{x_k} \frac{x_{b_1+1}}{x_k}, \ldots, \frac{x_{b_{n-1}+1}}{x_k}, \ldots, \frac{x_m}{x_k} \right)
\]

\[
= \sum_{h=1}^{n} \left( d_h - a_h \right) \frac{\log \left( \frac{\prod_{j \in I_h} x_j}{a_1} \right)}{a_1} + n \log \left( \frac{\prod_{j=1}^{n} x_j}{a_n} \right)
\]

\[
= K(1, x_1, \ldots, x_m) + a_1 \sum_{h=1}^{n} \frac{d_h - a_h}{a_h} + n \log \frac{1}{x_k}
\]

which implies

\[
K \left( 1, \frac{\gamma_{k-1}}{x_k}, \ldots, x_m \right)
\]

\[
= K(1, 1, \ldots, 1) + a_1 \sum_{h=1}^{n} \frac{d_h - a_h}{a_h} + n \log \frac{1}{x_k} \tag{9}
\]

On the other hand, by \( \sigma_{0,k} \)-invariance of \( \psi \), we have (see 2.2):

\[
\psi \left( 1, \frac{\gamma_{k-1}}{x_k}, \ldots, x_m \right)
\]

\[
= \psi \left( 1, \frac{1}{x_k} \sqrt[k]{x_{k+1}} \frac{1}{x_k} \frac{x_{b_1}}{x_k} \frac{x_{b_1+1}}{x_k} \frac{x_{b_2}}{x_k} \frac{x_{b_{n-1}+1}}{x_k} \frac{x_m}{x_k} \right) \tag{10}
\]

Now taking into account the \( \sigma_{i,j} \)-invariance of \( \psi \) for \( 1 \leq i < j \leq k \), we may assume that \( x_1 \leq \cdots \leq x_k \); thus,

\[
\gamma_{k-1} = \prod_{j=1}^{k-1} \frac{x_j}{x_k} \leq x_k \quad \text{and} \quad \zeta = \frac{\gamma_{k-1}}{x_k} \leq 1.
\]

Hence, by \((L_{k-1}), (9), (10)\) and (2) (written for \( k - 1 \) instead of \( k \)), we obtain

\[
-(K + \psi)(1, x_1, \ldots, x_m)
\]

\[
\leq -(K + \psi)(1, \frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{b_1}}{x_k} \frac{x_{b_1+1}}{x_k}, \ldots, \frac{x_{b_{n-1}+1}}{x_k}, \ldots, \frac{x_m}{x_k}) + a_1 \sum_{h=1}^{n} \frac{d_h - a_h}{a_h} \frac{1}{x_k} \tag{11}
\]

\[
\leq -(K + \psi) \left( 1, \frac{\gamma_{k-1}}{x_k}, \ldots, \frac{x_{b_1}}{x_k}, \frac{x_{b_1+1}}{x_k}, \ldots, \frac{x_{b_{n-1}+1}}{x_k}, \ldots, \frac{x_m}{x_k} \right) \frac{\zeta \gamma_{1/k}}{\zeta \gamma_{1/k}} \frac{\zeta \gamma_{1/k}}{\zeta \gamma_{1/k}} \frac{\zeta \gamma_{1/k}}{\zeta \gamma_{1/k}} \frac{\zeta \gamma_{1/k}}{\zeta \gamma_{1/k}} \frac{\zeta \gamma_{1/k}}{\zeta \gamma_{1/k}}
\]

by induction. We have to show that

$$
\zeta^c x_k = \left(\frac{y_k-1}{x_k}\right)^{(k-1)/k} x_k = \left(\prod_{j=1}^{k-1} x_j\right)^{1/k} x_k^{1/k} = y_k
$$

Consequently, since

$$
\zeta^c a_{js} x_k^{a_{js}} = \zeta^c x_k^{a_{js}} = y_k^{a_{js}}, \quad \text{for } h \geq 2 \text{ and } k \in I_h,
$$

we get:

$$
-(K + \varphi)(1, x_1, \ldots, x_m) \\
\leq -(K + \varphi)\left(\prod_{j=1}^{[k]} \frac{1}{y_k} x_{k+1} \frac{x_{b_1+1}}{y_{b_1+1}} \frac{x_{b_2+1}}{y_{b_2+1}} \ldots \frac{x_{b_n+\cdots+1}}{y_{b_n+\cdots+1}} \frac{x_m}{y_m}\right) + \sum_{h=1}^{n} \frac{d_h}{a_h} \log \frac{1}{y_k}
$$

(by $\sigma_{0, k}$-invariance of $\varphi$)

$$
= -(K + \varphi)(1, y_k^{[k]}, x_{k+1}, \ldots, x_m)
$$

(because the bracket involving the difference of the values taken by $K$ in two points is equal to $(-a_1 \sum_{h=1}^{n} \frac{d_h}{a_h} \log \frac{1}{y_k})$).

Finally, tracing through the inequalities, we see that we have obtained (I$_h$) as required.

**Proposition 3.** Let $\varphi \in C^\infty(X)$ be a $g$-admissible and $G$-invariant function. For $x_1, \ldots, x_m > 0$, we set

$$
\zeta_1 = \left(\prod_{j=1}^{d_1-1} x_j\right)^{1/(d_1-1)} \quad \text{and} \quad \zeta_h = \left(\prod_{j=b_{h-1}+1}^{d_1+\cdots+d_j-1} x_j\right)^{1/d_s}, \quad if \ 2 \leq h \leq n.
$$

When $\zeta \in \mathbb{R}$, $\zeta^{[d]}$ denotes the vector $(\zeta, \ldots, \zeta) \in \mathbb{R}^d$. Then, the following inequality is satisfied:

$$
-(K + \varphi)(1, x_1, \ldots, x_m) \leq -(K + \varphi)(1, \zeta_1^{[d_1-1]}, \zeta_2^{[d_2]}, \ldots, \zeta_n^{[d_n]}).
$$

**Proof.** Let $j \in \{1, \ldots, n\}$ and $h \in [2, \ldots, n]$. We suppose the inequality

$$
(*)_j \quad -(K + \varphi)(1, x_1, \ldots, x_m) \leq -(K + \varphi)(1, \zeta_1^{[d_1-1]}, \zeta_2^{[d_2]}, \ldots, \zeta_j^{[d_j]}, x_{b_{h-1}+1}, \ldots, x_m)
$$

valid when $j = h - 1$ and we prove $(*)_h$. Since $(*)_1$ is true according to the previous proposition, $(*)_h$ will thus be obtained by induction. We have to show that

$$
-(K + \varphi)(1, \zeta_1^{[d_1-1]}, \zeta_2^{[d_2]}, \ldots, \zeta_{h-1}^{[d_{h-1}]}, x_{b_{h-1}+1}, \ldots, x_m) \leq -(K + \varphi)(1, \zeta_1^{[d_1-1]}, \zeta_2^{[d_2]}, \ldots, \zeta_h^{[d_h]}, x_{b_h+1}, \ldots, x_m).
$$

We set $c = b_{h-1} + 1$. Using the change of parametrization $\psi_h^{-1} \circ \psi_1$ of $V$ (see 2.1(b)), the definition of $K$ and Proposition 2 written in parametrization $\psi_h$ of $V$ yields:

$$
-(K + \varphi)(1, \zeta_1^{[d_1-1]}, \zeta_2^{[d_2]}, \ldots, \zeta_{h-1}^{[d_{h-1}]}, x_c, \ldots, x_m)
$$

$$
= -(K + \varphi)\left(\frac{1}{x_c^{a_1/a_1}} \frac{\zeta_1^{[d_1-1]}}{x_c^{a_1/a_1}} \frac{\zeta_2^{[d_2]}}{x_c^{a_2/a_2}} \ldots \frac{\zeta_h^{[d_h-1]}}{x_c^{a_h/a_{h-1}}} \frac{c_{h+1}}{x_c} \ldots \frac{x_{b_{h-1}+1}}{x_c} \frac{x_m}{x_c^{a_h/a_{h+1}}}ight)
$$

This completes the proof of Proposition 3.
Thus we conclude that

\[ \frac{\xi^{[d]-1}}{x_c} - \frac{\xi^{[d]}}{x_c^n} = \frac{\xi^{[d]}}{x_c^n}, \]

which proves Proposition 4.

Proposition 4. Under the hypothesis \( a_1 = \cdots = a_n = a \) and \( d_1 = \cdots = d_n = d \), let \( \varphi \in C^\infty(X_{[d], [a_1]}^n) \) be a g-admissible, G-invariant function. Then, for \( x_1, \ldots, x_m > 0 \), we have

\[ -(K + \varphi)(1, x_1, \ldots, x_m) \leq -(K + \varphi)(1, 1, \ldots, 1) - \log x_1 \cdots x_m. \]

Proof. (a) First, let us prove that for \( \eta_1, \ldots, \eta_n > 0 \) and

\[ \eta = \left( \frac{1}{\eta_1^{1/(n-1)}} \right), \]

we have

\[ -(K + \varphi)(1, \eta_1, \ldots, \eta_n) \leq -(K + \varphi)(1, \eta_1, \ldots, \eta_1) - \log \frac{1}{\eta_h}. \]

Thus we conclude that

\[ -(K + \varphi)(x_1, \ldots, x_m) \leq -(K + \varphi)(1, 1, \ldots, 1) - \log x_1 \cdots x_m. \]

Proposition 2(2) to get an upper bound of the term in \(-(K + \varphi)\) at the right-hand side of the last inequality. Setting \( \alpha = (d_h - 1)/d_h \), we obtain:

\[ -(K + \varphi)(1, \xi^{[d_1]-1}, \eta_1, \ldots, \eta_1) = -(K + \varphi)(1, \xi^{[d_1]-1}, \xi^{[d_2]}, \ldots, \eta_1) - \log \frac{1}{\eta_h}. \]

Using once more the definition of \( K \) and the change \( \psi_1^{-1} \circ \psi_h \) of parametrization of \( V \), we remark that the term in \(-(K + \varphi)\) at the end of the last equality is equal to

\[ -(K + \varphi)(1, \xi^{[d_1]-1}, \xi^{[d_2]}, \ldots, \xi^{[d_k]}, \eta_1, \ldots, \eta_1) - \log \frac{1}{\eta_h}. \]

Thus we conclude that

\[ -(K + \varphi)(x_1, \ldots, x_m) \leq -(K + \varphi)(1, 1, \ldots, 1) - \log x_1 \cdots x_m. \]
(b) Proof of (12). As in the first step of the proof of Proposition 2 one proves that the function
\[ \psi(s) = s \frac{d}{ds} (K + \varphi)(1^{(h-1)d}, (s^h \xi)^d, s x_{hd}, \ldots, s x_m), \]
where
\[ \zeta = \zeta_{h+1}/t, \quad x_j = \frac{\zeta_{k+1}}{t} \text{ if } j \in I_k \text{ and } k = h + 1, \ldots, n - 1, \]
\[ x_{(n-1)d} = \frac{1}{t} \quad \text{and} \quad x_j = \frac{\zeta_1}{t} \text{ if } (n - 1)d + 1 \leq j \leq nd - 1, \]
is increasing. The derivatives of \((K + \varphi)\) being taken at point
\[ P = \big(1^{(h-1)d}, (s^h \xi)^d, s x_{hd}, \ldots, s x_m\big), \]
we have
\[ \psi(s) = h s^h \xi \sum_{j=(h-1)d}^{hd-1} \partial_j (K + \varphi) + s \sum_{j=hd}^m x_j \partial_j (K + \varphi). \]
So, for \(1 \leq s \leq s_0 = \zeta^{-1/b}\), we infer
\[ \frac{d}{ds} (K + \varphi)(1^{(h-1)d}, (s^h \xi)^d, s x_{hd}, \ldots, s x_m) \leq \frac{\psi(s_0)}{s}. \]  
(13)

Let us compute \(\psi(s_0)\). Defining
\[ P_0 = \big(1^{(h-1)d}, (s_0^h \xi)^d, s_0 x_{hd}, \ldots, s_0 x_m\big) = \big(1^{hd}, s_0 x_{hd}, \ldots, s_0 x_m\big), \]
we have
\[ \psi(s_0) = \left\{ h s_0^h \xi \sum_{j=(h-1)d}^{hd-1} \partial_j (K + \varphi) + s_0 \sum_{j=hd}^m x_j \partial_j (K + \varphi) \right\}(P_0) \]
\[ = \left\{ h \sum_{j=(h-1)d}^{hd-1} \partial_j (K + \varphi) + \sum_{j=hd}^m s_0 x_j \partial_j (K + \varphi) \right\}(P_0). \]
Then, by definition of \(K\), we get
\[ \left( h \sum_{j=(h-1)d}^{hd-1} \partial_j K + \sum_{j=hd}^m s_0 x_j \partial_j K \right)(P_0) = \]
\[ = \sum_{j=(h-1)d}^{hd-1} \frac{nh_a}{hd + s_0^h x_{hd} + \cdots + s_0^h x_{hm}} + \sum_{j=(h-1)d}^{hd-1} \frac{(d-a)h}{j} + \sum_{j=hd}^m \frac{na s_0^{a-1} x_j}{hd + s_0^h x_{hd} + \cdots + s_0^h x_{hm}} + (n-h)(d-a) = m + 1. \]  
(14)

On the other hand, let us show that
\[ \left( h \sum_{j=(h-1)d}^{hd-1} \partial_j \varphi + \sum_{j=hd}^m s_0 x_j \partial_j \varphi \right)(P_0) = 0. \]  
(15)

Since \(\varphi\) is \(G\)-invariant, we have, at point \(P_0\):
\[ \sum_{j \in I_h} \partial_j \varphi = \cdots = \sum_{j \in I_h} \partial_j \varphi \]
(let us remind that \(I_h = \{(h-1)d \leq j \leq hd - 1\} \text{ if } h \geq 2\) and, for \(u > 0\),
\[ \varphi(1^{(h-1)d}, u^d, s_0 x_{hd}, \ldots, s_0 x_m) = \varphi \left(1^d, \left(\frac{1}{u}\right)^{(h-1)d}, s_0 x_{hd}, \ldots, s_0 x_m\right) \].
Hence, since

\[ \sum_{j \in I_0} \partial_j \psi = - \sum_{j \in I_2} \partial_j \psi - \cdots - \sum_{j \in I_m} \partial_j \psi - \sum_{j = h}^m s_0 x_j \partial_j \psi, \]

which implies

\[ h \sum_{j = (h-1)d}^{hd-1} \partial_j \psi + \sum_{j = hd}^m s_0 x_j \partial_j \psi = 0 \]

and finally yields relation (15).

Let us integrate the previous inequality between 1 and \( s_0 \), and, of the proof, we deduce (12) and thus (11).

Combining (14) and (15), we have

\[ \psi(s_0) = 0 \]

and

\[ \int (K + \psi) \left( \frac{t}{\lambda}, \left( \frac{t}{\lambda} \right)^d \right) ds_0 = \int (K + \psi) \left( \frac{t}{\lambda}, \left( \frac{t}{\lambda} \right)^d \right) ds_0 = \int \left( \frac{t}{\lambda} \right)^d ds_0 = 0. \]

The \( G \)-invariance of \( \psi \) and the definition of \( K \) imply

\[ -(K + \psi) \left( 1, \frac{t}{\lambda} \right)^d, \ldots, \frac{t}{\lambda} \right)^d \right) \]

and

\[ -(K + \psi) \left( 1, \frac{t}{\lambda} \right)^d, \ldots, \frac{t}{\lambda} \right)^d \right) \]

Inserting these two equalities in (16) gives:

Hence, since

\[ \frac{\lambda}{t} \left( \frac{\xi h + 1}{t} \right)^{-1/h} = 1, \]

we deduce (12) and thus (11).

(c) Given a \( G \)-invariant admissible function \( \psi \in C^\infty(X) \) and \( x_1, \ldots, x_m > 0 \), combining Proposition 3 and inequality (11) leads to

\[ -(K + \psi)(1, x_1, \ldots, x_m) \leq -(K + \psi)(1, \xi^{[d-1]}, \eta^{[m+1-d]}). \]

where

\[ \xi = (x_1 \cdots x_{d-1})^{1/(d-1)} \quad \text{and} \quad \eta = (x_d \cdots x_m)^{1/(m+1-d)}. \]
Hence, to conclude the proof of Proposition 4, we have to bound from above
\[-(K + \psi)(1, \chi^{[d-1]}, \chi^{[n+1-d]})\]
by \(-(K + \psi)(1, \ldots, 1) - \log x_1 \cdots x_n\). A slight modification of the proof of Lemma 6 in [12] gives this result.

The proof of Theorem 2 also needs the two following Lemmas 1 and 2.

**Lemma 1.** There exists a constant \( C \) such that for any \( \psi \in C^\infty(X) \), \( g \)-admissible and \( G \)-invariant, we have
\[-(K + \psi)(1, \ldots, 1) \leq C.\]

**Proof.** It is analogous to the proof given in [12, pp. 682–683], to which we refer.

Now, we establish a result which expresses the integral over \( X \) of any \( G \)-invariant function as the sum of \( n \) integrals over the unit polydisc \( D \) of \( \mathbb{C}^m \).

**Lemma 2.** Let \( \psi \) be a \( G \)-invariant integrable function on \( X = X_{\lfloor d \rfloor}, \lfloor a \rfloor \). For any \( h = 1, \ldots, n \), we denote by
\[ \psi_h = \psi \circ \psi_h = \psi(z_0, \ldots, z_{b_{h-1}}, 1, z_{b_{h-1}+2}, \ldots, z_m) \]
the local expression of \( \psi \) in parametrisation \( \psi_h \) of \( V \) (see 2.1(b)). Then, if \( dv_h \) is the volume element of the metric \( \psi_h^*(g) \) on \( \mathbb{C}^m \), and if \( d'_h = \prod_{j \neq h} a_j \), we have
\[ \int_X \psi \, dv = \sum_{h=1}^n \frac{d_h}{d'_h} \int_{D_h} \psi_h \, dv_h. \]

**Remark.** If we impose that the argument of the first component of \( Z_j = (z_k)_{k \in I_j} \) belong to some interval of length \( 2\pi/a_j \) for any \( j \in \{1, \ldots, n\} - \{h\} \), we get a subset \( D_h \) of \( D \) and
\[ \int_X \psi \, dv = \sum_{h=1}^n d_h \int_{D_h} \psi_h \, dv_h. \]

**Proof.** If \( 1 \leq h \leq n \), we set \( Z_h = (z_k)_{k \in I_h} \) and \( X_h = (x_k = |z_k|^2)_{k \in I_h} \). The notation \( \tilde{Z}_h \) means that the first component \( z_{b_{h-1}+1} \) of \( Z_h \) is equal to one; on the other hand, \( X_h \leq 1 \) if \( 0 < x_k \leq 1 \) for any \( k \in I_h \). Since \( \psi_h(Z), \ldots, \tilde{Z}_h, \ldots, Z_n \) is invariant by multiplication of the \( z_k \) of index \( k \neq b_{h-1} + 1 \) by any \( e^{i\theta} \), \( \psi_h \) depends only on \( X_1, \ldots, \tilde{X}_h, \ldots, X_n \) and we write
\[ \psi_h = \psi(Z_1, \ldots, \tilde{Z}_h, \ldots, Z_n) = \psi(X_1, \ldots, \tilde{X}_h, \ldots, X_n). \]
We want to prove by induction on \( h \) that
\[ \int_X \psi \, dv = \sum_{j=1}^{b_h} \frac{d_j}{d'_j} \int_{X_1, \ldots, \tilde{X}_j, \ldots, X_h \leq 1} \psi(X_1, \ldots, \tilde{X}_j, \ldots, X_h, \ldots, X_n) \, dv_j. \]  \hspace{1cm} (18)
This equality will be labelled \((18)_h\).  

(a) To start the inductive process, we must show that
\[ \int_X \psi \, dv = \frac{d_1}{d'_1} \int_{X_1 \leq 1} \psi(\tilde{X}_1, X_2, \ldots, X_n) \, dv_1. \]
Since we work in parametrisation \( \psi_1 : \mathbb{C}^m \to V \), which covers \( a'_1 \)-times \( V \), if
\[ \Omega = \{(\tilde{X}_1, \ldots, X_n); \tilde{X}_1 \leq 1 \} \]
and
\[ \Omega' = \{(\tilde{X}_1, \ldots, X_n); x_1 \geq 1 \\text{ and } x_2, \ldots, x_{b_1} \leq x_1 \}, \]
we have
\[ a'_l \int_X \varphi \, dv = \int_{X} \varphi(\tilde{X}_1, \ldots, X_n) \, dv_1 = \int_{\Omega} \varphi(\tilde{X}_1, \ldots, X_n) \, dv_1 + (d_1 - 1) \int_{\Omega'} \varphi(\tilde{X}_1, \ldots, X_n) \, dv_1 \]

(using \(\sigma_{l,j}\)-invariance of \(\varphi\), for \(2 \leq j \leq d_1 - 1\)). Next, we take into account that \(\sigma_{0,1}\) is an involutive isometry of \((X, g)\) which keeps invariant \(V\). In parametrization \(\psi_1\), it exchanges the subsets \(\Omega\) and \(\Omega'\), since according to 2.2,

\[ \sigma_{0,1}(\tilde{Z}_1, Z_2, \ldots, Z_n) = \left( \frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{d_1}}{z_1}, \frac{Z_2}{z_{d_1}}, \ldots, \frac{Z_n}{z_{d_1}} \right) \]

Hence, \(\int_{\Omega'} \psi_1 \, dv_1 = \int_{\Omega} \psi_1 \, dv_1\) and (18) follows.

(b) Suppose now that (18)\(_{h-1}\) is true for some \(h = [2, \ldots, n]\). If \(1 \leq l \leq h - 1\), \(\delta_h = b_{h-1} + 1\) and

\[ A_{l,h} = \{(Z_1, \ldots, \tilde{Z}_l, \ldots, Z_n) : X_1, \ldots, \tilde{X}_l, X_{h-1} \leq 1, x_{\delta_h} \geq 1, \text{ and } \max_{j \in I_h} x_j = x_{\delta_h} \} \]

by virtue of the \(\sigma_{\delta, j}\)-invariance of \(\varphi\) for \(j \in I_h\), we get

\[ \int_{X_1, \ldots, \tilde{X}_l, X_{h-1} \leq 1} \psi_1 \, dv_l = \int_{X_1, \ldots, \tilde{X}_l, X_{h} \leq 1} \psi_1 \, dv_l + \int_{A_{l,h}} \psi_1 \, dv_l \]

Consequently, thanks to (18)\(_{h-1}\),

\[ \int_{X} \psi_1 \, dv_l = \sum_{l=1}^{h-1} \frac{d_l}{\alpha_l} \int_{X_1, \ldots, \tilde{X}_l, X_{h-1} \leq 1} \psi_1 \, dv_l + \sum_{l=1}^{h-1} \frac{d_l d_h}{\alpha_l} \int_{A_{l,h}} \psi_1 \, dv_l \]

and, to obtain (18)\(_h\), we have to prove that

\[ \sum_{l=1}^{h-1} \frac{d_l}{\alpha_l} \int_{A_{l,h}} \psi_1 \, dv_l = \frac{1}{a_{\delta_h}} \int_{X_1, \ldots, \tilde{X}_h, X_{h-1} \leq 1} \psi_1 \, dv_{h} \]  

(19)

Notice that \(\frac{1}{a_{\delta_h}} \int_{A_{l,h}} \psi_1 \, dv_l\) is the integral of \(\psi\), written in terms of parametrization \(\psi_l\), over the subset

\[ \tilde{A}_{l,h} = \{(Z_1, \ldots, \tilde{Z}_l, \ldots, Z_n) : (Z_1, \ldots, \tilde{Z}_l, \ldots, Z_n) \in X; (Z_1, \ldots, \tilde{Z}_l, \ldots, Z_n) \in A_{l,h} \} \]

which is also described as

\[ \tilde{A}_{l,h} = \left\{ \left[ \frac{Z_1}{z_{\delta_h/a_1}}, \ldots, \frac{\tilde{Z}_l}{z_{\delta_h/a_1}}, \ldots, \frac{Z_n}{z_{\delta_h/a_1}} \right] \left[ \frac{Z_{a_1}}{z_{\delta_h/a_2}}, \ldots, \frac{Z_{a_1}}{z_{\delta_h/a_2}}, \ldots, \frac{Z_{a_n}}{z_{\delta_h/a_2}} \right] \right\} \in X; \quad (Z_1, \ldots, \tilde{Z}_l, \ldots, Z_n) \in A_{l,h} \]

\[ = \left\{ \left[ \frac{Z_1}{z_{\delta_h/a_1}}, \ldots, \frac{\tilde{Z}_l}{z_{\delta_h/a_1}}, \ldots, \frac{Z_n}{z_{\delta_h/a_1}} \right] \right\} \in X; \quad (Z_1', \ldots, Z_{h-1}', \tilde{X}_h' \leq 1, \max_{0 \leq j \leq \delta_h-1} x_j = x_{\delta_h} \} \]

Thus, writing \(\int_{\tilde{A}_{l,h}} \psi_1 \, dv_l\) in terms of parametrization \(\psi_h\) yields

\[ \frac{d_l}{a_{\delta_h}} \int_{\tilde{A}_{l,h}} \psi_1 \, dv_l = \frac{d_l}{a_{\delta_h}} \int_{X_1, \ldots, \tilde{X}_h, X_{h-1} \leq 1, \max(x_0, \ldots, x_{\delta_h-1}) = x_{\delta_h}} \psi(Z_1, \ldots, \tilde{Z}_h, \ldots, Z_n) \, dv_h \]

\[ = \frac{1}{a_{\delta_h}} \int_{X_1, \ldots, \tilde{X}_h, X_{h-1} \leq 1, \max(x_0, \ldots, x_{\delta_h-1}) = x_{\delta_h}} \psi_h \, dv_h \]

where the last equality is obtained thanks to the \(\sigma_{\delta, j}\)-invariance of \(\varphi\) for \(j \in I_l\).

Finally, by summation over \(l = 1, \ldots, h - 1\), we get the requested equality (19).
3.2. End of the proof of Theorem 2

The proof of Theorem 2 uses an invariant introduced by Tian [26] and defined as follows:

$$\alpha_G(X) = \sup \left\{ \alpha > 0; \exists C \text{ such that } \forall \psi \in A_G, \int_X \exp(-\alpha \psi) \, dv \leq C \exp \left( -\frac{\alpha}{V} \int_X \varphi \, dv \right) \right\}. $$

Using a strategy initiated by Aubin in the fundamental work [2], and an inequality involving the integral of the exponential of plurisubharmonic functions due to type Bombieri [14], Skoda [25] and Hörmander [18], one shows (see Aubin [5], Tian [26]) that a lower bound of this invariant gives a C\(^0\) estimate of the solutions of the family of Monge–Ampère equations

$$\log M(\psi) = -tf + f, \quad t > 0,$$

where \(f\) is the geometric datum given by Ricci(\(\omega\)) - \(\omega = \frac{1}{2\pi} d^\pi \varphi \),

$$M(\psi) = \det \left( \frac{\psi}{\lambda} + \frac{\psi}{\lambda} \right)_{\lambda, \mu \in \{1, \ldots, m\}}, \quad \text{and} \quad \omega = \frac{1}{2\pi} g_{\lambda \mu} dz^\lambda \wedge d\bar{z}^\mu \in C_1(X).$$

In fact if \(\alpha_G > tm/(m + 1)\), one has the required C\(^0\) estimate for the previous equation. By higher order a priori estimates obtained by Aubin [3], the C\(^0\) estimate yields a solution of the Monge–Ampère equation. If we reach \(t = 1\), the manifold admits a Kähler–Einstein metric given by

$$g_{\lambda \mu}^0 = g_{\lambda \mu} + \theta_{\lambda \mu} \varphi.$$

And now, let us give the proof of Theorem 2. Let 0 < \(\alpha < 1\). Given any \(G\)-invariant, \(\varphi\)-admissible function \(\psi \in C^\infty(X)\) such that \(\int_X \varphi \, dv = 0\), we shall bound from above, independently of \(\varphi\), the integral \(\int_X \exp(-\alpha \psi) \, dv\). Consequently, Tian’s invariant \(\alpha_X(X)\) is \(\geq 1\), which proves the existence of an Einstein–Kähler metric on \(X\).

We work in parametrization \(\psi_1\) of \(V\). Thanks to Lemma 2, we have to bound from above

$$\int_D e^{-\alpha \varphi} \, dv,$$

where \(D\) is identified to \([1, z_1, \ldots, z_m]; \quad 0 < x_i, \ldots, x_m \leq 1\]. First, according to Proposition 1, the volume element of the metric \(g^*(\varphi)\) is such that

$$dv \leq C \prod_{h=2}^{n-1} \left( \sum_{j \in I_h} x_j \right)^{a-d} dx_1 \cdots dx_m \text{ on } D.$$

Next, since \(K = \log(2^n \prod_{h=1}^n (a_d - a) \, dx_1 \cdots dx_m)\) is the potential of \(g\) in the parametrization \(\psi_1\) of \(V\), we have

$$e^{aK} \leq C \prod_{h=2}^n (\lambda - a) \quad \text{on } D.$$

Then, according to Lemma 1, \(-(K + \varphi)(1, \ldots, 1) \leq \text{Const.} \) and, thanks to Proposition 4, we get

$$\int_D e^{-\alpha \varphi} \, dv = \int_D e^{-\alpha(K + \varphi) + aK} \, dv \leq e^{-\alpha(K + \varphi)(1, \ldots, 1)} \int_D \prod_{j=1}^n x^j \, dv$$

$$\leq C \int_D \left( \prod_{h=2}^n \left( \lambda - (a - \alpha - a) \right) \right) (x_1 \cdots x_m)^{-a} \, dx_1 \cdots dx_m.$$

Hence, since \(\alpha < 1\),

$$\int_D e^{-\alpha \varphi} \, dv \leq C \int_{0 < x_1, \ldots, x_m \leq 1} \left( \prod_{h=2}^n (\lambda - a_d) \right) (x_d \cdots x_m)^{-a} \, dx_d \cdots dx_m$$

$$= C \left\{ \int_{0 < y_1, \ldots, y_d \leq 1} \frac{dy_1 \cdots dy_d}{(y_1 + \cdots + y_d)^{(1-a)(d-a)}(y_1 \cdots y_d)^a} \right\}^{n-1} = C \left\{ \int_0^1 \frac{r^{d-1}}{r^{(1-a)(d-a)} r^a} \, dr \right\}^{n-1}.$$
where \( \alpha (1-a) < 0 \). Thus, the requested bound is obtained.

Appendix A

Proof of Proposition 1. (1) First, we explicit the terms of the matrix \( M \). We have

\[
\partial_\mu \log T = \frac{\alpha_\mu z_\mu}{T},
\]

and, if we denote the Kronecker symbols by \( \delta_{\lambda \mu} \),

\[
\partial_\lambda z_\mu = \frac{\alpha_\mu z_\mu}{T} \delta_{\lambda \mu} - \frac{\alpha_\mu z_\mu}{T} \delta_{\lambda \mu}.
\]

on the other hand, for \( \lambda, \mu \in J_h \),

\[
\partial_\lambda z_\mu = \delta_{\lambda \mu} - \frac{z_\mu z_\lambda}{T}.
\]

Thus, we can write

\[
M = D + P + Q,
\]

with

\[
D = \sum_{h=1}^n \operatorname{diag} \left( \frac{d_h - a_h}{t_h} + \frac{\alpha_h z_\mu}{T} \delta_{\lambda \mu} \right), \quad P = \sum_{h=1}^n \frac{d_h - a_h}{t_h} \delta_{\lambda \mu} \delta_{\lambda \mu}, \quad Q = \sum_{h=1}^n \frac{2}{T} \left( \nabla v_\mu \right) \delta_{\lambda \mu}.
\]

where \( v_\mu = \alpha_\mu z_\mu \) and \( f = (v_1, \ldots, v_m) \). If \( D_\mu, P_\mu, Q_\mu \) denote the columns of \( D, P, Q \) of index \( \mu \), since any \( Q_\mu \) is colinear to \( V \), we write

\[
\det M = \Delta_1 + \Delta_2.
\]

with

\[
\Delta_1 = \det(D_1 + P_1, \ldots, D_m + P_m)
\]

and

\[
\Delta_2 = \sum_{\mu=1}^m \det(D_1 + P_1, \ldots, D_{\mu-1} + P_{\mu-1}, Q_\mu, D_{\mu+1} + P_{\mu+1}, \ldots, D_m + P_m).
\]

Taking into account the decomposition by blocks of \( D + P \), and setting \( B_h = (\operatorname{diag} \delta_{\lambda \mu})_{\lambda \in J_h} + (p_{\lambda \mu})_{\lambda \mu \in J_h} \), we have

\[
\Delta_1 = \prod_{h=1}^n \det B_h.
\]

Now, if a matrix \( S \) of order \( d \) is sum of \( \operatorname{diag}(r_1, \ldots, r_d) \) and of the rank one matrix \( (b_1 W, \ldots, b_d W) \), where \( fW = (w_1, \ldots, w_d) \), then

\[
\det S = r_1 \cdots r_d + \sum_{l=1}^d r_1 \cdots r_{l-1} b_l w_l r_{l+1} \cdots r_d.
\]

Hence, since the rank of \( (p_{\lambda \mu})_{\lambda \mu \in J_h} \) is equal to one,

\[
\det B_h = \prod_{\lambda \in J_h} \delta_\lambda + \sum_{\lambda \in J_h} \left( \prod_{\lambda \in J_h} \delta_\lambda \right) p_{\lambda \lambda} = \left( \prod_{\lambda \in J_h} \delta_\lambda \right) \left( 1 + \sum_{\lambda \in J_h} \frac{p_{\lambda \lambda}}{\delta_\lambda} \right)
\]

and, using the explicit values of \( \delta_\lambda \) and \( p_{\lambda \lambda} \).
\[ \det B_h = \frac{1}{(t_h T)^d h!} \prod_{\lambda \in J_h} \left( (d_h - a_h)T + na_h^2 \lambda_h^{-1} \right) \left[ 1 - \frac{(d_h - a_h)T}{t_h} \sum_{\lambda \in J_h} \frac{x_{\lambda}}{(d_h - a_h)T + na_h^2 \lambda_h^{-1} t_h} \right]. \quad (A.3) \]

with \( d_h = d_1 - 1 \) and \( d_h = d_h \) if \( h \geq 2 \).

Let us now study \( \delta_2 \) which we expand as the sum of \( m \) terms. Collecting the terms which belong to the same subset \( J_h \), we write

\[ \Delta_2 = \sum_{h=1}^{n} \det B_1 \cdots \det B_{h-1} \left( \sum_{\mu \in J_h} \det \Gamma_{\mu}^h \right) \det B_{h+1} \cdots \det B_n. \quad (A.4) \]

For \( \mu \in J_h \), \( \Gamma_{\mu}^h \) is the matrix of order \( d_h^\mu \) whose column \( (\Gamma_{\mu})_v \) is the projection of \( Q_{\mu} \) on \( C_{J_h}^\mu \) in the decomposition \( C^m = \bigoplus_{h=1}^n C_{J_h}^\mu \), and whose column \( (\Gamma_{\mu})_v \), for \( v \in J_h, v \neq \mu \), is the projection \( \bar{D}_v + \bar{P}_v \) of \( D_v + P_v \) on \( C_{J_h}^\mu \) (which could be identified with \( D_v + P_v \), since the components of \( D_v + P_v \) of indices belonging to \( \bigcup_{J \neq J_i} J_i \) are equal to zero).

We want to compute \( \Gamma_{\mu}^h = \sum_{\mu \in J_h} \det \Gamma_{\mu}^h \). To simplify the notations, we suppose that \( h = 1 \); in the summations which occur, all the indices \( \mu, v, \rho \) belong to \( J_1 = \{1, \ldots, d_1^i\} \). We have

\[ \Gamma(1) = \sum_{\mu \in J_1} \det \Gamma_{\mu}^1 = \sum_{\mu \in J_1} \det(\bar{D}_1 + \bar{P}_1, \ldots, \bar{D}_{\mu-1} + \bar{P}_{\mu-1}, \bar{Q}_\mu, \bar{D}_{\mu+1} + \bar{P}_{\mu+1}, \ldots, \bar{D}_{d_1^i} + \bar{P}_{d_1^i}) \]

\[ = 1 + II + III, \quad (A.5) \]

where, since all the vectors \( \bar{P}_\rho \) are parallel,

\[ I = \sum_{\mu \in J_1} \det(\bar{D}_1, \ldots, \bar{D}_{\mu-1}, \bar{Q}_\mu, \bar{D}_{\mu+1}, \ldots, \bar{D}_{d_1^i}). \]

\[ II = \sum_{\mu < v} \sum_{\mu \in J_1} \det(\bar{D}_1, \ldots, \bar{D}_{\mu-1}, \bar{P}_v, \ldots, \bar{P}_\rho, \bar{Q}_\mu, \bar{D}_{\mu+1}, \ldots, \bar{D}_{d_1^i}). \]

\[ III = \sum_{\mu, v < \mu} \sum_{\mu \in J_1} \det(\bar{D}_1, \ldots, \bar{D}_{\mu-1}, \bar{Q}_\mu, \bar{D}_v, \bar{P}_v, \bar{D}_{\mu+1}, \ldots, \bar{D}_{d_1^i}). \]

Recall that the element \( \delta_v \) of the diagonal matrix \( D \) is given by

\[ \delta_v = \frac{d_1 - a_1}{t_1} + \frac{n a_1^2 x_v^{-1}}{T}. \]

First,

\[ I = - \sum_{\mu \in J_1} \left[ \frac{n a_1^2 x_v^{-1}}{T} \prod_{v \neq \mu} \delta_v \right]. \quad (A.6) \]

On the other hand,

\[ II + III = \sum_{\mu, v < \mu} \sum_{\mu \notin J_1, v} \left( \sum_{\mu, \nu} \delta_v \sum_{\mu < \nu} \delta_v \sum_{\mu < \nu} \delta_v \right) \gamma_{v \nu}. \]

For \( v < \mu \), the quantity \( \gamma_{v \mu} \) is defined by

\[ \gamma_{v \mu} = \frac{n a_1^2 x_v^{-1}}{T} \left( \frac{d_1 - a_1}{t_1} + \frac{n a_1^2 x_v^{-1}}{T} \prod_{\mu \neq \nu, \nu} \delta_v \right) \gamma_{v \nu}. \]

and, for \( \mu < v \),

\[ \gamma_{\nu \mu} = \frac{n a_1^2 x_v^{-1}}{T} \left( \frac{d_1 - a_1}{t_1} + \frac{n a_1^2 x_v^{-1}}{T} \prod_{\mu \neq \nu, \nu} \delta_v \right) \gamma_{v \mu}. \]

Thus,

\[ II + III = \sum_{\mu < \nu} \left( \sum_{\mu < \nu} \delta_v \gamma_{v \mu} + \gamma_{v \mu} \right) = \frac{n(d_1 - a_1) a_1^2}{t_1^2 T^2} \sum_{\mu < \nu} \left( \sum_{\mu < \nu} \delta_v \right) x_v \left( a_1^{-1} - x_v^{-1} \right)^2. \]
since, for \( \mu < \nu \),
\[
\gamma_{\mu \nu}^{(\mu)} + \gamma_{\mu \nu}^{(\nu)} = \frac{n(d_1 - a_1) a_1^2}{t h^2} \left[ a_1^2 x_\nu (x_\mu^{-1} - x_\nu^{-1}) + x_\mu x_\nu a_1^2 (x_\nu^{-1} - x_\mu^{-1}) \right] = \frac{n(d_1 - a_1) a_1^2}{t h^2} x_\mu x_\nu (x_\mu^{-1} - x_\nu^{-1})^2.
\]

Taking into account (A.6), (A.7) and (A.5), we obtain, for any \( h \),
\[
\Gamma(h) = \sum_{\mu \in J_h} \det \Gamma_{\mu} = - \sum_{\mu \in J_h} \frac{n a_\mu^2 x_\mu a_\mu^{-1}}{t h^2} \prod_{\nu \in J_h, \nu \neq \mu} \delta_{\nu} + \frac{n(d_h - a_h) a_h^2}{t h^2} \sum_{\mu, \nu \in J_h, \mu < \nu} \prod_{\rho \in J_h, \rho \neq \mu, \nu} \delta_{\rho} x_\mu x_\nu (x_\mu a_\mu^{-1} - x_\nu a_\nu^{-1})^2. \tag{A.7}
\]
recalling that \( \delta_{\lambda} = (d_h - a_h)/t h + na_\lambda^2 x_\lambda a_\lambda^{-1} / T \) if \( \lambda \in J_h \).

(2) We now look for any upper bound of \( \det M \) on the cube
\[
D = \{ Z = (z_k)_{1 \leq k \leq m} \in C^m : 0 < x_k \leq 1 \text{ for all } k \}.
\]

Let us study \( \det B_h \) defined in (A.3). We write \( \det B_h = B_h' B_h'' \), with
\[
B_h' = \prod_{\lambda \in J_h} \left( \frac{d_h - a_h}{t h} + \frac{n a_\lambda^2 x_\lambda a_\lambda^{-1}}{t h^2} \right)
\]
and, if \( b_h = na_\lambda^2/(d_h - a_h) \),
\[
B_h'' = 1 - \frac{1}{t h} \sum_{\lambda \in J_h} x_\lambda \left( 1 + b_h x_\lambda a_\lambda^{-1} / t h \right)^{-1}.
\]

Since \( 0 < x_\lambda \leq 1 \) on \( D \), and according to the definitions of \( t h \) and \( T \) (in particular, \( T \) and \( t_1 \) are \( > 1 \) on \( D \)), we have
\[
\left[ \frac{d_h - a_h}{t h} \right] < B_h' < \left[ \frac{(m + 1)(d_h - a_h) + na_\lambda^2 d_h}{t h^2} \right] a_\lambda^h
\]
and
\[
1 - \frac{1}{t h} \sum_{\lambda \in J_h} x_\lambda = 0 < B_h'' < 1 - \left( 1 + b_h t_h / T \right)^{-1} < b_h t_h.
\]

To get a more precise upper bound of \( B_h'' \) when \( h \gg 2 \), since \( x_\lambda \leq t_h = \sum_{\mu \in J_h} x_\mu \) if \( \lambda \in J_h \), we write:
\[
B_h'' = 1 - \frac{1}{t h} \sum_{\lambda \in J_h} \left( x_\mu - b_h x_\mu a_\mu / t h + o(x_\mu a_\mu) \right) = \frac{b_h x_\mu a_\mu}{t h} \left( 1 + o(x_\mu a_\mu) \right) \leq \text{Const.} \times t_h a_\mu.
\]

Hence, there exist positive constants \( C' \) and \( C'' \) such that, on \( D \),
\[
C' \leq \det B_1 \leq C''. \quad \frac{C'}{t_h a_{h-1}} \leq \det B_h \leq \frac{C''}{t_h a_{h-1}} \quad \text{if } h \geq 2
\]
and, consequently,
\[
0 < \prod_{h=1}^{n} \det B_h \leq \text{Const.} \frac{t_h a_{h-1}}{t_h a_{h-1}}. \tag{A.8}
\]

Next, we examine, on \( D \), \( \Gamma(h) \) as defined in (A.7). It is clear that \( |\Gamma(h)| \leq \text{Const.} \). Suppose \( h \geq 2 \). Since for \( \lambda, \mu, \nu \in J_h \),
\[
\frac{\text{Const.}}{t_h} \leq \delta_{\lambda} = \frac{(d_h - a_h) T + na_\lambda^2 x_\lambda a_\lambda^{-1}}{t_h T} \leq \frac{\text{Const.}}{t_h}
\]

Thus, according to (A.7),

\[ \left| x_\mu x_\nu \right| \leq \frac{2a_\mu - 2}{t_h^2} \cdot \frac{2a_\nu - 2}{t_h^2} \cdot \text{Const.} \]

we see that

\[ \left| \frac{n a_\mu^2 a_\nu^2}{T^2} \sum_{\mu \in J_h} \prod_{\nu \in J_h, \nu \neq \mu} \delta_\nu \right| \leq \text{Const.} \cdot \frac{2a_\mu - 2}{t_h^2} \cdot \frac{2a_\nu - 2}{t_h^2} \cdot \text{Const.} \]

and

\[ \frac{n(d_h - a_h)^2}{T^2} \sum_{\mu, \nu \in J_h, \mu \neq \nu} \left( \prod_{\nu \in J_h, \nu \neq \mu} \delta_\nu \right) \cdot \left| x_\mu x_\nu (x_\mu^{-1} - x_\nu^{-1}) \right|^2 \leq \text{Const.} \cdot \frac{2a_\mu - 2}{t_h^2} \cdot \frac{2a_\nu - 2}{t_h^2} \cdot \text{Const.} \]

Thus, according to (A.7),

\[ |F'_{(h)}| \leq \frac{\text{Const.}}{t_h^2 - 2a_h} \leq \frac{\text{Const.}}{t_h^2 - 2a_h} \]

and so

\[ \left| \sum_{h=1}^n \text{det } B_1 \cdots \text{det } B_{h-1} F'_{(h)} \text{ det } B_{h+1} \cdots \text{ det } B_n \right| \leq \frac{\text{Const.}}{t_h^{2n-2} - 2a_n} \cdot (A.9) \]

If we take into account the value of \( \text{det } M \) given in the statement of the proposition, (A.8) and (A.9) yield to the upper bound of \( \text{det } M \) we were seeking. \( \square \)

References