Note

Translation Planes of Order 16 Admitting a Baer 4-Group

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All translation planes of order 16, which admit a 4-group fixing a Baer subplane, are determined.

In [4] Johnson and Ostrom consider translation planes $P$ of order 16 admitting the group $L_2(7)$ as a collineation group. It can easily be shown that in this case $L_2(7)$ contains a 4-group $V$ fixing a Baer subplane of $P$ pointwise.

The purpose of this paper is to characterize translation planes of order 16 by this property. More precisely we prove the following.

**Theorem.** Let $P$ be a translation plane of order 16 admitting an elementary abelian 4-group $E$ such that $\text{Fix}(E)$ is a Baer subplane of $P$. Then $P$ is isomorphic to

(i) the Hall plane of order 16,

(ii) the Lorimer–Rahilly plane,

(iii) the Johnson–Walker plane,

(iv) a derived semifield plane with translation complement $Z_3 \times (Z_3 \times A_4) \cdot Z_2$,

(v) the Dempwolff plane.
We also have the following generalization of a result of Johnson [5, Theorem (3.3)].

**Corollary.** Let \( \pi \) be a semifield plane of order 16. If \( \pi' \) is a translation plane obtained by replacing a derivable net containing the shears axis, then \( \pi' \) is isomorphic to one of the planes of the above theorem.

**Proof of the corollary.** A derivable net containing the shears axis is left invariant by a 4-group of elations of the translation complement of \( \pi \). The inherited group induces on \( \pi' \) a 4-group of Baer involutions fixing a Baer subplane. Thus \( \pi' \) is isomorphic to one of the planes of the theorem. Conversely, each of these planes contains a derivable net (see Section 2).

**Remark.** Under the additional assumption that a translation plane of order 16 admits a nonsolvable group of collineations the assertions of the theorem may be deduced by a result of Johnson, who determines translation planes of order 16 by this property. The crucial point of the above theorem is that the small group \( E \) of collineations is sufficient for a classification. Finally we would like to remark that Johnson independently observed that the Dempwolff plane can be obtained by deriving the semifield plane of order 16 with kernel \( GF(2) \).

1. **Notation and Preparatory Results**

Let \( V \) be an \( n \)-dimensional \( GF(2) \)-vectorspace and \( W = V \oplus V \). Let \( \pi \) be a collection \( V_\infty = \{ (0, V) \mid v \in V \}, \ V_0 = \{(v, 0) \mid v \in V \}, \ V_1, \ldots, V_{2^n - 1} \) of \( n \)-dimensional subspaces (spread) such that \( W = V_\infty \cup \bigcup_{i=0}^{2^n - 1} V_i \). Then according to [2] \((W, \pi) \) becomes a translation plane of order \( 2^n \) and for \( i \geq 1 \) we may assume \( V_i = \{(v, vt_i) \mid v \in V \}, \) where \( t_i \) induces the identity on \( V \) and \( t_i t_j^{-1} \) acts fixedpointfree on \( V \) for \( i \neq j \) and \( 1 \leq i, j \leq 2^n - 1 \).

As \( GL(4, 2) \cong A_8 \), e.g., see [3, II, 2.5], we describe in our case the elements in \( M = \{ t_1, \ldots, t_{15} \} \) as permutations in \( A_8 \). This will give us an effective computational approach for classifying translation planes of order 16. For instance, the unpleasant matrix multiplication is replaced by simple multiplication of permutations. Note that we never make use of an explicit isomorphism between \( GL(4, 2) \) and \( A_8 \). The above condition imposed on the elements of \( M \) yields that \( t_i t_j^{-1} \) \((i \neq j) \) is a permutation of cycle structure

\[
(abc), \quad (abc)(defg), \quad (abcde), \quad \text{or} \quad (abcde)(fgh). \quad (*)
\]

Furthermore without loss of generality we label the subspaces \( V_i \) in such a way that \( \text{Fix}(E) \subseteq V_\infty \cup V_0 \cup V_1 \cup V_2 \cup V_3 \).

The following lemma is well known (e.g., see [1]).
Lemma 1.1. (a) Let \( p \) be a collineation of the translation complement of \((W, \pi)\).

(i) If \( p \) fixes \( V_\infty \) and \( V_0 \), then there are \( t, s \in GL(V) \), such that 
\[ t^{-1}Ms = M \text{ and } p: W \ni (v, w) \rightarrow (vt, ws) \in W. \]

(ii) Assumptions as in (i) and in addition \( p \) fixes \( V_1 \), then \( t = s \).

(b) (Changing coordinates.)

(i) Changing the notation, so that \( V_i \) becomes \( V_1 \) \( (2 \leq i \leq 15) \)
yields that \( M \) has to be replaced by \( t_i^{-1}M \).

(ii) Interchanging the roles of \( V_0 \) and \( V_\infty \) means that \( M \) is replaced by \( M^{-1} \).

(iii) Interchanging the roles of \( V_\infty \) and \( V_1 \) means that \( M \) is replaced by \( (Mt_i^{-1} + 1)^{-1} = \{ m \in M \mid m \in M - \{ t_i \} \} \cup \{ 1 \} \).

(iv) Replacing the pair \( V_0, V_\infty \) by the pair \( V_1, V_i \) gives a replacement of \( M \) by \( M^{-1}t_i + 1 = \{ m^{-1}t_i + 1 \mid m \in M - \{ t_i \} \} \cup \{ 1 \} \).

(c) Let \( M \) and \( M' \) (as above) define translation planes \((W, \pi)\) and 
\((W, \pi')\), where \( \pi' = V'_\infty, V'_0, \ldots \). If there is an \( x \in A_8 \) with \( x^{-1}Mx = M' \), then there is an isomorphism \( \varphi: (W, \pi) \rightarrow (W, \pi') \) with \( \varphi: V_i \rightarrow V'_i, V_i \rightarrow V'_i \) for \( i = \infty, 0, 1 \).

According to Lemma 1.1(a, ii) the group \( E \) can be regarded as a subgroup of \( GL(4, 2) \cong A_8 \) normalizing \( M \) and centralizing \( t_i, t_j, t_k \).

Lemma 1.2. Let \( i \) be an involution in \( A_8 \) normalizing \( M \) and centralizing 
\( \{ t_1, t_2, t_3 \} \), then \( i \) is of type \((ab)(cd)\). If \( E \) is a 4-group in \( A_8 \) normalizing \( M \) and centralizing \( \{ t_1, t_2, t_3 \} \), then \( E \) is of type \( \langle (ab)(cd), (ac)(bd) \rangle \).

Proof. Assume \( i \) is of type \( (ab)(cd)(ef')(gh) \). Then \( C_{A_8}(i) \) is a \( \{ 2, 3 \} \)-group and the 3-elements in this group are of type \( (ace)(bdf) \). So \( C_{A_8}(i) \cap M = 1 \), a contradiction. Let \( E \) be a 4-group whose nontrivial elements are of type \( (ab)(cd) \). Then \( E \) is conjugated in \( A_8 \) to either \( \langle (ab)(cd), (ac)(bd) \rangle \) or \( \langle (ab)(cd), (cd)(ef') \rangle \). However, in the second case \( C_{A_8}(E) \) is a 2-group which would imply \( C_{A_8}(E) \cap M = 1 \), a contradiction.

For the remainder of this article we assume without loss of generality that 
\( E = \langle (45)(67), (46)(57) \rangle \).

2. The Planes Occurring in the Theorem

For convenience we give a short description of the planes arising in the theorem in the language of permutations of \( A_8 \). We will also indicate how one obtains a large part of the collineation group (in fact in most cases the full collineation group) with the help of Lemma 1.1. We will show in the
proof of the theorem, that there are at most five nonisomorphic planes admitting a Baer 4-group. Then the identification of these planes described below via their automorphism group is obvious, since all five planes admit a Baer 4-group and are clearly nonisomorphic.

According to the preceding paragraph set \( t_1 = 1, \ t_2 = (123), \ t_3 = (132) \) and \( t^E = (ete \mid e \in E) \) for an element \( t \in A_8 \).

The Hall plane of order 16: \( M_1 = \{t_1, t_2, t, (45678)^E, (45786)^E, (46587)^E\} \). \( M_1 - \{t_1, t_2 t_3\} \) is a conjugacy class of elements of order 5 of \( A_5 \) acting on \( \{4, 5, 6, 7, 8\} \) and so \( L \simeq A_4 \) acting on \( \{4, 5, 6, 7, 8\} \) normalizes \( M_1 \) and induces a subgroup of \( \text{Aut}(W, \pi_4) \) fixing the components \( V_4,..., V_9 \). \( (123) \) centralizes \( M_1 \) and so the kernel of this plane is \( GF(4) \). As \( M_1 = M_1^{-1} \), there is an elation fixing \( V_4 \) and interchanging \( V_0 \) and \( V_1 \). Without verifying we mention, that for either \( i = 2 \) or \( i = 3 \) the map \( \sigma: W \ni (v, u) \rightarrow (u + vt_1, v) \) defines an automorphism of order 5 fixing \( V_4,..., V_9 \) and \( \langle \sigma \rangle L = \langle \sigma \rangle \times L \). (Note that \( i \) depends on the choice of the isomorphism \( \varphi: A_4 \rightarrow GL(4, 2) \).

Replacing \( \varphi \) by the automorphism \( \varphi^* \) the composition of \( \varphi \) with the transpose-inverse automorphism of \( GL(4, 2) \) means that the roles of \( t_2 \) and \( t_1 \) have to be interchanged.)

The Lorimer-Rahilly plane: \( M_2 = \{t_1, t_2, t_3, (345)^E, (246)^E, (147)^E\} \).

The Johnson-Walker plane: \( M_3 = \{t_1, t_2, t_3, (345)^E, (247)^E, (146)^E\} \). \( M_i \) \( (i = 2, 3) \) contains exactly fourteen 3-cycles. The support of the 3-cycles gives precisely seven sets of cardinality 3 which form a projective plane of order 2. Hence the permutation group \( X_i \) in \( A_8 \) leaving this plane fixed is isomorphic to \( GL(3, 2) \) and normalizes \( M_i \). We assume that we have picked the isomorphism \( \varphi: A_4 \rightarrow GL(4, 2) \) in such a way that \( X_i^G \) stabilizes a point. Then \( X_i^G \) stabilizes a hyperplane. In particular \( M_2 \) and \( M_3 \) can not coordinize isomorphic planes, since the full translation complement is isomorphic to \( S_3 \times GL(3, 2) \) in both cases (e.g., see [6]).

The derived semifield plane: \( M_4 = \{t_1, t_2, t_3, (148)^E, (248)^E, (348)^E\} \). Here \( H = \langle E, (123), (456), (23)(56) \rangle \simeq (Z_3 \times A_4) \cdot Z_2 \) obviously normalizes \( M_4 \). Since \( M_4 + 1 = M_4^{-1} \), there is by Lemma 1.1 a collineation \( \rho \) of order 3 leaving the components \( V_2,..., V_5 \) fixed and interchanging \( V_{10}, V_0 \) and \( V_1 \). We have that \( H\langle \rho \rangle = H \times \langle \rho \rangle \) is the full translation complement (see also [1]).

The Dempwolff plane: \( M_5 = \{t_1, t_2, t_3, (348)^E, (12483)^E, (14832)^E\} \). Since \((8)(56)M_5(34)(56) = M_5^{-1}\), there is a collineation \( \rho \) of order 2 such that \( L = \langle \rho, E \rangle \simeq A_4 \). \( M_5 = (123)M_5 \) means that there is a collineation \( \tau \) of order 3 fixing \( V_{10} \) pointwise, i.e., \( \tau \) is a homology.

Finally \((12)(56)M_5(12)(56) \) induces an involution \( \sigma \) and one readily checks in \( A_8 \) that \( L(\tau, \sigma) \simeq IG(2, 4) \) (e.g., see [1]).

Always the components \( V_{10}, V_0, V_1, V_2, V_3 \) with their translates form a derivable net. Derivation gives in the case of the Dempwolff plane the semifield plane with kern \( GF(2) \), in the case of the Hall plane the
3. Proof of the Theorem

Using the notation of the preceding sections we have that $E$ acts on $\mathcal{M}$ by conjugation fixing $t_1, t_2, t_3$. We determine all coordinate sets $\mathcal{M}$ with this property. Now $C_{A_4}(E) = E \times \langle (123), (128) \rangle$ and $C_{A_4}(E)/E \cong A_4$, so $t_2, t_3 \in C_{A_4}(E) \cap \mathcal{M}$ have order 3 or 6. $A_4$ acts 2-transitive on its Sylow 3-subgroups, so we may pick without loss of generality $t_3 = (123)a, t_3 = (132)b$ or $t_2 = (123)a, t_3 = (128)b$, where $a, b \in E$. By further conjugation with elements in $N_{A_4}(E)$ (and interchanging the roles of $t_2$ and $t_3$ if necessary), we may assume $a = (45)(67)$ and $b = (45)(67)$ or $(46)(57)$ if $a \neq 1$ and $b = 1$ or $(45)(67)$ if $a = 1$. Thus there are exactly eight different cases for picking $t_2$ and $t_3$. In each of these cases we compute all sets $\mathcal{M}' = \{ t_4, \ldots, t_{12} \}$ of elements of type $(\ast)$ of Section 1, such that $t_i^{-1}t_j$ also is an element of type $(\ast)$ for all choices $t_i \neq t_j$ out of $\mathcal{M}' = \{ t_1, t_2, t_3, \mathcal{M}' \}$ and such that $\mathcal{M}'$ is invariant under conjugation with $E$. Since $\mathcal{M}'$ splits into three orbits of length 4 under $E$, we only have to find representatives of each orbit in such a way, that all orbit members are compatible with $t_1, t_2, t_3$, among each other, and that each orbit member is compatible with one representative every other orbit.

For the search of the sets $\mathcal{M}'$ it is advisable to use a computer. The program we used was set up in the following pattern:

The list $L$ of 5824 permutations of type $(\ast)$ in $A_4$ is produced and totally ordered say by $<$. Then the sublist $L_1$ of permutations compatible with $t_1, t_2, t_3$ is computed. In a third step we run through the list $L_1$ following the order $<$ and establish all $E$-orbits $\mathcal{M}_i \subseteq L_1$ ($i = 1, 2, \ldots$) and collect them in a list $L_2$. In the final step all triples of sets $\mathcal{M}_i, \mathcal{M}_j, \mathcal{M}_k \subseteq L_2$ with $\mathcal{M}_i < \mathcal{M}_j < \mathcal{M}_k$ are established, such that they are mutually compatible. The result after less than 1 min computing time is that only in the case $t_2 = (123), t_3 = (132)$ are there compatible sets $\mathcal{M}'$ and that there are precisely 146 of them.

Using Lemma 1.1 we reduce these 146 cases to the examples $\mathcal{M}_1, \ldots, \mathcal{M}_8$, of Section 2 (the reduction process shows how these sets can be recovered from $\mathcal{M}_1, \ldots, \mathcal{M}_8$, so we do not write them down here): We call two sets "equivalent" if one is obtained by the other by an operation described in Lemma 1.1(b) or (c). Obviously equivalent sets coordinize isomorphic planes. Thus multiplying these 146 sets from the left with $(123)$ and $(132)$ we obtain 47 sets of equivalent triples of $\mathcal{M}'$s and 5 sets are fixed by this operation, e.g., the set $\mathcal{M}_5$. Inverting the elements in each set shows that there are 28 sets of equivalent $\mathcal{M}'$s. The operations $\mathcal{M} \rightarrow \mathcal{M} + 1$, desarguesian plane, and in all other cases the semifield plane with kern $GF(4)$ (for details also see [5]).
$M \rightarrow M + (123)$, and $M \rightarrow M + (132)$ give a further reduction to 9 sets of equivalent $M$'s with representatives $M_1,\ldots, M_9$, and

$$M'_6 = \{(346)^E, (245)^E, (147)^E\},$$
$$M'_7 = \{(346)^E, (247)^E, (145)^E\},$$
$$M'_8 = \{(347)^E, (245)^E, (146)^E\},$$
$$M'_9 = \{(347)^E, (246)^E, (145)^E\}.$$


References