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Integral degree of a ring, reduction numbers and uniform Artin–Rees numbers

José M. Giral [a], Francesc Planas-Vilanova [b,*,1]

[a] *Universitat de Barcelona, Departament d'Àlgebra i Geometria, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain*
[b] *Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada 1, Diagonal 647, ETSEIB, 08028 Barcelona, Spain*

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Abstract

The supremum of reduction numbers of ideals having principal reductions is expressed in terms of the integral degree, a new invariant of the ring, which is finite provided the ring has finite integral closure. As a consequence, one obtains bounds for the Castelnuovo–Mumford regularity of the Rees algebra and for the Artin–Rees numbers.

Keywords: Reduction number; Integral closure; Artin–Rees number

1. Introduction

Let A be a commutative noetherian ring with identity, let I be an ideal of A and let M be a finitely generated A-module. An ideal $J \subset I$ is said to be a reduction of I with respect to M if $I^{n+1}M = JI^nM$ for some integer $n \geqslant 0$. The least such integer $n \geqslant 0$ is called the *J-reduction number of I with respect to M* and is denoted by $\mathrm{rn}_J(I;M)$. If $M = A$, the phrase "with respect to M" is omitted and one writes $\mathrm{rn}_J(I)$. Clearly, if J is a reduction of I, then J is a reduction of I with respect to M and $\mathrm{rn}_J(I;M) \leqslant \mathrm{rn}_J(I)$. If I is regular (i.e. I contains a nonzero divisor)

* Corresponding author.
E-mail addresses: giral@ub.edu (J.M. Giral), francesc.planas@upc.edu (F. Planas-Vilanova).
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and J is a principal reduction, then $\mathrm{rn}_J(I)$ is independent of the given principal reduction and is denoted by $\mathrm{rn}(I)$ (more details are given at the beginning of Section 7).

In the last decade there has been a great deal of attention to finding bounds on the reduction number (see e.g. [1,7,11,12,17,30,35–37], by no means a complete list of references). If the regular ideals have principal reductions and the integral closure is finite, d'Anna, Guerrieri and Heinzer gave an absolute bound for the reduction number in terms of the minimal number of generators of the integral closure as a module over the base ring [11, Corollary 5.2]. In this paper we express the supremum of reduction numbers of regular ideals having principal reductions in terms of the following new invariant associated to the ring A, provided A contains the field of rational numbers $\mathbb{Q}$. If $A \subset B$ is a ring extension and $b \in B$ is integral over A, let the *integral degree* of b over A be

$$\mathrm{id}_A(b) = \min\{n \geqslant 1 \mid b \text{ satisfies an integral equation of degree } n\}.$$

If $A \subset B$ is an integral extension, the *integral degree* of B over A is defined as

$$\mathrm{d}_A(B) = \sup\bigl\{\mathrm{id}_A(b) \bigm| b \in B\bigr\}.$$

When B is taken to be $\overline{A}$, the integral closure of A in its total quotient ring, $\mathrm{d}_A(\overline{A})$ is just called the *integral degree* of A. We will prove that if A has finite integral closure then it has also finite integral degree and that the converse is not true in general. Our main result (see Theorem 7.1) is:

Theorem 1.1. *Let A be a noetherian ring, $A \supset \mathbb{Q}$. Then*

$$\mathrm{d}_A(\overline{A}) = \sup\bigl\{\mathrm{rn}(I) \bigm| I \textit{ a regular ideal having a principal reduction}\bigr\} + 1.$$

In Theorem 1.1 it is possible to replace $\mathrm{rn}(I) + 1$ by either $\mathrm{reg}(\mathcal{R}(I)) + 1$ or $\mathrm{rt}(I)$, $\mathrm{reg}(\mathcal{R}(I))$ and $\mathrm{rt}(I)$ being the Castelnuovo–Mumford regularity of the Rees algebra of I and the relation type of I, respectively.

It is known that Artin–Rees numbers are bounded by the relation type and that, in some particular cases, the relation type can be bounded by reduction numbers. Having in mind this idea and as a consequence of Theorem 1.1, we get the following results in the context of uniform Artin–Rees properties (see Theorem 8.1).

Theorem 1.2. *Let A be a noetherian ring with finite integral degree $\mathrm{d}_A(\overline{A}) = d$. Suppose that $A \supset \mathbb{Q}$. Let $N \subset M$ be two finitely generated A-modules. Let I be a regular ideal of A having a principal reduction that is generated by a d-sequence with respect to M/N. Then, for every integer $n \geqslant d$,*

$$I^n M \cap N = I^{n-d}\bigl(I^d M \cap N\bigr).$$

In other words, $\mathrm{d}_A(\overline{A})$ is a uniform Artin–Rees number for the pair $N \subset M$ and the whole set of regular ideals having principal reductions generated by a d-sequence with respect to M/N. Our ideal-theoretic version is the following (see Theorem 8.3).

Theorem 1.3. *Let A be a noetherian ring, $A \supset \mathbb{Q}$. Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ has finite integral degree $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}}) = d$. Let I be an ideal of A such that $IA/\mathfrak{a}$ has an $A/\mathfrak{a}$-regular principal reduction. Then, for every integer $n \geqslant d$,*

$$I^n \cap \mathfrak{a} = I^{n-d}\big(I^d \cap \mathfrak{a}\big).$$

If the integral degree of $\overline{A/\mathfrak{a}}$ is not finite or if $IA/\mathfrak{a}$ has no principal reduction, then there may not exist such a uniform Artin–Rees number (Example 8.5 and Example 8.7) even in the presence of regularity. This will be seen by using an example of Eisenbud and Hochster in [14], the work where they raised the uniform Artin–Rees conjecture, and an example of Wang in [38] (see also [13,21,26–28] for more information). On the other hand, it is well known that there exists a uniform Artin–Rees number for the set of principal ideals of a noetherian ring and that, in general, there does not exist a uniform Artin–Rees number for the set of three generated ideals (see the work of O'Carroll in [27] and the aforementioned example of Wang in [38]). Therefore, it remained to study if there exists a uniform Artin–Rees number for the whole set of two-generated ideals (without any other assumption on the ideals). We obtain a slightly weaker uniform Artin–Rees property for the set of two-generated regular ideals (which is not true anymore for the set of three-generated ideals, see Example 8.7). Specifically, we have the following (see Theorem 8.6).

Theorem 1.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring with infinite residue field. Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ has finite integral degree $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}}) = d$. Let I be a two-generated ideal of A such that $IA/\mathfrak{a}$ is $A/\mathfrak{a}$-regular. Then, for every $n \geqslant d$,*

$$I^n \cap \mathfrak{a} = I^{n-d}\big(I^d \cap \mathfrak{a}\big) + \mathfrak{m}I^n \cap \mathfrak{a}.$$

The paper is organized as follows. Sections 2–5 are devoted to the following four invariants and the relationship among them: Artin–Rees numbers modulo an ideal, relation type of a standard module, Castelnuovo–Mumford regularity and reduction number with respect to a module. Concretely, in Section 2 we introduce the Artin–Rees number $s_J(N, M; I)$ of an ideal I and two finitely generated A-modules $N \subset M$, modulo another ideal J. This number is the minimum integer $s \geqslant 0$ such that

$$I^n M \cap N = I^{n-s}\big(I^s M \cap N\big) + JI^n M \cap N$$

for all $n \geqslant s + 1$, and thus it controls the weaker Artin–Rees property of Theorem 1.4. Following the ideas in [28], in Section 3 we bound above the Artin–Rees number $s_J(N, M; I)$ by the relation type of the Rees module

$$\mathcal{R}_J(I; M/N) = \left(\bigoplus_{n\geqslant 0} I^n M/N\right) \otimes A/J.$$

Section 4 is dedicated to recalling some definitions concerning Castelnuovo–Mumford regularity and formulating an extension to modules of some results of Trung in [33] and [34]. In Section 5, we prove that the relation type of an ideal I with respect to a module M, $\mathrm{rt}(I; M)$, is bounded above by $\mathrm{rn}_J(I; M) + \mathrm{rt}(J; I^r M)$, where J is a reduction of I with respect to M and $r := \mathrm{rn}_J(I; M)$ is the J-reduction number of I with respect to M. If J is generated by a complete

d-sequence with respect to I and M (the terminology is explained in Section 5), then the relation type of J with respect to $I^r M$ satisfies $\mathrm{rt}(J; I^r M) = 1$ and $\mathrm{reg}(\mathcal{R}(I; M)) = \mathrm{rn}_J(I; M)$. Thus one has the inequality $\mathrm{rt}(I; M) \leqslant \mathrm{rn}_J(I; M) + 1$, which is well known for the case $M = A$ and J a principal reduction of a regular ideal I (see the work of d'Anna, Guerrieri and Heinzer [10], Huckaba [19,20], Schenzel [32] and Trung [33,34]). In Section 6, we introduce and study $\mathrm{d}_A(\overline{A})$, the integral degree of A, a new invariant associated to the ring A. We prove that if A has finite integral closure then it also has finite integral degree. The ingenious example of Akizuki ([2], see also [29]) provides us with an example of a one-dimensional noetherian local domain A with finite integral degree but infinite integral closure. In Section 7 we prove the main result of the paper, namely that $\mathrm{d}_A(\overline{A})$ is equal to the supremum of the reduction numbers plus one (or else the Castelnuovo–Mumford regularity of the Rees algebra plus one or the relation type: see Remark 7.2) of regular ideals having principal reductions. Finally, in Section 8 we prove all the results concerning Artin–Rees numbers.

All rings will be commutative and with identity. As usual, an element being M-regular will mean that the element is not contained in the set of zero divisors of M, and μ will stand for minimal number of generators.

2. Artin–Rees modulo an ideal

Let us introduce a slight variant of the Artin–Rees lemma which will be very useful. Let A be a noetherian ring, I an ideal of A and $N \subseteq M$ two finitely generated A-modules. The Artin–Rees lemma assures us that there exists an integer $s \geqslant 0$, depending on N, M and I, such that for all $n \geqslant s$,

$$I^n M \cap N = I^{n-s}\big(I^s M \cap N\big).$$

In particular, for any ideal J of A, one obtains what we will call the *Artin–Rees modulo J*:

$$I^n M \cap N = I^{n-s}\big(I^s M \cap N\big) + J I^n M \cap N.$$

For every integer $n \geqslant 1$, let

$$E_J(N, M; I)_n = \frac{I^n M \cap N}{I(I^{n-1} M \cap N) + J I^n M \cap N}.$$

For easy reference, and without proof, we state the Artin–Rees lemma modulo J.

Lemma 2.1. *Let A be a ring, I, J ideals of A and $N \subseteq M$ two A-modules. Set*

$$s_J(N, M; I) = \min\big\{s \geqslant 0 \bigm| E_J(N, M; I)_n = 0 \textit{ for all } n \geqslant s+1\big\}.$$

Then, the following conditions are equivalent:

(i) $I^n M \cap N = I^{n-s}(I^s M \cap N) + J I^n M \cap N$ *for all* $n \geqslant s+1$.
(ii) $s_J(N, M; I) \leqslant s$.

If A is a noetherian ring and $N \subseteq M$ are finitely-generated A-modules, then $s_J(N, M; I)$ is finite.

If $J = 0$, we recover the standard notion of Artin–Rees and simply write $s(N, M; I)$. Remark that if $J_1 \subset J_2$ are two ideals, for $n \geqslant 1$ there is a natural epimorphism $E_{J_1}(N, M; I)_n \to E_{J_2}(N, M; I)_n$ and thus $s_{J_2}(N, M; I) \leqslant s_{J_1}(N, M; I) \leqslant s(N, M; I)$.

Remark 2.2. If A is noetherian, $J \subset I$ are two ideals of A contained in the Jacobson radical of A, and $N \subset M$ are two finitely generated A-modules, then $s_J(N, M; I) = s(N, M; I)$.

Proof. Since $0 \subset J \subset I$, then $s_I(N, M; I) \leqslant s_J(N, M; I) \leqslant s(N, M; I)$. It is enough to see that $s(N, M; I) \leqslant s_I(N, M; I)$. Set $s = s_I(N, M; I)$, so $I^nM \cap N = I^{n-s}(I^sM \cap N) + I^{n+1}M \cap N$ for all $n \geqslant s + 1$. Then $I^{n+1}M \cap N = I^{n+1-s}(I^sM \cap N) + I^{n+2}M \cap N$ and substituting the second equality in the first, $I^nM \cap N = I^{n-s}(I^sM \cap N) + I^{n+1-s}(I^sM \cap N) + I^{n+2}M \cap N = I^{n-s}(I^sM \cap N) + I^{n+2}M \cap N$. Inductively, $I^nM \cap N = \bigcap_{k\geqslant 1}(I^{n-s}(I^sM \cap N) + I^{n+k}M \cap N) \subset \bigcap_{k\geqslant 1}(P + I^{n+k}M)$, where $P = I^{n-s}(I^sM \cap N) \subset I^nM \cap N \subset M$. But,

$$\frac{\bigcap_{k\geqslant 1}(P + I^{n+k}M)}{P} = \bigcap_{k\geqslant 1}\left(\frac{P + I^{n+k}M}{P}\right) = \bigcap_{k\geqslant 1} I^{n+k}(M/P),$$

which is zero by Krull's intersection theorem. Therefore, $\bigcap_{k\geqslant 1}(P + I^{n+k}M) = P$ and $I^nM \cap N = I^{n-s}(I^sM \cap N)$. □

3. Relation type modulo an ideal

A standard A-algebra is a commutative graded algebra $U = \bigoplus_{n\geqslant 0} U_n$, with $U_0 = A$ and U generated as an A-algebra by the elements of degree 1. The Rees algebra of I is the standard A-algebra $\mathcal{R}(I) = \bigoplus_{n\geqslant 0} I^n$. For any ideal J of A, the *Rees algebra of I modulo J* will be the standard A/J-algebra $\mathcal{R}_J(I) = \mathcal{R}(I) \otimes A/J = \bigoplus_{n\geqslant 0} I^n/JI^n$. Taking $J = I$, we recover the associated graded ring of I, $\mathcal{R}_I(I) = \mathcal{G}(I) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$, and taking $J = \mathfrak{m}$ a maximal ideal of A, we recover the corresponding fiber cone of I, $\mathcal{R}_\mathfrak{m}(I) = \mathcal{F}_\mathfrak{m}(I) = \bigoplus_{n\geqslant 0} I^n/\mathfrak{m}I^n$.

A standard U-module will be a graded U-module $F = \bigoplus_{n\geqslant 0} F_n$ such that $F_n = U_nF_0$ for all $n \geqslant 0$. The Rees module of I with respect to M is the standard $\mathcal{R}(I)$-module $\mathcal{R}(I; M) = \bigoplus_{n\geqslant 0} I^nM$. For any ideal J of A, the *Rees module of I with respect to M and modulo J* will be the standard $\mathcal{R}_J(I)$-module $\mathcal{R}_J(I; M) = \mathcal{R}(I; M) \otimes A/J = \bigoplus_{n\geqslant 0} I^nM/JI^nM$. Taking $J = I$, we recover the associated graded module of I with respect to M, $\mathcal{R}_I(I; M) = \mathcal{G}(I; M) = \bigoplus_{n\geqslant 0} I^nM/I^{n+1}M$ and taking $J = \mathfrak{m}$ a maximal ideal of A, we recover the corresponding fiber cone of I with respect to M, $\mathcal{R}_\mathfrak{m}(I; M) = \mathcal{F}_\mathfrak{m}(I; M) = \bigoplus_{n\geqslant 0} I^nM/\mathfrak{m}I^nM$.

Given two standard U-modules F, G and $\varphi : G \to F$, a surjective graded morphism of U-modules, put $E(\varphi)_n = \ker\varphi_n/U_1 \ker\varphi_{n-1}$ for $n \geqslant 2$. Consider $\gamma : \mathbf{S}(U_1) \otimes F_0 \overset{\alpha\otimes 1}{\to} U \otimes F_0 \to F$, where $\alpha : \mathbf{S}(U_1) \to U$ is the canonical symmetric presentation of U and $U \otimes F_0 \to F$ is the structural morphism. For $n \geqslant 2$, the *module of effective n-relations of F* is $E(F)_n = E(\gamma)_n = \ker\gamma_n/U_1 \ker\gamma_{n-1}$. The *relation type of F* is $\mathrm{rt}(F) = \min\{r \geqslant 1 \mid E(F)_n = 0 \text{ for all } n \geqslant r + 1\}$, which is finite if A is noetherian, U is a finitely generated algebra and F is a finitely generated U-module. It can be shown that the module of effective n-relations, $n \geqslant 2$, and the relation type do not depend on the chosen symmetric presentation ([28, Definition 2.4]; see also [10,37,38]). In particular, in order to find the effective relations of F and its relation type, one can always take a presentation of F as a quotient of a polynomial module with coefficients in F_0.

The *module of effective n-relations of I with respect to M* is $E(I;M)_n = E(\mathcal{R}(I;M))_n$ and the relation type of I with respect to M is $\operatorname{rt}(I;M) = \operatorname{rt}(\mathcal{R}(I;M))$. For any ideal J of A, the *module of effective n-relations of I with respect to M and modulo J* will be $E_J(I;M)_n = E(\mathcal{R}_J(I;M))_n$ and the relation type of I with respect to M and modulo J will be $\operatorname{rt}_J(I;M) = \operatorname{rt}(\mathcal{R}_J(I;M))$. If $M = A$, then we omit the phrase "with respect to M" and simply write $E(I)_n$, $\operatorname{rt}(I)$, $E_J(I)_n$ and $\operatorname{rt}_J(I)$.

Remark 3.1. Let A be a ring, J, I, $\mathfrak{a}$, ideals of A and M an A-module.

(1) Then $\operatorname{rt}(I;A/\mathfrak{a}) = \operatorname{rt}(IA/\mathfrak{a}) = \operatorname{rt}(IA/\mathfrak{a};A/\mathfrak{a})$.
(2) $\operatorname{rt}_J(I;M) \leqslant \operatorname{rt}(I;M)$.
(3) If A is noetherian, $J \subset I$ and M is finitely generated, then $\operatorname{rt}_J(I;M) = \operatorname{rt}(I;M)$.

Proof. $\mathcal{R}(I;A/\mathfrak{a}) = \mathcal{R}(IA/\mathfrak{a}) = \mathcal{R}(IA/\mathfrak{a};A/\mathfrak{a})$. Moreover, the relation type of the standard $\mathcal{R}(I)$-module $\mathcal{R}(I;A/\mathfrak{a})$, the relation type of the standard $A/\mathfrak{a}$-algebra $\mathcal{R}(IA/\mathfrak{a})$ and the relation type of the standard $\mathcal{R}(IA/\mathfrak{a})$-module $\mathcal{R}(IA/\mathfrak{a};A/\mathfrak{a})$ all coincide (see [28, Remark 2.5]). This proves (1). The proof of (2) and (3) follow from [28, Remark 2.7] (and in contrast to Remark 2.2, here we do not need I to be included in the Jacobson radical). $\square$

Next we show the relation between $E(I;M)_n$ and $E_J(I;M)_n$ and describe these modules for the two-generated regular case.

Proposition 3.2. *Let A be a ring, I and J ideals of A and M an A-module. Then, for every integer $n \geqslant 2$, there exists an exact sequence of A-modules*:

$$E(I;JM)_n \longrightarrow E(I;M)_n \longrightarrow E_J(I;M)_n \to 0.$$

In particular, if $I = (x,y)$ is two-generated and x is M-regular, then, for every $n \geqslant 2$,

$$E_J(I;M)_n = \frac{(xI^{n-1}M : y^n)}{(xI^{n-1}JM : y^n) \cap JM + (xI^{n-2}M : y^{n-1})}.$$

Proof. Let $f: P \to I$ be a presentation of I, with P a free A-module, and, for every $n \geqslant 2$, consider the following commutative diagram:

$$\begin{tikzcd}
\Lambda_2(P) \otimes I^{n-2}JM \arrow[r, "\partial'_{2,n}"] \arrow[d] & P \otimes I^{n-1}JM \arrow[r, "\partial'_{1,n}"] \arrow[d] & I^nJM \arrow[r] \arrow[d, tail] & 0 \\
\Lambda_2(P) \otimes I^{n-2}M \arrow[r, "\partial_{2,n}"] \arrow[d] & P \otimes I^{n-1}M \arrow[r, "\partial_{1,n}"] \arrow[d] & I^nM \arrow[r] \arrow[d] & 0 \\
\Lambda_2(P) \otimes (I^{n-2}M/I^{n-2}JM) \arrow[r, "\bar{\partial}_{2,n}"] & P \otimes (I^{n-1}M/I^{n-1}JM) \arrow[r, "\bar{\partial}_{1,n}"] & I^nM/I^nJM \arrow[r] & 0.
\end{tikzcd}$$

The top, middle and bottom rows of these diagrams represent the last three nonzero terms of the nth homogeneous part of the Koszul complexes induced by the $\mathbf{S}(P)$-linear forms $P \otimes \mathcal{R}(I; JM) \to \mathcal{R}(I; JM)$, $P \otimes \mathcal{R}(I; M) \to \mathcal{R}(I; M)$ and $P \otimes \mathcal{R}_J(I; M) \to \mathcal{R}_J(I; M)$. The differentials are defined as usual: $\partial_{2,n}((x \wedge y) \otimes z) = y \otimes xz - x \otimes yz$ and $\partial_{1,n}(x \otimes t) = xt$, $x, y \in P$, $z \in I^{n-2}M$, $t \in I^{n-1}M$; $\partial'_{i,n}$ and $\bar{\partial}_{i,n}$ are defined analogously (see e.g. [6, Definition 1.6.1]). The vertical morphisms are induced by the obvious inclusions and quotients and define morphisms of complexes. By a similar reasoning to that in [28, Proposition 2.6], the first homology groups of these complexes are, respectively, $\ker \partial'_{1,n} / \operatorname{im} \partial'_{2,n} = E(I; JM)_n$, $\ker \partial_{1,n} / \operatorname{im} \partial_{2,n} = E(I; M)_n$ and $\ker \bar{\partial}_{1,n} / \operatorname{im} \bar{\partial}_{2,n} = E_J(I; M)_n$. The exact sequence we seek is nothing else but the short exact sequence induced in homology.

If $I = (x, y)$ with x an M-regular element, take $P = A^2$ and $f : P \to I$ with $f(1, 0) = x$ and $f(0, 1) = y$. Then, the middle row becomes isomorphic to the complex:

$$I^{n-2}M \xrightarrow{\partial_{2,n}} I^{n-1}M \oplus I^{n-1}M \xrightarrow{\partial_{1,n}} I^n M \to 0,$$

with differentials $\partial_{2,n}(u) = (-yu, xu)$ and $\partial_{1,n}(z, t) = xz + yt$. Take $(z, t) = (\sum a_i u_i, \sum b_i v_i)$, $(z, t) \in \ker \partial_{1,n}$, with $a_i, b_i \in I^{n-1}$, $u_i, v_i \in M$ and $xz + yt = 0$. Write $b_i = c_i y^{n-1} + d_i x$, $c_i \in A$ and $d_i \in I^{n-2}$. Then

$$y^n \sum c_i v_i = y \sum b_i v_i - y \sum d_i x v_i = yt - x \sum d_i y v_i = -x\Big(z - \sum d_i y v_i\Big).$$

Thus $\sum c_i v_i \in (xI^{n-1}M : y^n)$. Consider

$$\varphi : \ker \partial_{1,n} \longrightarrow \frac{(xI^{n-1}M : y^n)}{(xI^{n-2}M : y^{n-1})},$$

defined by $\varphi(z, t) = \overline{\sum c_i v_i}$. It is not difficult to see that φ is well defined, surjective and $\operatorname{im} \partial_{2,n} \subset \ker \varphi$. Moreover, since x is M-regular, then $\ker \varphi \subset \operatorname{im} \partial_{2,n}$. Thus

$$E(I; M)_n = \frac{(xI^{n-1}M : y^n)}{(xI^{n-2}M : y^{n-1})}.$$

Using the former exact sequence of modules of effective relations, one deduces the expression of $E_J(I; M)_n$. $\square$

Next we compare the Artin–Rees number modulo J with the relation type modulo J.

Proposition 3.3. *Let A be a ring, I and J two ideals of A and $N \subset M$ two A-modules. Then*

$$s_J(N, M; I) \leqslant \operatorname{rt}_J(I; M/N) \leqslant \max\big(\operatorname{rt}_J(I; M), s_J(N, M; I)\big).$$

Proof. Take $F = \mathcal{R}_J(I; M/N)$, $G = \mathcal{R}_J(I; M)$ and $H = \mathbf{S}(I/JI) \otimes M$ and $\varphi : G \to F$ and $\gamma : H \to G$ induced by the natural surjective graded morphisms $\mathcal{R}(I; M) \to \mathcal{R}(I; M/N)$ and $\mathbf{S}(I) \otimes M \to \mathcal{R}(I; M)$. By [28, Lemma 2.3], for every integer $n \geqslant 2$, one has the short exact sequence of A-modules:

$$E(\gamma)_n \to E(\varphi \circ \gamma)_n \to E(\varphi)_n \to 0.$$

But $E(\gamma)_n = E_J(I; M)_n$ and $E(\varphi \circ \gamma)_n = E_J(I; M/N)_n$ and a short computation shows that $E(\varphi)_n = E_J(N, M; I)_n$. From the exact sequence we obtain the desired inequalities. $\square$

4. Castelnuovo–Mumford regularity

The purpose of this section is to recall some definitions and formulate, in order to use them subsequently, a generalization to modules of some results of Trung in [33,34]. Being natural extensions of his results, we omit or just sketch the proofs.

Let A be a noetherian ring and $U = \bigoplus_{n\geqslant 0} U_n$ a finitely generated standard A-algebra. Let $F = \bigoplus_{n\geqslant 0} F_n$ be a standard U-module. Define

$$a(F) = \begin{cases} \max\{n \geqslant 0 \mid F_n \neq 0\} & \text{if } F \neq 0, \\ -\infty & \text{if } F = 0. \end{cases}$$

Let $U_+ = \bigoplus_{n>0} U_n$ be the irrelevant ideal of U. If $i \geqslant 0$, denote by

$$a_i(F) = a\big(H^i_{U_+}(F)\big),$$

where $H^i_{U_+}(\cdot)$ denotes the ith local cohomology functor with respect to the ideal U_+. Then $a_i(F) < \infty$ and the *Castelnuovo–Mumford regularity of F* is defined to be

$$\operatorname{reg}(F) = \max\big\{a_i(F) + i \bigm| i \geqslant 0\big\}$$

(see e.g. [5, 15.2.9]; [32,34]). We shall mainly be concerned with the case $U = \mathcal{R}(I)$, the Rees algebra of an ideal I of A, and $F = \mathcal{R}(I; M)$, the Rees module of I with respect to a finitely generated A-module M. In particular, if $M \neq 0$, then $\operatorname{reg}(F) \neq -\infty$ (see e.g. [5, 15.2.13]).

A sequence $\mathbf{z} = z_1, \ldots, z_s$ of homogeneous elements of U is called *n-regular with respect to F* if, for all $i = 1, \ldots, s$,

$$\big((z_1, \ldots, z_{i-1})F : z_i\big)_n = \big((z_1, \ldots, z_{i-1})F\big)_n.$$

The least integer $m \geqslant 0$ such that $\mathbf{z}$ is n-regular for all $n \geqslant m + 1$ is denoted by $a(\mathbf{z})$ (see [33], Section 2). In other words,

$$a(\mathbf{z}) = \max\big\{a\big((z_1, \ldots, z_{i-1})F : z_i/(z_1, \ldots, z_{i-1})F\big) \bigm| i = 1, \ldots, s\big\}.$$

A sequence $\mathbf{z} = z_1, \ldots, z_s$ of homogeneous elements of U is called a *U_+-filter-regular sequence with respect to F* if $z_i \notin \mathfrak{p}$ for any associated prime ideal $\mathfrak{p}$ of $F/(z_1, \ldots, z_{i-1})F$, $\mathfrak{p} \not\supseteq U_+$, for all $i = 1, \ldots, s$ (see [33, Section 2]; [34, Section 2]).

Lemma 4.1. *(See [5, 18.3.8]; [33, 2.1].) Let $\mathbf{z} = z_1, \ldots, z_s$ be a sequence of homogeneous elements of U. Then $\mathbf{z}$ is a U_+-filter regular sequence with respect to F if and only if $a(\mathbf{z}) < \infty$.*

Lemma 4.2. *(See [33, 2.3].) Let $z \in U_1$ be a homogeneous U_+-filter-regular element with respect to F. Then, for all $i \geqslant 0$,*

$$a_{i+1}(F) + 1 \leqslant a_i(F/zF) \leqslant \max\bigl\{a_i(F), a_{i+1}(F) + 1\bigr\}.$$

Lemma 4.3. *(See [34, 2.2].) Let $\mathbf{z} = z_1, \dots, z_s$ be a U_+-filter-regular sequence with respect to F, $z_i \in U_1$ for all $i = 1, \dots, s$. Then*

$$a(\mathbf{z}) = \max\bigl\{a_i(F) + i \bigm| i = 0, \dots, s-1\bigr\}$$

and, for all $0 \leqslant t \leqslant s$,

$$\begin{aligned} &\max\bigl\{a_i(F) + i \bigm| i = 0, \dots, t\bigr\} \\ &\quad = \max\bigl\{a\bigl((z_1, \dots, z_i)F : U_+/(z_1, \dots, z_i)F\bigr) \bigm| i = 0, \dots, t\bigr\}. \end{aligned}$$

Proposition 4.4. *(See [34, 2.4].) Let $\mathbf{z} = z_1, \dots, z_s$ be a U_+-filter-regular sequence with respect to F, $z_i \in U_1$, $i = 1, \dots, s$, which generates a reduction Q of U_+ with respect to F. Then*

$$\operatorname{reg}(F) = \max\bigl\{a(\mathbf{z}), \operatorname{rn}_Q(U_+; F)\bigr\}.$$

Proof. By Lemma 4.3,

$$a(\mathbf{z}) = \max\bigl\{a\bigl((z_1, \dots, z_i)F : U_+/(z_1, \dots, z_i)F\bigr) \bigm| i = 0, \dots, s-1\bigr\}.$$

Further,

$$\operatorname{rn}_Q(U_+; F) = a(F/QF) = a\bigl((z_1, \dots, z_s)F : U_+/(z_1, \dots, z_s)F\bigr).$$

Therefore,

$$\begin{aligned} &\max\bigl\{a(\mathbf{z}), \operatorname{rn}_Q(U_+; F)\bigr\} \\ &\quad = \max\bigl\{a\bigl((z_1, \dots, z_i)F : U_+/(z_1, \dots, z_i)F\bigr) \bigm| i = 0, \dots, s\bigr\} \\ &\quad = \max\bigl\{a_i(F) + i \bigm| i = 0, \dots, s\bigr\}. \end{aligned}$$

Since $\operatorname{reg}(F) = \max\{a_i(F) + i \mid i \geqslant 0\}$, it is enough to show that $H^i_{U_+}(F) = 0$ for all $i > s$. If $s = 0$, then 0 is a reduction of U_+ with respect to F and F_n for all large n. So F is a U_+-torsion module and $H^i_{U_+}(F) = 0$ for all $i > 0$ (see e.g. [5, 2.1.7]). If $s \geqslant 1$, by induction, $H^i_{U_+}(F/z_1F) = 0$ for all $i > s-1$. So $a_i(F/z_1F) = -\infty$ for all $i > s-1$. By Lemma 4.2, $a_{i+1}(F) = -\infty$ and $H^{i+1}_{U_+}(F) = 0$ for all $i > s$. □

Now take A a noetherian ring, I an ideal of A and M a finitely generated A-module. Consider $\mathcal{R}(I) = \bigoplus_{n \geqslant 0} I^n t^n \subset A[t]$ as a subring of $A[t]$.

Lemma 4.5. *(See [34, 4.1].) Let A be a noetherian ring, let I be an ideal of A and let M be a finitely generated A-module. Let $x_1,\dots,x_s$ be a sequence of elements in I. Then $x_1t,\dots,x_st$ is a $\mathcal{R}(I)_+$-filter-regular sequence with respect to $\mathcal{R}(I;M)$ if and only if for all large $n\geqslant 1$,*

$$\big[(x_1,\dots,x_{i-1})I^nM : x_i\big]\cap I^nM=(x_1,\dots,x_{i-1})I^{n-1}M,\quad i=1,\dots,s. \tag{$*$}$$

If that is the case, $a(\mathbf{z})$ is the least integer r such that $(*)$ holds for all $n\geqslant r+1$.

Proof. $\mathbf{z}=x_1t,\dots,x_st$ is a $\mathcal{R}(I)_+$-filter-regular sequence with respect to $\mathcal{R}(I;M)$ if and only if $[(x_1t,\dots,x_{i-1}t)\mathcal{R}(I;M):x_it]_n$ is equal to

$$\big[(x_1t,\dots,x_{i-1}t)\mathcal{R}(I;M)\big]_n$$

for all large $n\geqslant 1$. But the first module is equal to

$$\big[(x_1,\dots,x_{i-1})I^nM : x_i\big]\cap I^nM$$

and the second is equal to $(x_1,\dots,x_{i-1})I^{n-1}M$. □

Proposition 4.6. *(See [34, 4.2].) Let A be a noetherian ring, let I be an ideal of A and let M be a finitely generated A-module. Let $J=(x_1,\dots,x_s)$ be a reduction of I with respect to M. Suppose that $\mathbf{z}=x_1t,\dots,x_st$ is a $\mathcal{R}(I)_+$-filter-regular sequence with respect to $\mathcal{R}(I;M)$. Then*

$$\operatorname{reg}\big(\mathcal{R}(I;M)\big)=\min\big\{r\geqslant 0 \;\big|\; r\geqslant \operatorname{rn}_J(I;M) \textit{ and } (*) \textit{ holds for } n\geqslant r+1\big\}.$$

Proof. Let $Q=(\mathbf{z})$ denote the ideal generated by $\mathbf{z}=x_1t,\dots,x_st$, $U=\mathcal{R}(I)$ the Rees algebra of I and $F=\mathcal{R}(I;M)$ the Rees module of I with respect to M. Since J is a reduction of I with respect to M, then Q is a reduction of U_+ with respect to F. Moreover, if $I^{r+1}M=JI^rM$, then $U_+^{r+1}F=QU_+^rF$ and $\operatorname{rn}_Q(U_+;F)=\operatorname{rn}_J(I;M)$. By Proposition 4.4, $\operatorname{reg}(F)=\max\{a(\mathbf{z}),\operatorname{rn}_J(I;M)\}$. The conclusion follows from Lemma 4.5. □

5. Relation type and reduction number

The first result of the section suggests the relationship subsisting between the relation type and the reduction number (see [37, p. 63]).

Proposition 5.1. *Let A be a ring, I an ideal of A and M an A-module. Let $J\subset I$ be a reduction of I with respect to M and with reduction number $\operatorname{rn}_J(I;M)=r$. Then*

$$\operatorname{rt}(I;M)\leqslant \operatorname{rn}_J(I;M)+\operatorname{rt}\big(J;I^rM\big).$$

Proof. Let us prove that $E(I;M)_n=0$ for all $n\geqslant r+\operatorname{rt}(J;I^rM)+1$. Write $n=r+k$, where $k\geqslant \operatorname{rt}(J;I^rM)+1(\geqslant 2)$. In particular, $I^nM=J^kI^rM=JI^{n-1}M$, $I^{n-1}M=J^{k-1}I^rM=$

$JI^{n-2}M$ and $I^{n-2}M = J^{k-2}I^rM = JI^{n-3}M$ (where $I^{n-3} = A$ if $n = 2$ and $r = 0$). Consider the following diagram:

$$\begin{array}{ccccccc}
\Lambda_2(J)\otimes J^{k-2}I^rM & \xrightarrow{\partial'_{2,k}} & J\otimes J^{k-1}I^rM & \xrightarrow{\partial'_{1,k}} & J^kI^rM & \longrightarrow & 0\\
\Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} & & \Big\| & & \\
\Lambda_2(I)\otimes I^{n-2}M & \xrightarrow[\partial_{2,n}]{} & I\otimes I^{n-1}M & \xrightarrow[\partial_{1,n}]{} & I^nM & \longrightarrow & 0.
\end{array}$$

The top row represents the last three nonzero terms of the kth homogeneous part of the Koszul complex induced by the $\mathcal{R}(J)$-linear form $J \otimes \mathcal{R}(J; I^rM) \to \mathcal{R}(J; I^rM)$ and the bottom row represents the last three nonzero terms of the nth homogeneous part of the Koszul complex induced by the $\mathcal{R}(I)$-linear form $I \otimes \mathcal{R}(I; M) \to \mathcal{R}(I; M)$. The Koszul differentials are defined as usual (e.g. [6, Definition 1.6.1]; see also the proof of Proposition 3.2). The vertical morphisms are induced by the inclusion $J \subset I$ and define a morphism of complexes. By [28, Proposition 2.6], the first homology groups of these complexes are $\ker\partial'_{1,k}/\operatorname{im}\partial'_{2,k} = E(J; I^rM)_k$ and $\ker\partial_{1,n}/\operatorname{im}\partial_{2,n} = E(I; M)_n$. Thus we want to prove $\ker\partial_{1,n} \subset \operatorname{im}\partial_{2,n}$. Take $u = \sum_i x_i \otimes m_i \in I \otimes I^{n-1}M$ such that $\partial_{1,n}(u) = \sum_i x_im_i = 0$. Write each $m_i = \sum_j y_{i,j}m_{i,j}$, $y_{i,j} \in J$, $m_{i,j} \in I^{n-2}M$. Take $v = \sum_{i,j} y_{i,j} \wedge x_i \otimes m_{i,j} \in \Lambda_2(I) \otimes I^{n-2}M$. Then $\partial_{2,n}(v) = u - w$, where $w = \sum_{i,j} y_{i,j} \otimes x_im_{i,j} \in I \otimes I^{n-1}M$. Consider $w' = \sum_{i,j} y_{i,j} \otimes x_im_{i,j} \in J \otimes J^{k-1}I^rM$. Remark that $\partial'_{1,k}(w') = \partial_{1,n}(f(w')) = \partial_{1,n}(w) = 0$. Since $k \geqslant \operatorname{rt}(J; I^rM) + 1$, then $E(J; I^rM)_k = 0$ and $w' \in \operatorname{im}\partial'_{2,k}$. Take $t' \in \Lambda_2(J) \otimes J^{k-2}I^rM$ such that $\partial_{2,k}(t') = w'$. Then $\partial_{2,n}(v + g(t')) = u - w + f(\partial'_{2,k}(t')) = u - w + f(w') = u$ and $u \in \operatorname{im}\partial_{2,n}$. □

The purpose now is to control the relation type of the reduction J with respect to I^rM. We will use the filter-regular conditions ($*$) of Lemma 4.5, which first appeared, to our knowledge, in a paper by Costa for $M = A$ and $J = I$ [9, p. 258].

Proposition 5.2. *Let A be a ring, let I be an ideal of A and let M be an A-module. Let $J = (x_1, \ldots, x_s) \subset I$ be a reduction of I with respect to M and with reduction number $\operatorname{rn}_J(I; M) = r$. Suppose that there exists $k \geqslant 1$ such that for all $n \geqslant r + k$ and all $i = 1, \ldots, s$,*

$$\big[(x_1, \ldots, x_{i-1})I^nM : x_i\big] \cap I^nM = (x_1, \ldots, x_{i-1})I^{n-1}M.$$

Then $\operatorname{rt}(J; I^rM) \leqslant k$. Moreover, if A is noetherian and M is finitely generated, then $\operatorname{rn}_J(I; M) \leqslant \operatorname{reg}(\mathcal{R}(I; M)) \leqslant \operatorname{rn}_J(I; M) + k - 1$.

Proof. Write $J_0 = 0$ and $J_i = (x_1, \ldots, x_i)$ for $i = 1, \ldots, s$. Let $m \geqslant k + 1$ and consider the last three nonzero terms of the mth homogeneous part of the Koszul complex induced by the $\mathcal{R}(J_i)$-linear form $J_i \otimes \mathcal{R}(J; I^rM) \to \mathcal{R}(J; I^rM)$:

$$\Lambda_2(J_i) \otimes J^{m-2}I^rM \xrightarrow{\partial_{2,m-2}} J_i \otimes J^{m-1}I^rM \xrightarrow{\partial_{1,m-1}} J^mI^rM \to 0.$$

If $i = s$, then $J_s = J$ and one has the Koszul complex

$$\Lambda_2(J) \otimes J^{m-2}I^r M \xrightarrow{\partial_{2,m-2}} J \otimes J^{m-1}I^r M \xrightarrow{\partial_{1,m-1}} J^m I^r M \to 0,$$

whose first homology group $\ker \partial_{1,m-1}/\operatorname{im} \partial_{2,m-2}$ is, by [28, Proposition 2.6], equal to the module of m-effective relations $E(J; I^r M)_m$. Thus, it is enough to prove by induction on $i = 1, \dots, s$, that $\ker \partial_{1,m-1} \subset \operatorname{im} \partial_{2,m-2}$ for all $m \geqslant k+1$ (remark that, in this case, $r + m - 1 \geqslant r + k$). If $i = 1$, let $z = x_1 \otimes c \in J_1 \otimes J^{m-1}I^r M$ such that $0 = \partial_{1,m-1}(z) = x_1 c$. Then $c \in (0 : x_1) \cap J^{m-1}I^r M = (0 : x_1) \cap I^{r+m-1}M = J_0 I^{r+m-2}M = 0$. Thus $z = 0$. If $i = s$, let $z = \sum_{i=1}^{s} x_i \otimes c_i \in J \otimes J^{m-1}I^r M$ such that $0 = \partial_{1,m-1}(z) = \sum_{i=1}^{s} x_i c_i$. Then $x_s c_s = -\sum_{i=1}^{s-1} x_i c_i$. Thus $c_s \in (J_{s-1}J^{m-1}I^r M : x_s) \cap J^{m-1}I^r M = (J_{s-1}I^{r+m-1}M : x_s) \cap I^{r+m-1}M = J_{s-1}I^{r+m-2}M$. Thus $c_s = \sum_{i=1}^{s-1} x_i \lambda_i$, $\lambda_i \in I^{r+m-2}M = J^{m-2}I^r M$. Take $u = \sum_{i=1}^{s-1} x_i \otimes (c_i + x_s \lambda_i) \in J_{s-1} \otimes J^{m-1}I^r M$. Then $\partial_{1,m-1}(u) = 0$. By induction hypothesis, there exists $v \in \Lambda_2(J_{s-1}) \otimes J^{m-2}I^r M$ such that $\partial_{2,m-2}(v) = u$. Take $w = v + \sum_{i=1}^{s-1}(x_j \wedge x_s) \otimes \lambda_i$ and one has $\partial_{2,m-2}(w) = z$. This proves $\operatorname{rt}(J; I^r M) \leqslant k$. The second assertion follows from Proposition 4.6. $\square$

Let A be a noetherian ring, $J \subset I$ two ideals of A and M a finitely generated A-module. Let $x_1, \dots, x_s$ be a system of generators of J. Write, as before, $J_0 = 0$ and $J_i = (x_1, \dots, x_i)$ for $i = 1, \dots, s$. The sequence $x_1, \dots, x_s$ is said to be a *d-sequence with respect to M* if any x_j is not contained in the ideal generated by the other x_i and for all $k \geqslant i+1$ and all $i \geqslant 0$, $(J_i M : x_{i+1}x_k) = (J_i M : x_k)$. It is known that this last condition is equivalent to $(J_i M : x_{i+1}) \cap JM = J_i M$ for all $i = 0, \dots, s-1$ (see e.g. [16, pp. 112, 113]). Let $\mathcal{G}(I; M)$ be the associated graded module of I with respect to M and denote by $x_1^*, \dots, x_s^*$ the images of $x_1, \dots, x_s$ in $I/I^2 \subset \mathcal{G}(I)$. The sequence $x_1, \dots, x_s$ is said to be a *complete d-sequence with respect to I and M* if $x_1, \dots, x_s$ is a d-sequence with respect to M and $x_1^*, \dots, x_{s-1}^*$ is a $\mathcal{G}(I; M)$-regular sequence (see [20] and [34]). If A is local, it can be shown that $x_1^*, \dots, x_{s-1}^*$ is a $\mathcal{G}(I; M)$-regular sequence if and only if $x_1, \dots, x_{s-1}$ is an M-regular sequence and, for all $n \geqslant 0$ and all $i = 1, \dots, s-1$, the nth Valabrega–Valla module $VV_{J_i}(I; M)_n = J_i M \cap I^{n+1}M/J_i I^n M$ is equal to zero (see e.g. [19, Lemma 2.2]; [8, Proposition 2.3]).

Huckaba proved that if A is noetherian local, if I is an ideal with analytic spread $l(I)$ equal to the height of the ideal $\operatorname{ht}(I)$ or $\operatorname{ht}(I) + 1$ and with a minimal reduction J generated by a complete d-sequence with respect to I, then $\operatorname{rt}(I) \leqslant \operatorname{rn}_J(I) + 1$ (see [19, Theorem 2.3] and [20, Theorem 1.4]). Later, Trung proved that, in general, $\operatorname{rt}(I) \leqslant \operatorname{reg}(\mathcal{R}(I)) + 1$ and that if I has a reduction J generated by a complete d-sequence with respect to I, then $\operatorname{reg}(\mathcal{R}(I)) = \operatorname{rn}_J(I)$ (see [34, Proposition 2.6 and Theorem 6.4]; for more related results on this topic see also [32,33]). From our Propositions 5.1 and 5.2, we obtain a generalization of these results. Our proof closely follows ideas of Trung in [34].

Theorem 5.3. *Let A be a noetherian ring, let I be an ideal of A and let M be a finitely generated A-module. Let $J = (x_1, \dots, x_s) \subset I$ be a reduction of I with respect to M and with reduction number* $\operatorname{rn}_J(I; M) = r$. *Suppose that*

(i) $x_1, \dots, x_s$ *is a d-sequence with respect to M.*
(ii) $x_1, \dots, x_{s-1}$ *is an M-regular sequence.*
(iii) $(x_1, \dots, x_i)M \cap I^{r+1}M = (x_1, \dots, x_i)I^r M$ *for all* $i = 1, \dots, s-1$.

Then $\operatorname{rt}(J; I^r M) = 1$, $\operatorname{rt}(I; M) \leqslant \operatorname{rn}_J(I; M) + 1$ *and* $\operatorname{rn}_J(I; M) = \operatorname{reg}(\mathcal{R}(I; M))$.

Proof. Write $J_0 = 0$ and $J_i = (x_1, \ldots, x_i)$ for $i = 1, \ldots, s$. Using (ii) and (iii), we obtain $(J_{i-1}M : x_i) \cap I^{r+1}M = J_{i-1}M \cap I^{r+1}M = J_{i-1}I^rM$ for $i = 1, \ldots, s-1$. By (i), $(J_{s-1}M : x_s) \cap JM = J_{s-1}M$. Since $I^{r+1}M = JI^rM$, then $(J_{s-1}M : x_s) \cap I^{r+1}M = J_{s-1}M \cap I^{r+1}M$ which, by (iii), is equal to $J_{s-1}I^rM$. Thus, for all $i = 1, \ldots, s$,

$$(J_{i-1}M : x_i) \cap I^{r+1}M = J_{i-1}I^rM.$$

A straightforward generalization to modules of Theorem 4.8(i) in [34], allows us to assert that for all integers $n \geqslant r+1$ and for all $i = 1, \ldots, s$,

$$(J_{i-1}M : x_i) \cap I^nM = J_{i-1}I^{n-1}M,$$

which clearly implies for all integers $n \geqslant r+1$ and for all $i = 1, \ldots, s$,

$$\left(J_{i-1}I^nM : x_i\right) \cap I^nM = J_{i-1}I^{n-1}M.$$

By Proposition 5.1, $\mathrm{rt}(I; M) \leqslant \mathrm{rn}_J(I; M) + \mathrm{rt}(J; I^rM)$ and, by Proposition 5.2, $\mathrm{rt}(I; J^rM) = 1$ and $\mathrm{rn}_J(I; M) = \mathrm{reg}(\mathcal{R}(I; M))$. □

6. Integral degree of a ring

In this section we introduce the *integral degree*, an invariant associated to the ring, which later will be used to bound the reduction number. Let $A \subset B$ be a ring extension. Recall that an element $b \in B$ is said to be integral over A if there exist $a_i \in A$ and an integral equation of degree $n \geqslant 1$:

$$b^n + a_1b^{n-1} + a_2b^{n-2} + \cdots + a_{n-1}b + a_n = 0.$$

If $b \in B$ is integral over A, we will call the integer

$$\mathrm{id}_A(b) = \min\{n \geqslant 1 \mid b \text{ satisfies an integral equation of degree } n\}$$

the *integral degree of b over A*.

Let $A \subset C \subset B$, C an A-submodule of B. Suppose the elements of C are integral over A. Then the *integral degree of C over A* is defined as the integer (possibly infinite):

$$\mathrm{d}_A(C) = \sup\left\{\mathrm{id}_A(c) \;\middle|\; c \in C\right\}.$$

Remark that $\mathrm{d}_A(C) = 1$ if and only if $A = C$.

As usual, $\mu_A(\cdot)$ stands for the minimal number of generators as an A-module.

Proposition 6.1. *Let $A \subset B$ be a ring extension, $b \in B$ and $n \geqslant 1$. Then the following conditions are equivalent*:

(i) *b is integral over A and* $\mathrm{id}_A(b) \leqslant n$.
(ii) *$A[b]$ is a finitely generated A-module and* $\mu_A(A[b]) \leqslant n$.

(iii) *There exists a ring C, $A \subset A[b] \subset C \subset B$, such that C is a finitely generated A-module and $\mu_A(C) \leqslant n$.*
(iv) *There exists a faithful $A[b]$-module M such that M is a finitely generated A-module and $\mu_A(M) \leqslant n$.*

Proof. This follows from [3, Proposition 5.1], just taking into account the definition of $\mathrm{id}_A(b)$. □

Next we prove that the integral degree of the sum or product of two integral elements is, in fact, bounded above by the product of their integral degrees.

Corollary 6.2. *Let $A \subset B$ be a ring extension and $b_1, \dots, b_n \in B$ integral over A. Then $A[b_1, \dots, b_n]$ is a finitely generated A-module, $A \subset A[b_1, \dots, b_n]$ is an integral extension and*:

$$\max\{\mathrm{id}_A(b_i)\} \leqslant \mathrm{d}_A(A[b_1, \dots, b_n]) \leqslant \mu_A(A[b_1, \dots, b_n]) \leqslant \prod_{i=1}^{n} \mathrm{id}_A(b_i).$$

In particular, if $b \in B$ is integral over A, then $\mathrm{id}_A(b) = \mathrm{d}_A(A[b]) = \mu_A(A[b])$.

Proof. Let $C = A[b_1, \dots, b_n]$ and $m = \prod_{i=1}^{n} \mathrm{id}_A(b_i)$. Then it is clear that $\mu_A(C) \leqslant m$. Now take any $b \in C \subset B$. So we have $A \subset A[b] \subset C \subset B$ with $\mu_A(C) = r \leqslant m$. By Proposition 6.1, (iii) $\Rightarrow$ (i), b is integral over A and $\mathrm{id}_A(b) \leqslant r$ and taking the supremum over all $b \in C$, then $\mathrm{d}_A(C) \leqslant r = \mu_A(C)$. As for the second assertion, just take $n = 1$. □

Corollary 6.3. *Let $A \subset B$ be a ring extension. If B is a finitely generated A-module, then $A \subset B$ is integral and*

$$\mathrm{d}_A(B) \leqslant \mu_A(B).$$

Proof. If $b \in B$, take $A \subset A[b] \subset B$, with B a finitely generated A-module. By Proposition 6.1, (iii) $\Rightarrow$ (i), b is integral over A and $\mathrm{id}_A(b) \leqslant \mu_A(B)$. Taking the supremum, $\mathrm{d}_A(B) \leqslant \mu_A(B)$. □

Let A be noetherian domain and let $\overline{A}$ be the integral closure of A in its quotient field. If $\dim A \leqslant 2$, then $\overline{A}$ is noetherian (see e.g. [23, 11.7] and [24, 33.12]). Nevertheless, $\overline{A}$ may be a nonfinitely generated A-module, as an example of Akizuki shows ([2] or [29, 9.5]). Next, we want to prove that the ring A in the example of Akizuki has at least finite integral degree $\mathrm{d}_A(\overline{A})$. Before that, and for easy reference, we state the following lemma.

Lemma 6.4. *Let A be a ring and $x, y \in A$, with x regular. The following are equivalent.*

(i) *y/x is integral over A and* $\mathrm{id}_A(y/x) \leqslant n$.
(ii) *(x) is a reduction of (x, y) and* $\mathrm{rn}_{(x)}(x, y) \leqslant n - 1$.
(iii) $x(x, y)^{n-1} : y^n = A$.

In particular, if y/x is integral over A, then $\mathrm{id}_A(y/x) = \mathrm{rt}(x, y) = \mathrm{rn}_{(x)}(x, y) + 1$.

Proof. Take $y/x \in \overline{A}$ with $\mathrm{id}_A(y/x) \leqslant n$. Then, there exist $a_i \in A$ such that $(y/x)^n + a_1(y/x)^{n-1} + \cdots + a_n = 0$. Multiplying by x^n, one has $y^n \in xI^{n-1}$, where $I = (x, y)$. Thus $I^n = xI^{n-1}$, $J = (x)$ is a reduction of I and $\mathrm{rn}_J(I) \leqslant n-1$. If $J = (x)$ is a reduction of $I = (x, y)$ with $\mathrm{rn}_J(I) \leqslant n-1$, then $I^n = xI^{n-1}$ and $y^n \in xI^{n-1}$. Thus $1 \in xI^{n-1} : y^n$ and $xI^{n-1} : y^n = A$. Finally, if $xI^{n-1} : y^n = A$, where $I = (x, y)$, then $y^n \in xI^{n-1}$ and $y^n = b_1xy^{n-1} + \cdots + b_nx^n$, for some $b_i \in A$. Dividing by x^n one obtains an integral equation of y/x over A of degree n. In particular, if $y/x \in \overline{A}$ with $\mathrm{id}_A(y/x) = n \geqslant 2$, then $xI^{n-2} : y^{n-1} \subsetneq A = xI^{n-1} : y^n$, where $I = (x, y)$. By Proposition 3.2, $E(I)_n \neq 0$ and $E(I)_{n+s} = 0$ for all $s \geqslant 1$. Thus $\mathrm{rt}(I) = n = \mathrm{id}_A(y/x)$. Moreover, (i) $\Leftrightarrow$ (ii) says that $J = (x)$ is a reduction of $I = (x, y)$ and that $\mathrm{rn}_J(I) = n-1$. $\square$

Now, let us prove that the example of Akizuki has finite integral degree. Denote by $e(A)$ the multiplicity of A.

Proposition 6.5. *Let A be a one-dimensional noetherian local ring. Then*

$$\mathrm{d}_A(\overline{A}) \leqslant e\big(A/H^0_{\mathfrak{m}}(A)\big) + \mathrm{length}\big(H^0_{\mathfrak{m}}(A)\big).$$

In particular, if A is a domain (as Akizuki's example is), then $\mathrm{d}_A(\overline{A}) \leqslant e(A)$.

Proof. Take $y/x \in \overline{A}$ with $x, y \in A$, x regular, and $I = (x, y)$ the ideal of A generated by x, y. By Lemma 6.4, $\mathrm{id}_A(y/x) = \mathrm{rt}(I)$. By [28, Lemma 6.1], $\mathrm{rt}(I) \leqslant \mathrm{rt}(IA/J) + \mathrm{length}(J)$, where $J = H^0_{\mathfrak{m}}(A)$. By [28, Lemma 6.3], then $\mathrm{rt}(IA/J) \leqslant e(A/J)$. $\square$

We next see that there exist one-dimensional noetherian domains with infinite integral degree. Remark that the ring in this example must be not local nor excellent so that one cannot apply the existence of a uniform bound for the relation type of all ideals (see [28, Proposition 6.5 and Theorem 3]). The next example is due to Sally and Vasconcelos (see [31, Example 1.4] and also [28, Remark 7.3]).

Example 6.6. There exist one-dimensional noetherian domains A with $\mathrm{d}_A(\overline{A})$ infinite.

Proof. Let $t_1, t_2, t_3, \ldots$ be infinitely many indeterminates over a field k. Let R be defined as $R = k[t_1^2, t_1^3, t_2^3, t_2^4, t_2^5, \ldots, t_n^{n+1}, t_n^{n+2}, \ldots, t_n^{2n+1}, \ldots]$. Take $\mathfrak{p}_n = (t_n^{n+1}, t_n^{n+2}, \ldots, t_n^{2n+1})$, which is a prime ideal of height 1. Let S be the multiplicative closed set $R - \bigcup \mathfrak{p}_n$ and $A = S^{-1}R$. One can prove that A is a one-dimensional noetherian domain and that t_n^{n+2}/t_n^{n+1} is in $\overline{A}$ and has integral degree n. Therefore $\mathrm{d}_A(\overline{A}) = \infty$. $\square$

We now give two more properties of the integral degree.

Proposition 6.7. *Let $A \subset B$ and $B \subset C$ be integral extensions. Then $A \subset C$ is an integral extension and*

$$\mathrm{d}_A(C) \leqslant \mathrm{d}_A(B)^{\mathrm{d}_B(C)} \cdot \mathrm{d}_B(C).$$

Proof. If $c \in C$, there exists an equation $c^n + b_1c^{n-1} + \cdots + b_{n-1}c + b_n = 0$, with $b_i \in B$ and $n \leqslant \mathrm{d}_B(C)$. Take $D = A[b_1, \ldots, b_n]$. Since $A \subset B$ is an integral extension, all b_i are integral over A and, by Corollary 6.2, D is a finitely generated A-module and $\mu_A(D) \leqslant$

$\prod_{i=1}^{n} \mathrm{id}_A(b_i) \leqslant \mathrm{d}_A(B)^{\mathrm{d}_B(C)}$. On the other hand, c is clearly integral over D and $D[c]$ is a finitely generated D-module with $\mu_D(D[c]) \leqslant n \leqslant \mathrm{d}_B(C)$. Since D is a finitely generated A-module and $D[c]$ is a finitely generated D-module, then $D[c]$ is a finitely generated A-module. So we have $A \subset A[c] \subset D[c] \subset C$ with $D[c]$ a finitely generated A-module with $\mu_A(D[c]) \leqslant \mu_A(D)\mu_D(D[c]) \leqslant \mathrm{d}_A(B)^{\mathrm{d}_B(C)} \cdot \mathrm{d}_B(C)$. Applying Proposition 6.1, (iii) ⇒ (i), we deduce that c is integral over A and $\mathrm{id}_A(c) \leqslant \mathrm{d}_A(B)^{\mathrm{d}_B(C)} \cdot \mathrm{d}_B(C)$. □

Proposition 6.8. *Let $A \subset B$ be an integral extension and S a multiplicatively closed subset of A. Then $S^{-1}A \subset S^{-1}B$ is an integral extension and*

$$\mathrm{d}_{S^{-1}A}\big(S^{-1}B\big) \leqslant \mathrm{d}_A(B).$$

In particular, if A is reduced, $\mathfrak{p}$ is a prime ideal of A and $\overline{A}$ and $\overline{A_{\mathfrak{p}}}$ are the integral closures of A and $A_{\mathfrak{p}}$ in their total quotient rings, then

$$\mathrm{d}_{A_{\mathfrak{p}}}(\overline{A_{\mathfrak{p}}}) \leqslant \mathrm{d}_A(\overline{A}).$$

Proof. Let $b/s \in S^{-1}B$, $b \in B$, $s \in S$. Then b/s is integral over $S^{-1}A$ and $\mathrm{id}_{S^{-1}A}(b/s) \leqslant \mathrm{id}_A(b) \leqslant \mathrm{d}_A(B)$. If A is reduced, then $S^{-1}\overline{A} = \overline{S^{-1}A}$ (see e.g. [15, Lemma 2.1]). Therefore, if $S = A - \mathfrak{p}$, $\overline{(A_{\mathfrak{p}})} = \overline{S^{-1}A} = S^{-1}\overline{A}$ and $\mathrm{d}_{A_{\mathfrak{p}}}(\overline{A_{\mathfrak{p}}}) = \mathrm{d}_{S^{-1}A}(S^{-1}\overline{A}) \leqslant \mathrm{d}_A(\overline{A})$. □

If A is not reduced, Proposition 6.8 may fail. The next example is taken from [15].

Example 6.9. Let k be a field and $A = k[\![x,y,z]\!]/(x^3 - y^2)(x,y,z)$. Since the maximal ideal annihilates the nonzero element $x^3 - y^2$, A is its own classical ring of quotients and so is integrally closed and $\mathrm{d}_A(\overline{A}) = 1$. Set S the multiplicatively closed set $\{z^n, n \geqslant 0\}$. Then $S^{-1}A = K[\![x,y]\!]/(x^3 - y^2)$, where K is the quotient field of $k[\![z]\!]$, and one can prove that $\mathrm{d}_{S^{-1}A}(\overline{S^{-1}A}) = 2$.

7. Integral degree and reduction number

We now prove the main result of the paper. Recall that if I is a regular ideal having principal reductions J_1 and J_2 with $\mathrm{rn}_{J_1}(I) = n$ and $\mathrm{rn}_{J_2}(I) = m$, then Huckaba proved that $n = m$ (see [18], where the local assumption is not needed; it could also be deduced from Theorem 5.3). We will denote $\mathrm{rn}(I)$ to the J-reduction number of I for any principal reduction J of I.

Theorem 7.1. *Let A be a noetherian ring, $A \supset \mathbb{Q}$. Then*

$$\mathrm{d}_A(\overline{A}) = \sup\big\{\mathrm{rn}(I) \;\big|\; I \text{ a regular ideal having a principal reduction}\big\} + 1.$$

Proof. Set $\sigma = \sup\{\mathrm{rn}(I) \mid I$ a regular ideal having a principal reduction$\} + 1$ and $d = \mathrm{d}_A(\overline{A})$. Take I any regular ideal of A having a principal reduction $J = (x)$, which is also regular. Then $I^{n+1} = xI^n$ for some $n \geqslant 0$. Set $H = x^{-1}I$. Then H is a fractional ideal of A with $H^{n+1} = H^n$. If $y \in I$, $(y/x)H^n \subset H^{n+1} = H^n$. Thus H^n is a faithful $A[y/x]$-module. By Proposition 6.1, y/x is integral over A. Thus $\mathrm{id}_A(y/x) \leqslant d$. By Lemma 6.4, $x(x,y)^{d-1} : y^d = A$ and $y^d \in x(x,y)^{d-1} \subseteq xI^{d-1}$. Therefore $I^{[d]} \subset xI^{d-1}$, where $I^{[d]}$ stands for the ideal generated by the dth powers of all elements of I. If $A \supset \mathbb{Q}$, then $I^{[d]} = I^d$ (see e.g. [4, A1, §8, n°2, p. 95]). Thus $\mathrm{rn}(I) \leqslant d-1$ and $\sigma \leqslant d$. Now take $x, y \in A$, with x regular, such that y/x is integral over A. By Lemma 6.4, $\mathrm{id}_A(y/x) = \mathrm{rn}(x, y) + 1 \leqslant \sigma$. Therefore $d \leqslant \sigma$. □

Remark 7.2. Let A be a noetherian ring, $A \supset \mathbb{Q}$. If I is a regular ideal of A having a principal reduction, by Theorem 5.3, $\mathrm{rt}(I) \leqslant \mathrm{rn}(I) + 1$ and $\mathrm{reg}(\mathcal{R}(I)) = \mathrm{rn}(I)$. Moreover, by Lemma 6.4, $\mathrm{id}_A(y/x) = \mathrm{rt}(x, y)$ for any $x, y \in A$, with x regular and such that y/x is integral over A. In other words, $\mathrm{d}_A(\overline{A})$ is less than or equal to the supremum of the relation type of two-generated regular ideals of A having principal reductions. Therefore, in Theorem 7.1, one can replace $\mathrm{rn}(I) + 1$ by either $\mathrm{reg}(\mathcal{R}(I)) + 1$ or else $\mathrm{rt}(I)$. In addition, one can take the supremum just over the set of two-generated regular ideals having principal reductions.

We state a particular version of Theorem 7.1 which will be used later. Note that here we do not need the hypothesis $A \supset \mathbb{Q}$.

Proposition 7.3. *Let $(A, \mathfrak{m})$ be a noetherian local ring with infinite residue field. Then*

$$\mathrm{d}_A(\overline{A}) = \sup\bigl\{\mathrm{rt}_{\mathfrak{m}}(I) \bigm| I \text{ a two-generated regular ideal of } A\bigr\}.$$

Proof. Set $\sigma = \sup\{\mathrm{rt}_{\mathfrak{m}}(I) \mid I$ a two-generated regular ideal of $A\}$ and $d = \mathrm{d}_A(\overline{A})$. Take I a two-generated regular ideal of A. Since A is noetherian local with infinite residue field, I has a minimal reduction J generated by as many elements as the analytic spread $l(I)$ of I (see [25] or [22]). If $l(I) = 1$, by Remark 3.1 and Theorems 5.3 and 7.1, $\mathrm{rt}_{\mathfrak{m}}(I) \leqslant \mathrm{rt}(I) \leqslant \mathrm{rn}(I) + 1 \leqslant d$. If $l(I) = 2$, then I is generated by two analytically independent elements x, y and the fiber cone of I, $\mathcal{R}_{\mathfrak{m}}(I) = \mathcal{F}_{\mathfrak{m}}(I) = \bigoplus_{n \geqslant 0} I^n/\mathfrak{m}I^n$ is isomorphic to a polynomial ring $(A/\mathfrak{m})[X, Y]$. Thus $\mathrm{rt}_{\mathfrak{m}}(I) = \mathrm{rt}(\mathcal{F}_{\mathfrak{m}}(I)) = 1 \leqslant d$ and $\sigma \leqslant d$. Now take $x, y \in A$, with x regular, such that y/x is integral over A. Set $\mathrm{id}_A(y/x) = n$. By Lemma 6.4, $xI^{n-2} : y^{n-1} \subsetneq xI^{n-1} : y^n = A$. By Proposition 3.2, $E_{\mathfrak{m}}(I)_n = A/\mathfrak{m}$ and $E_{\mathfrak{m}}(I)_{n+s} = 0$ for all $s \geqslant 1$. Thus $\mathrm{rt}_{\mathfrak{m}}(I) = n$. Therefore $\mathrm{id}_A(y/x) = n = \mathrm{rt}_{\mathfrak{m}}(I) \leqslant \sigma$. Thus $d \leqslant \sigma$. □

Remark 7.4. Clearly, Theorem 7.1 is no longer true for ideals having reductions generated by regular sequences of length $l \geqslant 2$. For instance, in the power series ring $A = k[\![x, y]\!]$ over a field k, the ideals $I_n = (x^n, y^n, x^{n-1}y)$ have reductions (x^n, y^n) with reduction number $n-1$ (see [21, Remark 5.8]). However, (x^n, y^n) does not verify condition (iii) of Theorem 5.3.

8. Uniform Artin–Rees numbers

We now can prove all the results related to Artin–Rees properties.

Theorem 8.1. *Let A be a noetherian ring with finite integral degree $\mathrm{d}_A(\overline{A}) = d$. Suppose that $A \supset \mathbb{Q}$. Let $N \subset M$ be two finitely generated A-modules. Let I be a regular ideal of A having a principal reduction that is generated by a d-sequence with respect to M/N. Then, for every integer $n \geqslant d$,*

$$I^nM \cap N = I^{n-d}\bigl(I^dM \cap N\bigr).$$

Proof. Since I is regular and has a principal reduction, by Theorem 7.1, $\mathrm{rn}(I) \leqslant d-1$. It is enough to prove that $s(N, M; I) \leqslant \mathrm{rn}(I)+1$. Let $J=(x)$ be a principal reduction of I, set $r=\mathrm{rn}(I)$ and take $k \geqslant 1$. Then $I^{r+k}M \cap N = x^k I^r M \cap N$. Since x is a d-sequence with respect to M/N, $x^k I^r M \cap N = x^{k-1}(x I^r M \cap N) \subseteq I^{k-1}(I^{r+1}M \cap N)$. Thus $s(N, M; I) \leqslant r+1$. □

Remark 8.2. By Proposition 3.3, $s(N, M; I) \leqslant \mathrm{rt}(I; M/N)$. If I has a principal reduction J generated by a d-sequence with respect to M/N, then J is also a principal reduction of I with respect to M/N and, by Theorem 5.3, $\mathrm{rt}(I; M/N) \leqslant \mathrm{rn}_J(I; M/N)+1 \leqslant \mathrm{rn}_J(I)+1$. Since I is a regular ideal having a principal reduction, by Theorem 7.1, $\mathrm{rn}(I) \leqslant d-1$. Therefore, $s(N, M; I) \leqslant d$, which also proves Theorem 8.1.

Our ideal-theoretic version of Theorem 8.1 is the following.

Theorem 8.3. *Let A be a noetherian ring, $A \supset \mathbb{Q}$. Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ has finite integral degree $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}})=d$. Let I be an ideal of A such that $IA/\mathfrak{a}$ has an $A/\mathfrak{a}$-regular principal reduction. Then, for every integer $n \geqslant d$,*

$$I^n \cap \mathfrak{a} = I^{n-d}\big(I^d \cap \mathfrak{a}\big).$$

Proof. By Proposition 3.3, $s(\mathfrak{a}, A; I) \leqslant \mathrm{rt}(I; A/\mathfrak{a})$. By Remark 3.1, $\mathrm{rt}(I; A/\mathfrak{a}) = \mathrm{rt}(IA/\mathfrak{a}) = \mathrm{rt}(IA/\mathfrak{a}; A/\mathfrak{a})$. Since $IA/\mathfrak{a}$ is $A/\mathfrak{a}$-regular and has principal reduction $JA/\mathfrak{a}$, by Theorem 5.3, $\mathrm{rt}(IA/\mathfrak{a}; A/\mathfrak{a}) \leqslant \mathrm{rn}_{JA/\mathfrak{a}}(IA/\mathfrak{a}; A/\mathfrak{a})+1 = \mathrm{rn}_{JA/\mathfrak{a}}(IA/\mathfrak{a})+1$. By Theorem 7.1, $\mathrm{rn}_{JA/\mathfrak{a}}(IA/\mathfrak{a}) \leqslant d-1$. So $s(\mathfrak{a}, A; I) \leqslant d$. □

As a corollary of Theorem 8.3 we obtain a particular version of the main result in [28].

Corollary 8.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring, $A \supset \mathbb{Q}$. Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ has finite integral degree $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}})=d$. Suppose that* $\dim(A/\mathfrak{a}) \leqslant 1$. *Then, for every integer $n \geqslant d$ and for every ideal I of A such that $IA/\mathfrak{a}$ is $A/\mathfrak{a}$-regular,*

$$I^n \cap \mathfrak{a} = I^{n-d}\big(I^d \cap \mathfrak{a}\big).$$

Proof. Since $\dim(A/\mathfrak{a}) \leqslant 1$, every ideal I of A is such that $IA/\mathfrak{a}$ has a principal reduction. Then apply Theorem 8.3. □

Remark that by a result of Krull, if $(R, \mathfrak{n})$ is a noetherian local nonreduced ring such that $\mathfrak{n}$ contains a regular element, then the integral closure $\overline{R}$ is not a finite R-module (see e.g. [23, §33]). In particular, in Theorem 8.3 and in Corollary 8.4 (as well as in Theorem 8.6), setting $R=A/\mathfrak{a}$, if $A/\mathfrak{a}$ has a finite integral closure and $\mathfrak{m}/\mathfrak{a}$ has a regular element, one deduces that $\mathfrak{a}$ is forced to be a radical ideal.

The next example, taken from Eisenbud and Hochster in [14], shows that if the integral degree is not finite, then the conclusion of Theorem 8.3 may be false.

Example 8.5. There exist A, a two-dimensional noetherian domain, $\mathfrak{a}$, a prime ideal of A, and $\{I_n\}_n$, a family of two-generated ideals of A such that $I_nA/\mathfrak{a}$ has an $A/\mathfrak{a}$-regular principal reduction, but, for every integer $n \geqslant 1$,

$$I_n^n \cap \mathfrak{a} \supsetneq I_n\big(I_n^{n-1} \cap \mathfrak{a}\big).$$

Proof. Let k be an algebraically closed field and $\{X_n\}$, $\{Y_n\}$ two countable families of indeterminates. Set $f_n = X_n^n - Y_n^{n+1}$ and I_n the ideal in $T_n = k[X_1, Y_1, \dots, X_n, Y_n]$ generated by $f_2 - f_1, \dots, f_n - f_1$. Set $S_n = T_n/I_n$ and $U_n = S_n - \bigcup_{i=1}^n (X_i, Y_i)S_n$. U_n is a multiplicatively closed subset of S_n. Set $A_n = U_n^{-1}S_n$, $A = \varinjlim A_n$ and x_n, y_n and f the images of X_n, Y_n and f_n in A. Then A is a two-dimensional noetherian regular factorial ring whose maximal ideals $I_n = (x_n, y_n)$ form a countable set. Their intersection $\bigcap_n I_n$ is a prime principal ideal $\mathfrak{a} = (f)$ whose generator f is in I_n^n. Then $I_n A/\mathfrak{a}$ is $A/\mathfrak{a}$-regular and $y_n A/\mathfrak{a}$ is a principal reduction of $I_n A/\mathfrak{a}$. Moreover,

$$I_n\big(I_n^{n-1} \cap \mathfrak{a}\big) = I_n\mathfrak{a} \subsetneq \mathfrak{a} = I_n^n \cap \mathfrak{a}.$$

In particular, by Theorem 8.3, $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}}) = \infty$. □

We now prove that there exists a uniform Artin–Rees modulo $\mathfrak{m}$ number for the set of two-generated regular ideals. Here, we do not need $A \supset \mathbb{Q}$.

Theorem 8.6. *Let $(A, \mathfrak{m})$ be a noetherian local ring with infinite residue field. Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ has finite integral degree $\mathrm{d}_{A/\mathfrak{a}}(\overline{A/\mathfrak{a}}) = d$. Let I be a two-generated ideal of A such that $IA/\mathfrak{a}$ is $A/\mathfrak{a}$-regular. Then, for every $n \geqslant d$,*

$$I^n \cap \mathfrak{a} = I^{n-d}\big(I^d \cap \mathfrak{a}\big) + \mathfrak{m}I^n \cap \mathfrak{a}.$$

Proof. Let I be a two-generated ideal of A such that $IA/\mathfrak{a}$ is $A/\mathfrak{a}$-regular. By Proposition 3.3, $s_{\mathfrak{m}}(\mathfrak{a}, A; I) \leqslant \mathrm{rt}_{\mathfrak{m}}(I; A/\mathfrak{a})$. By Remark 3.1, $\mathrm{rt}_{\mathfrak{m}}(I; A/\mathfrak{a}) = \mathrm{rt}_{\mathfrak{m}/\mathfrak{a}}(IA/\mathfrak{a})$, which is d or less by Proposition 7.3. □

The next example, taken from Wang in [38], shows that even this weaker uniform Artin–Rees property of Theorem 8.6 is not true anymore for the set of three-generated ideals. It also shows that if in Theorem 8.3 one changes the set of ideals having principal reductions for the set of ideals having reductions generated by regular sequences of length two, then there may not exist a uniform Artin–Rees (modulo $\mathfrak{m}$) number.

Example 8.7. There exist $(A, \mathfrak{m})$, a three-dimensional noetherian local ring with infinite residue field, $\mathfrak{a}$, a prime ideal of A such that $A/\mathfrak{a}$ has finite integral closure $\overline{A/\mathfrak{a}}$, and $\{I_n\}_n$, a family of three-generated ideals of A such that $I_n A/\mathfrak{a}$ is $A/\mathfrak{a}$-regular, but, for every $n \geqslant 1$,

$$I_n^n \cap \mathfrak{a} \supsetneq I_n\big(I_n^{n-1} \cap \mathfrak{a}\big) + \mathfrak{m}I_n^n \cap \mathfrak{a}.$$

Proof. Take $(A, \mathfrak{m})$, a three-dimensional regular local ring with infinite residue field, $\mathfrak{m} = (x, y, z)$, the maximal ideal generated by a regular system of parameters x, y, z, and $\mathfrak{a} = (z)$. Let $I_n = (x^n, y^n, x^{n-1}y + z^n)$. Since $x^n, y^n, x^{n-1}y + z^n$ is a regular sequence of A, the relation type of I_n is $\mathrm{rt}(I_n) = 1$. It is not difficult to prove that the relation type of $I_n A/\mathfrak{a}$ and that the relation type of its fiber cone are given by $\mathrm{rt}(I; A/\mathfrak{a}) = \mathrm{rt}_{\mathfrak{m}}(I; A/\mathfrak{a}) = n$. Then, by Proposition 3.3, $s_{\mathfrak{m}}(\mathfrak{a}, A; I_n) = \mathrm{rt}_{\mathfrak{m}}(I_n; A/\mathfrak{a}) = n$. Remark that $(x^n, y^n)A/\mathfrak{a}$ is a reduction of $I_n A/\mathfrak{a}$ generated by a regular sequence of length two. □

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